This is a postprint version of the following published document:


DOI: 10.1109/ISWCS49558.2021.9562137

© 2021 IEEE. Personal use of this material is permitted. Permission from IEEE must be obtained for all other uses, in any current or future media, including reprinting/republishing this material for advertising or promotional purposes, creating new collective works, for resale or redistribution to servers or lists, or reuse of any copyrighted component of this work in other works.
Abstract—The Fisher-Snedecor $F$ distribution has been recently proposed as an experimentally verified and tractable turbulence induced fading model (TIFM) for free space optical (FSO) communications. This paper provides outage probability (OP) and higher-order (HO) performance analysis of the product of $N$ independent but not identically distributed (i.n.i.d) Fisher-Snedecor $F$ random variates (RVs). Accurate and closed-form (C-F) expressions for cumulative distribution function (CDF), level crossing rate (LCR) and average fade duration (AFD) of $N$-Fisher-Snedecor $F$ distribution are successfully derived. The general property of a Laplace approximation approach for evaluation of $N$-folded complex integral-form (I-F) LCR expressions has been applied. The obtained statistical results are directly related to the performance evaluation of $N$-hop FSO communication links over weak, moderate and strong atmospheric turbulence conditions.

Index Terms—Fisher-Snedecor $F$ distribution, FSO communications, Higher order statistics, Outage probability.

I. INTRODUCTION

FSO communications are envisioned to be incorporated in emerging 5G and beyond 5G (B5G) communication systems [1]-[2]. FSO communications can provide many advantages in wireless communications. In particular, FSO links are cost effective and spectrum license free. Moreover, FSO communications provide large bandwidth and due to narrow beam widths, FSO communications can enable high level of security, interference immunity and increase energy efficiency of wireless transmission systems. On the other hand, atmospheric turbulence can drastically impact propagation during FSO transmission and can cause the system performance deterioration. Atmospheric weather conditions as well as pointing errors between transmitter and receiver are other relevant phenomena that can cause additional FSO system performance deterioration.

The Fisher-Snedecor $F$ distribution has been recently proposed as an accurate and experimentally verified TIFM for FSO communications [3]. The Fisher-Snedecor $F$ TIFM for FSO communications scenario with pointing errors in terms of the first-order statistical measures is considered in [4], whereas the hybrid mmWave/FSO transmission system over the Fisher-Snedecor $F$ turbulence induced fading channel is addressed in [5]. The Fisher-Snedecor $F$ distribution has been initially introduced in [6]. In [7], authors introduced cascaded N-Fisher-Snedecor $F$ distribution modeled as the product of $N$ independent but not identically distributed (i.n.i.d) RVs for feasible application in performance assessment of multi-hop relay communications. The first-order performance analysis of Fisher-Snedecor $F$ fading distribution have been reported in [3]-[7].

The HO statistics of composite $F$ fading model are provided in [8] while the HO measures of bivariate Fisher-Snedecor $F$ RVs are considered in [9]. In order to get broader insight into the behaviour of FSO systems over rapidly time-variant turbulence induced fading channels, the HO statistical analysis is needed. Indeed, a level crossing rate (LCR) is time rate of change of the turbulence induced faded signal, while average fade duration (AFD) is the mean time of the turbulence induced faded signal being below a specified threshold. Moreover, the HO statistics provide useful knowledge of time-variant turbulence induced fading channels for a variety of 5G and beyond 5G FSO communication scenarios [10]-[11]. The HO performance measures over turbulence induced fading channels for FSO communications have been investigated in [12]-[14]. However, there are no reported results on LCR and AFD over N-Fisher-Snedecor $F$ turbulence induced fading channels and its application to N-Hop FSO relay communications over weak, moderate and strong turbulence conditions.

This paper provides accurate and fast-computing, C-F HO statistical expressions such as LCR and AFD of N-Fisher-Snedecor $F$ distribution obtained by general Laplace approximation method (GLAM). The obtained results for OP and HO statistics are related to the system performance of N-hop FSO amplify and forward relay communication system over N-Fisher-Snedecor $F$ turbulence channels. Namely, we investigate the impact of the number of hops under weak, moderate and strong turbulence severity conditions on OP and HO performance measures of N-hop relay FSO system.
II. FISHER-SNEDECOR $F$ COMPOSITE FADEING MODEL

