Genuine Multipartite Nonlocality Is Intrinsic to Quantum Networks

Patricia Contreras-Tejada,[1] Carlos Palazuelos,[2,1] and Julio I. de Vicente[3]

[1] *Instituto de Ciencias Matemáticas, E-28049 Madrid, Spain*

[2] *Departamento de Análisis Matemático y Matemática Aplicada, Universidad Complutense de Madrid, E-28040 Madrid, Spain*

[3] *Departamento de Matemáticas, Universidad Carlos III de Madrid, E-28911 Leganés (Madrid), Spain*

(Received 20 May 2020; accepted 5 January 2021; published 27 January 2021)

Quantum entanglement and nonlocality are inextricably linked. However, while entanglement is necessary for nonlocality, it is not always sufficient in the standard Bell scenario. We derive sufficient conditions for entanglement to give rise to genuine multipartite nonlocality in networks. We find that any network where the parties are connected by bipartite pure entangled states is genuine multipartite nonlocal, independently of the amount of entanglement in the shared states and of the topology of the network. As an application of this result, we also show that all pure genuine multipartite entangled states are genuine multipartite nonlocal in the sense that measurements can be found on finitely many copies of any genuine multipartite entangled state to yield a genuine multipartite nonlocal behavior. Our results pave the way toward feasible manners of generating genuine multipartite nonlocality using any connected network.

DOI: 10.1103/PhysRevLett.126.040501

Correlations between quantum particles may be much stronger than those between classical particles. Their applications are manifold: cryptography [1,2], randomness extraction, amplification and certification [3], communication complexity reduction [4], etc., and the study of these *nonlocal* correlations has led to the growing field of device-independent quantum information processing [5–7] (see also Ref. [8]).

While bipartite nonlocality has been well researched in the past three decades, much less is known about the multipartite case. Still, correlations in quantum multicomponent systems have gained increasing attention recently, with applications in multiparty cryptography [9], the understanding of condensed matter physics [10,11], and the development of quantum networks [12–18], particularly for quantum computation [19–21] and correlating particles which never interacted [22,23].

A necessary condition for nonlocality is quantum entanglement. Indeed, this is one reason why entangled states are useful for communication-related tasks. However, not all entangled states are nonlocal: some bipartite entangled states only yield local distributions [24,25]. Still, for pure bipartite states, entanglement *is* sufficient for nonlocality, which is the content of Gisin's theorem [26,27], and multipartite entangled pure states are never fully local [28,29]. Interestingly, distributing certain bipartite entangled states in certain multipartite networks yields nonlocality even if the involved states are individually local [12,14,30–33].

Multipartite nonlocality is in principle harder to generate than bipartite nonlocality. By exploring the relationship between entanglement and nonlocality in the multipartite regime, in this Letter we show that networks simplify the job considerably: distributing arbitrarily low node-to-node entanglement is sufficient to observe truly multipartite nonlocal effects involving all parties in the network independently of its geometry. Added to its practical consequences for applications, this fact points to a deep property of quantum networks.

The multipartite setting has a richer structure than the bipartite one, as different forms of entanglement and nonlocality can be identified. Full separability (full locality) refers to systems that do not display any form of entanglement (locality) whatsoever. However, falsifying these models need not imply truly multipartite quantum correlations since spreading them among two parties is sufficient. Hence, a *genuine multipartite* notion which inextricably relates all parties together is more often considered. Here, a state is genuine multipartite entangled (GME) if it is not a tensor product of states of two subsets of parties, M and its complement $\bar{M}$, i.e., of the form $|\psi\rangle = |\psi_M\rangle \otimes |\psi_{\bar{M}}\rangle$, or a convex combination of such states $|\psi\rangle\langle\psi|$ across all bipartitions. Analogously, a probability distribution $\{P(\alpha_1\alpha_2\ldots\alpha_n|\chi_1\chi_2\cdots\chi_n)\}_{\alpha_1,\ldots,\alpha_n,\chi_1,\ldots,\chi_n}$ (with input χ_i and output α_i for party i) which is not of the form

$$\begin{aligned} P(\alpha_1\alpha_2\ldots\alpha_n|\chi_1\chi_2\cdots\chi_n) &= \sum_{M\subsetneq[n]}\sum_{\lambda} q_M(\lambda) P_M(\{\alpha_i\}_{i\in M}|\{\chi_i\}_{i\in M}) \\ &\quad \times P_{\bar{M}}(\{\alpha_i\}_{i\in\bar{M}}|\{\chi_i\}_{i\in\bar{M}}), \end{aligned} \tag{1}$$

where $q_M(\lambda)\geq 0\,\forall\lambda,M$, $\sum_{\lambda,M} q_M(\lambda)=1$, and $[n]\coloneqq\{1,\ldots,n\}$, is genuine multipartite nonlocal (GMNL) [34–36], and a state is GMNL if measurements giving rise to a GMNL distribution exist. The original definition [34] leaves the distributions $P_M, P_{\bar{M}}$ in Eq. (1) unrestricted; however, this has been shown to lead to operational problems [37–42]. Hence, like most recent works on the topic, we assume these distributions are nonsignaling, which captures most physical situations better [43,44]. This means the marginal distributions for any subset of parties are independent of the inputs outside this subset, which is guaranteed if this holds for the marginals corresponding to ignoring just one party [45]:

$$\sum_{\alpha_j} P_M(\{\alpha_i\}_{i\in M, i\neq j}, \alpha_j | \{\chi_i\}_{i\in M, i\neq j}, \chi_j) = \sum_{\alpha_j} P_M(\{\alpha_i\}_{i\in M, i\neq j}, \alpha_j | \{\chi_i\}_{i\in M, i\neq j}, \chi'_j), \qquad (2)$$

for all $\chi_j \neq \chi'_j$ and all parties j, and similarly for $P_{\bar{M}}$.

