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THE MEASUREMENT OF OPPORTUNITY INEQUALITY; A CARDINALITY-BASED APPROACH

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Abstract

We consider the problem of ranking distributions of opportunity sets on the basis of equality. First, conditional on agents' preferences over individual opportunity sets, we formulate the analogues of the notions of the Lorenz partial ordering, equalizing Dalton transfers, and inequality averse social welfare functionals - concepts which play a central role in the literature on income inequality. For the particular case in which agents rank opportunity sets on the basis of their cardinalities, we establish an analogue of the fundamental theorem of inequality measurement: one distribution Lorenz dominates another if and only if the former can be obtained from the latter by a finite sequence of equalizing transfers, and if and only if the former is ranked higher than the latter by all inequality averse social welfare functionals. In addition, we characterize the smallest monotonic and transitive extension of the cardinality-based Lorenz inequality ordering.

Keywords: Opportunity Inequality, Equalizing Transfers, Lorenz Domination.

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1 Introduction

It is well known that the distribution of income affords a useful, but incomplete, means of evaluating social inequities. Indeed, Amartya Sen is very clear on this point:[1]

> "An important and frequently encountered problem arises from concentrating on inequality of *incomes* as the primary focus of attention in the analysis of inequality. The extent of real inequality of opportunities that people face cannot be readily deduced from the magnitude of inequality of *incomes*, since what we can or cannot do, can or cannot achieve, do not depend just on our incomes but also on the variety of physical and social characteristics that affect our lives and make us what we are."

Consequently, various authors have suggested that we expand or shift our focus to other indicators of well-being. For example, Rawls (1971) has argued that we should concentrate on the distribution of *primary goods* such as rights, liberties and social determinants of self-respect. Dworkin (1981) and Roemer (1986) have suggested that we focus on the distribution of *resources*. And Sen (1980, 1992) has proposed the distribution of *freedoms* and *capabilities* for the same purpose.

In this paper, we abstract from the issue of the appropriate equalisandum. Rather, we refer to the collective determinants of well-being as opportunities, and we consider the following question: Assuming individuals have preferences over their alternative sets of opportunities, how can we compare different distributions of opportunity sets? In particular, which distribution affords the greater degree of equality?

In the unidimensional case of income inequality, this question has been studied extensively. From the seminal contributions of Kolm (1969), Atkinson (1970), Dasgupta et al (1973), and Rotschild and Stiglitz (1973), a minimal consensus has emerged. A ranking of income distributions should be consistent with the Lorenz partial ordering, for this ordering is supported by all inequality averse welfare functionals, and it is the only ordering which is consistent with our intuition that rank-preserving transfers from rich to poor should decrease inequality.

We wish to establish analogous results for the multidimensional problem of ranking distributions of opportunity sets. First, to distill the argument to its essential form, we concentrate on the two-agent case.[2] We then formulate an analog of equalizing (Dalton) transfers relative to a given *ordering of individual opportunity sets.* This

[1] Sen (1992), p. 28. (Emphasis in the original.)

[2] Two remarks on this concentration are in order. First. an 'agent' may also be thought of as a 'class' rather than an individual. For example, if A and B are subsets of a given universal set of opportunities, (A, B) may represent an opportunity distribution of a certain society where A is the opportunity set of women in the population and B is that of men. Second, we stress that the measurement of opportunity inequality is a far more difficult matter than that of income inequality,

lets us define the corresponding concepts of Lorenz partial ordering and inequality averse social welfare functionals (both of which are again defined relative to a given opportunity set ordering).

To demonstrate the usefulness of this approach, we consider the example in which agents rank opportunity sets on the basis of their cardinalities (or in other words, where they attach the same welfare weight to each and every opportunity).[3] In this particular case, we establish an analogue of the fundamental theorem of income inequality: one distribution of opportunity sets Lorenz dominates another if and only if the former can be obtained from the latter by a finite sequence of equalizing transfers, and if and only if the former is ranked higher than the latter by all inequality averse social welfare functionals.

As in the standard theory, our generalization of the Lorenz ordering is well-defined only when the set of aggregate opportunities is fixed. In the case of income inequality, this is easily accommodated by normalizing total income and focusing on *relative* inequality. Here, however, no such normalization is available, and we must study appropriate extensions of the Lorenz ordering directly. We do this by characterizing the smallest monotonic and transitive extension of the induced Lorenz inequality ordering.

The paper is organized as follows. In the next section, we review the relevant results of the theory of income inequality measurement and state a fundamental theorem. In Section 3, we formulate the principle of rank preserving equalizations conditional on agent's ranking of opportunity sets. In Sections 4 and 5, we consider the case where agents rank the opportunity sets on the basis of their cardinalities, and within this framework, develop the analysis of Lorenz partial ordering and inequality averse social welfare functionals. The results obtained in these sections yield an analogue of the fundamental theorem of inequality measurement which is discussed in Section 6. Section 7 is then devoted to a study of monotonic and transitive extensions of the induced Lorenz ordering. The paper concludes with identifying an agenda for future research.

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and that the problem is by no means trivial when we restrict our attention to two-agent situations. Indeed, since a theory of opportunity inequality is not readily available, it only makes sense to study the problem in its simplest possible form and then study possible extensions.

[3]We do not, of course, suggest that the cardinality ordering is the best way of ranking opportunity sets. The advantage of this ordering lies in its analytical simplicity allowing us to concentrate on the inherent difficulties of opportunity inequality measurement in the simplest possible framework. For an extensive treatment of the cardinality ordering, we refer the reader to Pattanaik and Xu (1990) and Puppe (1995a) where it is axiomatically studied as an ordering which focuses *only* on the degree of freedom of choice an opportunity set provides to an agent. See, Sen (1991) and Klemisch-Ahlert (1993) for critical discussions.

2 Preliminaries

2.1 The Fundamental Theorem of Inequality Economics

This subsection contains a brief review of some of the well-known concepts of the theory of inequality measurement. For exhaustive reviews of the topic, we refer the reader to Foster (1985) and Lambert (1993).

Let $\mathbb{R}^+_H := \{t \in \mathbb{R}^H_+ : t_1 \leq t_2 \leq ... \leq t_H\}$, $H \geq 2$. The elements of $\mathbb{R}^+_H$ are interpreted as the illfare ordered distributions of income.

The **ordinary strict Lorenz partial ordering** $>_L \subset \mathbb{R}^H_+ \times \mathbb{R}^H_+$ is defined as $x >_L y$ if and only if

$$\sum_{i=1}^{s} x_{(i)} \geq \sum_{i=1}^{s} y_{(i)} \text{ for all } s \in \{1,2,...,H-1\} \text{ and } \sum_{i=1}^{H} x_i = \sum_{i=1}^{H} y_i,$$

and

$$\sum_{i=1}^{s} x_{(i)} > \sum_{i=1}^{s} y_{(i)} \text{ for some } s \in \{1,2,...,H\},$$

where, for any $t \in \mathbb{R}^H_+$, $t_{(\,)} = (t_{(1)}, t_{(2)}, ..., t_{(H)})$ is a permutation of t such that $t_{(\,)} \in \mathbb{R}^+_H$. The **ordinary Lorenz ordering** is then defined as

$$\geq_L := >_L \cup \{(x,x) : x \in \mathbb{R}^H_+\}.$$

A function $W : \mathbb{R}^H_+ \rightarrow \mathbb{R}$ is said to be **S-concave (Schur-concave)** if, for any $x, y \in \mathbb{R}^H_+$,

$$x \geq_L y \text{ implies } W(x) \geq W(y).$$

The usual interpretation of $W(.)$ is to treat it as an individualistic social welfare function where all the individuals have the same preferences for income. For instance, $W : x \longmapsto \sum_{i=1}^{H} u(x_i)$, where $u(.)$ is any increasing function on $\mathbb{R}_+$ is the classical *utilitarian* social welfare function. If we assume that $u(.)$ is concave on $\mathbb{R}_+$, then this welfare function is necessarily S-concave (cf. Atkinson (1970) and Dasgupta et al. (1973)).

