2022-15
Working paper. Economics
ISSN 2340-5031

AN ERGODIC THEORY OF SOVEREIGN DEFAULT

Damian Pierri and Hernán D. Seoane

uc3m | Universidad Carlos III de Madrid

Serie disponible en http://hdl.handle.net/10016/11

Web: http://economia.uc3m.es/

Correo electrónico: departamento.economia@eco.uc3m.es

An Ergodic Theory of Sovereign Default*

Damian Pierri †   Hernán D. Seoane ‡

September 20, 2023

Abstract

We present the conditions under which the dynamics of a sovereign default model of private external debt are stationary, ergodic and globally stable. As our results are constructive, the model can be used for the accurate computation of global long run stylized facts. We show that default can be used to derive a stable unconditional distribution (i.e., a stable stochastic steady state), one for each possible event, which in turn allows us to characterize globally positive probability paths. We show that the stable and the ergodic distribution are actually the same object. We found that there are 3 type of paths: non-sustainable and sustainable; among this last category, trajectories can be either stable or unstable. In the absence of default, non-sustainable and unstable paths generate explosive trajectories for debt. By deriving the notion of stable state space, we show that the government can use the default of private external debt as a stabilization policy.

Keywords: Default; Private external debt; Ergodicity; Stability.

JEL Codes: F41, E61, E10, C02

*We thank the participants of the 22nd SAET Conference (2023) and SNDE Symposium in (2023) for comments and suggestions. We also thank Kevin Reffett and Rodrigo Raad for comments. We thank the excellent research assistance of Sebastián Manzi and Jano Acosta. Damian Pierri ackowledges financial support from Ministerio de Asuntos Económicos y Transformación Digital through grant Number ECO2016-76818-C3-1-P, and PDI2021-122931NB-I00. Hernán Seoane gratefully acknowledges financial support by MICIN/ AEI/10.13039/501100011033, grants CEX2021-001181-M, ECO2016-76818-C3-1-P, and Comunidad de Madrid, grants EPUC3M11 (V PRICIT).

†Maria Zambrano Fellow at Universidad Autónoma de Madrid and Instituto Interdisciplinario de Economía Política. Email: damian.pierri@gmail.com.

‡Universidad Carlos III de Madrid. Email: hseoane@eco.uc3m.es.

1 Introduction

This paper presents the conditions under which the dynamics of a sovereign default model of private external debt are stationary, ergodic and globally stable. We derive restrictions on the primitives of the model that guarantee that the equilibrium is compact and stationary. Then, we strengthen on the previous set of assumptions to show uniqueness and introduce further restrictions to prove ergodicity. Economically, our results have a very strong interpretation. On top of the results in Zame (1993), where defaults complete markets, we show that default is a tool to stabilize economies where private external debt would otherwise generate unstable and unsustainable debt paths. As a by-product of these findings, we prove, constructively and in minimal state space, that we can use the model to compute global long run stylized facts as well as local short run moments. Thus, our simulations are obtained from accurate and numerically efficient methods.

We model an endowment small open economy populated by a set of atomistic consumers, a government and competitive risk-neutral international investors. We assume that households issue foreign debt while the government can induce households to default. That is, in our economy borrowing is decentralized, as in Bianchi (2011), but instead of a collateral constraint we assume that private external debt is defaultable, and the default choice is centralized. In the case of Bianchi (2011) a crisis is defined as a hit to a price dependent occasionally binding inequality constraint, which produces a massive deleverage. In our case, a default triggers deleverage. Moreover, contrarily to the default literature (see for instance Arellano (2008)) and because of the constructiveness of the existence proof, the default punishment is endogenously determined. Thus, default thresholds are endogenous. This implies that in our model deleverage is also generated by a crisis represented by an occasionally binding constraints that is determined in general equilibrium.

Empirically, private debt is an important share of defaultable external debt in most emerging economies. Thus, this environment is extremely tractable and at the same time empirically meaningful. Table 1 presents evidence along this dimension for Argentina and Chile. The table shows that total public debt with private lenders is 87 and 46 billion U$S while private debt is 73 and 192 billion U$S, both respectively. The sum of these items account for the defaultable debt (excluding debt with senior lenders, which is virtually non-defaultable). It turns out that 46% for Argentina and 81% for Chile of total defaultable debt is private. For this reason, we focus on private debt and we abstract from sovereign debt, which in turn allow us to derive theoretical results that the literature could not address in other frameworks. In particular, an accurate approximation of the stationary equilibrium that relies on an endogenous default punishment and an ergodic refinement of the previous result that allow us to simulate long run stylized facts safely.

Additionally, considering centralized default of private external debt is empirically relevant and plausible. Arellano et al. (2016) shows that the ability of the government to directly influence private external debt varies across countries. Also, policy makers can indirectly affect the ability of the private sector to repay its debt through market mechanisms.[1] Nevertheless, the literature has not studied this phenomenon so far. Hence, our paper fills this gap.

Table 1: External debt in Argentina and Chile (2021)

bln of Dollars	$Public_{PV}$ (I)	$Public_{IO}$ (II)	I+II	$Private$ (III)	I+III	III/(I+III)
Argentina	87	74	161	73	160	46%
Chile	46	0	46	192	238	81%

Note: "$Public_{PV}$" refers to public external debt with private lenders, "IO" stands for international organisations, "I+ II" is total public debt. $Private$ is private external debt. "I+III" is the total debt subject to default, which excludes debt with IO. The last column computes the share of private debt to total defaultable debt. Source: Central Bank of Chile and National Institute of Public Statistics (INDEC) for Argentina.

We highlight three implications of stationarity, ergodicity and global stability for the small open economy with centralized default. (1) Every time a country defaults, it generates a new ergodic and stable distribution. This means that even transitory defaults have permanent economic effects. Hence, default matters for unconditional moments of key macroeconomic variables such as the mean of external debt to GDP. (2) A recession generates default and re-entry to international capital markets depends on the level of output. (3) In our model, the default cost (the penalty) is essential to achieve stationarity and ergodicity. In the first type of equilibrium, as there can be multiplicity, the penalty works as a selection mechanism. After imposing a strengthening on preferences, we

[1]There are several reasons that account for their significance for the aggregate behavior of the economy. For instance, if the private sector issues debt in foreign currency a nominal depreciation may imply a de-facto massive private default, a case along these lines happened for Argentina in 2001 with the abandonment of the currency peg. On the other hand, we can think of the imposition of capital controls: in 2001, the Argentinean government unilaterally changed the conditions of the private credit market, while in 2021 the Central Bank of Argentina forced firms to renegotiate their external debt obligations, even without passing a bill through Congress. Each of these episodes represent an event where the government induces a default of privately issued external debt.

show uniqueness. Then the default cost can be used to determine the stable state space (i.e., the ergodic support of the steady state distribution).

We calibrate the model to match the private defaults in Argentina 2001. Our findings imply a quantitatively strong result that 100 basis point increase in the international risk free rate more than triples yearly capital debt services and almost doubles the probability of default. Hence private defaultable debt works as an important transmission channel of systemic shocks in Emerging Markets in the short run. Nevertheless, it is well known that the extreme behavior of economies during macroeconomic crisis does not represent its long run dynamics. Table 2 shows how different are short and long run moments for the case of Argentina using a full sample and a fraction of it, between default episodes. As our equilibrium is ergodic, we show that we can also target long run "global" stylized facts.

Table 2: Global and local moments for Argentina

CA/GDP		External Assets	
1983-2001	1960-2017	1983-2001	1960-2017
−2.4%	−0.7%	−36.3%	−25.7%

Note: CA/GDP stands for current account to GDP. Net external assets refer to the overall figure, which includes total public and private debt, including international organisms. In the body of the paper we present evidence in favor of these differences for private and defaultable external debt.

1.1 Relation with the literature

Our main contribution is to derive an ergodicity result in the small open economy under financial frictions literature. Because ergodicity is deeply connected with global stochastic stability, we can also provide a novel rational behind default: it stabilizes an inherently unstable economy. Ergodicity is not new in macro theory. For the standard RBC model, there is an extensive discussion in Lucas et al. (1989). However, these results depend on the continuity of the equilibrium. When the stationary equilibrium maybe discontinuous, either because there are multiple decentralized equilibria or since there is a planner which induces a discontinuous equilibrium rule, there are very few results. For general equilibrium models with incomplete markets, Duffie et al. (1994) show the existence of an ergodic equilibrium. However, those results depend on the existence of convexifying sunspots, a fact that affects severely the computability of equilibrium. To our best knowledge, the only other paper that provides conditions to guarantee that a computable equilibrium is ergodic even though it maybe discontinuous is Pierri and Reffett (2021). In line with Duffie et al. (1994), the authors use an expanded state space to obtain an ergodic representation and apply this technology to a decentralized equilibrium to models of balance of payment crises. Contrarily, in our paper we do not need additional state variables and thus it is the first one to show that it is possible to derive an ergodic equilibrium, even in the presence of discontinuities, in minimal state space. This fact is relevant not only for the numerical efficiency of our accurate algorithm, but also because minimal state space recursive equilibrium is the most widely used equilibrium concept in macro.