In this Letter we show that the nonlocality arising from networks of bipartite pure entangled states is a generic property and manifests in its strongest form, GMNL. Specifically, we obtain that any connected network of bipartite pure entangled states is GMNL. It was already known that a star network of maximally entangled states is GMNL [12], but we provide a full, qualitative generalization of this result by making it independent of both the amount of entanglement shared and the network topology. Thus, we show GMNL is an intrinsic property of networks of pure bipartite entangled states.

Further, there are known mixed GME states that are not GMNL [46,47]—some are even fully local [48]. Still, it is not known whether Gisin's theorem extends to the genuine multipartite regime. Recent results show that, for pure n-qubit symmetric states [49] and all pure 3-qubit states [50], GME implies GMNL (at the single-copy level) [51]. Our result above shows that all pure GME states that have a network structure are GMNL; interestingly, we further apply this property to establish a second result: all pure GME states are GMNL in the sense that measurements can be found on finitely many copies of any GME state to yield a GMNL behavior. We thus tighten the relationship between multipartite entanglement and nonlocality.

Our construction exploits the fact that the set of non-GME states is not closed under tensor products; i.e., GME can be superactivated by taking tensor products of states that are unentangled across different bipartitions. Thus, GME can be achieved by distributing bipartite entangled states among different pairs of parties. To obtain our results, we extend the superactivation property [52–54] from the level of states to that of probability distributions; i.e., GMNL can be superactivated by taking Cartesian products of probability distributions that are local across different bipartitions. In fact, when considering copies of quantum states, we only consider local measurements performed on each copy separately, thus pointing at a stronger notion of superactivation to achieve GMNL.

Definitions and preliminaries.—We consider distributions arising from GME states, and ask whether they satisfy Eq. (1). The set of distributions of the form (1) is a polytope: indeed, the set of local distributions across each bipartition $M|\bar{M}$ is a polytope, and convex combinations preserve that structure. We call this n-partite polytope $\mathcal{B}_n$. We call an inequality,

$$\sum_{\substack{\alpha_i\chi_i \\ i\in[n]}} c_{\alpha_1\ldots\alpha_n\chi_1,\ldots,\chi_n} P(\alpha_1\ldots\alpha_n|\chi_1\ldots\chi_n) \leq c_0, \qquad (3)$$

which holds for all P of the form (1) a GMNL inequality.

We use results from Ref. [55] to lift inequalities to account for more parties, inputs, and outputs. They consider the fully local polytope $\mathcal{L}$, which only includes distributions

$$P(\alpha\beta|\chi\upsilon) = \sum_\lambda q(\lambda) P_A(\alpha|\chi,\lambda) P_B(\beta|\upsilon,\lambda), \qquad (4)$$

where each party may have different numbers of inputs and outputs (more parties may be considered by adding more distributions correlated only by λ). Polytope $\mathcal{B}_n$ includes convex combinations of distributions that are local across different bipartitions $M|\bar{M}$ of the parties, but the lifting results in Ref. [55] still hold. Indeed, to check an inequality holds for a polytope, it is sufficient by convexity to check the extremal points. As all extremal points in $\mathcal{B}_n$ are contained in some polytope $\mathcal{L}$, lifting results for $\mathcal{L}$ can be straightforwardly extended to $\mathcal{B}_n$.

We also use the Eliztur-Popescu-Rohrlich (EPR2) decomposition [56] and its multipartite extension [57]: any distribution P can be expressed (nonuniquely) as

$$P(\alpha_1\ldots\alpha_n|\chi_1\ldots\chi_n) = \sum_{M\subsetneq[n]} p_L^M P_L^M(\alpha_1\ldots\alpha_n|\chi_1\ldots\chi_n) + p_{\rm NS} P_{\rm NS}(\alpha_1\ldots\alpha_n|\chi_1\ldots\chi_n), \qquad (5)$$

where $\sum_{M\subsetneq[n]} p_L^M + p_{\rm NS} = 1$, P_L^M is local across $M|\bar{M}$ [i.e., satisfies Eq. (4) with parties grouped as per $M|\bar{M}$], and $P_{\rm NS}$ is nonsignaling. P is GMNL if all such decompositions have $p_{\rm NS} > 0$, and fully GMNL if all such decompositions have $p_{\rm NS} = 1$. A state ρ is fully GMNL if, $\forall\, \varepsilon > 0$, there exist local measurements giving rise to some P such that any decomposition (5) has $p_{\rm NS} > 1 - \varepsilon$. Bipartite distributions and states may be nonlocal or fully nonlocal [58] analogously.