The **equalizing (Dalton) transfer operator** $T : \mathbb{R}^+_H \rightarrow 2^{\mathbb{R}^+_H}$ is defined as $x \in T(y)$ if and only if there exists an $\epsilon \geq 0$ and $i, j \in \{1, ..., H\}$, $i < j$, such that

$$x_i = y_i + \epsilon, \quad x_j = y_j - \epsilon \quad \text{and} \quad x_k = y_k \text{ for all } k \in \{1,2,...,H\} \backslash \{i,j\}.^4$$

If $x \in T(y)$, we understand that either x is equal to y or it can be obtained from y by transfering a certain amount of money from a richer individual to a poorer individual such that the relative positions of these individuals remain unchanged. One of the

[4]Let A be any set. The power set of A is denoted by 2^A, that is, $2^A := \{B : B \subseteq A\}$.

most basic premises of the theory of income inequality measurement is to regard x as *less unequal than* y whenever $x \in T(y)$. A natural extension of this premise is to regard x as less unequal than y if, for some $n \in \mathbb{N}$, there exist $x_1, x_2, ..., x_n \in \mathbb{R}_H^+$ such that

$$x_1 \in T(y),\ x_2 \in T(x_1),\ ...,\ x_n \in T(x_{n-1}) \ \text{ and } \ x \in T(x_n).$$

We write, for any $n = 2, 3, ...,$

$$T^n(y) = T(\{z \in \mathbb{R}_H^+ : z \in T^{n-1}(y)\}),^5$$

and define the **ordinary Dalton ordering** $\geq_D \subset \mathbb{R}_+^H \times \mathbb{R}_+^H$ as

$$x \geq_D y \quad \text{if and only if} \quad x_{(\ )} \in T^n(y_{(\ )}) \text{ for some } n \geq 1.$$

In the terminology of Fields and Fei (1978), if $x \geq_D y$, we understand that either x is equal to y or it can be obtained from y by means of a finite sequence of *rank preserving equalizations.* The main idea goes back to Dalton who noted that "... in comparing two distributions, in which both the total income and the number of income receivers are the same, we may see that one might be able to be evolved from the other by means of a series of transfers of this kind. In such a case we could say with certainty that the inequality of one was less than that of the other" (Dalton (1920), p. 351).

The following well-known theorem which is due to Hardy et al. (1934) brings all of these key concepts together in a unifying manner. It is in this sense that we refer to this result as the *fundamental theorem of inequality economics.*

THEOREM 2.1. **(Hardy-Littlewood-Polya)** *Let $x, y \in \mathbb{R}_+^H$, $H \geq 2$. The following statements are equivalent:*

(i) $x \geq_L y$.
(ii) $x \geq_D y$.
(iii) $W(x) \geq W(y)$ *for all S-concave* $W : \mathbb{R}_+^H \to \mathbb{R}$.

For extensive discussions and alternative proofs of this theorem, we refer the reader to Dasgupta et al. (1973), Rotschild and Stiglitz (1973), Fields and Fei (1978), Foster (1985), Marshall and Olkin (1979), Arnold (1987) and Pecaric et al. (1992).

The main purpose of this paper is to study a possible analogue of this famous theorem in the fundamentally different framework of opportunity distributions to which we now turn.

[5] Let $f(.)$ be a correspondence on set A and $m \geq 1$. We define $f(B) := \bigcup_{b \in B} f(b)$ for all $B \subseteq A$.

2.2 The Opportunity Distributions in a Dual Economy

Let L be a universal set of opportunities. For instance, one might think of the elements of L as the *freedoms* and *functionings* a lá Sen but, of course, other interpretations ((such as rights, basic liberties, choice of education and occupation, and so on) are possible. We shall maintain throughout this paper that each element of L is in some sense *desirable.*

Define

$$\mathbf{\Omega} := \{\mathbf{O} = (O_1, O_2) \in 2^L \times 2^L : O_1 \cup O_2 \neq \emptyset \text{ and } \max\{\#O_1, \#O_2\} < \infty\}.$$

We interpret $\mathbf{\Omega}$ as the set of all possible opportunity distributions in a two-agent society. (If, for instance, L is interpreted as the set of all functionings with respect to a given aspect of well-being, then the members of $\mathbf{\Omega}$ can be thought of as the induced distributions of capability sets (cf. Sen (1992), p. 39-42).)

We want to study an "*equality*" partial ordering on $\mathbf{\Omega}$ which we shall denote by $\succcurlyeq$. For $\mathbf{O}, \mathbf{U} \in \mathbf{\Omega}$, we interpret $\mathbf{O} \succcurlyeq \mathbf{U}$ as "$\mathbf{O}$ *is at least as equal as* $\mathbf{U}$".

The following two properties are rather unexceptionable conditions to posit on $\succcurlyeq$.

AXIOM A. $\succcurlyeq$ *is a preorder on* $\mathbf{\Omega}$*; that is, it is a reflexive and transitive binary relation on* $\mathbf{\Omega}$.

AXIOM B. (Anonymity) *For any* $\mathbf{O} \in \mathbf{\Omega}$, $\mathbf{O} \sim (O_2, O_1)$.

3 A Principle of Equalizing Transfers

In this section we aim to find an analog of the "*principle of equalizing (Dalton) transfers*" of the unidimensional inequality measurement theory. To this end, one needs to uncover the idea of an equalizing operator in the present context. This notion, of course, presupposes that we can decide from which agent a transfer should be made so as to decrease inequality. In other words, given an opportunity distribution $\mathbf{O} \in \mathbf{\Omega}$, we need to know who actually is the "richer" agent in this distribution to define and apply an analog of the principle of equalizing transfers. This leads us to consult the preferences of the agents over L.

We assume that all agents have the same preferences over L, and these are represented by a given complete preorder $\mathbf{R} \subset L \times L$. (The strict form of $\mathbf{R}$ is denoted by $\mathbf{P}$.) Let $\mathcal{R}_0(\mathbf{R})$ be the class of all preorders (quasi-orders) on $2^L \times 2^L$ which extends $\mathbf{R}$, that is

$$\mathcal{R}_0(\mathbf{R}) := \{\succeq \subset 2^L \times 2^L : \succeq \text{ is reflexive, transitive and } \forall x, y \in L : [x\mathbf{R}y \implies \{x\} \succeq \{y\}]\}.$$

(The asymmetric and the symmetric components of $\succeq$ is denoted by $\succ$ and $\simeq$.) Many interesting examples of orderings that are in this class are axiomatically characterized in the literature (see, for instance, Barbera and Pattanaik (1984), Barbera et al. (1984), Pattanaik and Peleg (1984), Nitzan and Pattanaik (1984), Bossert (1989), Pattanaik and Xu (1990), Klemisch-Ahlert (1993), Bossert et al. (1994) and Puppe (1995a,b)). Since it seems quite reasonable to declare an agent "richer" (or, in a more advantageous situation) if every option that is available to the other party is available to it, the following refinement of $\mathcal{R}_0(\mathbf{R})$ is of interest:

$$\mathcal{R}(\mathbf{R}) := \{\succeq \in \mathcal{R}_0(\mathbf{R}) : \forall A, B \in 2^L : [A \supseteq B \Longrightarrow A \succeq B]\}.$$

That is, $\mathcal{R}(\mathbf{R})$ is a subclass of $\mathcal{R}_0(\mathbf{R})$ such that all of its members are extensions of the subsethood relation $\supseteq$.[6] For any given $\succeq \in \mathcal{R}(\mathbf{R})$, we shall let

$$\Lambda(\succeq) := \{\mathbf{O} \in \mathbf{\Omega} : (O_1, O_2) \notin \succ\}.$$

If $\succeq$ is complete, then $\Lambda(\succeq)$ is the class of all opportunity distributions which are illfare ordered.

We can now define the **equalizing transfer operator** $T_{\succeq} : \Lambda(\succeq) \to 2^{\Lambda(\succeq)}$ with respect to any given complete $\succeq \in \mathcal{R}(\mathbf{R})$ as follows:

$$T_{\succeq}(\mathbf{U}) := \begin{cases} \{(U_1 \cup \{x\}, U_2) \in \Lambda(\succeq) : x \in U_2 \backslash U_1\}, & \text{if } U_2 \succ U_1 \\ \{(U_1 \cup \{x\}, U_2 \cup \{y\}) \in \Lambda(\succeq) : (x, y) \in U_2 \backslash U_1 \times U_1 \backslash U_2\}, & \text{if } U_2 \simeq U_1 \end{cases}.$$

Notice that since *Range* $T_{\succeq} = 2^{\Lambda(\succeq)}$, the transfers embodied in $T_{\succeq}(.)$ does not alter the intra-group ranking.[7]

Let $\mathbf{U} \in \mathbf{\Omega}$ and suppose that $U_2 \succ U_1$. Therefore, from the perspective of the extension $\succeq$ which all agents are assumed to agree upon, the second agent is 'richer' than the first one. Consequently, an equalizing transfer adds to the opportunity set of the first class an opportunity $x \in U_2 \backslash U_1$. If, on the other hand, $U_1 \simeq U_2$, then both agents are equally rich, and to make the distribution more equal would require that we make the sets U_1 and U_2 more "similar". Thus, an equalizing transfer adds to the opportunity set of the first agent an opportunity $x \in U_2 \backslash U_1$ and that of the second agent an opportunity $y \in U_1 \backslash U_2$. If $U_1 = U_2$, an equalization cannot take place, for the inital distribution is already perfectly egalitarian. The following example gives a more concrete illustration.

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[6] As innocent as it may seem, this is a considerable refinement. A major part of the literature concerning the extensions of an order on a set to the power set uses the Gärdenfors principle as a basic axiom (see Kannai and Peleg (1984)). A much weaker axiom than the Gärdenfors principle is an axiom introduced in Barbera (1977): $\forall x, y \in L : [x\mathbf{R}y \Rightarrow \{x\} \succ \{x, y\}$ and $\{x, y\} \succ \{y\}]$ (see also Barbera and Pattanaik (1984)). It is obvious that no relation in $\mathcal{R}(\mathbf{R})$ can satisfy this condition. See also Gaertner (1990) for a criticism of the noted refinement from an alternative perspective.