Theoretical results in the sovereign default literature are rare apart from the notable exceptions of Auclert and Rognlie (2016), Aguiar and Amador (2019) and Feng and Santos (2021). The first paper shows that if there is an equilibrium in the Eaton and Gersovitz (1981) model, that equilibrium is unique. Then, Aguiar and Amador (2019) prove existence and uniqueness of the Markov Equilibria of the one-period-bond model as in Eaton and Gersovitz (1981). They show this by rewriting the model in a dual form that allows for characterizing the Markov Equilibria as a fixed point of a contraction mapping. Feng and Santos (2021) show existence of a stationary equilibrium in a model with capital and labor. All these, however, are silent as regards the global stochastic dynamics of those models.

In terms of modeling choices, we are not the first ones to build a model with decentralized debt and centralized default. Kim and Zhang (2012) designs a model along those lines and similar to ours, where they assume that households issue private defaultable debt and do not internalize the impact of debt accumulation in the price of debt. However, they consider an exogenous default costs along the lines of Aguiar and Gopinath (2006) and Arellano (2008) while in order to prove our theoretical results we need to carefully select the default costs. Moreover, the authors use prices instead of interest rates, something that affects the definition of equilibrium that they can use in that model.[2] These differences prevent the authors to address the points we address here about existence, unicity and ergodicity of equilibrium.

[2] By assuming that bonds pay an interest rate instead of being purchased by a below-parity price, we are able to write the equilibrium in minimal state space (i.e., exogenous shocks and net external assets).

The remainder of this paper goes as follow. In section 2 we present the model. Section 3 describes the main theoretical results. Section 4 contains the quantitative implementation and the main numerical results. Section 5 concludes. All proofs are available in the appendix.

2 The model

We consider a small open economy populated by households and a government. Households are atomistic risk averse agents that issue foreign debt denominated in real terms, consume and receive an exogenously determined endowment. There is a unique good and the households borrow or lend using a non-contingent asset. The benevolent government decides every period to allow the private sector to repay the foreign debt or forces them to default.[3] The international investors are deep-pockets, risk neutral agents that purchase external debt. Their objective is to break even in expectation and they understand the default risk. We next describe the mathematical environment of the economy.

2.1 The international investors

International investors in this model are risk neutral, deep pockets agents whose objective is to break even in expectations. In line with the standard assumptions in the literature, these agents price debt internalizing the default probability. Denoting the risk free rate by R^*, π for the transition probabilities, the price of debt is:

$$R(B', y) = R^* \left[\sum_{y' \in [Y_{LB}, Y_{UB}]} \pi(y', y) \mathbb{I} \left\{ V^c(B_+(B, y), y') > V^{def}(y') \right\} \right]^{-1} . \tag{1}$$

Here $B_+(B, y)$ is the aggregate debt in equilibrium and $\mathbb{I} \left\{ V^c(b_+(B, y), y') > V^{def}(y') \right\}$ is an indicator function that takes the value of 1 in states in which the government does not induce a default. As it is clear in the equation, the assumption of incomplete markets imply that the investors cannot condition the return on debt on the future states. Hence, the equilibrium interest

[3] The Government can prevent excessive borrowing in any state by forcing the private sector to default on its debt. The intuition for this is that in emerging economies this is typically achieved by a a drastic change in the economic environment surrounding private indebtedness such as a domestic currency depreciation, suspending the access to the exchange rate markets, direct capital controls and other means.

rate does not depend on future income, and depends on current income only if the income process is persistent.

2.2 The households

The domestic economy is populated by a large number of identical households that can have access to a one period real debt, $b < 0$, with a non-contingent gross interest rate R and receive a positive stochastic endowment y that follows an i.i.d process.[4] Preferences are standard and represented by an increasing concave and differential instantaneous return function u. There is a single consumption good c. As there is centralized default risk and the endowment is i.i.d., the interest rate is decreasing in aggregate assets and does not depend on y. We then write the market interest rate as a function only of aggregate assets, $R(B)$. As it is discussed in the appendix, the monotonicity of R on B does not depend on the assumption of i.i.d. shocks. [5] Thus, the recursive problem of the agent is:

$$V^c(b, B, y; h) = \max_{b_+ \geq -\bar{b}} u(F(b, B, y) - b_+) + \beta \mathbb{E}[V(b_+, h(B, y), y'; h)]. \tag{2}$$

Here $F(b, B, y) = y + bR(B)$ and h is the aggregate law of motion for assets B_+, V^c is the value function under repayment and V is the option value between default and repay. We will define these last 2 functions in the next subsection. The policy function for this problem is $b^*_+(b, B, y; h)$. One of our objectives is to show the existence of at least one h and characterize globally the dynamic stochastic equilibrium induced by it.

Taking R as given, as the households are atomistic, the characterization of this problem follows almost from standard results. We give details about the first order condition in the appendix (see the subsection containing preliminary remarks). The Euler equation is given by:

---

[4] We only need the i.i.d. assumption to prove ergodicity of the equilibrium. Nevertheless, we introduce this assumption here because it allows us to simplify notation. In section 4.2.2 we solve a stationary version of the model with Markov shocks.

[5] Note that as the default is centralized, our environment does not have the type of multiplicity in Ayres et al. (2018) as we do not have their Laffer curve. Moreover, the assumption of i.i.d. endowment shocks imply that the interest rate does not depend on past values of the endowment autonomously, in contrast to Arellano (2008). Finally, as in Arellano (2008), incomplete markets imply that the interest rate does not depend on future endowment. The preliminary remarks associated with the proofs for section 3, in the appendix, contains details about the interest rate, the Euler equation and the operator used to constructively prove the existence of equilibria.

$$u'(c(b,B,y;h)) \geq R(h(B,y))\beta\mathbb{E}[u'(c_+(b_+(b,B,y;h),h(B,y),y';h))]. \tag{3}$$

Equation (3) may hold with strict inequality if the upper bound on debt is binding and $c(b,B,y;h) = y - b_+(b,B,y;h) + bR(B)$. This is the same Euler equation used by Kim and Zhang (2012). The details in the appendix describe the usefulness of this equation for this paper (i.e., prove compactness, monotonicity of the equilibrium policy function and constructive existence of the stationary equilibrium). Note that as in Kim and Zhang (2012) we are modeling a perceived law of motion h that maps $(B \times Y) \mapsto B'$. Thus, as the integral in the expectation operator is taken with respect to y', $B' = h(B,y)$ is constant with respect to that variable and thus can be factored out in the Euler equation. This is one important difference with respect to Arellano (2008): as there is decentralized credit, the effects of borrowing on the interest rate are not internalized by the private agent. Thus, given (B,y) the agent expects a constant interest rate in the future, which is a reflection of the price taking assumption.

If B is sufficiently low, which implies that $R(B)$ is sufficiently large, we may have:[6]

$$u'(c(b,B,y;h)) \geq \mathbb{E}[u'(c_+(b_+(b,B,y;h),h(B,y),y';h))]. \tag{4}$$

Equation (4) is at the heart of the instability of the decentralized equilibrium. Section 3.1 presents positive probability paths which illustrate this behavior. Taking the behavior of the households as given, the government decides to allow for repayment or force to default the private sector. We specify the government's problem in the following section.

[6] To obtain equation (4) it suffices to impose $R(h(B,y))\beta \geq 1$. Note that $R(B)$ could be arbitrary large if repayment occurs with probability zero. However, as we show in lemma 1, this may happen in off-equilibrium paths which are not relevant for the solution of the model. Even in the most general Markov equilibrium notions, such as Duffie et al. (1994) and Phelan and Stacchetti (2001), the consistency of general equilibrium implies restricting the stationary equilibrium to those paths that satisfy individual optimization with positive probability. Due to the restrictions in preferences, u is bounded above and unbounded below which in turn implies the existence of a uniform lower bound for consumption. Thus, as in the proof of lemma 1, the fact that $R(B')$ can be factored out, implies that any unbounded endogenous interest rate would violate individual optimization. Thus, these paths cannot be considered a part of the recursive stationary equilibria with minimal state space. This type of equilibria, which is the notion used in this paper, is a subset of the recursive equilibrium notions in Duffie et al. (1994) and Phelan and Stacchetti (2001).