GMNL from bipartite entanglement.—Our first result shows that any connected network of pure bipartite entanglement (see Fig. 1) is GMNL.

Theorem 1: Any connected network of bipartite pure entangled states is GMNL.

We now outline the proof for a tripartite network where A_1 is entangled to each of A_2 and A_3, and leave the general case to Ref. [59]. Since it turns out sufficient to measure

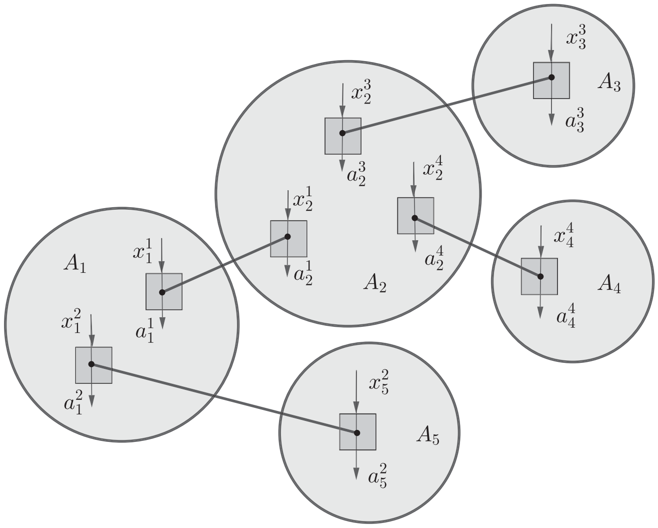

FIG. 1. Connected network of bipartite entanglement. For each $i \in [n]$, party A_i has input x_i^k and output a_i^k on the particle at edge k. Particles connected by an edge are entangled.

individually on each party's different particles (see Fig. 1 for the n-partite structure), the shared distribution $P(a_1^1 a_1^2, a_2^1, a_3^2 | x_1^1 x_1^2, x_2^1, x_3^2)$ takes the form

$$P_1(a_1^1 a_2^1 | x_1^1 x_2^1) P_2(a_1^2 a_3^2 | x_1^2 x_3^2), \tag{6}$$

where parties A_i, A_j are connected by edge k (we label vertices and edges independently), and $P_k(a_i^k a_j^k | x_i^k x_j^k)$ is the distribution arising from the state at edge k.

The proof considers three cases, depending on whether the shared states are maximally entangled. If none are, we devise inequalities to detect bipartite nonlocality at each edge of the network, and combine them to form a multipartite inequality. Then, we find measurements on the shared states to violate it. If both states are maximally entangled, existing results show the network is fully GMNL [12,57]. Combining these two cases for a heterogeneous network completes the proof.

To prove the first case, we take bipartite inequalities between A_1 and each other party, lift them to three parties, and combine them using Refs. [55,60], to obtain the following GMNL inequality:

$$\begin{aligned} I_3 = I^1 + I^2 + P(00,0,0|00,0,0) \\ - \sum_{a_1^2=0,1} P(0a_1^2,0,0|00,0,0) \\ - \sum_{a_1^1=0,1} P(a_1^1 0,0,0|00,0,0) \leq 0. \end{aligned} \tag{7}$$

Here,

$$\begin{aligned} I^1 = \sum_{a_1^2=0,1} [P(0a_1^2,0,0|00,0,0) - P(0a_1^2,1,0|00,1,0) \\ - P(1a_1^2,0,0|10,0,0) - P(0a_1^2,0,0|10,1,0)] \leq 0, \end{aligned} \tag{8}$$

$$\begin{aligned} I^2 = \sum_{a_1^1=0,1} [P(a_1^1 0,0,0|00,0,0) - P(a_1^1 0,0,1|00,0,1) \\ - P(a_1^1 1,0,0|01,0,0) - P(a_1^1 0,0,0|01,0,1)] \leq 0, \end{aligned} \tag{9}$$

are liftings of

$$I = P(00|00) - P(01|01) - P(10|10) - P(00|11) \leq 0 \tag{10}$$

to three parties with A_1 having 4 inputs and 4 outputs. Inequality (10) is equivalent to the CHSH inequality [61] for nonsignaling distributions [60]. Thus, inequalities (8) and (9) are satisfied by distributions that are local across $A_1|A_2$ and $A_1|A_3$, respectively. To see that Eq. (7) is a GMNL inequality it is sufficient to check it holds for distributions that are local across some bipartition. This is straightforwardly done by observing the cancellations that occur when I^1 or I^2 are ≤ 0.

Since both states are less-than-maximally entangled, A_1 can satisfy Hardy's paradox [62,63] with each other party, achieving

$$P_k(00|00) > 0 = P_k(01|01) = P_k(10|10) = P_k(00|11) \tag{11}$$

for both k (the proof for qubits in Refs. [62,63] is extended to qudits by measuring on a two-dimensional subspace; see Ref. [59]). Then, each negative term in I^1 and I^2 is zero, as

$$\sum_{a_1^2=0,1} P(0a_1^2,1,0|00,1,0) = P_1(01|01) \sum_{a_1^2=0,1} P_2(a_1^2 0|00), \tag{12}$$

and similarly for the others. Hence, only

$$P(00,0,0|00,0,0) = P_1(00|00) P_2(00|00) > 0 \tag{13}$$

survives, violating the inequality.