[7] In the definition of $T_{\succeq}(.)$, we adopt the convention that $T_{\succeq}(\mathbf{U}) = \{\mathbf{U}\}$, if $U_1 = U_2$.

EXAMPLE 3.1. The **cardinality ordering** $\succeq_{\#} \in \mathcal{R}(\mathbf{R})$ is defined as

$$A \succeq_{\#} B \Longleftrightarrow \#A \geq \#B$$

(cf. Pattanaik and Xu (1990)). On the other hand, the **cardinality-first lexicographic ordering** $\succeq_c \in \mathcal{R}(\mathbf{R})$ is defined as

$$A \succeq_c B \Leftrightarrow [\#A > \#B] \text{ or } [\#A = \#B \text{ and } [\forall (x,y) \in \max(A) \times \max(B) : [x\mathbf{R}y]]$$

where, for any $S \subseteq L$, $\max(S)$ denotes the **R**-*greatest* elements of S (cf. Bossert et al. (1994)). Let $a, b, c \in L$ such that $a \, \mathbf{P} \, b \, \mathbf{R} \, c$, and consider the opportunity distribution $\mathbf{U} = (\{a\}, \{b, c\})$. In this case,

$$T_{\succeq_{\#}}(\mathbf{U}) = \{(\{a,b\}, \{b,c\}), (\{a,c\}, \{b,c\})\}$$

and

$$T_{\succeq_{\#}}(\mathbf{O}) = \{(a,b,c\}, \{a,b,c\})\} \text{ for all } \mathbf{O} \in T_{\succeq_{\#}}(\mathbf{U}),$$

whereas

$$T_{\succeq_c}(\mathbf{U}) = \emptyset.^{8} \square$$

We now formulate the principle of equalizing transfers in the present context as an axiom. Since the equalizing transfer operator defined above is parametric over the extension chosen from $\mathcal{R}(\mathbf{R})$, so will be such an axiom:

AXIOM C($\succeq$). **(Principle of $\succeq$-Equalization)** *For any* $\mathbf{U} \in \Lambda(\succeq)$,

$$\mathbf{O} \succcurlyeq \mathbf{U} \quad \textit{for all } \mathbf{O} \in T_{\succeq}(\mathbf{U}).$$

For any $\mathbf{U} \in \Lambda(\succeq)$ and $n = 2, 3, ...$, let us write

$$T_{\succeq}^{n}(\mathbf{U}) := T_{\succeq}(\{\mathbf{O} : \mathbf{O} \in T_{\succeq}^{n-1}(\mathbf{U})\}).$$

Moreover, let us define, for any $\mathbf{O} \in \Omega$, $\mathbf{O}_{(\,)} := (O_{(1)}, O_{(2)})$ as the permutation of $\mathbf{O}$ such that $\mathbf{O}_{(\,)} \in \Lambda(\succeq)$.

We are now ready to state the following

[8] $T_{\succeq_c}(\mathbf{U}) = \emptyset$ means that, given the data of the example, we can never obtain a distribution from $\mathbf{U}$ by an equalizing transfer. For instance, the perfectly egalitarian opportunity distribution $(\{a,b,c\}, \{a,b,c\})$ can never be reached from $\mathbf{U}$ by means of any number of equalizing transfers. This is, of course, a very troubling observation which poses extremely difficult questions with regard to a potential theory of opportunity inequality measurement.

DEFINITION 3.2. *The* **Dalton ordering induced by** $\succeq \in \mathcal{R}(\mathbf{R})$ *is defined as the reflexive relation on* $\mathbf{\Omega}$ *such that, for any* $\mathbf{O}, \mathbf{U} \in \mathbf{\Omega}$, $\mathbf{O} \neq \mathbf{U}$,

$$\mathbf{O} \succcurlyeq_D^{\succeq} \mathbf{U} \text{ if and only if } \mathbf{O}_{(\,)} \in T_{\succeq}^{n}(\mathbf{U}_{(\,)}) \text{ for some } n \geq 1.$$

The following result is immediate:

PROPOSITION 3.3. *Let* $\succeq \in \mathcal{R}(\mathbf{R})$. $\succcurlyeq_D^{\succeq}$ *is the smallest binary relation on* $\mathbf{\Omega}$ *which satisfies Axioms A, B and* $\mathcal{C}(\succeq)$.

In view of Definition 3.2 and the equivalence of the statements (i) and (ii) in Theorem 2.1, there is a natural way of defining an analogue of the celebrated Lorenz ordering in the present context: the **Lorenz ordering induced by** $\succeq$ is a binary relation over $\mathbf{\Omega}$ which is not defined recursively and yet is equal to $\succcurlyeq_D^{\succeq}$. The following section will elaborate on this definition in a more specific framework.

4 The Lorenz Ordering Induced by $\succeq_\#$

In what follows we shall examine the implications of using the simplest member of $\mathcal{R}(\mathbf{R})$, namely the *cardinality ordering* (defined in Example 3.1) in making inequality comparisons of opportunity distributions.[9] Put more concretely, we shall study the *Lorenz ordering induced by the cardinality ordering.*

NOTATION. The Dalton partial ordering induced by $\succeq_\#$ is denoted by $\succcurlyeq_D^{\#}$. Furthermore, from now on, for any $\mathbf{O} \in \mathbf{\Omega}$, $\mathbf{O}_{(\,)} := (O_{(1)}, O_{(2)})$ will stand for the permutation of $\mathbf{O}$ such that $\#O_{(1)} \leq \#O_{(2)}$. (Thus, for any $\mathbf{O} \in \mathbf{\Omega}$, $\mathbf{O}_{(\,)} \in \Lambda(\succeq_\#)$.)

DEFINITION 4.1. *The* **(strict) Lorenz ordering induced by** $\succeq_\#$, $\succ_{\mathbf{L}}^{\#} \subset \mathbf{\Omega} \times \mathbf{\Omega}$, *is defined as follows:* $\mathbf{O} \succ_{\mathbf{L}}^{\#} \mathbf{U}$ *if and only if* $\mathbf{O} \neq \mathbf{U}$, $U_1 \cup U_2 = O_1 \cup O_2$ *and either*

$$\#O_{(1)} = \#O_{(2)} \text{ and } U_{(i)} \subseteq O_{(i)},\ i = 1, 2,$$

or

$$\#O_{(1)} < \#O_{(2)},\ U_{(1)} \subset O_{(1)} \text{ and } U_{(2)} = O_{(2)}.$$

The weak form of $\succ_{\mathbf{L}}^{\#}$ *is defined as*

$$\succcurlyeq_{\mathbf{L}}^{\#} = \succ_{\mathbf{L}}^{\#} \cup \{(\mathbf{O}, \mathbf{O}) : \mathbf{O} \in \mathbf{\Omega}\}.$$

[9] See Sen (1991) and Klemisch-Ahlert (1993) for critical discussions about the cardinality ordering. In our opinion, the major shortcoming of this ordering is due to the fact that it is essentially independent of the underlying preferences of the agents, i.e. $\mathbf{R}$. Indeed, as shown by Pattanaik and Xu (1990), $\succeq_\#$ is the only member of $\mathcal{R}(\mathbf{R})$ which satisfies a weak independence condition and which declares all the singleton sets indifferent. (See also Puppe (1995a).)

REMARK. We stress that $\succ_{\mathbf{L}}^{\#}$ is not the ordinary strict Lorenz ordering simply applied to the cardinality distributions. To illustrate, let $\#L \geq 3$ and consider $\mathbf{U} = (\{a\}, \{a,b,c\})$ and $\mathbf{O} = (\{a,b\}, \{b,c\})$. Clearly, $(\#O_1, \#O_2) = (2,2) >_L (1,3) = (\#U_1, \#U_2)$ whereas $(\mathbf{O}, \mathbf{U}), (\mathbf{U}, \mathbf{O}) \notin \succ_{\mathbf{L}}^{\#}$. □

Why do we trust the ordinary Lorenz ordering when making income inequality comparisons? Because, by Theorem 2.1, we know that an income distribution Lorenz dominates the other if, and only if, the former distribution can be obtained from the latter by means of a finite sequence of equalizing (Dalton) transfers. The following theorem justifies Definition 4.1 in an analogous way.

THEOREM 4.2. *The Dalton ordering induced by $\succeq_{\#}$ is equivalent to $\succcurlyeq_{\mathbf{L}}^{\#}$;* that is $\succcurlyeq_{\mathbf{L}}^{\#} = \succcurlyeq_{D}^{\#}$.