2.3 Centralized default

Suppose that default decisions depend on a benevolent government that may prohibit the private sector to repay its' debts. As in Kim and Zhang (2012) we assume that it is possible to default in any state and we abstract from the specific instruments that could lead to a massive default of the private sector. Nevertheless, we provided some examples in the introduction. To focus on stationary equilibrium, we set $b = B$. As the government does not choose the level of debt but only the decision to induce a private default, the problem of the government is:

$$\text{Default if:} \quad V^c(B, y) \leq V^{def}(y). \tag{5}$$

Here, $V^c(B, y)$ represents the continuation value, i.e., the reward for repayment the outstanding debt, that satisfies

$$V^c(B, y) = u(y - b_+(y, B; h) + BR(B)) + \beta\mathbb{E}\max\left\{V^c(b_+(y, B; h), y'), V^{def}(y')\right\}, \tag{6}$$

while $V^{def}(y)$ stands for the value of default and satisfies

$$V^{def}(y) = u(y^{def}(y)) + \beta\mathbb{E}\left\{\theta V^c(0, y') + (1 - \theta)V^{def}(y')\right\}. \tag{7}$$

Here θ is the probability of re-gaining access to the market after default occurs.

Note that, *if consumption and assets in the next period are both increasing in B for each y*, we have the following characterization of default sets:

$$\left\{\begin{matrix} \text{Repay} & if\ B > \bar{B}(y) & \text{as this implies} & V^c(B, y) > V^{def}(y) \\ \text{Default} & if\ B \leq \bar{B}(y) & \text{as this implies} & V^c(B, y) \leq V^{def}(y) \end{matrix}\right\}. \tag{8}$$

Given the existence of a stationary equilibrium, equation (8) shows that, if consumption and assets are both increasing in aggregate states, *private debt induces state dependent default sets* as in Arellano (2008). This fact will allow us to characterize stochastic dynamics following a traditional approach in the literature. However, this type of analysis was used only in models with public debt, which are not suitable for macro-prudential analysis.

This characterization depends on the existence of a stationary equilibrium, which in turn defines h. In the next section we will show that there exist at least one set of functions $(c, R, V^c, V^{def}, \bar{B})$ that defines h as follows:

$$\left\{\begin{array}{c} if\ B \geq \bar{B}(y),\ h(B,y) = b_+(B,y;h)\ and\ c(B,y;h) = y + BR(B) - h(B,y) \\ if\ B < \bar{B}(y),\ \text{with probability}\ \theta,\ h(B,y) = b_+(0,y;h)\ and\ c(B,y;h) = y^{def}(y) - h(0,y) \\ if\ B < \bar{B}(y),\ \text{with probability}\ 1-\theta,\ h(B,y) = 0\ and\ c(B,y;h) = y^{def}(y) \end{array}\right\} \quad (9)$$

Note that h is discontinuous even if it is unique. So the tools used to prove the existence of equilibrium must be robust to the presence of discontinuities. Fortunately, we will show that equation (3) induces an order structure, which will allow us to use suitable theorems. It turns out that Coleman (1991), Mirman et al. (2008) and Aguiar and Amador (2019) serve this purpose. Moreover, the default restrictions associated with (9) are not internalized by the household. Thus, as $\bar{b}$ can be assumed to be arbitrarily large, we can prove the results using a standard Euler operator without taking into account inequality constraints. As it is typical in the default literature, the model assumes that the government has an enforcement technology to keep the private economy away from individual optimization (as described by equation (3) and formally captured by h when $b_+(B,y;h) < \bar{B}(y)$) as long as re-entry is not possible.

3 Existence and characterization of equilibria

We characterize the solution of the model introduced in the previous section. To show stationarity, we use the results from Aguiar and Amador (2019) and Mirman et al. (2008). We define a nested fixed point operator combining these two papers. We use the former to show the existence of R, V^c, V^{def} for each c. The latter allows us to update c using private optimization. In this sense, we show that government decisions can be "nested" into private optimization generating a unique fixed point for R, V^c, V^{def} parametrized by c. We then show that standard Coleman-Reffett operator borrowed from Aguiar and Amador (2019) and Mirman et al. (2008) converges to a fixed point of the Euler equation characterizing private optimization. Critically, the government's decisions do not alter the monotonicity of R with respect to B, which in turn allow us to keep a stable topological structure for any sequence of c generated by the Euler equation. This property is then essential to derive a stable uniformly convergent algorithm. Finally, we use a result in Pierri and Reffett (2021) to show that this model contains at least 1 ergodic equilibrium. Moreover, as existence proofs are constructive we derive an algorithmic procedure and we use it to characterize all equilibria (stationary, unique or ergodic). Our numerical results accurately characterize all equilibrium due to the constructive nature of the equilibrium proofs.

3.1 Properties of the private economy

We now globally characterize the private economy. We show that c, b_+ and R are uniformly bounded. Contrarily to the results in Aguiar and Amador (2019), by slightly restricting preferences, we derive these bounds from primitive conditions. The additional assumptions on preferences are not restrictive for most of the literature as a standard CRRA function with a lower bound on the risk aversion parameter satisfies them. Finally, we provide sufficient conditions to bound interest rate near default. This is a major advantage with respect to the standard practice, where interest rates explode around default, as combined with our ergodicity result will allow us to derive a stable distribution for interest rates; typically displaying fat-tails.

We then show that consumption c and savings b_+ are both jointly increasing in b along the equilibrium paths (i.e., when $b = B$). This property is important numerically and empirically. As shown in Coleman (1991), the sequence of consumption functions generated by the Euler equation converge using the sup-norm; which is typically used in practice to terminate algorithms. More to the point, as the private economy is characterized by a rather standard savings problem, we can globally characterize stochastic paths starting from an arbitrary initial condition; a fact that is essential to prove ergodicity.

Finally, we show that if the level of private debt is sufficiently high, the economy will default with positive probability and characterize these paths. We call these paths "unstable". Note however, that this property is not at odds with the compactness of the equilibrium. Under risk neutral pricing, the interest rate is unbounded at B if $V^c(B, y) \leq V^{def}(y)$ for all y. That is, $b_+(B, y; h) \leq \bar{B}(y)$ for all y. We show that $b_+(B(y), y; h) \leq \bar{B}(y)$ almost everywhere; which is a milder condition. We now introduce the basic assumptions.

Assumption 1 (Finite i.i.d. endowments)**.** *All* $y \in [Y_{LB}, Y_{UB}] \equiv Y$ *with* $Y_{LB} > 0,\ , Y_{UB} < \infty$ *and* $\pi(y) > 0$*; where* π *is a probability measure.*

Assumption 2 (Preferences)**.** $u : \mathbb{X} \to \mathbb{R}$*, where* $\mathbb{X}$ *is the consumption space,* u *is once differentiable with derivative given by* $u'(c)$*, strictly increasing, strictly concave, unbounded below and bounded above. Moreover,* u' *satisfies Inada:* $lim_{c\to\infty}\ u'(c) = 0$ *and* $lim_{c\to 0}\ u'(c) = \infty$*. Finally,* $\beta R^* < 1$

Assumption 1 states that shocks are bounded, positive and i.i.d. This assumption is only required for the ergodic equilibrium. We will show that the stationary equilibrium is unique using an endogenous punishment (see assumption 3) and an operator (see definition 2) that can be defined for Markov shocks. As this equilibrium is not continuous, we cannot prove ergodicity using standard results (see Futia (1982)). Thus, we need to show the existence of a meaningful atom, which in turn requires a *point* (i.e., a set composed by a single element) to regenerate the process. This point is associated with default. Thus, the state space must be the same every time the government decides to default (i.e., $[b, B, y] = [0, 0, y^{def}]$, see assumption 5). Thus, as ergodicity is deeply connected with the stability of the equilibrium, we can not allow for Markov shocks.

Assumption 2 is standard except for its bounds. A sufficient condition for assumption 2 is setting $u(c) = c^{1-\sigma}/(1-\sigma)$ with $\sigma > 1$ and $\mathbb{X}$ bounded below by zero. We require $\beta R^* < 1$ to guarantee the compactness of the equilibrium. We will state a remark below on the relevance of this assumption.

We now show that c, b_+, and R are bounded. Given assumption 1, it is possible to define a process $(\Omega, \Sigma, \mu_{y_0})$ with a typical element in the sequence space $[y_0, y_1, ...]$ and an associated process in the space of random variables for $[c, b_+, R](\omega)$ mapping Ω to $\mathbb{R}^3$ (see Lucas et al. (1989), chapters 7 to 9).