If, instead, $A_1 A_2$ share a maximally entangled state, and $A_2 A_3$ share a less-than-maximally entangled state, then $A_1 A_3$ can measure so that P_2 satisfies Hardy's paradox; hence $\exists\, \varepsilon > 0$ such that its local component in any EPR2 decomposition satisfies

$$p_{L,2} \leq 1 - \varepsilon. \tag{14}$$

Since the maximally entangled state is fully nonlocal [64], for this ε, $A_1 A_2$ can measure such that any EPR2 decomposition of P_1 satisfies

$$p_{L,1} < \varepsilon. \tag{15}$$

Then, we assume for a contradiction that $P(a_1^1 a_1^2, a_2^1, a_3^2 | x_1^1 x_1^2, x_2^1, x_3^2)$ is not GMNL and decompose it in its bipartite splittings,

$$\begin{aligned} &P(a_1^1 a_1^2, a_2^1, a_3^2 | x_1^1 x_1^2, x_2^1, x_3^2) \\ &= \sum_\lambda [p_L(\lambda) P_{A_1A_2}(a_1^1 a_1^2, a_2^1 | x_1^1 x_1^2, x_2^1, \lambda) P_{A_3}(a_3^2 | x_3^2, \lambda) \\ &\quad + q_L(\lambda) P_{A_1A_3}(a_1^1 a_1^2, a_3^2 | x_1^1 x_1^2, x_3^2, \lambda) P_{A_2}(a_2^1 | x_2^1, \lambda) \\ &\quad + r_L(\lambda) P_{A_1}(a_1^1 a_1^2 | x_1^1 x_1^2, \lambda) P_{A_2A_3}(a_2^1, a_3^2 | x_2^1, x_3^2, \lambda)], \end{aligned} \tag{16}$$

where $\sum_\lambda [p_L(\lambda) + q_L(\lambda) + r_L(\lambda)] = 1$.

Summing Eq. (16) over a_1^2, a_3^2 and using Eq. (6), we get an EPR2 decomposition of P_1 with local components q_L, r_L. By Eq. (15), this entails $\sum_\lambda [q_L(\lambda) + r_L(\lambda)] < \varepsilon$, so

$$\sum_\lambda p_L(\lambda) > 1 - \varepsilon. \tag{17}$$

Summing, instead, Eq. (16) over a_1^1, a_2^1, we obtain an EPR2 decomposition of P_2 whose only non-negligible component, $\sum_\lambda p_L(\lambda)$, is local in $A_1|A_3$, contradicting Eq. (14). Therefore, P must be GMNL.

GMNL from GME.—By Theorem 1, a star network whose central node shares pure entanglement with all others is GMNL. We now ask whether all GME states are GMNL (i.e., the genuine multipartite extension of Gisin's theorem). We show $(n-1)$ copies of any pure GME n-partite state suffice to generate n-partite GMNL. We do this by generating a distribution from these copies that mimics the star network configuration.

Theorem 2: Any GME state $|\Psi\rangle \in \mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_n \cong (\mathbb{C}^d)^{\otimes n}$ is such that $|\Psi\rangle^{\otimes(n-1)}$ is GMNL.

The full proof is given in Ref. [59], and we presently outline the tripartite case. Hence, we consider two copies of the state. For each copy, we derive measurements for Bob1 and Bob2 that leave Alice bipartitely entangled with Bob2 and Bob1, respectively. This yields a network as in Eq. (6) but postselected on the inputs and outputs of these measurements. We generalize Theorem 1 to show this network is also GMNL.

For $i, j = 1, 2$, on copy i, B_j's measurements have input y_j^i and output b_j^i and Alice's measurement has input x_i and output a_i. We denote B_j's inputs and outputs in terms of their digits as $\upsilon_j = y_j^1 y_j^2$ and $\beta_j = b_j^1 b_j^2$. Then, after measurement, the parties share a distribution:

$$\begin{aligned} &P(\alpha\beta_1\beta_2 | \chi\upsilon_1\upsilon_2) \\ &\quad = P_1(a_1, b_1^1 b_2^1 | x_1, y_1^1 y_2^1) P_2(a_2, b_1^2 b_2^2 | x_2, y_1^2 y_2^2). \end{aligned} \tag{18}$$

For each $i, j = 1, 2$, $i \neq j$, we assume B_j uses input 0_j^i and output 0_j^i to project the ith copy of $|\Psi\rangle$ onto $|\phi_i\rangle_{AB_i}$, as shown in Fig. 2 for n parties. Then, Refs. [28,29] and a

