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TESTING SERIAL INDEPENDENCE USING THE SAMPLE DISTRIBUTION FUNCTION

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Abstract

This paper presents and discusses a nonparametric test for detecting serial dependence. We consider a Cramèr-v.Mises statistic based on the difference between the joint sample distribution and the product of the marginals. Exact critical values can be approximated from the asymptotic null distribution or by resampling, randomly permuting the original series. The approximation based on resampling is more accurate and the corresponding test enjoys, like other bootstrap based procedures, excellent level accuracy, with level error of order $T^{-3/2}$. A Monte Carlo experiment illustrates the test performance with small and moderate sample sizes. The paper also includes an application, testing the random walk hypothesis of exchange rate returns for several currencies.

Key Words
Serial independence test; Multivariate sample distribution; Hoeffding-Blum-Kiefer-Rosenblatt empirical process; Random permutation test; Ergodic alternatives.

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1. INTRODUCTION

Testing independence of a sequence of random variables is well motivated in statistics. Inference on probability models is frequently based on the assumption that the observations form a random sample. In time series, it is not always obvious whether a pattern is present in the data, and testing independence is often the first step in model building.

Traditionally, serial dependence has been tested by means of the sample correlation coefficient, or, in Gaussian cases, the Lagrange Multiplier principle. There are also a number of nonparametric tests based on ranks like the runs test, turning point test, and Spearman-Wald-Wolfowitz serial rank correlation coefficient test. These procedures have been applied to time series problems by Knoke (1977), Bartels (1982), Dufour (1981), Hallin et al (1985), Hallin and Puri (1988), and Hallin and Mélard (1988), to mention only a few.

The above mentioned test procedures are designed for specific alternative hypotheses, and they work well under commonly used dependence structures, like ARMA models. However, the usual serial correlation or serial rank-based procedures fail to detect subtle nonlinear underlying dependence structures.

A number of independence tests have been constructed based on the fact that the null hypothesis holds, if and only if, the joint distribution equals the product of the marginals. In these procedures, the test statistic is a distance between the estimated joint distribution (or joint density) and the product of the estimated marginal distributions (or marginal densities). Hoeffding (1948) and Blum et al (1961) proposed to estimate the joint and marginal distributions by the empirical distribution function, and introduced statistics based on the L_2 and L_∞ distances. Rosenblatt (1975) (see also Nadaraya 1989, Chap. 3.5) and Robinson (1991) used smooth kernel estimates of the densities. Rosenblatt's statistic is based on the L_2 distance and has a χ^2 limiting null distribution; Robinson's is based on the Kublack-Leibler information criterion and achieves a normal asymptotic null distribution. Only Robinson proved consistency in a time series context. Recently Chan and Tran (1992) have proposed to estimate the joint and marginal densities by the histogram, and have introduced a statistic based on the L_1 distance. They did not obtain the asymptotic null distribution, but they proved consistency under general alternatives. The critical values were obtained by resampling, randomly permuting the original series. Procedures based on the kernel and histogram methods share the disadvantage of depending on the choice of a smoothing number. Statistics based on the empirical distribution function, or the empirical characteristic function, avoid estimation by smoothing procedures, but the limiting null distribution is not standard. Brock et. al (1987) developed a test based on concepts that arise in the theory of chaotic processes. The consistency of this test depends on the choice of a kernel and smoothing number (see discussion in Robinson (1991)). Pinkse (1993) has proposed a statistic based on the squared difference of the empirical characteristic function and the product of the marginals, weighted by a suitable kernel function. This procedure depends also on the choice of a kernel and the statistic achieves a limiting χ^2 distribution under the null hypothesis.

This paper proposes and discusses a serial independence test based on the empirical distribution function. We apply the Hoeffding-Blum-Kiefer-Rosenblatt (HBKR) statistic (Hoeffding (1948) and Blum, Kiefer and Rosenblatt (1961)) to test serial dependence. Skaug and Tjøstheim (1992) have shown that the asymptotic null distribution of this statistic, when testing first order dependence (one lag), is the same as that of the HBKR empirical process used for testing independence of two random vectors, which was tabulated by Blum, Kiefer and Rosenblatt (1961). We derive the asymptotic distribution in the higher dimensional case (more than one lag), which differs from HBKR in a non obvious way. We propose to approximate the exact critical points by means of a resampling technique based on random permutations. The test is consistent under a wide range of dependence structures. Like other bootstrap based tests, it enjoys exceptional level accuracy, with level error of order $T^{-3/2}$, rather than T^{-1}, the error of the asymptotic test, where T is the sample size. The test uses critical points which differ also from the critical points of the exact test by a magnitude of order $T^{-3/2}$ under the null, while the asymptotic critical points differ by an order T^{-1}. Reported simulations support that the test enjoys good power and level properties in small and moderate samples.

The rest of the paper is organized as follows. Section 2 presents the statistics. Section 3 discusses asymptotic properties under the null and fixed alternatives. Section 4 presents and justifies a resampled version of the test. Section 5 summarizes a simulation study and an empirical application of the test procedure to testing serial independence using daily, monthly, and quarterly data on pound, yen, and deutschmark exchange rates changes.