Lemma 1 (Bounds)**.** *Under assumptions 1 and 2,* $[c, b_+, R](\omega) \in \mathbb{K}$ *almost everywhere in* Ω*, where* $\mathbb{K} \in \mathbb{R}^3$ *and is compact. Moreover,* $c(\omega)$ *is bounded below almost everywhere in* Ω *by* $\underline{c} > 0$*.*

Proof. See the Appendix □

We can now characterize the policy function induced by equation (3).

Lemma 2 (Policy Functions)**.** *Under assumptions 1 and 2, if $R(B)$ is decreasing in B, then $c(b, B, y; h)$ and $b_+(b, B, y; h)$ are both weakly increasing in b when $b = B$ for any $y \in [Y_{LB}, Y_{UB}]$ and h.*[7] *Moreover, either c or b_+ is strictly increasing.*

Proof. See the Appendix □

We will use Lemma 2 to characterize the dynamic properties of the equilibrium as we need that at least 1 policy function is strictly increasing to prove the existence of equilibria (Aguiar and Amador (2019)).

It is possible to use lemma 2 to characterize "unstable" paths. Let $\left[\bar{B}(y_{LB}), ..., \bar{B}(Y_{UB})\right] \equiv \bar{B}$ be the set default thresholds defined in equation (9). We say that a path of shocks is weakly decreasing if $y_s \leq y_t$ with $s > t$. We denote such a path $[y \downarrow, ..., y_T]$. Note that as $T < \infty$, a weakly decreasing path has positive probability. The qualitative properties of the policy function described in lemma 2 suggests that the private economy is inherently unstable. That is, it is possible to generate a persistent recession (i.e.,$[y \downarrow, ..., y_T]$) that leads to a sufficiently high debt level (i.e., $B < \bar{B}(y)$ for some y) which, coupled with a poor growth prospect, leads to a default. The supplementary appendix to this section formally states this result (see lemma 8). Of course, there is a "limit" to the degree of instability that allows us to keep the equilibrium tractable. The remark below connects the restrictions on β and R^* in assumption 2 that keeps the equilibrium tractable in this sense.

Remark 1 (Unbounded consumption and debt)**.** *If $\beta R^* \geq 1$ and $-\bar{b} = +\infty$, then consumption and debt are unbounded.*

Remark 1 follows from the standard sub-martingale theorem (see for instance Ljungqvist and Sargent (2012)).

[7] Both c and h are contained in s set of function C defined in the appendix.

If $\beta R^* \geq 1$ equation (4) holds with full probability. The results in Ljungqvist and Sargent (2012) implies that $u'(c)$ follows a non-negative supermartingale. This last result in turn implies that, if $-\bar{b} = +\infty$, $u'(c) \to 0$. Due to the Inada condition in assumption 2, consumption is unbounded. The results in Ljungqvist and Sargent (2012) apply to an economy with no debt. Thus, in that economy, the only possibility is that agents accumulate infinite positive net wealth. This is not possible in our model as we show in lemma 1 that net assets are bounded above. As the interest rate is bounded above by the uniform lower bound on consumption (see lemma 1), equation (4) still holds if the agent prefers an unbounded consumption path generated by an arbitrary large level of debt. Thus, if $\beta R^* \geq 1$ and $-\bar{b} = +\infty$, agents will prefer an unbounded level of consumption and debt. To prevent these paths we must *jointly* restrict β and b_+. Thus, we require $\beta R^* < 1$ (in assumption 2) and $b_+ \leq -\infty$ (in equation (2)).

3.2 Existence of stationary equilibria

To understand the nature of our existence proof, we first have to list which are the elements involved in any recursive equilibrium. First the policy functions, c and b_+ coming from equations (2) and (3). These functions are defined for any R given by equation (1). Finally, we need value functions V^c and V^{def} given by equations (6) and (7). All these elements must form an operator that has at least 1 fixed point given by h and satisfying equation (9). The following definition formally states these requirements.

Definition 1 (Recursive equilibria)**.** *A set of elements* $(c, b_+, R, V^c, V^{def}, h) \equiv H$ *form a recursive equilibrium if:*

- c *and* b_+ *solve* (2) *give* R*.*
- R *satisfies equations* (5) *and* (1)*.*
- V^c *and* V^{def} *are given by* (6) *and* (7)*.*
- h *satisfies* (9)*.*

As any set of elements satisfying definition 1 are time independent, we call them a *stationary equilibria*. We will say that the equilibrium is *ergodic* if, in addition of being stationary, it has an ergodic invariant measure. The appendix containing the proofs for section 3.3 formally defines an ergodic equilibrium. Moreover, we will say that the equilibrium is *stable* if it generates paths that are *recurrent* (i.e., they don't diverge to the boundary of the equilibrium set, which is well-defined due to lemma 1) and the state space is *connected* (i.e., the process does not break into "islands" or equivalently it is irreducible). [8]

To show existence we will use a nested fix point argument. We show that equation (3) induce a Coleman-Reffett operator on c, b_+ satisfying the properties in lemma 1 and 2 for any R that is decreasing in B. This fact depends in turn on V^c, V^{def}. We show, using the results in Aguiar and Amador (2019), that these functions have a unique fixed point for any triple (c, b_+, R). Moreover, we can use the fixed point of V^c, V^{def} to update R and then use (3) to update c, b_+. As the bounds on the policy functions and interest rates are uniform and depend on assumptions on the primitives, equations (5), (6) and (7) preserve the monotonicity of R. The Coleman-Reffett operator induces a sequence of ordered policy functions given an interest rate R that converges to a fixed point. Then Aguiar-Amador operator obtains the associated value functions. Notice that this proof induces an algorithmic procedure:

Definition 2 (Nested fixed Point Operator)**.** *The existence of a stationary equilibrium is proved using the following Nested fixed Point Operator:*

- *Coleman-Reffett. Given R, equation (3) generates an operator A with $c_n \to Ac_n = c_{n+1}$.*
- *Aguiar-Amador.*
  - *Equations (6) and (7) induce an operator $\mathbb{T}$ with: $\left[V_j^c, V_j^{def}\right](c_n) \to \mathbb{T}\left[V_j^c, V_j^{def}\right](c_n) = \left[V_{j+1}^c, V_{j+1}^{def}\right](c_n)$.*
  - *This operator has a fixed point $\left[V_*^c, V_*^{def}\right](c_n)$.*
- *Equations (5) and (1) update $R\left(\left[V_*^c, V_*^{def}\right](c_n)\right)$.*
- *The Coleman-Reffett operator updates c using $R\left(\left[V_*^c, V_*^{def}\right](c_n)\right)$.*
- *Continue until convergence with $R\left(\left[V_*^c, V_*^{def}\right](c_*)\right) \equiv R_*$ and c_* is a fixed point of A.*

[8] The appendix containing the proofs of section 3.3 formally defines an accessible atom which jointly ensures recurrence and irreducibility. As the atom is also useful to show ergodicity, this concept is deeply connected with the stability of paths.

Notice that the first step of definition 2 requires an initial condition for R, which is typically assumed to be $R_0 = R^*$. Moreover, the Coleman-Reffett operator converges to a different fixed point depending on the initial condition c_0. Further, definition 2 does not depend on h, the equilibrium law of motion for debt with default. As mentioned before, we first show the existence of (c_*, R_*) and then use equation (9) to define h. Finally, there must be a consistency requirement between $V_0^c, V_0^{def}, R_0, c_0$ given by equations (5), (6) and (7). Fortunately, there is one degree of freedom: $y^{def}(y)$. Under the following assumption we show that an equilibrium exists and depend on the initial condition c_0.

Assumption 3 (Stationary punishment)**.** *Let $\mathbb{C}$ be the space of consumption functions*[9] *and $\mathbb{B}$ the compact set containing any B, both obtained in lemma 1. Let $c_0 \in \mathbb{C}$. Then $y^{def}(y)$ with $y \in Y$ satisfies:*

1. $V_0^c(B, y) = u(c_0(B, y)) + \beta\mathbb{E}\left\{V_0^c(y + R^*B - c_0(B, y), y')\right\}$
2. $V_0^c(B, y) \geq V_0^{def}(y) = u(y^{def}(y)) + \beta\mathbb{E}\left\{(1-\theta)V_0^{def}(y') + (\theta)V_0^c(0, y')\right\}$ *for all* $y, B \in Y \times \mathbb{B}$.
3. c_0 *satisfies* $\bar{c}_0 = SUP(\mathbb{C})$ *or* $\underline{c}_0 = INF(\mathbb{C})$.
4. $y \geq y^{def}(y)$ *for all* $y \in Y$

Remark 2. *Note that assumption 3.4 allows us to model asymmetric default costs (i.e., $y^{def}(y) = \hat{y}$ if $y > \hat{y}$ and $y^{def}(y) = y$ if $y \leq \hat{y}$), which are typical in the literature.*

Aguiar and Amador (2019) use a similar restriction for y^{def}.[10] Notice importantly that, as u is unbounded below because of assumption 2 and V_0^c, V_0^{def} have a fixed point under standard arguments, assumption 3 is rather mild. The last requirement on $y^{def}(y)$ is standard in the literature (see for instance Arellano (2008)).