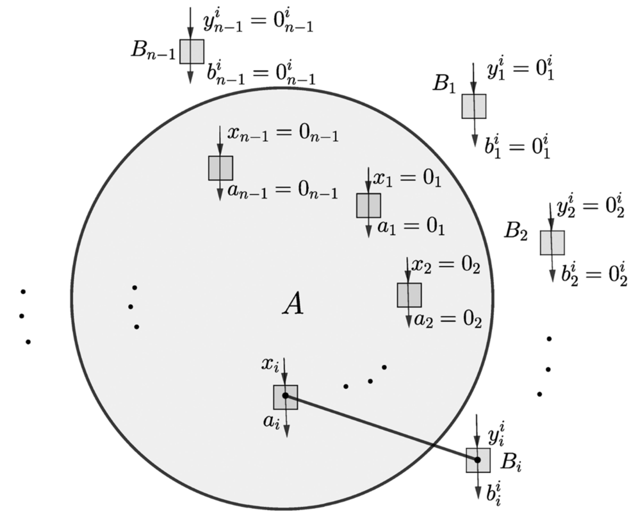

FIG. 2. Element $i \in [n-1]$ of the star network of bipartite entanglement created from a GME state $|\Psi\rangle$. Parties $\{B_j\}_{j\in[n-1], j\neq i}$ have already measured $|\Psi\rangle$ and are left unentangled. Alice and party B_i share a pure bipartite entangled state. Alice has input x_i and output a_i while each party B_j, $j \in [n-1]$, has input y_j^i and output b_j^i.

continuity argument serve to show we only have two possibilities for each i: either there exists an input and output per party such that $|\phi_i\rangle_{AB_i}$ is less-than-maximally entangled, or there exists an input per party such that, for all outputs, $|\phi_i\rangle_{AB_i}$ is maximally entangled. In each case we generalize the proof in Theorem 1 to show $|\Psi\rangle^{\otimes 2}$ is GMNL.

If both $|\phi_i\rangle_{AB_i}$, $i = 1, 2$ are less-than-maximally entangled, we use the following expression, which is a GMNL inequality by the same reasoning as in Theorem 1:

$$\begin{aligned} I_3 = &\sum_{i=1}^{2} I^i + P(00, 00, 00 | 00, 00, 00) \\ &- \sum_{i=1}^{2} \sum_{\substack{a_j, b_i^j, \\ b_j^j = 0,1, \\ j \neq i}} P(0_i a_j, 0_i^i b_i^j, 0_j^i b_j^j | 0_i 0_j, 0_i^i 0_i^j, 0_j^i 0_j^j) \leq 0, \end{aligned} \tag{19}$$

where

$$\begin{aligned} I^i = \sum_{\substack{a_j, b_i^j, b_j^j = 0,1, \\ j \neq i}} &[P(0_i a_j, 0_i^i b_i^j, 0_j^i b_j^j | 0_i 0_j, 0_i^i 0_i^j, 0_j^i 0_j^j) \\ &- P(0_i a_j, 1_i^i b_i^j, 0_j^i b_j^j | 0_i 0_j, 1_i^i 0_i^j, 0_j^i 0_j^j) \\ &- P(1_i a_j, 0_i^i b_i^j, 0_j^i b_j^j | 1_i 0_j, 0_i^i 0_i^j, 0_j^i 0_j^j) \\ &- P(0_i a_j, 0_i^i b_i^j, 0_j^i b_j^j | 1_i 0_j, 1_i^i 0_i^j, 0_j^i 0_j^j)]. \end{aligned} \tag{20}$$

Evaluating the inequality on the distribution (18), we find again that all negative terms in each I^i can be sent to zero. For each i we get, for example,

$$\sum_{\substack{a_j,b_i^j,b_j^j\\=0,1}} P(0_i a_j, 1_i^i b_i^j, 0_j^i b_j^j | 0_i 0_j, 1_i^i 0_i^j, 0_j^i 0_j^j)$$
$$= P_i(0_i 1_i^i 0_j^i | 0_i 1_i^i 0_j^i), \tag{21}$$

as the sum over P_j is 1. But, conditioned on B_j's input and output being 0_j^i, parties AB_i can measure so P_i satisfies Hardy's paradox, hence this term is zero, and similarly for the other two negative terms. This means all terms in I_3 are zero except $P(00,00,00|00,00,00) > 0$, violating the inequality. Therefore, $|\Psi\rangle^{\otimes 2}$ is GMNL.

If, for both $i = 1, 2$, there exists a local measurement for party B_j, $j \neq i$ such that, for *all* outputs, $|\Psi\rangle$ is projected onto a maximally entangled state $|\phi_i\rangle_{AB_i}$, then $|\Psi\rangle$ satisfies Theorem 2 in Ref. [57], so $|\Psi\rangle$ itself is GMNL. Therefore so is $|\Psi\rangle^{\otimes 2}$.