The recently introduced Fisher-Snedecor $F$ TIFM for FSO communications can be modeled as the product of independent but not identically distributed (i.i.d) Gamma and normalized inverse Gamma (I-Gamma) random variates (RVs) [3]. The Fisher-Snedecor $F$ fading model $Z_F(t)$ can be written as:

$$Z_F(t) = X_i(t)Y_i(t) = X_i(t)\frac{1}{y_i(t)}$$  \hspace{1cm} (1)

where $X_i(t)$ and $Y_i(t)$ are Gamma and normalized I-Gamma random processes (RPs), respectively. Since the normalized I-Gamma RV can be expressed as $Y_i = \frac{Z}{x_i}$, the probability density functions (pdfs) of $X_i$ and $Y_i$ are given as, respectively:

$$p_{X_i}(x_i) = \left(\frac{m_o}{\Gamma(m_o)}\right)^{m_o}x_i^{m_o-1}e^{-x_i/m_o}$$ (2)

$$p_{Y_i}(y_i) = \left(\frac{(m_o-1)/\Gamma(m_o)}{\Gamma(m_o)}\right)^{m_o}y_i^{m_o-1}e^{-y_i/m_o}$$ (3)

whose shape parameters are $m_o$, and $m_o$, respectively, whereas the average powers are $\Omega_x$ and $\Omega_y$, respectively. The $\Gamma()$ is the Gamma function [15, Eq. (8.310.1)].

The pdf of $Z_F$ can be obtained as:

$$p_{Z_F}(z) = \int_0^\infty \frac{dz}{dz} p_{X_i}(x_i) p_{Y_i}(y_i) dx_i dy_i$$ (4)

where $\frac{dz}{dz} = y_i$. After substitutions (2) and (3) in (4), and using transformations [15, Eq. (3.326.2)] and [15, Eq. (8.384.1)], respectively, the pdf of $Z_F$ can be written as:

$$p_{Z_F}(z) = \left(\frac{(m_o-1)/\Gamma(m_o)}{\Gamma(m_o)}\right)^{m_o} \left(\frac{m_o}{\Gamma(m_o)}\right)^{m_o} \left(\frac{1}{\Omega_x \Omega_y} + 1\right)^{m_o-1}$$ (5)

where $\beta(i,\cdot)$ is the Beta function [15, Eq. (8.380.1)]. It can be observed that $p_{Z_F}(z)$ in (5) for $\Omega_x = 1$, $\Omega_y = 1$, $m_o = a$ and $m_o = b$ reduces to the pdf of Fisher-Snedecor $F$ distribution given by [3, Eq. (6)].

III. N-FISHER-SNEDECOR $F$ TURBULENCE INDUCED FADEING MODEL

The cascaded N-Fisher-Snedecor $F$ TIFM can be modeled as the product of $N$ i.i.d Gamma and normalized I-Gamma RVs:

$$Z_{F,i}(t) = \prod_{i=1}^N X_i(t)Y_i(t) = \prod_{i=1}^N X_i(t)\frac{1}{y_i(t)}$$ (6)

where Gamma and normalized Gamma pdfs, $p_{X_i}(x_i)$ and $p_{Y_i}(y_i)$ are:

$$p_{X_i}(x_i) = \frac{a_i^\alpha x_i^{\alpha-1}e^{-a_i x_i}}{\Gamma(a_i)}, \quad i = 1, N$$ (7)

$$p_{Y_i}(y_i) = \frac{b_i^\beta y_i^{\beta-1}e^{-b_i y_i}}{\Gamma(b_i)}$$ (8)

where $a_i = \frac{1}{\Omega_x}$, $b_i = \frac{1}{\Omega_y}$. The $\alpha$, and $\beta$ are N-Fisher-Snedecor $F$ small-scale and large-scale characteristics, respectively related to TIFM severity conditions.

The scintillation index of N-Fisher-Snedecor $F$ TIFM can be expressed as [3, Eq. (10)]:

$$\sigma^2_{S_{F,i}} = \left(1 + \frac{1}{a_i}\right) \left(1 + \frac{1}{b_i}\right) - 1, \quad b_i > 2$$ (9)