We will now show the main theorem of the paper, which states the existence of stationary equilibria in definition 1. Notice remarkably that definition 2 will allow us to compute *directly* the

[9] See the appendix.
[10] See Assumption 4.

equilibrium without the need of a heuristic updating rule for prices.[11] Thus, we call this equilibrium *computable.*

Theorem 1 (Existence of stationary computable equilibria (SCE))**.** *Under assumptions 1, 2 and 3, there exist at least 2 SCE,* $H(\bar{c}_0)$ *and* $H(\underline{c}_0)$*, with* $c_*(\underline{c}_0)(B,y) \leq c_*(\bar{c}_0)(B,y)$ *for all* $B \in \mathbb{B}$ *given* y*.*

Proof. See the appendix □

Notice that we show the existence of multiple ordered equilibria. The economy can coordinate in any of these 2 equilibria, depending on the initial condition of the iterative process. In this sense, private debt induces a *coordination failure that may generate a permanently low consumption level.* The Government has then incentives to break this coordination failure by providing conditions that guarantee the uniqueness of equilibrium. The next result gives these conditions.

3.3 Uniqueness and Ergodicity

In this section we show two further properties of the equilibria: i) the equilibrium is unique under a strengthening of assumption 2,[12] and, ii) by imposing an additional restriction to assumption 3, the equilibrium is ergodic. This section establishes, then, the main theoretical result of this paper: proposing the first available proof of ergodicity in default models, something that is required based on the different short and long run behavior observed around default.

Remarkably, as these two additional assumptions are independent, we can get multiple ergodic equilibrium. Contrarily to the standard ergodicity proof (see Futia (1982)), we can dispense with the continuity of equilibrium; which is typically associated with uniqueness (see Duffie et al. (1994)). As the preferences frequently used in practice satisfy the additional assumption which ensures uniqueness, we do not investigate the behavior of different ergodic steady states.

Assumption 4 (Pseudo-Concavity of the utility function)**.** *In addition to assumption 2, suppose that* $u'(c_1 c_2) = u'(c_1)u'(c_2)$ *for all* $c_1, c_2 \in \mathbb{C}$ *and* $c_1, c_2 > \overrightarrow{0} \in \mathbb{R}^2$*.*

[11]The proof of existence requires that every iteration preserves the monotonicity of interest rates. Thus, not every updating rule will serve this purpose.

[12]For a model with centralized default and public debt, uniqueness was shown by Aguiar and Amador (2019). However, this paper offers the first uniqueness proof for a model with private debt.

Remark 3 (Constant Relative Risk Aversion Preferences)**.** *Note that if* $u(c) = \frac{c^{1-\sigma}}{1-\sigma}$ *and* $\sigma > 1$ *assumptions 2 and 4 are simultaneously satisfied.*

Assumption 2 guarantees that equation (3) defines a pseudo-concave operator A. In particular, given some $\alpha \in (0, 1)$ for all possible consumption functions c we have that: $A(\alpha c) > \alpha A(c)$. Coupled with the uniform positive lower bound for consumption in lemma 1 and the ordered structure of the set of fixed points in theorem 1, this assumption is enough to show uniqueness. Remark 3 implies that the preferences which are frequently used (i.e., represented by constant relative risk aversion functions) and the typical parameter values which arise from calibrations (i.e., $\sigma > 1$) will usually lead to a unique equilibrium under i.i.d. shocks (see Arellano (2008) among others).

Theorem 2 (Uniqueness of Stationary Computable Equilibria)**.** *Under assumptions 1, 3 and 4, there is at most 1 SCE* c_*.

Proof. See the appendix. □

We now turn to the dynamic global behavior of the model. Note that, even though theorems 1 and 2 offers a global characterization of stationary equilibria, we have been silent as regards the simulations. Moreover, lemma 8 provides a characterization of local dynamics, as we have to condition on a particular initial level of debt to characterize the stochastic paths. However, the empirical evidence suggests that there is a striking difference between the local behavior around the default and the global characteristics of the time series as summarized by standard statistics (i.e. correlation coefficients, standard deviations, etc.). To keep on characterizing the model's dynamics, we have to connect long run simulations with the model's statistics. Ideally, these simulations should contain relevant information as regards the stochastic steady state of the model. For that, we need a law of large numbers and an ergodic steady state represented by an invariant probability measure (see Pierri and Reffett (2021) for a related discussion).

We now use Definition 1 to define an equilibrium Markov process in (B, y, c, R). One of the most important characteristics of the default literature is that, given the existence of a SCE, we can derive an ergodic equilibrium in a *minimal state space*. That is, we can describe the dynamic behavior of the model using an arbitrary sequence of shocks and the law of motion for aggregate debt, h, as the rest of the variable is the state space can be recovered using two stationary functions.

In other incomplete market models, as in the sudden stop literature, it is not possible to use this parsimonious state space. Thus, the model presented in this paper constitutes a unique opportunity to study ergodic dynamics in a tractable and numerically efficient framework.

Let $Z_1 \subset \mathbb{B} \times Y$ be the state space defined in the appendix. Then, for each $(B, y) \in Z_1$ and $y_+ \in Y$, we can get $R_*(B, y), R_*(h(B, y), y_+)$ from equations (5) and (1) using $\left[V_*^c, V_*^{def}\right](c_*)$ and $c_*(B, y), c_*(h(B, y), y_+)$ using the budget equation. Note that this procedure allows us to define a system φ mapping $(B, y, c, R) \to (B_+, y_+, c_+, R_+)$. Let Z be the state space containing any (B, y, c, R) satisfying definition 1. Thus, we can project Z_1 onto Z using φ, which in turn allows us to derive the following Markov kernel:

$$P_\varphi(z, A) = \left\{\pi\left(y' \in Y : \left[c_*(h(B, y), y'), R_*(h(B, y)), h(B, y), y'\right] \in A\right)\right\} \tag{10}$$

We now derive the stochastic steady state for the model summarized by (Z, P_φ). Formally, we show that P_φ has an ergodic invariant measure, which is at the same time the stable distributions and the stochastic steady state for any equilibrium vector (B, y, c, R). It turns out that if we restrict the number of possible distinct values that y can take to be finite, we can prove the existence of an ergodic probability measure. Using equation (9) and restricting assumption 3.4 such that $y^{def}(y) = y^{def}$ for all $y \in Y$ we can construct a *point* z_* which the process hits with positive probability starting from any initial condition. This point will be called *atom* and belongs to Z. The discussion below and in the next subsection shows how z_* creates an orbit which endows the dynamical system with a recurrent and connected structure, which in turn implies that: i) there will be a unique (and thus ergodic) invariant measure for each atom, ii) the stochastic process represented by (Z, P_φ) is globally stable. Note importantly, this implies *that there could be at most 1 default for each stable distribution, associated with* y^{def}*, which in turn implies that that this type of events are so extreme that generate a change in the entire stable distribution of the economy.*

Once we find z_*, we construct a stable state space. That is, any meaningful (i.e. with positive measure) subset of this state space will be hit by the process in finite time. This property, called irreducibility, guarantees the uniqueness and ergodicity of the process together with the global stochastic stability of the process. If we allow for discontinuous equilibrium function φ, we can construct a phase diagram such that the process jumps to the atom every time there is a default.

The results in Meyn and Tweedie (1993) give us the tools to prove all the intermediate steps required to go from the existence of a SCE to its ergodicity.[13]

Figure 1 represents the way in which default induces global stochastic stability.