2. THE TEST STATISTIC.

Let $\{X_1, X_2, \ldots, X_{T+p}\}$ be T+p observations of the real valued strictly stationary process $\{X_t\}$. Define $Z_t = (X_t, X_{t+1}, \ldots, X_{t+p})$. Assume that Z_t has a continuous distribution. We test serial p-dependence by testing

$$H_0:\ S(x) = 0 \text{ for all } x = (x_1, \ldots, x_{p+1}) \in \mathbb{R}^{p+1},$$

versus

$$H_1:\ S(x) \neq 0 \text{ for some } x = (x_1, \ldots, x_{p+1}) \in \mathbb{R}^{p+1},$$

where

$$S(x) = F(x) - \prod_{j=1}^{p+1} F_1(x_j), \qquad (2.1)$$

and

$$F(x) = \Pr\{X_1 \le x_1, X_2 \le x_2, \ldots, X_{p+1} \le x_{p+1}\},\ F_1(x_1) = \Pr\{X_1 \le x_1\}.$$

The null hypothesis states that X_t and X_u are independent for all $t \neq u$, and $t < u \le t+p$, $t \ge 1$.

Note that,

$$S(x) = E\{\prod_{j=1}^{p+1} 1(X_j \le x_j)\} - \prod_{j=1}^{p+1} E\{1(X_j \le x_j)\},$$

where $1(A)$ is the indicator function of the event A. Then, the sample analog of $S(x)$ is

$$S_T(x) = T^{-1} \sum_{t=1}^{T} \prod_{j=1}^{p+1} 1(X_{t+j-1} \le x_j) - \prod_{j=1}^{p+1} \left\{ T^{-1} \sum_{t=1}^{T} 1(X_{t+j-1} \le x_j) \right\}. \qquad (2.2)$$

Since $S_T(x)$ takes small values under H_0 and larger values under H_1, it forms a basis for testing H_0. We consider the statistic

$$B_T = \sum_{t=1}^{T} S_T(Z_t)^2, \tag{2.3}$$

constructed in the spirit of the Cramér-v.Mises statistic. An alternative Kolmogorov-Smirnov type statistic is,

$$A_T = T^{1/2} \sup_{\alpha} |S_T(\alpha)|.$$

These statistics were introduced by Hoeffding (1948) and Blum et al (1961) for testing independence of several random variables.

The statistic B_T is computationally more attractive than A_T. In order to compute B_T, we need to evaluate $S_T(.)$ at the T data points $Z_1, \ldots, Z_T$, while computation of A_T requires evaluation of $S_T(.)$ at T^{p+1} points. Note also that, unlike kernel based procedures, B_T needs not choose any smoothing number.

3. ASYMPTOTIC PROPERTIES OF B_T.

Let $F_1^{-1}(.)$ be the inverse function of the marginal distribution $F_1(.)$, $\alpha_i = F_1^{-1}(\omega_i)$, $i = 1, \ldots, p+1$, $\omega_i \in [0,\ 1]$, $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_{p+1})$ and $\omega = (\omega_1, \ldots, \omega_{p+1}) \in [0,\ 1]^{p+1} \equiv I^{p+1}$. As usual, it is convenient to write $S_T(\alpha)$ and $S(\alpha)$ in terms of $F_1^{-1}(.)$, that is

$$S_T(\alpha) = S_T(\omega) = T^{-1}\sum_{t=1}^{T} \prod_{j=1}^{p+1} 1(X_{t+j-1} \le F_1^{-1}(\omega_j)) - \prod_{j=1}^{p+1}\left\{T^{-1}\sum_{t=1}^{T} 1(X_{t+j-1} \le F_1^{-1}(\omega_j))\right\}$$

and

$$S(\alpha) = S(\omega) = E\{\prod_{j=1}^{p+1} 1(X_j \le F_1^{-1}(\omega_j))\} - \prod_{j=1}^{p+1} E\{1(X_j \le F_1^{-1}(\omega_j))\}.$$

Under H_0,

$$S_T(\omega) = T^{-1} \sum_{t=1}^{T} R_t(\omega) + O(T^{-1}) \text{ a.s.}, \tag{3.1}$$

where

$$R_t(\omega) = \prod_{j=1}^{p+1} 1(X_{t+j-1} \le F_1^{-1}(\omega_j))$$

$$- \sum_{i=1}^{p+1} 1(X_{t+j-1} \le F_1^{-1}(\omega_j)) \prod_{j=1,\, j\ne i}^{p+1} \omega_j + p \prod_{j=1}^{p+1} \omega_j. \qquad (3.2)$$