Finally, if $|\phi_1\rangle_{AB_1}$ is maximally entangled for all of B_2's outputs, and $|\phi_2\rangle_{AB_2}$ is less-than-maximally entangled, using Refs. [28,57] we deduce that the bipartite EPR2 components of $P_{1,2}$ across $A|B_{1,2}$, respectively, are bounded like in Theorem 1. That is, $\exists\, \varepsilon > 0$ such that the local component of any EPR2 decomposition across $A|B_2$ satisfies

$$p_{L,2}^{A|B_2} \leq 1 - \varepsilon, \tag{22}$$

and, given this ε, parties AB_1 can measure locally such that all bipartite EPR2 decompositions across $A|B_1$ have a local component:

$$p_{L,1}^{A|B_1} < \varepsilon. \tag{23}$$

Then, we assume $P(\alpha\beta_1\beta_2|\chi\upsilon_1\upsilon_2)$ is not GMNL and decompose it in local terms across different bipartitions, like in Eq. (16) in Theorem 1. Summing over a_2, b_j^2, $j = 1, 2$ gives an EPR2 decomposition of P_1 whose local components can be bounded using Eq. (23). Summing over a_1, b_j^1, $j = 1, 2$ instead gives an EPR2 decomposition of P_2. But the bound on the local component of P_1 entails a bound on that of P_2 which contradicts Eq. (22), proving P is GMNL.

Conclusions.—We have shown that GMNL can be obtained by distributing arbitrary pure bipartite entanglement, which paves the way toward feasible generation of GMNL from any network. In fact, our results imply that, given a set of nodes, distributing entanglement in the form of a tree is sufficient to observe GMNL. In practical applications, the entanglement shared by the nodes would unavoidably degrade to mixed-state form. By continuity, the GMNL in the networks of pure bipartite entanglement considered here must be robust to some noise. Quantifying this tolerance is interesting for future work.

Further, we have shown that a tensor product of finitely many GME states is always GMNL. The question whether all single-copy pure GME states are GMNL remains open.

The assumption that the distributions $P_M, P_{\tilde{M}}$ are nonsignaling in the GMNL definition is physically natural. Still, removing it raises the stakes to achieve nonlocality, and establishing analogous results with the stronger definition is an open question.

Very recently, Ref. [65] proposed the concept of "genuine network entanglement," a stricter notion than GME which rules out states which are a tensor product of non-GME states. One might hope that states that are GME but not genuine network entangled might be detected device independently by not passing GMNL tests. However, our results show this will not work. Any distribution of pure bipartite states, even with arbitrarily weak entanglement, always displays GMNL as long as all parties are connected. This further motivates searching for an analogous concept of genuine network nonlocality that may detect genuine network entanglement.

The authors thank Aleksander M. Kubicki and Elie Wolfe for enlightening discussions. This research was funded by the Spanish MINECO through Grant No. MTM2017-88385-P and by the Comunidad de Madrid through Grant No. QUITEMAD-CMS2018/TCS-4342. We also acknowledge funding from the Spanish MINECO, through the "Severo Ochoa Programme for Centres of Excellence in R&D" SEV-2015-0554 and from the Spanish National Research Council, through the "Ayuda extraordinaria a Centros de Excelencia Severo Ochoa" 20205CEX001 (P. C.-T. and C. P.) and the Spanish MINECO Grant No. MTM2017-84098-P (J. I. d. V.). P. C.-T. is grateful for the hospitality of Perimeter Institute where part of this work was carried out. Research at Perimeter Institute is supported in part by the Government of Canada through the Department of Innovation, Science and Economic Development Canada and by the Province of Ontario through the Ministry of Economic Development, Job Creation and Trade.

[1] N. Gisin, G. Ribordy, W. Tittel, and H. Zbinden, Quantum cryptography, Rev. Mod. Phys. **74**, 145 (2002).

[2] S. Pirandola, U. L. Andersen, L. Banchi, M. Berta, D. Bunandar, R. Colbeck, D. Englund, T. Gehring, C. Lupo, C. Ottaviani, J. Pereira, M. Razavi, J. S. Shaari, M. Tomamichel, V. C. Usenko, G. Vallone, P. Villoresi, and P. Wallden, Advances in quantum cryptography, Adv. Opt. Photonics **12**, 1012 (2020).

[3] A. Acín and L. Masanes, Certified randomness in quantum physics, Nature (London) **540**, 213 (2016).

[4] H. Buhrman, R. Cleve, S. Massar, and R. de Wolf, Nonlocality and communication complexity, Rev. Mod. Phys. **82**, 665 (2010).

[5] D. Mayers and A. Yao, Quantum cryptography with imperfect apparatus, in *Proceedings of the 39th Annual Symposium on Foundations of Computer Science (FOCS '98)* (IEEE Computer Society, Palo Alto, 1998), p. 503,

https://doi.ieeecomputersociety.org/10.1109/SFCS.1998.743420.

[6] A. Acín, N. Brunner, N. Gisin, S. Massar, S. Pironio, and V. Scarani, Device-Independent Security of Quantum Cryptography against Collective Attacks, Phys. Rev. Lett. **98**, 230501 (2007).

[7] R. Colbeck, Quantum and relativistic protocols for secure multi-party computation, Ph.D. thesis, University of Cambridge, 2011.

[8] N. Brunner, D. Cavalcanti, S. Pironio, V. Scarani, and S. Wehner, Bell nonlocality, Rev. Mod. Phys. **86**, 419 (2014).

[9] L. Aolita, R. Gallego, A. Cabello, and A. Acín, Fully Nonlocal, Monogamous, and Random Genuinely Multipartite Quantum Correlations, Phys. Rev. Lett. **108**, 100401 (2012).