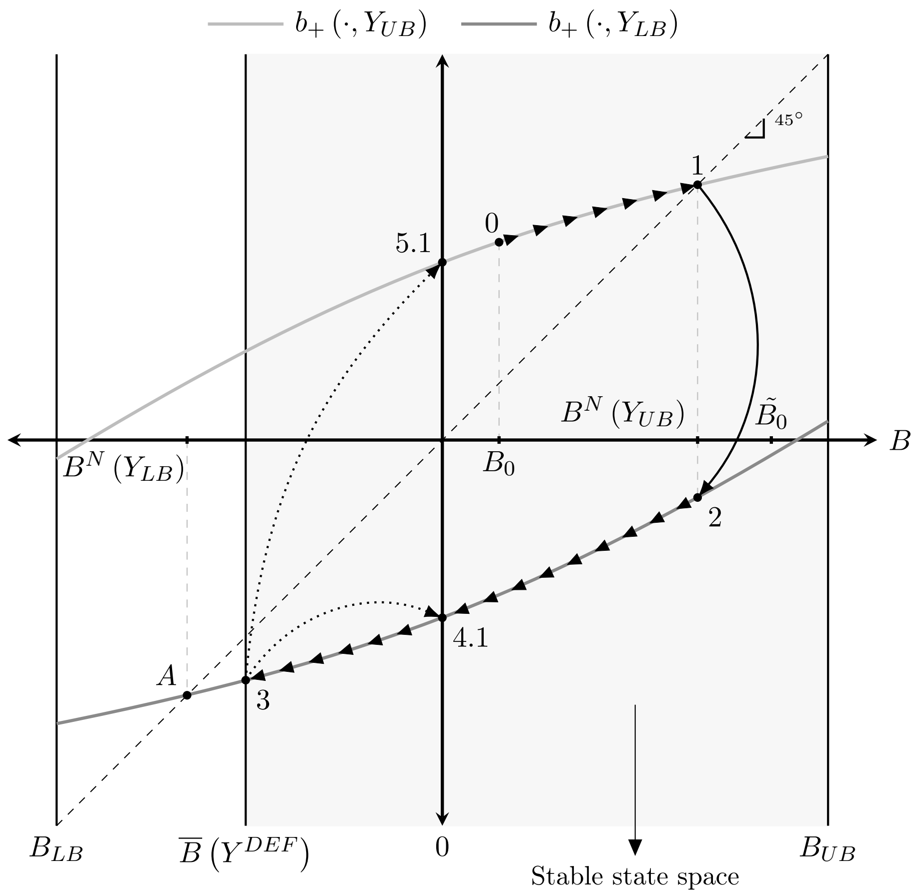

Figure 1: Non-empty and (saddle path) stable default set

Note: Start with the initial state B_0, Y_{UB} in point 0, an initial condition which implies $b_{+,*}(B_0, Y_{UB})$. A good endowment shock increase assets accumulation to point 1, where a bad endowment realization occurs inducing the economy to issue debt to smooth consumption (point 2). If a sequence of Y_{LB} occurs, the economy goes straight to default in point 3. The figure illustrates that when returning to asset markets the economy can transition to point 5.1 if Y_{UB} or 4.1 if Y_{LB}.

In the figure, we present the transition for the case of 2 shocks and one of the 2 possible initial conditions $B_0 : b_{+,*}(B_0, Y_{UB}) > B_0$. We only need to show that starting from any initial condition $B_0, y_0, c_*(B_0, y), R_*(B_0, y_0)$ the process hits $Z_* = (0, Y_{LB}, Y^{DEF}, R^*)$ with positive probability and

[13]See Meyn and Tweedie (1993), chapters 5, 8 and 10 for a detailed discussion of the implications of the existence of an atom for the existence of an invariant probability measure.

in finite time. Under assumption 4, it is possible to show that definition 2 and theorem 1 generate a continuous (in B) and unique demarcation curves for each y.[14] These curves also contain a candidate for a non-stochastic steady state, i.e. a point B^N satisfying $B^N = b_{+,*}(B^N, y)$ for any $y \in Y$,[15] which allows us to establish that the default set is non-empty and stable.[16] Thus, the only discontinuity point is associated with the occurrence of default, z_*, and it suffice to characterize dynamically the state space Z_1 and to find an ergodic invariant measure for (Z, P_φ). Starting from B_0, Y_{UB} the economy transitions to point 0, then to 1 for the same shock. When $B_\tau = b_{+,*}(B_\tau, Y_{UB})$, note that this point exists due to the continuity property generated by the uniqueness of the nested fixed point operator, then we choose y_{LB} and jump to point 2, transitioning to 3 under $y_s = y_{LB}$ for $s \geq \tau + 1$. We can also start the iteration from $\tilde{B}_0$, with $b_{+,*}(\tilde{B}_0, Y_{UB}) < \tilde{B}_0$ and obtain a decreasing sequence until we hit the non-stochastic steady state B^N.

In figure 2 we illustrate another possible trajectory. It is also possible to observe that the economy is accumulating debt in the "good state", Y_{UB}, which implies that we will observe that a country is frequently a net debtor. Notice that this happens even though $b_{+,*}(., Y_{UB})$ is always above $b_{+,*}(B_\tau, Y_{LB})$. That is, the model can generate debt accumulation is "good times" and at the same time it keeps an increasing relationship between net external assets B' and GDP y. This is not the case in Arellano (2008), that suggest that net external assets are decreasing in the GDP. In our model the interest rate $R(B)$ is independent of the current shock and thus the counter-cyclicality of this variable, which is critical for the results in Arellano (2008), is absent. The difference between figures 1 and 2, as we will show in the calibration section, could be due to a lower discount factor β for the second figure. As this parameter goes down, demarcation curves rotates to the south-east; generating the observed change between 2 figures.

[14]We state this fact in remark 4 below.

[15]Note that as we are considering an equilibrium, we have $b = B$. On the top of that $B_+ = b_{+,*}(B, y)$ and along the 45º line $B = b_{+,*}(B, y)$.

[16]This means that it does not contain transient sets. A transient set A satisfies $P^n_\varphi(z, A) \to 0$ for all $z \in A$ and in practice are eliminated by throwing away the first 1,000/10,000 simulations before computing any long run average.

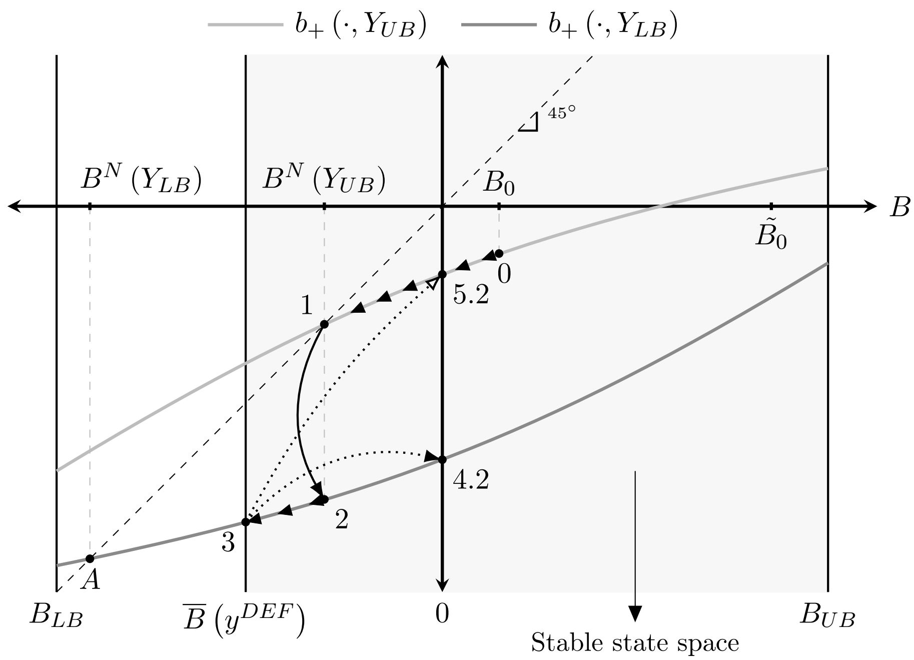

Figure 2: Transitions for an initial net debtor

Note: Start with the initial state B_0, Y_{UB} in point 0, an initial condition which implies $b_{+,*}(B_0, Y_{UB})$. A good endowment shock increase country debt to point 1, where a bad endowment realization occurs inducing the economy to issue debt to smooth consumption (point 2). If a sequence of Y_{LB} occurs, the economy goes straight to default in point 3. The figure illustrates that when returning to asset markets the economy can transition to point 5.2 if Y_{UB} or 4.2 if Y_{LB}.

One important take away point from figures 1 and 2 is the following: in "good times" a country could be accumulating assets or debt. More to the point, this could also happen regardless of the fact that a nation is initially a creditor or a debtor (i.e., it is possible to construct increasing/decreasing paths with a high shock in figure 1 / 2 with a positive or with a negative value of B_0). The distinctive fact for the relationship between net assets and GDP is the position of demarcation curves with respect to the 45° line, which in turn determines if the non-stochastic steady state $B_{NSS}(y) \equiv B = b_{+,*}(B, y)$ for both shocks: a) are inside the stable state space, b) are associated with a net debt position (i.e., $B_{NSS}(Y_{LB}) < 0, B_{NSS}(Y_{UB}) < 0$). This is the case in figure 2 but not in 1. In these figures, non-stochastic steady sates, points 1 and A in figure 2, act as an "attraction point" because demarcation curves are sufficiently flat. In the next section, we show that this may not be the case and some non-stochastic steady states can be "unstable", implying that a country could be either accumulating assets or debt for the same shock depending on the value of B.