Let us define $\omega' = (\omega'_1, \ldots, \omega'_{p+1}) \in I^{p+1}$ and $\omega'_i = F_1(x'_i)$. Routine calculations show that

$$E(R_t(\omega)) = 0,$$

$$E(R_t(\omega)\ R_s(\omega')) = 0, \text{ all } t-p \le s \le t+p.$$

Then,

$$\Xi(\omega, \omega') = T^{-1} \sum_{t=1}^{T} \sum_{s=1}^{T} E(R_t(\omega)\ R_s(\omega'))$$

$$= \sum_{s=1-p}^{p-1} E(R_p(\omega)\ R_{p+s}(\omega'))$$

$$= \sum_{s=1-p}^{p-1} \gamma_s, \qquad (3.3)$$

where $\gamma_s = \gamma_s(\omega, \omega')$, $\gamma_{-s} = \gamma_s(\omega', \omega)$, and $\gamma_s(\omega, \omega') = E(R_p(\omega)\ R_{p+s}(\omega'))$,

$$\gamma_0 = \prod_{j=1}^{p+1} \min\{\omega_j, \omega'_j\} - \sum_{i=1}^{p+1} \min\{\omega_i, \omega'_i\} \prod_{j=1,\, j\ne i}^{p+1} \omega_j\, \omega'_j + p \prod_{j=1}^{p+1} \omega_j\, \omega'_j, \qquad (3.4)$$

and for $p > 1$, $\ell = 1, \ldots, p-1$,

$$\gamma_\ell(\omega, \omega') = \prod_{j=1}^{p+1-\ell} \min\{\omega_j, \omega'_{j+\ell}\} \prod_{j=1}^{\ell} \omega'_j \prod_{j=p+2-\ell}^{p+1} \omega_j$$

$$- \sum_{i=1}^{p+1} \min\{\omega_i, \omega'_{i+\ell}\} \prod_{j=1,\, j\ne i}^{p+1} \omega_j \prod_{j=1,\, j\ne i+\ell}^{p+1} \omega'_j$$

$$+ (p-\ell) \prod_{j=1}^{p+1} \omega_j\, \omega'_j. \qquad (3.5)$$

Then, $T^{1/2}\, S_T(\omega)$ has zero asymptotic mean and the asymptotic covariance between $T^{1/2}\, S_T(\omega)$ and $T^{1/2}\, S_T(\omega')$ is $\Xi(\omega, \omega')$.

Let us introduce the p+1 dimensional Brownian bridge $W^0(\omega)$, $\omega = (\omega_1, \ldots, \omega_{p+1}) \in I^{p+1} \equiv [0, 1]^{p+1}$; i.e.

$$W^{0}(\omega)= W(\omega) - W(1,1,\dots,1) \prod_{j=1}^{p+1} \omega_j,$$

where $W(.)$ is the standard Wiener random field on I^{p+1}. The process

$$W^{\dagger}(\omega) = W^{0}(\omega) - \sum_{i=1}^{p+1} W^{0}(\omega_{(i)}) \prod_{j=1,\, j\neq i}^{p+1} \omega_j,$$

where $\omega_{(i)}= (1,1,\dots,1,\omega_i,1,\dots 1)$ is a vector of ones with ω_i in the i-th place, is a separable Gaussian process with $E(W^{\dagger}(\omega))= 0$, and for $\omega' = (\omega'_1,\dots,\omega'_{p+1})$, $E(W^{\dagger}(\omega)\, W^{\dagger}(\omega')) = \gamma_0$. That is, when p= 1, the empirical process $T^{1/2}\, S_T(\omega)$ has the same asymptotic mean and covariances as $W^{\dagger}(\omega)$. Furthermore, Skaug and Tjøstheim (1992) have proved that

$$B_T \xrightarrow{d} B \equiv \int_{I^2} W^{\dagger}(\omega)^2\, d\omega, \text{ when } p = 1.$$

Then, in the case p = 1, B_T has the same asymptotic distribution as the HBKR statistic for testing independence of two random variables. In this case, Blum et al (1961) have found the characteristic function and have tabulated the corresponding distribution.

When p≥ 2, the asymptotic covariances of $T^{1/2}\, S_T(\omega)$ differ from those of $W^{\dagger}(\omega)$. Let $W^{*}(\omega)$, $\omega \in I^{p+1}$, be a separable Gaussian process with $E(W^{*}(\omega))= 0$, and for $\omega' \in I^{p+1}$, $E(W^{*}(\omega)\, W^{*}(\omega')) = \Xi(\omega, \omega')$. The following theorem, proved in the appendix, states the asymptotic null distribution of B_T in the general case.

Theorem 1: Under H_0, $B_T \xrightarrow{d} B \equiv \int_{I^{p+1}} W^{*}(\omega)^2\, d\omega$, for any $p \geq 1$ ∎.

Let us define B_α as $\Pr\{B > B_\alpha\}= \alpha$. Whenever B_α is available, an asymptotic test at the α-level of significance consists of rejecting H_0 when the observed value of B_T, for a given sample, is greater than B_α.

The consistency of the test follows whenever a Glivenko-Cantelli theorem under the alternative hypothesis is available. The following theorem, proved in the appendix, states that the test is consistent when the observed sample is an ergodic sequence.

Theorem 2: Under H_1, assuming that $\{X_t,\ t\geq 1\}$ is an ergodic sequence, then

$$\lim_{n\to\infty} \Pr\{B_T > B_\alpha\} = 1 \ \blacksquare.$$

Calculation of the critical values B_α does not seem an easy task. Cotterill and Csörgö (1985) have found the characteristic function of $\int_{I^{p+1}} W^{\dagger}(\omega)^2\, d\omega$, and have tabulated its distribution, for p> 1. The resampling procedure discussed in next section is easy to implement and enjoys better level accuracy than its asymptotic counterpart.