Proof of Theorem 4.2. That $\succcurlyeq_{D}^{\#} \subseteq \succcurlyeq_{\mathbf{L}}^{\#}$ can easily be verified. To see the converse containment, let $\mathbf{O} \succ_{\mathbf{L}}^{\#} \mathbf{U}$ and assume that $\mathbf{O}_{(\,)} = \mathbf{O}$ and $\mathbf{U}_{(\,)} = \mathbf{U}$ without loss of generality. We must then have $U_1 \cup U_2 = O_1 \cup O_2$, and one of the following mutually exclusive cases must be true:

$$\#O_1 = \#O_2,\ \#U_1 < \#U_2 \text{ and } U_i \subseteq O_i,\ i = 1,2,$$

$$\#O_1 < \#O_2,\ U_1 \subset O_1 \text{ and } U_2 = O_2,$$

$$\#O_1 = \#O_2,\ \#U_1 = \#U_2 \text{ and } U_i \subseteq O_i,\ i = 1,2.$$

We only examine the first case, the others being entirely similar. The hypotheses $U_1 \cup U_2 = O_1 \cup O_2$ and $U_i \subseteq O_i$, $i = 1,2$, together imply that

$$O_1 = U_1 \cup \{x_1, ..., x_{n-m}\} \cup \{y_1^2, ..., y_m^2\} \text{ for some } m \geq 0,$$

and

$$O_2 = U_2 \cup \{y_1^1, ..., y_m^1\} \text{ for some } m \geq 0,$$

where $n - m = \#U_2 - \#U_1 > 0$, $x_j \in U_2$, $j = 1, ..., n-m$, and $y_j^i \in U_i$, $i = 1,2$, $j = 1, ..., m$, with the convention that $\{y_1^i, y_0^i\} = \emptyset$, $i = 1,2$. But then, since $U_2 \succ_{\#} U_1$,

$$(V_1, U_2) := (((...(U_1 \cup \{x_1\}) \cup ...) \cup \{x_{n-m}\}), U_2) \in T_{\succeq_{\#}}^{n-m}(\mathbf{U}),$$

and since $V_1 \simeq_{\#} U_2$,

$$\mathbf{O} = (((...(V_1 \cup \{y_1^2\}) \cup ...) \cup \{y_m^2\}), ((...(U_2 \cup \{y_1^1\}) \cup ...) \cup \{y_m^1\})) \in T_{\succeq_{\#}}^{m}(V_1, U_2)$$

so that $\mathbf{O} \in T_{\succeq_{\#}}^{m}(T_{\succeq_{\#}}^{n-m}(\mathbf{U})) = T_{\succeq_{\#}}^{n}(\mathbf{U})$. Therefore, $\mathbf{O} \succcurlyeq_{D}^{\#} \mathbf{U}$. Since, for any $\mathbf{O} \in \mathbf{\Omega}$, $\mathbf{O} \succcurlyeq_{D}^{\#} \mathbf{O}$ by Definition 3.2, we conclude that $\succcurlyeq_{\mathbf{L}}^{\#} \subseteq \succcurlyeq_{D}^{\#}$, and the proof is complete. □

COROLLARY 4.3. *$\succcurlyeq_{\mathbf{L}}^{\#}$ is the smallest binary relation on Ω which satisfies Axioms A, B and $C(\succeq_{\#})$.*

REMARK. Kranich (1994) characterizes the **cardinality difference ordering** $\succcurlyeq_{CD} \subset \Omega \times \Omega$:

$$\mathbf{O} \succcurlyeq_{CD} \mathbf{U} \text{ if and only if } |\#O_1 - \#O_2| \leq |\#U_1 - \#U_2|.$$

Let the asymmetric factor of $\succcurlyeq_{CD}$ be denoted by $\succ_{CD}$. Since this ordering is obviously cardinality based, one might justly wonder if $\succ_{CD}$ is $\succ_{\mathbf{L}}^{\#}$*-consistent,* i.e. if $\succ_{CD} \supseteq \succ_{\mathbf{L}}^{\#}$ or not. Let $\#L \geq 3$, $\mathbf{U} = (\{a,b\},\{b,c\})$ and $\mathbf{O} = (\{a,b,c\},\{a,b,c\})$. Clearly, $\mathbf{O} \succ_{\mathbf{L}}^{\#} \mathbf{U}$ and *not* $\mathbf{O} \succ_{CD} \mathbf{U}$. Consequently, we conclude that $\succ_{CD}$ is not $\succ_{\mathbf{L}}^{\#}$-consistent. □

For any binary relation $\succcurlyeq$ on Ω, we denote the set of $\succcurlyeq$*-greatest,* $\succcurlyeq$*-maximal,* $\succcurlyeq$*-least,* and $\succcurlyeq$*-minimal* elements of Ω by $G(\succcurlyeq)$, $g(\succcurlyeq)$, $L(\succcurlyeq)$ and $\ell(\succcurlyeq)$, respectively.[10] Our next proposition computes the greatest, maximal, least and minimal elements of Ω with respect to the Lorenz ordering induced by $\succeq_{\#}$.

PROPOSITION 4.4. *Let $\#L \geq 2$. Then,*
(i) $G(\succcurlyeq_{\mathbf{L}}^{\#}) = \emptyset$
(ii) $g(\succcurlyeq_{\mathbf{L}}^{\#}) = \{\mathbf{O} \in \Omega : O_1 = O_2\}$
(iii) $L(\succcurlyeq_{\mathbf{L}}^{\#}) = \emptyset$
(iv) $\ell(\succcurlyeq_{\mathbf{L}}^{\#}) = \{\mathbf{O} \in \Omega : O_{(1)} = \emptyset\}$.
Proof of Proposition 4.4. Define, for any $\mathbf{U} \in \Omega$,

$$\mathcal{A}(\mathbf{U}) := \{\mathbf{O} \in \Omega : O_1 \cup O_2 = U_1 \cup U_2\}.$$

Fix an arbitrary $\mathbf{U} \in \Omega$ and notice that if $\mathbf{O} \notin \mathcal{A}(\mathbf{U})$, then $\mathbf{O}$ and $\mathbf{U}$ are not $\succcurlyeq_{\mathbf{L}}^{\#}$-connected. This readily establishes (i) and (iii). On the other hand, (ii) immediately follows from the fact that,

$$(U_1 \cup U_2, U_1 \cup U_2) \succcurlyeq_{\mathbf{L}}^{\#} \mathbf{O} \quad \text{for all } \mathbf{O} \in \mathcal{A}(\mathbf{U}).$$

Finally, to see (iv), let $\mathbf{U} = (\emptyset, U_2) \in \Omega$ and notice that, for any $\mathbf{O} \in \Omega$, $\mathbf{U} \succ_{\mathbf{L}}^{\#} \mathbf{O}$ implies that $O_{(1)} \subset \emptyset$ which is absurd. Therefore, $\mathbf{U} \succcurlyeq_{\mathbf{L}}^{\#} \mathbf{O}$ entails that $\mathbf{O} = \mathbf{U}$. □

[10] Let $\succcurlyeq \subset \Omega \times \Omega$. We define

$$G(\succcurlyeq) := \{\mathbf{O} \in \Omega : [\forall \mathbf{U} \in \Omega : \mathbf{O} \succcurlyeq \mathbf{U}]\}$$

and

$$g(\succcurlyeq) := \{\mathbf{O} \in \Omega : [\forall \mathbf{U} \in \Omega : \mathbf{U} \succcurlyeq \mathbf{O} \Rightarrow \mathbf{U} = \mathbf{O}]\}.$$

$L(\succcurlyeq)$ and $\ell(\succcurlyeq)$ are defined dually.

5 $S^{\#}$-Concavity

We now turn our attention to *welfare* related properties of $\succsim_{\mathbf{L}}^{\#}$. The following definition formulates the concept of *inequality averse* welfare functions in our setting.

DEFINITION 5.1. *$W : \mathbf{\Omega} \to \mathbb{R}$ is said to be* **$S^{\#}$-concave** *if, for any* $\mathbf{O}, \mathbf{U} \in \mathbf{\Omega}$, $\mathbf{O} \succsim_{\mathbf{L}}^{\#} \mathbf{U}$ *implies* $W(\mathbf{O}) \geq W(\mathbf{U})$. *The set of all $S^{\#}$-concave functions is denoted by* $\mathcal{S}^{\#}$.

The following lemma gives some examples of $S^{\#}$-concave functions which will be of use later on.