Note that 3 is to the right of point A as under assumption 3, the default set is non-empty *and we have assumed* $B^N(y_{LB}) \equiv B = b_{+,*}(B, y_{LB}) < 0$. After default, the economy jumps to either points 4 or 5 *after spending a finite number of periods in exclusion* according to θ. Note that the system between points 3 and 4/5 *behave as an i.i.d process*. It turns out that this is the distinctive characteristic of an atom: a point in which the conditional and the unconditional distribution are equal $P_\varphi(z_*, A) = \mu(A)$. Then, after we obtain a re-entry draw from θ, we can either go to 4 or 5 depending on y. In this sense, the zero at the vertical axes defines the appropriate initial condition for the economy after the re-entry. We call this behavior *saddle path ergodic stability*. The appendix contains the proof for the theorem above and additional technical details to keep the paper self-contained.

Assumption 5 (Ergodic punishment)**.** *In addition to assumption 3.4, assume that* $y^{def}(y) = y^{def}$ *for all* $y \in Y$*. Let* $B^N(y_{LB}) \equiv B = b_{+,*}(B, y_{LB})$*. Assume that* $B^N(y_{LB}) < 0$*.*

Theorem 3. *Under assumptions 1, 4, 5, there exist* y^{def} *such that* $B^N(y_{LB}) < \bar{B}(y^{def}) < 0$ *and* (Z, P_φ) *has an unique ergodic probability measure* μ_*^{def}*.*

Proof. See the appendix. □

Remark 4 (Existence of non-stochastic steady state B^N)**.** *Under assumption 4, we can refine the existence result to show uniqueness in theorem 2. Theorem 10 in Mirman et al. (2008) shows that the equilibrium of the private economy without considering default is continuous, which then implies that there is 1 non-stochastic steady state for each shock in* Y *by changing the probability distribution in assumption 1 such that 1 shock accumulates all the mass (i.e.,* $\pi(y) = 1, \pi(y') = 0$ *for all* $y \neq y'$*).*

Note that the assumption on $B^N(y_{LB})$ in 5 is a minimal consistency requirement: as we are modeling default, that occurs when net assets are negative, it is reasonable to assume that in the worst possible scenario (i.e., $y = y_{LB}$), households choose to hold debt in the non-stochastic steady state of this economy. Then, as lemma 2 shows that $b_{+,*}$ is increasing in B, the definition of y^{def}

will allow us to construct point A satisfying: $B^N(y_{LB}) < \bar{B}(y_{LB}) < 0$, where $\bar{B}(y_{LB})$ is defined in equation (8). Below we discuss the implications of the results in the previous subsections.

3.4 Discussion of the results

In this section we discuss the qualitative and quantitative implications of the results presented above. We comment 5 facts (I to V) and leave 2 (VI and VII) to the supplementary appendix.

I) Only one default is possible for each ergodic and stable distribution. The empirical evidence suggests that there are significant differences in the values of descriptive statistics computed locally, around the default, and globally, for the whole sample. The implications of theorem 3 give us an explanation for this behavior. As, for instance, Argentina and Ecuador experienced more than 1 default between 1960 and 2017, the pooled average across the whole sample may contain information of multiple different steady states. For the case of Argentina, the events in 1982 and 2001 implied very different levels of GDP, thus y^{def} should reflect this fact. As there could be only 1 atom for each stable and ergodic distributions, μ_*^{82}, μ_*^{01}, the cumulative average from 1960 to 2017 can't converge to $\mathbb{E}(z;\mu_*^{82})$ and to $\mathbb{E}(z;\mu_*^{01})$.[17] As long as we don't change y^{def}, the state space Z is the same which in turn implies $\mathbb{E}(z;\mu_*^{82}) \neq \mathbb{E}(z;\mu_*^{01})$.

II) Interest rates, current account and ergodic kernels. Note that $\bar{B}(y_{UB}) < \bar{B}(y_{LB})$. The figure above suggests that if we set the lower bound of the state space Z_1 to be $\bar{B}(y_{LB})$,[18] then interest rates are bounded; a result which follows formally from lemma 1. Typically, the default literature does not compute kernels for unconditional measures, especially the interest rate, because it tends to explode around the default. The results in this paper allows us to construct well defined kernels as the ergodic equilibrium is bounded almost everywhere. Based on these results we can target the current account, instead of the trade balance as is typically done in the literature. Moreover, we can study the concentration of the process around the mean; a fact that is deeply connected with the stability of the distribution. We will address these issues in the next section.

[17] The notation intends to make it clear that one stable distribution corresponds to the default in 1982 (μ_*^{82}) while the other corresponds to the default in 2001 (μ_*^{01}).

[18] In figure 1 this can be done without loss of generality as the is no intersection between the upper demarcation curve and the 45° line. However, in next section, when we derive the empirical phase diagram, $\bar{B}(y_{UB})$ must be the lower bound of the stable state space.

III) Stochastic stability, default probability and stylized facts. The figure above shows that the process does not contain divergent paths but it hits the upper bounds for debt associated with default with positive probability starting from any initial condition. Thus, by definition, we are improving the ability of the model to reproduce any observed probability of default based on changes in y^{def}. Moreover, as the distribution is ergodic, we can match multiple empirical moments as f in $\sum f(z)/N \rightarrow \mathbb{E}(z; \mu_*^{def})$ can be chosen arbitrarily as long as it is continuous.

IV) Non-stochastic steady state, continuity of the equilibrium and stochastic paths. Remark 4 allows us to identify clearly that the only source of discontinuity in this model is the default. Thus, we can get a continuous equilibrium for the private economy as depicted in the figure above by the intersection of the 2 demarcation curves with the 45^{o} degree line. That is, we have a well defined non-stochastic steady state. Because of its local nature, it can't be used to approximate the actual behavior of the economy. However, as we will discuss in the next section, the non-stochastic steady state is useful to classify stochastic paths as stable, unstable and non-sustainable.

V) Frequency of defaults. One of the main targeted values in the literature is the number of defaults in a given time spell. This is called frequency of defaults. To ensure that our model is capable of generating meaningful events (i.e., simulations that replicate the observed frequency of default), we impose assumption 5 and add an additional restriction on endogenous variables in theorem 3. Figure 3 shows an equilibrium which does not satisfy the restrictions on $B^N(y_{LB})$ and $\bar{B}(y_{LB})$ imposed by theorem 3 (i.e., in point A$'$ we have $B^N(y_{LB}) < 0$ but $\bar{B}(y_{LB}) < B^N(y_{LB})$, which does not satisfy these requirements). Additionally, in point A$''$ we observe $B^N(y_{LB}) > 0$, which violates assumption 5. These facts imply that the frequency of defaults that the model will generate will be negligible. While the depicted equilibrium has a non-empty default set, the model can't generate a stable default process. To see this, pick B_0^{DEF} as an initial condition. For the low shock the planner will default in the first period (we will not observe default for the high shock) and then the economy will move to either points 4.3 or 5.3 and default will never be observed again. In this sense, the default set contains only transient elements, a fact that will severely affect the ability of the model to match the empirical probability of default (which is around 3%). Technically, if the default set is transient, it is not possible to generate a recurrent atom and thus, the equilibrium cannot be shown to be either ergodic nor stable.