4. COMPUTING CRITICAL VALUES BY RANDOM PERMUTATION.

An exact test based on the statistic B_T consists on rejecting H_0 when the observed statistic value exceeds $B_{T\alpha}$ defined as,

$$\Pr\{B_T \leq B_{T\alpha} \mid H_0\} = 1-\alpha.$$

The resampling based estimate of $B_{T\alpha}$ presented in this Section is more accurate than the asymptotic critical value B_α.

Let $\{\eta_1,\ldots,\eta_{T+p}\}$ be a random permutation of the integers $\{1,\ldots,T+p\}$. So, we construct the sample $\mathcal{X}^* = \{Z_1^*,\ldots,Z_T^*\}$, where $Z_t^* = \{X_{\eta_t},\ldots,X_{\eta_{t+p}}\}$, and from this sample we calculate the statistic

$$B_T^* = \sum_{t=1}^{T} S_T^*\,(Z_t^*)^2, \qquad (4.1)$$

where

$$S_T^*(x) = T^{-1}\sum_{t=1}^{T}\prod_{j=1}^{p+1} 1(X_{\eta_{t+j-1}} \leq x_j) - \prod_{j=1}^{p+1}\{T^{-1}\sum_{t=1}^{T} 1(X_{\eta_{t+j-1}} \leq x_j)\}. \qquad (4.2)$$

We are, in fact, sampling from the distribution

$$\hat{F}_{0T}(x) = \prod_{j=1}^{p+1} \hat{F}_{1T}(x_j), \tag{4.3}$$

where $\hat{F}_{1T}(x_j) = (T + p)^{-1} \sum_{t=1}^{T} 1(X_t \le x_j)$ is an estimate of the marginal distribution $F_1(x)$, based on the observed sample $\mathcal{E} = \{X_1, X_2, \ldots, X_{T+p}\}$. That is, for each sample we can obtain (T+p)! different samples $\mathcal{X}_i^+ = \{Z_{i1}^+, \ldots, Z_{iT}^+\}$, where $Z_{it}^+ = \{X_{\xi_t^i}, \ldots, X_{\xi_{t+p}^i}\}$ and $\{\xi_t^i, \ldots, \xi_{t+p}^i\}$, $i=1, \ldots, (T+p)!$ are all possible permutations with integers $(1, \ldots, (T+p)!)$. Then, for fixed x, both under H_0 and under H_1,

$$E\{S_T^*(x) | \hat{F}_{0T}\} = E\{S_T^*(x) | \mathcal{E}\} = ((T+p)!)^{-1} \sum_{i=1}^{(T+p)!} S_{iT}^+(x) = 0, \tag{4.4}$$

where $S_{iT}^+(x)$ is obtained as $S_T^*(x)$ but using the sample $\{Z_{i1}^+, \ldots, Z_{iT}^+\}$. The resampled estimate of $B_{T\alpha}$ based on the observed sample $\mathcal{E}$ is $\hat{B}_{T\alpha}$, defined as

$$1-\alpha = \Pr\{B_T^* \le \hat{B}_{T\alpha} | \mathcal{E}\} = ((T+p)!)^{-1} \sum_{i=1}^{(T+p)!} 1(B_{iT}^+ \le \hat{B}_{T\alpha}), \tag{4.5}$$

where $B_{iT}^+ = \sum_{t=1}^{T} S_{iT}^+(Z_{it}^+)^2$.

The resampling test rejects H_0 in favor of H_1 if B_T is greater than $\hat{B}_{T\alpha}$. In the same way, the exact p-values are approximated by

$$\Pr\{B_T^* > B_T | \mathcal{E}\} = ((T+p)!)^{-1} \sum_{i=1}^{(T+p)!} 1(B_{iT}^+ > B_T). \tag{4.6}$$

Calculation of the critical values $\hat{B}_{T\alpha}$ and p-values is computationally demanding, and they are approximated by repeated resampling, using random permutation. That is, M random permutations of the integers $(1, \ldots, T+p)$, $\{\eta_t^i, \ldots, \eta_{t+p}^i\}$, $i = 1, \ldots, M$ are drawn and, for each permutation, we compute B_{iT}^*, $i = 1, \ldots, M$, in the same way as B_T^* but using samples $\mathcal{X}_i^* = \{Z_{i1}^*, \ldots, Z_{iT}^*\}$, and $Z_{i1}^* = \{X_{\eta_t^i}, \ldots, X_{\eta_{t+p}^i}\}$. Then, for fixed x,

$$M^{-1} \sum_{i=1}^{M} S_{iT}^*(x) \xrightarrow{P} 0,$$

and, for fixed b,

$$M^{-1} \sum_{i=1}^{M} 1(B_{iT}^* \le b) \xrightarrow{P} \Pr\{B_T^* \le b | \mathcal{E}\}.$$

Then $\hat{B}_{T\alpha}$ is approximated as accurately as desired by $\hat{B}^{(M)}_{T\alpha}$, where

$$M^{-1} \sum_{j=1}^{M} 1(B^{*}_{jT} \leq \hat{B}^{(M)}_{T\alpha}) = 1 - \alpha. \tag{4.7}$$

Similarly, the p-value is approximated by

$$M^{-1} \sum_{j=1}^{M} 1(B^{*}_{jT} > B_{T}) = 1 - \alpha.$$

This random permutation technique for approximating critical values is usually employed for implementing randomization tests; see eg. Noreen (1989) and Edgington (1987). In a time series context, this technique has been used before by Chan and Tran (1990).