LEMMA 5.2. *Let $\alpha : L \to \mathbb{R}_{++}$ be any function and $u : [0,1] \to \mathbb{R}$ be any concave function. The following functions on $\mathbf{\Omega}$ are $S^{\#}$-concave:*

$$W_i^{\alpha}(\mathbf{O}) = \sum_{x \in O_{(i)}} \alpha(x), \ i = 1, 2, \tag{1}$$

$$W_3^{u}(\mathbf{O}) = u\left(\frac{\#O_1}{\#O_1 + \#O_2}\right) + u\left(\frac{\#O_2}{\#O_1 + \#O_2}\right), \tag{2}$$

$$W_4(\mathbf{O}) = \begin{cases} \#(O_{(1)} \backslash O_{(2)}), & \text{if } \#O_{(1)} < \#O_{(2)} \\ \\ \#(O_{(1)} \cup O_{(2)}), & \text{if } \#O_{(1)} = \#O_{(2)} \end{cases} . \tag{3}$$

Proof of Lemma 5.2. That the functions given in (1) are $S^{\#}$-concave follows immediately from the hypothesis that $\alpha(a) > 0$ for all $a \in L$. To see the second assertion, fix a concave $u : [0,1] \to \mathbb{R}$ and let $\mathbf{O} \succ_{\mathbf{L}}^{\#} \mathbf{U}$. As noted in Section 2, the concavity of u implies the S-concavity of $(x, y) \mapsto u(x) + u(y)$, and hence, either $\#O_1 = \#O_2$ or $\#U_{(1)} < \#O_{(1)} < \#O_{(2)} = \#U_{(2)}$ (and these are the only possibilities) so that

$$\left(\frac{\#O_{(1)}}{\#O_1 + \#O_2}, \frac{\#O_{(2)}}{\#O_1 + \#O_2}\right) \geq_L \left(\frac{\#U_{(1)}}{\#U_1 + \#U_2}, \frac{\#U_{(2)}}{\#U_1 + \#U_2}\right) \Longrightarrow W_3^u(\mathbf{O}) \geq W_3^u(\mathbf{U}).$$

To prove the final proposition, let $\mathbf{O} \succ_{\mathbf{L}}^{\#} \mathbf{U}$ again, and observe that the assertion is trivial when $\#O_1 = \#O_2$ and $\#U_{(1)} \leq \#U_{(2)}$. If, on the other hand, $\#O_{(1)} < \#O_{(2)}$, we have $U_{(1)} \subset O_{(1)}$ and $U_{(2)} = O_{(2)}$, and hence $O_1 \cup O_2 = U_1 \cup U_2$ implies that $W_4(\mathbf{O}) = \#(O_{(1)} \backslash O_{(2)}) = \#(U_{(1)} \backslash U_{(2)}) = W_4(\mathbf{U})$. □

The next lemma is a crucial step toward our main theorem.

LEMMA 5.3. *If* $\mathbf{O}, \mathbf{U} \in \mathbf{\Omega}$ *and*

$$\forall\, W \in \mathcal{S}^{\#} : [W(\mathbf{O}) \geq W(\mathbf{U})], \tag{4}$$

then $U_{(i)} \subseteq O_{(i)}$, $i = 1, 2$, *and* $O_1 \cup O_2 = U_1 \cup U_2$.

NOTATION. Let $\{A_1, ..., A_m\}, : m \geq 2$, be a collection of sets such that every two distinct sets in the collection are disjoint. The union of these sets is then written as $A_1 \uplus A_2 \uplus ... \uplus A_m$.

Proof of Lemma 5.3. By Lemma 5.2, $W_i^{\alpha} \in \mathcal{S}^{\#}$, $i = 1, 2$, for any $\alpha : L \rightarrow \mathbb{R}_{++}$ so that by (4),

$$\forall \alpha : L \rightarrow \mathbb{R}_{++} : [\sum_{x \in O_{(i)}} \alpha(x) \geq \sum_{x \in U_{(i)}} \alpha(x)],$$

$i = 1, 2$. Therefore, by Lemma 2.1 of Klemisch-Ahlert (1993), we must have $U_{(i)} \subseteq O_{(i)}$, $i = 1, 2$.

To prove the second assertion, define, for any $\mathbf{O} \in \mathbf{\Omega}$

$$W_+(\mathbf{O}) = \#(O_1 \cup O_2) \text{ and } W_-(\mathbf{O}) = -\#(O_1 \cup O_2).$$

One can easily check that both $W_+(.)$ and $W_-(.)$ are both $\mathrm{S}^{\#}$-concave. Therefore, if (4) holds for $\mathbf{O}, \mathbf{U} \in \mathbf{\Omega}$, we must have $\#(O_1 \cup O_2) = \#(U_1 \cup U_2)$, that is

$$\#O_1 + \#O_2 - \#(O_1 \cap O_2) = \#U_1 + \#U_2 - \#(U_1 \cap U_2)$$

so that

$$(\#O_1 - \#U_1) + (\#O_2 - \#U_2) = \#(O_1 \cap O_2) - \#(U_1 \cap U_2). \tag{5}$$

Now, since $U_{(i)} \subseteq O_{(i)}$, $i = 1, 2$, we may write

$$O_{(1)} = U_{(1)} \uplus A \text{ and } O_{(2)} = U_{(2)} \uplus B, \tag{6}$$

for some $A, B \in 2^L$. Let

$$X = (O_1 \cup O_2) \backslash (U_1 \cup U_2).$$

(We wish to show that $X = \emptyset$.) Let $Z_1 = X \cap A$ and $Z_2 = X \cap B$. We first establish the following

Claim. $A \cap B = Z_1 \cap Z_2$.

Proof of Claim. That $Z_1 \cap Z_2 = X \cap (A \cap B)$ is immediate from the definitions. Also recall that $U_1 \cap A = U_2 \cap B = \emptyset$ so that $(U_1 \cup U_2) \cap (A \cap B) = \emptyset$. Therefore, $A \cap B \subseteq X$ and the claim follows.

Now, by (6),

$$\#O_{(1)} = \#A + \#U_{(1)} \quad \text{and} \quad \#O_{(2)} = \#B + \#U_{(2)},$$

and combining this with (5),

$$\#A + \#B = \#(O_1 \cap O_2) - \#(U_1 \cap U_2). \tag{7}$$

On the other hand, one can easily verify that

$$O_1 \cap O_2 = (A \cap B) \uplus (U_2 \cap A) \uplus (U_1 \cap B) \uplus (U_1 \cap U_2)$$

and thus, by the claim above,

$$\#(O_1 \cap O_2) = \#(Z_1 \cap Z_2) + \#(U_2 \cap A) + \#(U_1 \cap B) + \#(U_1 \cap U_2).$$

Hence, by (7)

$$\#A + \#B = \#(U_2 \cap A) + \#(U_1 \cap B) + \#(Z_1 \cap Z_2),$$

that is

$$(\#A - \#(U_2 \cap A)) + (\#B - \#(U_1 \cap B)) = \#(Z_1 \cap Z_2),$$

which yields

$$\#(A \backslash U_2) + \#(B \backslash U_1) = \#(Z_1 \cap Z_2). \tag{8}$$

But, by definition of Z_1 and Z_2,

$$Z_1 \cap Z_2 \subseteq A \backslash U_2 \quad \text{and} \quad Z_1 \cap Z_2 \subseteq B \backslash U_1$$

so that (8) cannot hold unless $\#(Z_1 \cap Z_2) = 0$. On the other hand, $\#(Z_1 \cap Z_2) = 0$ implies that $\#(A \backslash U_2) = \#(B \backslash U_1) = 0$, that is, $A \backslash U_2 = \emptyset$ and $B \backslash U_1 = \emptyset$, and in view of (6), this yields that $O_1 \cup O_2 \subseteq U_1 \cup U_2$. Since we have already shown that $U_{(i)} \subseteq O_{(i)}$, $i = 1, 2$, $O_1 \cup O_2 \supseteq U_1 \cup U_2$ also holds, and hence the proof. □

The following theorem is the main result of this section.

THEOREM 5.4. *For any* $\mathbf{O}, \mathbf{U} \in \boldsymbol{\Omega}$, *we have*

$$\mathbf{O} \succcurlyeq_{\mathbf{L}}^{\#} \mathbf{U} \quad \textit{if and only if} \quad W(\mathbf{O}) \geq W(\mathbf{U}) \quad \textit{for all } W \in \mathcal{S}^{\#}.$$

Proof of Theorem 5.4. Necessity follows from the definition of S$^{\#}$-concavity. To prove sufficiency, let $\mathbf{O}, \mathbf{U} \in \boldsymbol{\Omega}$ such that $\mathbf{O} \neq \mathbf{U}$ and

$$\forall\, W \in \mathcal{S}^{\#} : [W(\mathbf{O}) \geq W(\mathbf{U})]. \tag{9}$$

(If $\mathbf{O} = \mathbf{U}$, the result is trivial.) We may assume that $\mathbf{O}_{(\,)} = \mathbf{O}$ and $\mathbf{U}_{(\,)} = \mathbf{U}$ without loss of generality. Therefore, by Lemma 5.3, we have $U_i \subseteq O_i$, $i = 1, 2$, and

$O_1 \cup O_2 = U_1 \cup U_2$. Consequently, if $\#O_1 = \#O_2$, that $\mathbf{O} \succ_{\mathbf{L}}^{\#} \mathbf{U}$ is immediate. We should therefore examine the case where $\#O_1 < \#O_2$.

Assume that $\#O_1 < \#O_2$. By Lemma 5.2, (2) and (9), we have

$$W_3^u(\mathbf{O}) \geq W_3^u(\mathbf{U}), \text{ for all concave } u : [0,1] \to \mathbb{R}.$$

By a well-known result due to Hardy et al. (1934), p. 89, we must then have

$$\left(\frac{\#O_1}{\#O_1 + \#O_2}, \frac{\#O_2}{\#O_1 + \#O_2}\right) \geq_L \left(\frac{\#U_1}{\#U_1 + \#U_2}, \frac{\#U_2}{\#U_1 + \#U_2}\right). \tag{10}$$

Therefore, since $\#U_1 = \#U_2$ would contradict this observation, we conclude that $\#U_1 < \#U_2$.