[10] J. Tura, R. Augusiak, A. B. Sainz, T. Vértesi, M. Lewenstein, and A. Acín, Detecting nonlocality in many-body quantum states, Science **344**, 1256 (2014).

[11] J. Tura, G. De las Cuevas, R. Augusiak, M. Lewenstein, A. Acín, and J. I. Cirac, Energy as a Detector of Nonlocality of Many-Body Spin Systems, Phys. Rev. X **7**, 021005 (2017).

[12] D. Cavalcanti, M. L. Almeida, V. Scarani, and A. Acín, Quantum networks reveal quantum nonlocality, Nat. Commun. **2**, 184 (2011).

[13] N. Gisin, Q. Mei, A. Tavakoli, M. O. Renou, and N. Brunner, All entangled pure quantum states violate the bilocality inequality, Phys. Rev. A **96**, 020304(R) (2017).

[14] I. Šupić, P. Skrzypczyk, and D. Cavalcanti, Measurement-device-independent entanglement and randomness estimation in quantum networks, Phys. Rev. A **95**, 042340 (2017).

[15] A. Tavakoli, M. O. Renou, N. Gisin, and N. Brunner, Correlations in star networks: From Bell inequalities to network inequalities, New J. Phys. **19**, 073003 (2017).

[16] M.-O. Renou, E. Bäumer, S. Boreiri, N. Brunner, N. Gisin, and S. Beigi, Genuine Quantum Nonlocality in the Triangle Network, Phys. Rev. Lett. **123**, 140401 (2019).

[17] N. Gisin, J.-D. Bancal, Y. Cai, P. Remy, A. Tavakoli, E. Z. Cruzeiro, S. Popescu, and N. Brunner, Constraints on nonlocality in networks from no-signaling and independence, Nat. Commun. **11**, 2378 (2020).

[18] T. Kriváchy, Y. Cai, D. Cavalcanti, A. Tavakoli, N. Gisin, and N. Brunner, A neural network oracle for quantum nonlocality problems in networks, npj Quantum Inf. **6**, 70 (2020).

[19] J. I. Cirac, A. K. Ekert, S. F. Huelga, and C. Macchiavello, Distributed quantum computation over noisy channels, Phys. Rev. A **59**, 4249 (1999).

[20] M. Howard and J. Vala, Nonlocality as a benchmark for universal quantum computation in Ising anyon topological quantum computers, Phys. Rev. A **85**, 022304 (2012).

[21] M. Howard, J. J. Wallman, V. Veitch, and J. Emerson, Contextuality supplies the magic for quantum computation, Nature (London) **510**, 351 (2014).

[22] C. Branciard, N. Gisin, and S. Pironio, Characterizing the Nonlocal Correlations Created via Entanglement Swapping, Phys. Rev. Lett. **104**, 170401 (2010).

[23] C. Branciard, D. Rosset, N. Gisin, and S. Pironio, Bilocal versus non-bilocal correlations in entanglement swapping experiments, Phys. Rev. A **85**, 032119 (2012).

[24] R. F. Werner, Quantum states with Einstein-Podolsky-Rosen correlations admitting a hidden-variable model, Phys. Rev. A **40**, 4277 (1989).

[25] J. Barrett, Nonsequential positive-operator-valued measurements on entangled mixed states do not always violate a Bell inequality, Phys. Rev. A **65**, 042302 (2002).

[26] N. Gisin, Bell's inequality holds for all non-product states, Phys. Lett. A **154**, 201 (1991).

[27] N. Gisin and A. Peres, Maximal violation of Bell's inequality for arbitrarily large spin, Phys. Lett. A **162**, 15 (1992).

[28] S. Popescu and D. Rohrlich, Generic quantum nonlocality, Phys. Lett. A **166**, 293 (1992).

[29] M. Gachechiladze and O. Gühne, Completing the proof of Generic quantum nonlocality, Phys. Lett. A **381**, 1281 (2017).

[30] A. Sen(De), U. Sen, Č. Brukner, V. Bužek, and M. Żukowski, Entanglement swapping of noisy states: A kind of superadditivity in nonclassicality, Phys. Rev. A **72**, 042310 (2005).

[31] D. Cavalcanti, R. Rabelo, and V. Scarani, Nonlocality Tests Enhanced by a Third Observer, Phys. Rev. Lett. **108**, 040402 (2012).

[32] M.-X. Luo, Nonlocality of all quantum networks, Phys. Rev. A **98**, 042317 (2018).

[33] M.-X. Luo, A nonlocal game for witnessing quantum networks, npj Quantum Inf. **5**, 1 (2019).

[34] G. Svetlichny, Distinguishing three-body from two-body nonseparability by a Bell-type inequality, Phys. Rev. D **35**, 3066 (1987).

[35] M. Seevinck and G. Svetlichny, Bell-Type Inequalities for Partial Separability in N -Particle Systems and Quantum Mechanical Violations, Phys. Rev. Lett. **89**, 060401 (2002).

[36] D. Collins, N. Gisin, S. Popescu, D. Roberts, and V. Scarani, Bell-Type Inequalities to Detect True n-Body Nonseparability, Phys. Rev. Lett. **88**, 170405 (2002).