The resampling testing procedure can be formally justified in the same way as any other bootstrap type test, see Hall and Hart (1990) or Hall (1992) Chap. 3. From Theorem 1, and applying results in Götze (1985) on asymptotic expansions in functional limit theorems, under H_0,

$$\Pr\{B_T \leq \mathfrak{z}\} = \Pr\{B \leq \mathfrak{z}\} + T^{-1} P_1(\mathfrak{z}, F_1) + O(T^{-2}), \tag{4.8}$$

where $P_1(\mathfrak{z}, F_1)$ is a function of the quantile $\mathfrak{z}$ and the marginal distribution function F_1 of X. Then, under H_0,

$$B_{T\alpha} = B_{\alpha} - T^{-1} P_1(B_{\alpha}, F_1)\, b(B_{\alpha})^{-1} + O_p(T^{-2}), \tag{4.9}$$

where $b(.)$ is the density of B.

Noting that B^{*}_{T}, conditional on the sample $\mathfrak{C}$, has the distribution that B_T would have if $\{Z_t,\ t \geq 1\}$ were drawn from a population with distribution $\hat{F}_{0T}$ in (4.3), the sample counterpart of (4.8) is

$$\Pr\{B^{*}_{T} \leq \mathfrak{z} | \mathfrak{C}\} = \Pr\{B \leq \mathfrak{z}\} + T^{-1} P_1(\mathfrak{z}, \hat{F}_{T1}) + O_p(T^{-2}). \tag{4.10}$$

Since, under H_0, $\sup_{x} |\hat{F}_{1T}(x) - F_1(x)| = O_p(T^{-1/2})$, and $P_1(.,.)$ is a smooth function,

$$\Pr\{B^{*}_{T} \leq \mathfrak{z} | \mathfrak{C}\} = \Pr\{B \leq \mathfrak{z}\} + T^{-1} P_1(\mathfrak{z}, F_1) + O_p(T^{-3/2}).$$

By (4.8),

$$\Pr\{B^*_T \leq \chi | \mathcal{E}\} = \Pr\{ B_T \leq \chi \} + O_p(T^{-3/2}), \tag{4.11}$$

and

$$\hat{B}_{T\alpha} = B_{T\alpha} + O_p(T^{-3/2}). \tag{4.12}$$

The asymptotic test has level error of order T^{-1}, while the error of the resampled test is of order $T^{-3/2}$. Under H_0, the exact critical values $B_{T\alpha}$ are approximated by $\hat{B}_{T\alpha}$ with error of order $T^{-3/2}$, while the asymptotic critical values B_α approximate $B_{T\alpha}$ with error of order T^{-1}.

Under the alternative hypothesis, B^*_T is the statistic obtained with a sample drawn at random from the joint distribution $\hat{F}_{0T}$ defined in (4.3). Then, (4.10) is still valid under H_1. Under alternative hypotheses as in Theorem 2, $\sup_x|\hat{F}_{1T}(x) - F_1(x)| = o_p(1)$ by the Glivenko-Cantelli Theorem for ergodic sequences (see Stute and Schumann (1980)), and from (4.10), $\Pr\{B^*_T \leq \chi|\mathcal{E}\} = \Pr\{B_T \leq \chi \} + o_p(1)$. So, $\hat{B}_{T\alpha} = B_\alpha + o_p(1)$. That is, the resampled critical point consistently estimates the asymptotic critical points under ergodicity.

5. MONTE CARLO EXPERIMENTS AND AN EMPIRICAL EXAMPLE USING EXCHAGE RATES DATA

In these Monte Carlo experiments we compare the empirical power function of the test based on the statistic B_T and the popular Ljung-Box Q statistic (Ljung and Box (1978)), based on the squared correlation coefficients. Both tests are only computed for p = 1.

The empirical power function is computed as the proportion of times that H_0 is rejected in 5000 replications of the time series model. We consider tests where the critical values are approximated from the quantiles of the asymptotic null distribution of the corresponding statistic, and from the random permutation procedure presented in section 3. We only consider first order dependence alternatives (p = 1) in the following two models,

$$X_t = b\, X_{t-1} + \varepsilon_t, \text{ where } \varepsilon_t \sim \text{iid } N(0,\ 1), \tag{5.1}$$

$$X_t = b\ \varepsilon^2_{t-1} + \varepsilon_t, \text{ where } \varepsilon_t \sim \text{iid } N(0,\ 1). \qquad (5.2)$$

The Q test is more powerful than any other when the dependence structure is linear. Then, the Q test is expected to work better than the empirical distribution function (EDF) test in the AR(1) model (5.1), while the EDF test is expected to perform better in the nonlinear MA(1) model (5.2).

The empirical power function has been computed for T= 20, 50, 100 and different values of the parameter b in the two models. The critical values in the random permutation test are computed using 500 random permutations. The results for model (5.1) are in Table 1 and for model (5.2) are in Table 2. Both tables indicate that the random permutation test enjoys good level,even for the smallest sample size, and the power is not worse than the test based on asymptotic critical values. The resampled procedure is time consuming, since at each replication the generated sample has to be permuted 500 times. Of course, level accuracy is expected to increase as the number of random permutation increases.