We now wish to establish that $U_1 \subset O_1$ and $U_2 = O_2$. By (10), we have

$$\frac{\#O_1}{\#O_1 + \#O_2} \geq \frac{\#U_1}{\#U_1 + \#U_2} \Longrightarrow \#O_1 \#U_2 \geq \#U_1 \#O_2.$$

Thus, if $U_1 = O_1$, then $\#U_2 \geq \#O_2$ and $U_2 \subseteq O_2$ yields $U_2 = O_2$ which contradicts the hypothesis that $\mathbf{O} \neq \mathbf{U}$. Hence we must have $U_1 \subset O_1$. To complete the proof, assume that $U_2 \subset O_2$. Since $O_1 \cup O_2 = U_1 \cup U_2$, there must exist non-empty A, B with $A \subset U_2$ and $B \subset U_1$ such that $O_1 = U_1 \cup A$ and $O_2 = U_2 \cup B$. But then

$$O_1 \backslash O_2 = (U_1 \cup A) \backslash (U_2 \cup B) = U_1 \backslash (U_2 \cup B) \subset U_1 \backslash U_2$$

and hence, recalling (3),

$$W_4(\mathbf{O}) = \#(O_1 \backslash O_2) < \#(U_1 \backslash U_2) = W_4(\mathbf{U})$$

where $W_4 \in \mathcal{S}^{\#}$ by Lemma 5.2. This contradicts (9) and thus, we conclude that $U_2 = O_2$ which, in turn, establishes that $\mathbf{O} \succ_{\mathbf{L}}^{\#} \mathbf{U}$. □

6 A Benchmark Result

In the previous sections, we have confined our attention to two-agent societies where both agents have the same preferences over the attributes in L which are represented by the cardinality ordering. We attempted to extend the definitions of the key notions of income inequality measurement theory to this admittedly restrictive framework in a natural way. This led us to Definitions 3.2, 4.1 and 5.1. The merit of these definitions is that they allow us to establish an exact analogue of the fundamental theorem of inequality economics in our framework. Indeed, Theorems 4.2 and 5.4 yield the following benchmark result. (Compare with Theorem 2.1.)

THEOREM 6.1. *Let* $\mathbf{O}, \mathbf{U} \in \mathbf{\Omega}$. *The following statements are equivalent:*
(i) $\mathbf{O} \succcurlyeq_{\mathbf{L}}^{\#} \mathbf{U}$
(ii) $\mathbf{O} \succcurlyeq_{D}^{\#} \mathbf{U}$
(iii) $W(\mathbf{O}) \geq W(\mathbf{U})$ *for all* $\mathcal{S}^{\#}$*-concave* $W : \mathbf{\Omega} \to \mathbb{R}$.

7 Monotonic Extensions of $\succcurlyeq^{\#}_{\mathbf{L}}$

As an indication that $\succcurlyeq^{\#}_{\mathbf{L}}$ is too incomplete, we observe its inability of comparing perfectly egalitarian distributions. For example, the opportunity distributions $(\{a\},\{a\})$ and $(\{a,b\},\{a,b\})$, $a,b \in L$, cannot be ranked by $\succcurlyeq^{\#}_{\mathbf{L}}$, although it is intuitively clear that these distributions are equally unequal since both of them are perfectly egalitarian. Indeed, the following axiom seems unexceptionable.

AXIOM D. *Let* $\mathbf{O} \in \mathbf{\Omega}$ *and* $O_1 = O_2$. *Then,* $\mathbf{O} \succcurlyeq \mathbf{U}$ *for all* $\mathbf{U} \in \mathbf{\Omega}$.

Although $\succcurlyeq^{\#}_{\mathbf{L}}$ does not satisfy Axiom D, we can extend it in a trivial way to solve the problem:

$$\succeq^{\#}_{\mathbf{L}} \;:=\; \succcurlyeq^{\#}_{\mathbf{L}} \cup \{(\mathbf{O},\mathbf{U}) \in \mathbf{\Omega} \times \mathbf{\Omega} : O_1 = O_2\}.$$

$\succeq^{\#}_{\mathbf{L}}$ is an extension of $\succcurlyeq^{\#}_{\mathbf{L}}$ which allows us to conclude that a perfectly egalitarian distribution is less unequal than any other distribution. Consequently, $(\{a\},\{a\}) \succeq^{\#}_{\mathbf{L}} (\{a,b\},\{a,b\})$ and $(\{a,b\},\{a,b\}) \succeq^{\#}_{\mathbf{L}} (\{a\},\{a\})$. Moreover, $\succeq^{\#}_{\mathbf{L}}$ satisfies Axioms A, B and C($\succeq_{\#}$). Nevertheless, $\succeq^{\#}_{\mathbf{L}}$ is still *too* incomplete.

One of the axioms introduced in Kranich (1994), namely *monotonicity*, is extremely compelling:

AXIOM E. (Monotonicity) *For all* $(O_1,O_2),(O_1,U_2) \in \mathbf{\Omega}$, $O_1 \subseteq O_2 \subset U_2$ *implies that* $(O_1,O_2) \succ (O_1,U_2)$.

Let $\mathbf{O} \in \mathbf{\Omega}$. If $O_1 \subseteq O_2$ then the second party is clearly enjoying greater opportunities than the first one. Monotonicity axiom says that if the second party's opportunity set expands further, then the degree of inequality should increase. Nevertheless, because of its incompleteness, $\succeq^{\#}_{\mathbf{L}}$ is not monotonic. Before we attempt to remedy this shortcoming, let us first clarify that this result is far from unexpected.

Consider the following income distributions of a two-person society: $z = (1,10)$ and $w = (1,15)$. Which distribution is more unequal? Presumably, a unanimous answer would be that z is more equal than w. However, the Lorenz ordering does not capture this clear intuition: $(z,w),(w,z) \notin \geq_L$. (Recall Subsection 2.1.) This is precisely because $z_1 + z_2 \neq w_1 + w_2$. We have an analogous situation in the present context. Let $\#L \geq 3$, $\mathbf{O} = (\{a\},\{a,b\})$ and $\mathbf{U} = (\{a\},\{a,b,c\})$. Observe that $(\mathbf{O},\mathbf{U}),(\mathbf{U},\mathbf{O}) \notin \succeq^{\#}_{\mathbf{L}}$, and this is precisely because $O_1 \cup O_2 \neq U_1 \cup U_2$. Consequently, if we are after an analogue of the standard Lorenz ordering in our framework of the measurement of opportunity inequality, that our candidate preorder is not monotonic is hardly surprising.

How do we deal with cases like comparing $z = (1,10)$ and $w = (1,15)$ in the standard theory? The most common way is to invoke the axiom of *scale invariance* (i.e. $x \sim \lambda x$, for all $x \in \mathbb{R}^H_+$, $\lambda > 0$) and rather use the minimal scale invariant and *transitive* extension of $\succeq_L$ (cf. Fields and Fei (1978)). (In fact, many authors take this extension as the *definition* of the Lorenz ordering.) The resulting ordering $\geq^L$ on $\mathbb{R}^H_+$, $H \geq 2$, satisfies

$$x \geq^L y \quad \text{if and only if} \quad \sum_{i=1}^{s} \left(x_{(i)} / \sum_{i=1}^{H} x_i \right) \geq \sum_{i=1}^{s} \left(y_{(i)} / \sum_{i=1}^{H} y_i \right), \quad s = 1,..,H-1.$$

This preorder captures our intuition and notes $(1,10) >^L (1,15)$.

Unfortunately, to formulate an analog of the axiom of scale invariance in the present context does not seem possible. Therefore, we are led to study the monotonic extensions of the Lorenz ordering directly. Now, since $\geq^L$ is the smallest scale invariant and transitive extension of $\geq_L$, by analogy, what we are after is the smallest monotonic and transitive extension of $\succcurlyeq^{\#}_{\mathbf{L}}$. Fortunately, formulating this extension explicitly turns out to be a rather simple task. To do this, we define $\succ_{\mathbf{M}} \subset \mathbf{\Omega} \times \mathbf{\Omega}$ as

$$\mathbf{O} \succ_{\mathbf{M}} \mathbf{U} \quad \text{if and only if} \quad U_{(1)} \subseteq O_{(1)} \subseteq O_{(2)} \subset U_{(2)},$$

and then extend $\succcurlyeq^{\#}_{\mathbf{L}}$ to

$$\succcurlyeq^{\#}_{\mathbf{ML}} := \succcurlyeq^{\#}_{\mathbf{L}} \cup \succ_{\mathbf{M}} .$$

We shall refer to $\succcurlyeq^{\#}_{\mathbf{ML}}$ as the **monotonic Lorenz ordering induced by** $\geq_{\#}$. (Notice that $\succcurlyeq^{\#}_{\mathbf{ML}}$lets us rank $(\{a\},\{a,b\})$ and $(\{a\},\{a,b,c\})$ in accordance with our intuition:

$$(\{a\},\{a,b\}) \succcurlyeq^{\#}_{\mathbf{ML}} (\{a\},\{a,b,c\}).$$

The transitivity of $\succcurlyeq^{\#}_{\mathbf{ML}}$ is proved next.