[37] F. Buscemi, All Entangled Quantum States Are Nonlocal, Phys. Rev. Lett. **108**, 200401 (2012).

[38] R. Gallego, L. E. Würflinger, A. Acín, and M. Navascués, Operational Framework for Nonlocality, Phys. Rev. Lett. **109**, 070401 (2012).

[39] J.-D. Bancal, J. Barrett, N. Gisin, and S. Pironio, Definitions of multipartite nonlocality, Phys. Rev. A **88**, 014102 (2013).

[40] J. Geller and M. Piani, Quantifying non-classical and beyond-quantum correlations in the unified operator formalism, J. Phys. A **47**, 424030 (2014).

[41] J. I. de Vicente, On nonlocality as a resource theory and nonlocality measures, J. Phys. A **47**, 424017 (2014).

[42] R. Gallego and L. Aolita, Nonlocality free wirings and the distinguishability between Bell boxes, Phys. Rev. A **95**, 032118 (2017).

[43] D. Schmid, D. Rosset, and F. Buscemi, The type-independent resource theory of local operations and shared randomness, Quantum **4**, 262 (2020).

[44] E. Wolfe, D. Schmid, A. B. Sainz, R. Kunjwal, and R. W. Spekkens, Quantifying Bell: The resource theory of nonclassicality of common-cause boxes, Quantum **4**, 280 (2020).

[45] E. Hänggi, Device-independent quantum key distribution, Ph. D. thesis, ETH Zürich, 2010.

[46] R. Augusiak, M. Demianowicz, J. Tura, and A. Acín, Entanglement and Nonlocality Are Inequivalent for Any Number of Particles, Phys. Rev. Lett. **115**, 030404 (2015).

[47] R. Augusiak, M. Demianowicz, and J. Tura, Constructions of genuinely entangled multipartite states with applications to local hidden variables (LHV) and states (LHS) models, Phys. Rev. A **98**, 012321 (2018).

[48] J. Bowles, J. Francfort, M. Fillettaz, F. Hirsch, and N. Brunner, Genuinely Multipartite Entangled Quantum States with Fully Local Hidden Variable Models and Hidden Multipartite Nonlocality, Phys. Rev. Lett. **116**, 130401 (2016).

[49] Q. Chen, S. Yu, C. Zhang, C. H. Lai, and C. H. Oh, Test of Genuine Multipartite Nonlocality without Inequalities, Phys. Rev. Lett. **112**, 140404 (2014).

[50] S. Yu and C. H. Oh, Tripartite entangled pure states are tripartite nonlocal, arXiv:1306.5330.

[51] Hidden GMNL for three parties beyond qubits can be shown if some form of preprocessing is allowed.

[52] M. Navascués and T. Vértesi, Activation of Nonlocal Quantum Resources, Phys. Rev. Lett. **106**, 060403 (2011).

[53] C. Palazuelos, Superactivation of Quantum Nonlocality, Phys. Rev. Lett. **109**, 190401 (2012).

[54] P. Caban, A. Molenda, and K. Trzcińska, Activation of the violation of the Svetlichny inequality, Phys. Rev. A **92**, 032119 (2015).

[55] S. Pironio, Lifting Bell inequalities, J. Math. Phys. (N.Y.) **46**, 062112 (2005).

[56] A. C. Elitzur, S. Popescu, and D. Rohrlich, Quantum nonlocality for each pair in an ensemble, Phys. Lett. A **162**, 25 (1992).

[57] M. L. Almeida, D. Cavalcanti, V. Scarani, and A. Acín, Multipartite fully nonlocal quantum states, Phys. Rev. A **81**, 052111 (2010).

[58] Not to be confused with “nonfully local,” which is the opposite of “fully local.” “Fully nonlocal” is a particular case of nonfully local.

[59] See Supplemental Material at http://link.aps.org/supplemental/10.1103/PhysRevLett.126.040501 for full proofs of the results in the main text.

[60] F. J. Curchod, M. L. Almeida, and A. Acín, A versatile construction of Bell inequalities for the multipartite scenario, New J. Phys. **21**, 023016 (2019).

[61] J. F. Clauser, M. A. Horne, A. Shimony, and R. A. Holt, Proposed Experiment to Test Local Hidden-Variable Theories, Phys. Rev. Lett. **23**, 880 (1969).

[62] L. Hardy, Quantum Mechanics, Local Realistic Theories, and Lorentz-Invariant Realistic Theories, Phys. Rev. Lett. **68**, 2981 (1992).

[63] L. Hardy, Nonlocality for Two Particles without Inequalities for Almost All Entangled States, Phys. Rev. Lett. **71**, 1665 (1993).

[64] J. Barrett, A. Kent, and S. Pironio, Maximally Nonlocal and Monogamous Quantum Correlations, Phys. Rev. Lett. **97**, 170409 (2006).

[65] M. Navascues, E. Wolfe, D. Rosset, and A. Pozas-Kerstjens, Genuine Network Multipartite Entanglement, Phys. Rev. Lett. **125**, 240505 (2020).