In model (5.1) the EDF and Q test behave similarly, both in their asymptotic and resampled versions, for any value of b and any significance level α. However, the EDF test overperforms the Q test in model (5.2), which exhibits a more subtle dependence structure.

Table 3 summarizes an empirical application of the test, in the asymptotic and resampled versions, to testing serial independence of exchange rates changes using New York stock market data. The observations are $X_t = \log P_t - \log P_{t-1}$, where P_t is the exchange rate in period t. The data are recorded, from January 1977 to April 1988, daily, monthly, and quarterly for three currencies against the US dollar: sterling pound, deutschmark, and japanese yen.

The effectiveness of several serial independence tests has been tried with this sort of data. Whistler (1990) found, using UK data recorded in a similar period, that based on parametric autoregressive conditional heteroskedasticity (ARCH) models, the independence hypothesis is rejected employing daily data but it cannot be rejected with monthly data. Based on the same data, Robinson (1991) rejected the independence null hypothesis in all cases, using his entropy based nonparametric test. With the same data, Pinkse (1993) also rejected the null hypothesis in all cases using a nonparametric test based on the empirical characteristic function.

With daily observations, our test always rejects the null hypothesis. However, the Q test only detects linear dependence in the deutschmark case. Using monthly observations, our statistic takes values above the asymptotic and resampled critical values at $\alpha = 0.05$ level of significance in all cases. However, at the $\alpha = 0.01$ level, the null hypothesis is not rejected by our test in any of its versions for all currencies except the yen. In the yen case, the observed value of the statistic is greater than the resampled critical value. This is not a surprise. Plotting the squared series, a structural change is clearly observed. There is more volatility in the first part of the sample. This may induce rejection of the null hypothesis. This argument also explains why, using quarterly data, the independence hypothesis cannot be rejected, with all tests, for the pound and deutschmark, but it is rejected for the yen. Using quarterly data for the yen, our test rejects clearly the null hypothesis, and the resampled p-value of the Q test is only 0.105.

APPENDIX

Proof of Theorem 1:

In order to obtain the infinite dimensional asymptotic distribution of $T^{1/2}\, S_T(\omega)$ in the general case, we first note that

$$T^{-1} \sum_{t=1}^{T} R_t(\omega) = (p + 1)^{-1} \sum_{j=0}^{p} S_{jN}(\omega), \tag{a.1}$$

where $R_t(.)$ is defined in (3.2) and

$$S_{jN}(\omega) = N^{-1} \sum_{t=1}^{N} R_{pt+\,t-j}(\omega),\ j = 0,\dots,p, \tag{a.2}$$

N= T/(p+1), and assuming, without loss of generality, that N is an integer. Under H_0, the random vectors $\{Z_{pt+t-j}, t=1,\dots,N\}$ are independent, for all j = 0,...,p. Then, using Blum et al (1961) results, the infinite dimensional asymptotic distribution of each $S_{jN}(\omega)$ is that of $W_j^{\dagger}(\omega)$, j= 0,...,p, where each $W_j^{\dagger}(\omega)$ is distributed as $W^{\dagger}(\omega)$. Furthermore, using Theorem 3 in Csörgö (1979)

$$\sup_{I^{p+1}} |N^{1/2} S_{jN}(\omega) - W_{jN}^{\dagger}(\omega)| = O(N^{[-1/2(p+2)]}\ (\log N)^{3/2}),\ \text{a.s.} \quad p \ge 1,\ j= 0,\dots,p, \tag{a.3}$$

and

$$\sup_{I^{p+1}} |N\, S_{jN}(\omega)| = O((N \log \log N)^{1/2}),\ \text{a.s.}\ p \ge 1,\ j= 0,\dots,p, \tag{a.4}$$

where each $\{W_{jN}^{\dagger}(\omega),\ \omega \in I^{p+1}\}$, j= 0,...,p is a sequence of separable Gaussian processes distributed as $W_j^{\dagger}(\omega)$, j=0,...,p. Therefore, from (a.3),

$$\sup_{I^{p+1}} |T^{-1/2} \sum_{t=1}^{T} R_t(\omega) - (p + 1)^{-1/2} \sum_{j=0}^{p} W_{jN}^{\dagger}(\omega)|$$

$$= \sup_{I^{p+1}} |(p + 1)^{-1/2} \sum_{j=0}^{p} (N^{1/2} S_{jN}(\omega) - W_{jN}^{\dagger}(\omega))|$$

$$\le (p + 1)^{-1/2} \sum_{j=0}^{p} \sup_{I^{p+1}} |N^{1/2} S_{jN}(\omega) - W_{jN}^{\dagger}(\omega)|$$

$$= O(N^{[-1/2(p+2)]}\ (\log N)^{3/2})\ \text{a.s.} \tag{a.5}$$

Note that $W^*_T(\omega)= (p + 1)^{-1/2}\sum_{j=0}^{p} W^\dagger_{jN}(\omega)$ is a separable Gaussian process with mean and covariances as $T^{-1/2} \sum_{t=1}^{T} R_t(\omega)$. That is, $E(W^*(\omega))= 0$, and for $\omega' \in I^{p+1}$, $E(W^*(\omega)\, W^*(\omega')) = \Xi(\omega, \omega')$. From (a.4)

$$\sup_{I^{p+1}} \left|\sum_{t=1}^{T} R_t(\omega)\right| = O((T \log\log T)^{1/2}) \quad \text{a.s.} \tag{a.6}$$