PROPOSITION 7.1. *$\succcurlyeq^{\#}_{\mathbf{ML}}$ is the smallest transitive and monotonic extension of $\succcurlyeq^{\#}_{\mathbf{L}}$ over $\mathbf{\Omega}$.*

Proof of Proposition 7.1. That $\succcurlyeq^{\#}_{\mathbf{ML}}$ is the smallest monotonic extension of $\succcurlyeq^{\#}_{\mathbf{L}}$ is obvious; all we have to show is that it is also transitive. Let $\mathbf{O} \succcurlyeq^{\#}_{\mathbf{ML}} \mathbf{U}$ and $\mathbf{U} \succcurlyeq^{\#}_{\mathbf{ML}} \mathbf{V}$ for some $\mathbf{O},\mathbf{U},\mathbf{V} \in \mathbf{\Omega}$. If $\mathbf{O} \succcurlyeq^{\#}_{\mathbf{L}} \mathbf{U} \succcurlyeq^{\#}_{\mathbf{L}} \mathbf{V}$ or $\mathbf{O} \succ_{\mathbf{M}} \mathbf{U} \succ_{\mathbf{M}} \mathbf{V}$, then there is nothing to prove, for both $\succcurlyeq^{\#}_{\mathbf{L}}$ and $\succ_{\mathbf{M}}$ are transitive. Suppose $\mathbf{O} \succcurlyeq^{\#}_{\mathbf{L}} \mathbf{U} \succ_{\mathbf{M}} \mathbf{V}$ holds. If $O_1 = O_2$, the case is trivial so assume that $O_{(1)} \neq O_{(2)}$. This necessitates that $U_1 \neq U_2$ and that $O_1 \cup O_2 = U_1 \cup U_2$. Furthermore, $\mathbf{U} \succ_{\mathbf{M}} \mathbf{V}$ implies that $U_{(1)} \subset U_{(2)}$. Now let $\#O_1 = \#O_2$ and $U_{(i)} \subseteq O_{(i)}, i = 1,2$. Then, since $O_1 \cup O_2 = U_1 \cup U_2$ and $U_{(1)} \subset U_{(2)}$ entail that $O_{(1)} \subseteq U_{(1)} \cup U_{(2)} = U_{(2)}$, we must have $U_{(1)} \subseteq O_{(1)} \subseteq U_{(2)} \subseteq O_{(2)}$. In fact, $O_{(2)} \subseteq O_1 \cup O_2 = U_1 \cup U_2 = U_{(2)}$ so that $U_{(2)} = O_{(2)}$. But since, $\mathbf{U} \succ_{\mathbf{M}} \mathbf{V}$, $V_{(1)} \subseteq U_{(1)} \subseteq U_{(2)} \subset V_{(2)}$, and hence $V_{(1)} \subseteq U_{(1)} \subseteq O_{(1)} \subseteq O_{(2)} = U_{(2)} \subset V_{(2)}$. Therefore, $\mathbf{O} \succ_{\mathbf{M}} \mathbf{V}$ and $\mathbf{O} \succcurlyeq^{\#}_{\mathbf{ML}} \mathbf{V}$ follows. If $\#O_{(1)} < \#O_{(2)}$, $U_{(1)} \subset O_{(1)}$ and $U_{(2)} = O_{(2)}$, the result is obtained similarly. Finally, assume that $\mathbf{O} \succ_{\mathbf{M}} \mathbf{U} \succcurlyeq^{\#}_{\mathbf{L}} \mathbf{V}$.

$\#U_{(1)} < \#U_{(2)}$ must hold since $U_{(1)} \subseteq O_{(1)} \subseteq O_{(2)} \subset U_{(2)}$. But then $\mathbf{U} \succcurlyeq^{\#}_{\mathbf{L}} \mathbf{V}$ implies that $V_{(1)} \subset U_{(1)}$ and $V_{(2)} = U_{(2)}$. Therefore, $V_{(1)} \subset U_{(1)} \subseteq O_{(1)} \subseteq O_{(2)} \subset U_{(2)} = V_{(2)}$ and $\mathbf{O} \succ_{\mathbf{M}} \mathbf{V}$ follows. We conclude that $\mathbf{O} \succcurlyeq^{\#}_{\mathbf{ML}} \mathbf{V}$. □

By Corollary 4.3, this proposition yields

COROLLARY 7.2. *$\succcurlyeq^{\#}_{\mathbf{ML}}$ is the smallest binary relation on $\mathbf{\Omega}$ that satisfies Axioms A, B, C($\geq_{\#}$), D and E.*

We would like to argue that $\succcurlyeq^{\#}_{\mathbf{ML}}$ is still too incomplete. Consider the opportunity distributions $\mathbf{O} = (\{a\}, \{b, c\})$ and $\mathbf{U} = (\emptyset, \{a, b, c\})$. To conclude that $\mathbf{O}$ is "more equal than" $\mathbf{U}$ is clearly in nature of things. However, $(\mathbf{O}, \mathbf{U}) \notin \succcurlyeq^{\#}_{\mathbf{ML}}$. This observation leads us to consider the following strengthening of Axiom E:

AXIOM E*. **(#-Monotonicity)** *For all* $(O_1, O_2), (O_1, U_2) \in \mathbf{\Omega}$, $\#O_1 \leq \#O_2$ and $O_2 \subset U_2$ *imply that* $(O_1, O_2) \succ (O_1, U_2)$.

Our final task is, therefore, to study the #-monotonic extensions of $\succcurlyeq^{\#}_{\mathbf{L}}$. Of course, one immediate way to do this is to define $\succ^{\#}_{\mathbf{M}} \subset \mathbf{\Omega} \times \mathbf{\Omega}$ as

$$O \succ^{\#}_{\mathbf{M}} U \quad \text{if and only if} \quad U_{(1)} \subseteq O_{(1)},\ \#O_{(1)} \leq \#O_{(2)} \ \text{ and } O_{(2)} \subset U,$$

and then extend $\succcurlyeq^{\#}_{\mathbf{L}}$ to $\succcurlyeq^{\#}_{\mathbf{L}} \cup \succ^{\#}_{\mathbf{M}}$. Indeed, this would let us rank $(\{a\}, \{b, c\})$ and $(\emptyset, \{a, b, c\})$ in exacty the way we wanted:

$$(\{a\}, \{b, c\})\, (\succcurlyeq^{\#}_{\mathbf{L}} \cup \succ^{\#}_{\mathbf{M}})\, (\emptyset, \{a, b, c\}).$$

However, there is a rather serious problem with this extension; it is not transitive:

EXAMPLE 7.3. Let $\#L \geq 5$ and let $\mathbf{O} = (\{a, b, c\}, \{c, a, d\})$, $\mathbf{U} = (\{b, c\}, \{a, d\})$ and $\mathbf{V} = (\{b\}, \{a, d, e\})$. We have $\mathbf{O} \succ^{\#}_{\mathbf{L}} \mathbf{U}$ and $\mathbf{U} \succ^{\#}_{\mathbf{M}} \mathbf{V}$ but *not* $\mathbf{O} \succcurlyeq^{\#}_{\mathbf{L}} \mathbf{V}$ and *not* $\mathbf{O} \succ^{\#}_{\mathbf{M}} \mathbf{V}$. Therefore, $\mathbf{O}\, (\succcurlyeq^{\#}_{\mathbf{L}} \cup \succ^{\#}_{\mathbf{M}}) \mathbf{U}$ and $\mathbf{U}\, (\succcurlyeq^{\#}_{\mathbf{L}} \cup \succ^{\#}_{\mathbf{M}}) \mathbf{V}$ but *not* $\mathbf{O}\, (\succcurlyeq^{\#}_{\mathbf{L}} \cup \succ^{\#}_{\mathbf{M}}) \mathbf{V}$. We conclude that $\succcurlyeq^{\#}_{\mathbf{L}} \cup \succ^{\#}_{\mathbf{M}}$ is not transitive. □

Example 7.3 suggests that we should rather study larger #-monotonic extensions of $\succcurlyeq^{\#}_{\mathbf{L}}$, or more precisely, that we should focus on transitive extensions of $\succcurlyeq^{\#}_{\mathbf{L}} \cup \succ^{\#}_{\mathbf{M}}$. Any transitive and #-monotonic extension of $\succcurlyeq^{\#}_{\mathbf{L}}$ is not, however, presumably suitable for our purposes, for such an extension may very well lead us to rank opportunity distributions which we would not like to. In other words, one would like to find an extension of $\succcurlyeq^{\#}_{\mathbf{L}}$ which respects *only* the properties of #-monotonicity and transitivity. An arbitrary #-monotonic and transitive extension of $\succcurlyeq^{\#}_{\mathbf{L}}$, after all, might respect some additional properties which may not be desirable. Consequently, we need to focus on the smallest #-monotonic and transitive extension of $\succcurlyeq_{\mathbf{L}}^{\#}$.