The $\alpha_i$ and $\beta_i$ of N-Fisher-Snedecor $F$ TIFM can be written as [4, Eq. (2)]:

$$\alpha_i = \frac{1}{\beta_i}, \quad \beta_i = \frac{1}{\alpha_i} + 1$$ (10)

where $\alpha_i$ and $\beta_i$ are normalized log-irradiance variances of $X_i$ and $Y_i$, respectively. Under the assumption of spherical propagation, the $\sigma^2_{S_{F,i}}$ is [4, Eq. (3)]:

$$\sigma^2_{S_{F,i}} = \frac{0.514 \sigma^2_{SP,i} (1 + 0.632 \gamma_{SP,i})}{1 + 0.904 \gamma_{SP,i}^{1/2} + 0.632 \gamma_{SP,i}^{1/2}}$$ (11)

where $\sigma^2_{SP,i}$ represents the spherical scintillation index and by assuming further weak fluctuation conditions, $\sigma^2_{SP,i}$ is:

$$\sigma^2_{SP,i} = 9.65\sigma^2_{SP,i} \left(\frac{0.4+(1+\gamma_{SP,i})^{1/2}}{\sin^2(\frac{\gamma_{SP,i}}{4}) \tan(Q_i)} + \frac{2.61}{(9+Q_i^2)^{1/4} \sin^2(\frac{\gamma_{SP,i}}{4}) \tan(Q_i)} - \frac{0.52}{(9+Q_i^2)^{1/4} \sin^2(\frac{\gamma_{SP,i}}{4}) \tan(Q_i)}\right)$$ (12)

where $Q_i = 10.89 L_i/(\beta_i^2 L_i^2)$, $L_i$ is the $i$-th propagation distance, $\beta_i = \frac{2\pi}{\lambda_i}$ is $i$-th wave number ($\lambda_i$-wavelength) and $b_i$ is $i$-th inner-scale. Furthermore, $d_i = S/L_i^2 Q_i$, and $\sigma^2_{SP,i}$ is the $i$-th refractive index. The $\sigma^2_{S_{F,i}}$ can be written as [3, Eq. (15)]:

$$\sigma^2_{S_{F,i}} = \sigma^2_{S_{F,i}}(b_i) - \sigma^2_{S_{F,i}}(L_i)$$ (13)

where $\sigma^2_{S_{F,i}}(b_i)$ and $\sigma^2_{S_{F,i}}(L_i)$ are inner and outer large-scale log-irradiance variances, respectively that can be expressed as:

$$\sigma^2_{S_{F,i}}(b_i) = 0.04 \gamma_{SP,i} \left(\frac{\eta(b_i)}{\eta(b_i) + Q_i^2} + \eta(b_i)\gamma_{SP,i}^{1/2}\right) + 1.75(\frac{\eta(b_i)}{\eta(b_i) + Q_i^2})$$ (14)

where $\eta(b_i) = \frac{\eta(L_i)}{\rho_i}$, and $\rho_i = \frac{\gamma_{SP,i}^{1/2}}{\rho_i}$.
A. Level Crossing Rate

Level crossing rate (LCR) of cascaded N-Fisher-Snedecor $F$ distribution for a given threshold $z_{th}$ can be written as:

$$N_{F_{z_{th}}}(z_{th}) = \int_{0}^{\infty} z_{th} P_{z_{th}}(z_{th}, \bar{z}_{th}) d\bar{z}_{th}$$

(15)

where, $P_{z_{th}}$ is the joint distribution of $Z_{F_{z_{th}}}$ and its first derivative $Z_{F_{z_{th}}}$ evaluated as:

$$Z_{F_{z}}\text{ expression and in the case of independent RVs can be}
$$

$\text{additional and individual pdfs R}$ as (20), provided on the next page, where

$$\text{writes from (16), Eq. (12)}$,

$\text{the N-Fisher-Snedecor distribution for a given threshold $z_{th}$ is:}$

$$N_{F_{z_{th}}}(z_{th}) = \frac{1}{\sqrt{2\pi}} \sum_{i=1}^{N} \sigma_{x_{th}}^{*} \sigma_{x_{th}}^{*} E_{i}$$

(24)

$\text{is the fixed gain of}$

$\text{and inverse Nakagami-m RPs has been recently provided in}$

$\text{product of N independent Nakagami-m}$

$\text{and inverse Gamma RPs can be written as (27), provided in terms of}$

$\text{of the Meijer’s G function}$

$\text{is evaluated in the Appendix.}$

B. Average Fade Duration

Average fade duration (AFD) of N-Fisher-Snedecor $F$ distribution can be defined as:

$$A_{F_{z_{th}}}(z_{th}) = \frac{P_{z_{th}}(z_{th})}{N_{F_{z_{th}}}(z_{th})}$$

(26)

$\text{A}$

$\text{where}$

$$\text{is the optical power can be represented by the optical intensity under the assumption that the area at reception is normalized to unity}$