Then, (3.1), (a.5) and (a.6) show that

$$\sup_{I^{p+1}} \left|T^{1/2}S_T(\omega)- W^*_T(\omega)\right|= O(T^{[-1/2(p+2)]} (\log T)^{3/2}), \text{ a.s.} \tag{a.7}$$

$$\sup_{I^{p+1}} \left|T\, S_T(\omega)\right| = O((T \log\log T)^{1/2}) \quad \text{a.s.} \tag{a.8}$$

From (a.7) and (a.8), and using same arguments as (4.57) in Csörgö (1979),

$$\int_{\mathbb{R}^{p+1}} S_T^2(x)\, d \prod_{j=1}^{p+1} F_1(x_j) \xrightarrow{d} B \equiv \int_{I^{p+1}} W^*(\omega)^2\, d\omega.$$

Applying the Lemma in Kiefer (1959) as in Blum et al (1961),

$$B_T- \int_{\mathbb{R}^{p+1}} S_T^2(x)\, d \prod_{j=1}^{p+1} F_1(x_j) = o_p(1). \ \square$$

Proof of Theorem 2:

It suffices to prove that

$$T^{-1} B_T - E(S(Z_1)^2) \longrightarrow 0 \text{ a.s.},$$

which is proved from

$$T^{-1} \sum_{t=1}^{T} \left(S_T(Z_t)^2 - S(Z_t)^2\right) \longrightarrow 0 \text{ a.s}, \tag{a.7}$$

$$T^{-1} \sum_{t=1}^{T} S(Z_t)^2 - E(S(Z_1)^2) \longrightarrow 0 \text{ a.s.} \tag{a.8}$$

Stute and Schumann (1980) theorem proves (a.7) and (a.8) follows from $\{S(Z)_t^2,\ t\geq 1\}$ ergodic. $\square$

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TABLE 1

Empirical power function of B_T based on 5000 replications of the model $x_t = bx_{t-1} + \varepsilon_t$, $\tau = 1, \ldots, T$, where $\varepsilon_t \sim$ iid N(0,1), for T = 20, 50, 100.

α=0.05												
	EDF Permutation Test			EDF Asymptotic Test			Q Permutation Test			Q Asymptotic Test		
	T= 20	T= 50	T=100	T= 20	T= 50	T=100	T= 20	T= 50	T=100	T=20	T= 50	T=100
b= 0	0.0566	0.0518	0.0512	0.0626	0.0532	0.0512	0.0520	0.0528	0.0524	0.0668	0.0558	0.0508
b= 0.1	0.0786	0.1192	0.1555	0.0860	0.1224	0.1524	0.0502	0.0872	0.1320	0.0610	0.0902	0.1344
b= 0.2	0.1424	0.2554	0.4526	0.1592	0.2628	0.4562	0.0798	0.2264	0.4550	0.0982	0.2330	0.4552
b= 0.3	0.2230	0.4902	0.7928	0.2400	0.4940	0.7940	0.1536	0.4810	0.8204	0.1874	0.4896	0.8214
b= 0.4	0.3479	0.7300	0.9540	0.3708	0.7370	0.9544	0.2676	0.7364	0.9686	0.3044	0.7512	0.9678
b= 0.5	0.4930	0.8978	0.9948	0.5136	0.9004	0.9950	0.4204	0.9056	0.9966	0.4676	0.9126	0.9972
b= 0.6	0.6216	0.9686	0.9996	0.6380	0.9698	0.9996	0.5760	0.9776	1.0000	0.6156	0.9802	1.0000
b= 0.7	0.7470	0.9920	1.0000	0.7658	0.9922	1.0000	0.7200	0.9950	1.0000	0.7518	0.9945	1.0000
b= 0.8	0.8292	0.9976	1.0000	0.8416	0.9982	1.0000	0.8106	0.9998	1.0000	0.8320	0.9998	1.0000
b= 0.9	0.8950	0.9998	1.000	0.9042	0.9998	1.0000	0.8862	0.9994	1.0000	0.9034	0.9994	1.0000
α=0.01												
	EDF Permutation Test			EDF Asymptotic Test			Q Permutation Test			Q Asymptotic Test		
	T= 20	T= 50	T=100	T= 20	T= 50	T=100	T= 20	T= 50	T=100	T= 20	T= 50	T=100
b= 0	0.0128	0.0128	0.0122	0.0112	0.0112	0.0090	0.0122	0.0126	0.0110	0.0124	0.0114	0.0092
b= 0.1	0.0246	0.0459	0.0570	0.0220	0.0388	0.0494	0.0124	0.0248	0.0456	0.0120	0.0226	0.0393
b= 0.2	0.0476	0.1166	0.2542	0.0466	0.1094	0.2392	0.0232	0.0890	0.2596	0.0244	0.0830	0.2368
b= 0.3	0.0938	0.2932	0.5946	0.0902	0.2796	0.5754	0.0488	0.2780	0.6298	0.0502	0.2596	0.6192

TABLE 2

Empirical power function of B_T based on 5000 replications of the model $x_t = b\varepsilon^2_{t-1} + \varepsilon_t$, t= 1,..., T, where $\varepsilon_t \sim$ iid N(0,1) for T= 20, 50, 100.