To find the smallest transitive and #-monotonic extension of $\succcurlyeq_{\mathbf{L}}^{\#}$, we shall first briefly digress on abstract relation theory. Let X be an arbitrary set and let $\Pi(X)$ be the class of all *transitive* relations on X. Define the binary relation $\oplus$ on $\Pi(X)$ as

$$\oplus : (\succ^1, \succ^2) \longmapsto \succ^\circ$$

where

$$x \succ^\circ y \quad \text{if and only if} \quad x\,(\succ^1 \cup \succ^2)\,y \ \text{ or } \ \exists z \in X : [x \succ^1 z \succ^2 y].$$

The following lemma will be quite useful for our purposes.

LEMMA 7.4. *Let X be any set and $\succ^i \in \Pi(X)$, $i = 1, 2$. If, for any $x, y, z \in X$,*

$$x \succ^2 z \ \text{ and } \ z \succ^1 y \ \text{ implies } \ x \succ^2 y, \tag{11}$$

then $\succ^1 \oplus \succ^2$ is the smallest transitive extension of $\succ^1 \cup \succ^2$.

NOTATION. For any $x, y \in X$, if there exists a $z \in X$ such that $x \succ^1 z \succ^2 y$, we write $x \succ^{12} y$. Therefore, $\succ^1 \oplus \succ^2 = (\succ^1 \cup \succ^2) \cup \succ^{12}$.

Proof of Lemma 7.4. Let us first demonstrate that under the hypotheses of the Lemma, $\succ^1 \oplus \succ^2$ needs to be transitive. Let $x\,(\succ^1 \oplus \succ^2)z$ and $z\,(\succ^1 \oplus \succ^2)y$, $x, y, z \in X$. Now, if $x \succ^1 z \succ^2 y$, then $x\,(\succ^1 \oplus \succ^2)y$ is immediate, and if $x \succ^1 z \succ^1 y$, or $x \succ^2 z \succ^2 y$, or $x \succ^1 z \succ^{12} y$, or $x \succ^{12} z \succ^2 y$, $x\,(\succ^1 \oplus \succ^2)y$ follows from the transitivity of $\succ^i, i = 1, 2$. In addition, by (11), $x \succ^2 z \succ^1 y$ implies $x\,(\succ^1 \oplus \succ^2)y$, and by (11) and the transitivity of $\succ^i, i = 1, 2$, there exist $w, w' \in X$ such that

$$x \succ^2 z \succ^{12} y \Rightarrow x \succ^2 z \succ^1 w \succ^2 y \Rightarrow x \succ^2 w \succ^2 y \Rightarrow x\,(\succ^1 \oplus \succ^2)y,$$

$$x \succ^{12} z \succ^1 y \Rightarrow x \succ^1 w \succ^2 z \succ^1 y \Rightarrow x \succ^1 w \succ^2 y \Rightarrow x\,(\succ^1 \oplus \succ^2)y$$

and

$$x \succ^{12} z \succ^{12} y \Rightarrow x \succ^1 w \succ^2 z \succ^1 w' \succ^2 y \Rightarrow x \succ^1 w \succ^2 y \Rightarrow x\,(\succ^1 \oplus \succ^2)y.$$

By these observations and the fact that $(\succ^1 \cup \succ^2) \subset (\succ^1 \oplus \succ^2)$, we conclude that $\succ^1 \oplus \succ^2$ is a transitive extension of $\succ^1 \cup \succ^2$.

Let $\mathcal{J}$ be the class of all transitive relations on X which contain $\succ^1 \cup \succ^2$. (By the above argument, $(\succ^1 \oplus \succ^2) \in \mathcal{J}$ so that $\mathcal{J} \neq \emptyset$.) Define $\rhd := \bigcap_{\succ \in \mathcal{J}} \succ$. One can easily check that $\rhd$ is transitive and $(\succ^1 \cup \succ^2) \subset \rhd$. Therefore, $\rhd$ is the smallest transitive extension of $\succ^1 \cup \succ^2$, and we have $\rhd \subseteq (\succ^1 \oplus \succ^2)$. To prove the converse containment, let $x\,(\succ^1 \oplus \succ^2)y$. If $x \succ^i y$, for some $i = 1, 2$, then since

$(\succ^1 \cup \succ^2) \subset \rhd$, we have $x \rhd y$, and if $x \succ^1 z \succ^2 y$, for some $z \in X$, then since $\succ^i \subset \rhd$, $i = 1, 2$, we have $x \rhd z \rhd y$, and by the transitivity of $\rhd$, $x \rhd y$. Thus, we conclude that $(\succ^1 \oplus \succ^2) \subseteq \rhd$, and hence the lemma. □

Let us now turn back to our main topic and state the following

DEFINITION 7.5. *The* **#-monotonic Lorenz ordering induced by** $\succeq_\#$, $\succcurlyeq^\#_{monL} \subset \Omega \times \Omega$, *is defined as*

$$\succcurlyeq^\#_{monL} = \succcurlyeq^\#_{L} \oplus \succ^\#_{M}.$$

In other words, for any $\mathbf{O}, \mathbf{U} \in \Omega$, we have

$$\mathbf{O} \succcurlyeq^\#_{monL} \mathbf{U} \iff \mathbf{O} \succcurlyeq^\#_{L} \mathbf{U} \text{ or } \mathbf{O} \succ^\#_{M} \mathbf{U} \text{ or } (\exists \mathbf{V} \in \Omega : [\mathbf{O} \succcurlyeq^\#_{L} \mathbf{V} \succ_{M} \mathbf{U}]).$$

Therefore, that $\succcurlyeq^\#_{monL}$ is a #-monotonic extension of $\succcurlyeq^\#_{L}$ is clear. (Hence, $\succcurlyeq^\#_{monL}$ is $\succcurlyeq^\#_{L}$-consistent, and lets us rank distributions like $(\{a\}, \{b, c\})$ and $(\emptyset, \{a, b, c\})$ in accordance with our intuition.) $\succcurlyeq^\#_{monL}$ is, in fact, the *smallest* #-monotonic extension of $\succcurlyeq^\#_{L}$. To prove this, all we need is the following

LEMMA 7.6. *Let* $\mathbf{O}, \mathbf{V}$ *and* $\mathbf{U}$ *be in* Ω. *If* $\mathbf{O} \succ^\#_{M} \mathbf{V}$ *and* $\mathbf{V} \succcurlyeq^\#_{L} \mathbf{U}$, *then* $\mathbf{O} \succ^\#_{M} \mathbf{U}$.

Proof of Lemma 7.6. If $\mathbf{O} \succ^\#_{M} \mathbf{V}$, then $V_{(1)} \subseteq O_{(1)}$, $\#O_{(1)} \leq \#O_{(2)}$ and $O_{(2)} \subset V_{(2)}$ so that $\#V_{(1)} < \#V_{(2)}$. Therefore, by Definition 4.1, $U_{(1)} \subset V_{(1)}$ and $U_{(2)} = V_{(2)}$. We must then have $U_{(1)} \subset O_{(1)}$ and $O_{(2)} \subset U_{(2)}$, and that $\mathbf{O} \succ^\#_{M} \mathbf{U}$ follows. □

By Lemmata 7.5 and 7.6, we therefore have the desired result:

THEOREM 7.7. $\succcurlyeq^\#_{monL}$ *is the smallest transitive and #-monotonic extension of* $\succcurlyeq^\#_{L}$ *over* Ω.

Equivalently,

COROLLARY 7.8. $\succcurlyeq^\#_{monL}$ *is the smallest binary relation on* Ω *that satisfies Axioms A, B, C*($\succeq_\#$)*, D and E**.

Our final proposition computes the greatest, maximal, least and minimal elements of Ω with respect to the #-monotonic Lorenz ordering induced by $\succeq_\#$. (Compare with Proposition 4.4.) The proof is easy, and thus omitted.

PROPOSITION 7.9. *Let* $\infty > \#L \geq 2$. *Then,*

(i) $G(\succcurlyeq^\#_{monL}) = g(\succcurlyeq^\#_{monL}) = \{\mathbf{O} \in \Omega : O_1 = O_2\}$,

(ii) $L(\succcurlyeq^\#_{monL}) = \ell(\succcurlyeq^\#_{monL}) = \{(\emptyset, L)\}$.

8 Conclusion

In our opinion there is a clear need for a more extensive analysis of inequality than on the basis of income differences alone. In particular, the opportunities available to an individual might be a more appropriate determinant of his or her well-being, and the policy implications of this might be profound. For example, policies designed to address measured educational differences might look markedly different if each individual is free and able to choose his or her level of participation. Or, in general, the significance of *ex post* differences might depend on the extent of *ex ante* similarities.

In this paper, we have tried to formulate a (hopefully) promising start of a potential theory of opportunity inequality measurement by demonstrating that an exact analogue of the fundamental theorem of inequality economics can be established in the framework of two-agent societies where each agent evaluates the opportunity sets in terms of their cardinalities. An analog of the familiar Lorenz ordering is discovered and its smallest monotonic and transitive extension is characterized.

Our study provides a straightforward research agenda: first, to generalize our main result (Theorem 6.1) to n-person societies; second, to generalize Theorem 6.1, if at all possible, to an arbitrary ordering of the individual sets of opportunities and its induced Lorenz and Dalton orderings; third, to investigate more complete orderings that are consistent with the Lorenz ranking. These will be the subject of future research.

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