$\text{is the transmitted information bit}$

$\text{is responsivity of the photo-detector}$

$\text{Fisher-Snedecor F TIFM can be written as}$

$$y_{i} = x_{i} z_{F_{i}} R_{i,1} + n_{i}$$

(28)

$\text{where the optical power can be represented by the optical intensity under the assumption that the area at reception is normalized to unity}$

$\text{is the fixed gain of}$

$$y_{N} = \prod_{i=1}^{N} s_{i} z_{F_{i}} R_{D,1} + \sum_{i=1}^{N} n_{i}$$

(30)

$\text{are Z-M G RVs. Based on (22) and after some mathematical manipulations, the variance of}$

$\text{Z_{F_{z}}}$

$\text{i = 1, N are the variances of } Z_{F_{i}} i = 1, N, \text{respectively. After substituting (21) in (20) and then (7), (8) in (20), the LCR of}$

$\text{Fisher-Snedecor F distribution for a given threshold}$

$\text{is:}$

$$Z_{F_{z}} = \prod_{i=1}^{N} Z_{F_{i}} Z_{F_{i}} R_{D,1} \text{ denote as, respectively}$$

$\text{are Z-M G RVs. Based on (22) and after some mathematical manipulations, the variance of}$

$\text{Z_{F_{z}}}$
Based on (30), the total channel gain is $S = \sum_{i=1}^{N} \gamma_{r_i} \cdot X_{r_i} \cdot R_{r_i}$. Furthermore, the received signal-to-noise ratio (SNR) of $i$-th link and average signal-to-noise ratio (SNR) of $i$-th link of the considered model are, respectively, $\gamma_{r_i} = \frac{\sum_{j=1}^{N} x_{r_j}^2 \cdot y_{r_j}^2}{\sum_{j=1}^{N} \sum_{j=1}^{N} x_{r_j}^2 \cdot y_{r_j}^2}$, $\bar{\gamma}_{r_i} = \frac{\sum_{j=1}^{N} x_{r_j}^2 \cdot y_{r_j}^2}{\sum_{j=1}^{N} \sum_{j=1}^{N} x_{r_j}^2 \cdot y_{r_j}^2}$, where $N_0$ is noise power of $i$-th link. Without loss of generality and in accordance with some previous works we define $\alpha_0 = \alpha_i = 1$ [17]. Under the assumption that the faded signal of $N$ hop FSO relay link can be directly estimated at destination, the N-Fisher-Snedecor $F$ faded signal at reception can be considered as cascaded one as in [17].

**IV. NUMERICAL RESULTS**

Numerical results for OP and HO statistical measures of N-Fisher-Snedecor $F$ distribution under weak ($\alpha_i = 4.5916$, $b_i = 7.0941$), moderate ($\alpha_i = 2.3378$, $b_i = 4.5323$) and strong ($\alpha_i = 1.4321$, $b_i = 3.4948$) TIFM conditions are presented in Figs. 1-3, respectively. It can be noticed that the approximate C-F expressions for LCR and A/FD fit well with the exact F-F expressions, especially for higher thresholds.

Fig. 1 presents OP of N-Fisher-Snedecor $F$ distribution. The OP is obtained by $P_{\text{out}}(\nu_{th}) = 1 - P_{\text{in}}(\nu_{th}) = \sum_{i=1}^{N} P_{\text{out}}(\nu_{th_{i-1}})$ where the SNR threshold at the output is defined as $\nu_{th} = \sum_{i=1}^{N} \nu_{th_i}$, whereas $\nu_{th_i} = \frac{\gamma_{r_i}}{\bar{\gamma}_{r_i}}$. It can be observed that $P_{\text{out}}(\nu_{th})$ takes smaller values for smaller number of hops (e.g., by decreasing number of hops from $N=4$ to $N=3$) and by shifting from strong-to-weak N-Fisher-Snedecor $F$ TIFM severity conditions. Moreover, Fig. 1 shows that the results for $P_{\text{out}}(\nu_{th})$ for $N=4$ and under weak turbulence severity conditions coincide with the reported results for OP of Fisher-Snedecor $F$ distribution [1].
Additionally, $\tau_i$ is quasi frequency of the $i$-th FSO path [12, eq. (15)].