α=0.05												
	EDF Permutation Test			EDF Asymptotic Test			Q Permutation Test			Q Asymptotic Test		
	T= 20	T= 50	T=100	T= 20	T= 50	T=100	T= 20	T= 50	T=100	T=20	T= 50	T= 100
b= 0	0.0566	0.0518	0.0512	0.0626	0.0532	0.0512	0.0552	0.0528	0.0524	0.0668	0.0558	0.0508
b= 0.1	0.0500	0.0608	0.0578	0.0568	0.0620	0.0570	0.0544	0.0622	0.0596	0.0692	0.0660	0.0584
b= 0.2	0.0554	0.0680	0.0954	0.0652	0.0698	0.0936	0.0660	0.0786	0.0822	0.0816	0.0822	0.0814
b= 0.3	0.0602	0.0938	0.1736	0.0698	0.0958	0.1740	0.0844	0.0944	0.1056	0.1014	0.0974	0.1040
b= 0.4	0.0728	0.1250	0.2472	0.0786	0.1290	0.2508	0.0982	0.1190	0.1288	0.1150	0.1222	0.1298
b= 0.5	0.0820	0.1388	0.3326	0.0936	0.1378	0.3292	0.1020	0.1362	0.1388	0.1226	0.1406	0.1382
b= 0.6	0.0806	0.1474	0.3772	0.0910	0.1508	0.3740	0.1116	0.1356	0.1460	0.1230	0.1364	0.1414
b= 0.7	0.0926	0.1698	0.4052	0.1030	0.1762	0.4084	0.1202	0.1526	0.1600	0.1286	0.1490	0.1572
b= 0.8	0.0872	0.1672	0.4414	0.0954	0.1726	0.4366	0.1078	0.1488	0.1654	0.1142	0.1424	0.1552
b= 0.9	0.1008	0.1740	0.4410	0.1074	0.1780	0.4370	0.1290	0.1444	0.1556	0.1228	0.1368	0.1476
α=0.01												
	EDF Permutation Test			EDF Asymptotic Test			Q Permutation Test			Q Permutation Test		
	T= 20	T= 50	T=100	T= 20	T= 50	T=100	T= 20	T= 50	T=100	T= 20	T= 50	T= 100
b= 0	0.0128	0.0128	0.0122	0.0112	0.0112	0.0090	0.0122	0.0126	0.0110	0.0124	0.0114	0.0092
b= 0.1	0.0152	0.0160	0.0120	0.0138	0.0142	0.0102	0.0122	0.0174	0.0134	0.0116	0.0152	0.0114
b= 0.2	0.0110	0.0174	0.0246	0.0108	0.0156	0.0206	0.0166	0.0218	0.0232	0.0164	0.0198	0.0188

TABLE 3

Statistic values, resampled critical values and p-values from exchange rates changes data of sterling pound, yen, and deutschmark with respect to US dollar in New York Stock Market from January 1977 to April 1988, (2,832 daily, 158 monthly, and 53 quarterly observations). Resampled critical values are computed using 1,000 random permutations of the original series.

A. Sterling Pound/US Dollar

	Daily		Monthly		Quarterly	
	EDF	Q	EDF	Q	EDF	Q
Statistic	0.140	0.076	0.071	0.416	0.016	0.001
Critical Values:						
$\alpha = 0.05$						
Asymptotic	0.058	3.840	0.058	3.840	0.058	3.840
Resampled	0.060	3.836	0.055	3.468	0.060	4.211
$\alpha = 0.01$						
Asymptotic	0.087	6.630	0.087	6.630	0.087	6.630
Resampled	0.093	6.171	0.079	5.544	0.087	6.406
P-Value (Resampled)	0.002	0.796	0.015	0.507	0.771	0.986

B. Deutschmark/US Dollar

	Daily		Monthly		Quarterly	
	EDF	Q	EDF	Q	EDF	Q
Statistic	0.158	7.326	0.071	0.170	0.017	0.053
Critical Values:						
$\alpha = 0.05$						
Asymptotic	0.058	3.840	0.058	3.840	0.058	3.840
Resampled	0.059	3.582	0.057	3.758	0.057	3.713
$\alpha = 0.01$						
Asymptotic	0.087	6.630	0.087	6.630	0.087	6.630
Resampled	0.082	5.645	0.085	6.664	0.084	6.427
P-Value (Resampled)	0.000	0.004	0.0269	0.880	0.712	0.842

TABLE 3 (Cont.)

C. Yen/US Dollar

	Daily		Monthly		Quarterly	
	EDF	Q	EDF	Q	EDF	Q
Statistic	0.112	2.389	0.081	0.978	0.083	2.447
Critical Values:						
α = 0.05						
Asymptotic	0.058	3.840	0.058	3.840	0.058	3.840
Resampled	0.061	3.990	0.054	3.664	0.057	3.539
α = 0.01						
Asymptotic	0.087	6.630	0.087	6.630	0.087	6.630
Resampled	0.100	6.230	0.073	5.673	0.081	5.691
P-Value (Resampled)	0.003	0.115	0.007	0.338	0.005	0.105