Assumed, strong turbulence causes $N_{\tau_i}(\tau_i)$ to take higher values. By shifting from strong-to-moderate or moderate-to-weak turbulence conditions, $N_{\tau_i}(\tau_i)$ decreases. Contrary, the increase in the value of $N$ causes $N_{\tau_i}(\tau_i)$ to increase. It can be further noticed that for lower $\tau_i$ dB values, N-Fisher-Snedecor \( F \) TIFM severity conditions have stronger impact on $N_{\tau_i}(\tau_i)$ than $N$, while for higher $\tau_i$ dB values, the number of hops has stronger impact on $N_{\tau_i}(\tau_i)$ than $N$.

![Graphs showing FAD and LCR for different turbulence conditions and $N$.](image)

Fig. 2 shows $N_{\tau_i}(\tau_i)$. The variances in (23) are evaluated as $\sigma_{\tau_i}^2 = \int_0^\infty 2\pi_{\tau_i}^2 P_{\tau_i}(\tau_i)$ [12, Eq. (13)], where $\langle \tau_i \rangle = 1$. Furthermore, the $\tau_i = \frac{1}{\sqrt{\pi}}$ is turbulence correlation time, $\lambda_i$ is the wavelength, $L_i$ is optical propagation distance and $U_i$ is average wind speed of the $i$-th transmission link. As expected, strong turbulence causes $N_{\tau_i}(\tau_i)$ to take higher values. By shifting from strong-to-moderate or moderate-to-weak turbulence conditions, $N_{\tau_i}(\tau_i)$ decreases. Contrary, the increase in the value of $N$ causes $N_{\tau_i}(\tau_i)$ to increase. It can be further noticed that for lower $\tau_i$ dB values, N-Fisher-Snedecor \( F \) TIFM severity conditions have stronger impact on $N_{\tau_i}(\tau_i)$ than $N$, while for higher $\tau_i$ dB values, the number of hops has stronger impact on $N_{\tau_i}(\tau_i)$ than $N$.

Additionally, a Laplace approximation formula (GLAF) has been initially proven by authors in [18], whereas in [16] authors have applied the GLAF for the cases where real-valued parameter $\gamma$ takes small values. The GLAF that can be used to evaluate $2N$-1 I-F expression is [16, Eq. (1)]:

$$\int_0^\infty dz_{\tau_i,3} \int_0^\infty dz_{\tau_i,3} \ldots \int_0^\infty dz_{\tau_i,N}$$

$$\times \int_0^{\lambda_{\tau_i,1}} d\rho_{\tau_i,1} \int_0^{\lambda_{\tau_i,2}} d\rho_{\tau_i,2} \ldots \int_0^{\lambda_{\tau_i,N-1}} d\rho_{\tau_i,N-1}$$

$$\times \int_0^L f_{\lambda_{\tau_i,1},-\tau_i,3,\ldots,\tau_i,N-1,\rho_{\tau_i,N}}$$

V. CONCLUSION

The paper provides novel C-F expressions for OP, LCR and AFD of N-Fisher-Snedecor \( F \) distribution. The accurate and fast-computing HO statistical C-F approximate results are fitting well with the exact I-F results. In general, the system performance improvement in lower dB output threshold regime can be achieved by shifting from strong-to-weak TIFM severity conditions and by decreasing number of hops.

APPENDIX

The general Laplace approximation formula (GLAF) has been initially proven by authors in [18], whereas in [16] authors have applied the GLAF for the cases where real-valued parameter $\gamma$ takes small values. The GLAF that can be used to evaluate $2N$-1 I-F expression is [16, Eq. (1.5)]:
\[
\begin{align*}
\nabla^2 f &= \begin{bmatrix}
\frac{\partial^2 f_1(x_1, x_2, y, y)}{\partial x_1^2} & \frac{\partial^2 f_1(x_1, x_2, y, y)}{\partial x_1 \partial y} & \frac{\partial^2 f_1(x_1, x_2, y, y)}{\partial x_1 \partial y} \\
\frac{\partial^2 f_1(x_1, x_2, y, y)}{\partial x_2 \partial x_1} & \frac{\partial^2 f_1(x_1, x_2, y, y)}{\partial x_2^2} & \frac{\partial^2 f_1(x_1, x_2, y, y)}{\partial x_2 \partial y} \\
\frac{\partial^2 f_1(x_1, x_2, y, y)}{\partial y \partial x_1} & \frac{\partial^2 f_1(x_1, x_2, y, y)}{\partial y \partial x_2} & \frac{\partial^2 f_1(x_1, x_2, y, y)}{\partial y^2}
\end{bmatrix} \\
\end{align*}
\]

(33)