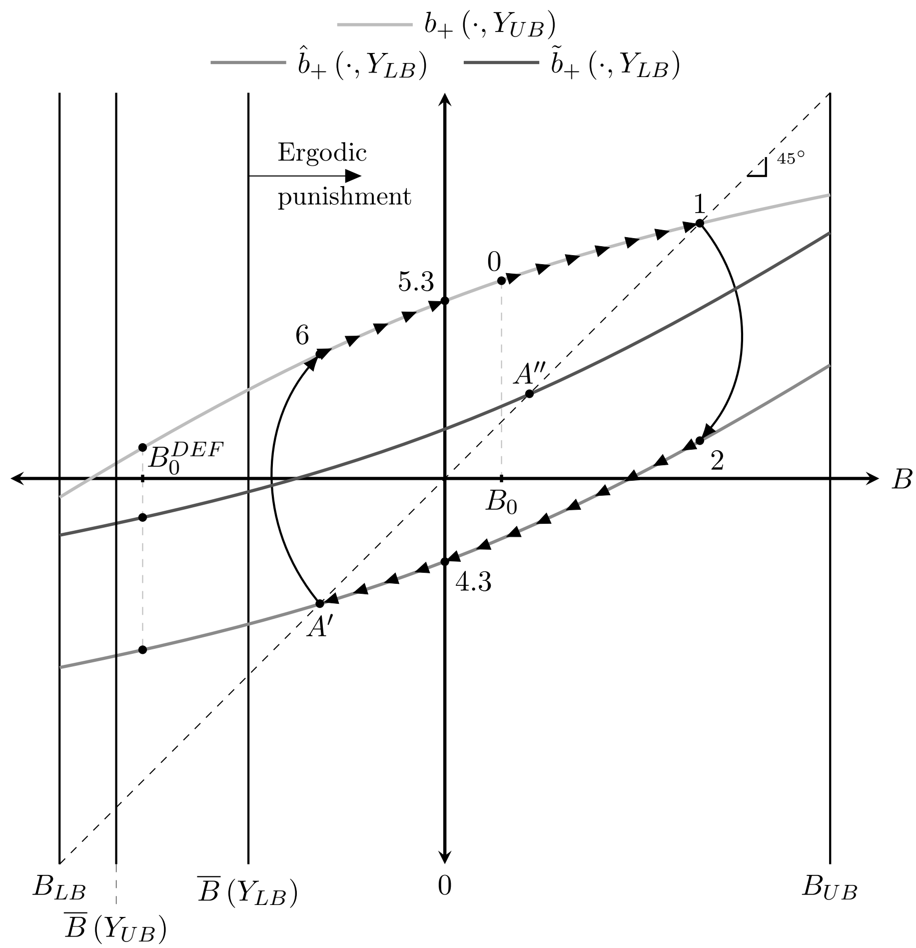

Figure 3: Non-empty default set

Note: $\hat{b}_+(\cdot, Y_{LB})$ is associated with A' whilst $\tilde{b}_+(\cdot, Y_{LB})$ is associated with A''.

4 A numerical example

In this section we test the model empirically and calibrate it to match ergodic moments of the data. Based on previous sections, we need to define the length of the sample as there must be at most 1 default for stable distribution. We choose Argentina between 1982 and 2016 as this sample includes only 1 event, the default of 2001.[19] Using 2 unconditional statistics, we estimate by the simulated method of moments 2 parameters. We borrow the remaining parameters from the literature. We

[19]Between 2014 and 2016 the country was affected by a court ruling which took Argentina out of the international capital markets. The results are similar if we choose 1983-2013 instead of 1983-2016.

then test the empirical fit of the model by comparing non-targeted moments with their empirical analogous.

Arellano (2008) targets the macro-dynamics in an interval before default (1983-2001). We call this approach stationary or local. Instead, our strategy targets a longer sample (1983-2016), including observations during and after default, that will be in line with the ergodic long run moments of the model. In that sense the typical strategy is to calibrate the model locally using moments calculated around the default which, as will be seen in this section, can differ markedly with respect to ergodic global ones.

To calibrate the model, we use a novel algorithm based on definition 2, refining the initial condition of the iterative process using assumption 3. That is, we initialize the Nested Fixed Point Operator in definition 2 in the supremum of the space of candidate functions $\bar{c}_0$ and compute the endogenous punishment $y^{def}(y)$ according to the procedure described in assumption 3. We then take the minimum value of $y^{def}(y)$ and compute the ergodic punishment. Theorem 2 guarantees the uniqueness and computability of the stationary equilibrium using the Nested Fixed Point Operator described in definition 2. Theorem 3 guarantees its ergodicity.

We target yearly interest payments with respect to GDP, $(R(B)B)/Y$, and the frequency of default using β and θ. The non-targeted moments are yearly capital payments with respect to GDP, B/Y, the current account to GDP, CA/Y, and the standard deviation of CA/Y. As we can bound interest payments, we can target the current instead of the trade balance as in Arellano (2008). We compute external private debt using a novel database which classifies external indebtedness according to: international organisms (i.e., non-subject-to default), subject-to default public and subject-to default private. We target the interest payments of last category. The remaining parameters are borrowed from the Kim and Zhang (2012) and Arellano (2008). The results, parameters and moments are contained in the tables below.

Table 3: Results

Variable	$(R(B)B)/Y$*	Def. freq.*	B/Y	CA/Y	$C.V.(CA/Y)$
Data	-0.6%	3.0%	-1.4%	-0.8%	3.6
Model	-0.6%	2.4%	-2.0%	-1.3%	4.2

Note: * denotes moments that are matched using the simulated method of moments. The rest of the statistics are non-targeted moments. $(R(B)B)/Y$ are (yearly) interest payments of private external debt with respect to GDP. "Def. freq." is the frequency of default for events that were preceded by 19 years (between 1983 and 2001) of open access to the international credit markets. B/Y are yearly capital payments (i.e., amortizations) of foreign private debt over GDP. CA/Y is the current account to GDP and C. V. is the coefficient of variation of CA/Y, its standard deviation divided by its mean.

Table 4: Parameters

Parameter	Value	Kim and Zhang (2012)	Arellano (2008)	Description
σ	2.0	2.0	2.0	Risk aversion param.
θ*	0.0725	0.10	0.28	Re-entry prob.
β*	0.935	0.97	0.953	Discount factor
ρ_e	0.001	0.945	0.945	Persis. (endowment)
STD_e	0.02	0.02	0.02	St. dev. (endowment)
r*	1.7%	1.7%	1.7%	Net risk-free interest rate

Note: the second column contains the values of the parameters used in this paper as a benchmark calibration. The third and forth columns contain the analogous set of parameters in Kim and Zhang (2012) and Arellano (2008) respectively. * denotes parameters that are used in the simulated method of moments. The remaining parameters, as can be seen from columns 3 and 4, are borrowed from the literature. ρ_e and STD_e are the coefficients of the AR(1) process that was discretized using a grid of 15 points. $\beta R^* = \beta(1 + r^*) < 1$ as required by assumption 2.

Tables 3 and 4 show that the model replicates well the non-targeted moments using only 2 estimated parameters and. As expected, the model does not get the persistence of the income process due to the i.i.d. assumption. In subsection 4.2.2 we calibrate the model using Markov shocks. As the equilibrium is stationary but not ergodic, which relies on the i.i.d. assumption, we target local moments.

From table 5 it is clear that the structure of the model and the results in the theory section affect the value of the moments to be targeted for 2 reasons: i) fundamental macro variables behave remarkably different around the default, which is a "local behavior", and in the whole sample. This is the case of the current account: the mean around the default implies a deficit almost 3 times bigger than in the whole sample and the dispersion is much lower. ii) The "refinement" process for debt statistics imply that the targeted level of debt varies from -34.0% to -1.4%: first we remove

Table 5: Data with respect to GDP

Date / Percent	Stock of net external assets			Net private debt services		Current account	
	Total	Defaultable	Private	Capital	Interests	Mean	STD
83-01	-36.5%	-31.0%	-6.7%	-1.1%	-0.5%	-2.3%	0.8
83-16	-34.0%	-28.9%	-8.2%	-1.4%	-0.6%	-0.8%	3.6

Note: The second row contains the "local" sample, between the default episodes of 1982 and 2002. The third row shows the "global" sample, which includes the default of 2002. The second column contains total external assets divided by the GDP. The third one shows private plus public external assets, excluding loans granted by international and multilateral organisms which are not subject to a hair-cut. The forth column denotes private external assets only. The fifth and sixth columns show yearly capital payments (i.e., amortizations) and interests of private external debt. "Mean" and STD denote the average of the current account to GDP and its standard deviation divided by the mean, respectively.

the multilateral organizations from the sample (the average for the whole sample goes from -34.0% to -28.9%). Then, we remove public debt and the average goes down to -8.2% and then we use the average duration of debt (6 years) to derive the yearly capital payments -1.4%. As the model only contains 1 period bonds, we follow Arellano (2008) and target yearly debt services.

4.1 The dynamics of the model

Figure 4 shows the global stochastic stable dynamics of debt for the estimated phase diagram. As before, the horizontal axis depicts the stock of debt. The demarcation curves, one for each level of exogenous endowment, show the equilibrium dynamics for debt choice. The arrows in each demarcation curve indicate whether an equilibrium is a stable point or unstable. The $\bar{B}(Y^{DEF}, Y_{Dj})$ indicate the level of debt that will trigger a default if the j endowment decile is realized. Notice that, as indicated in the figure, that level of assets is positive if $Y \in [Y_{D1}, Y_{D5})$, defining what we call "Exclusion area".

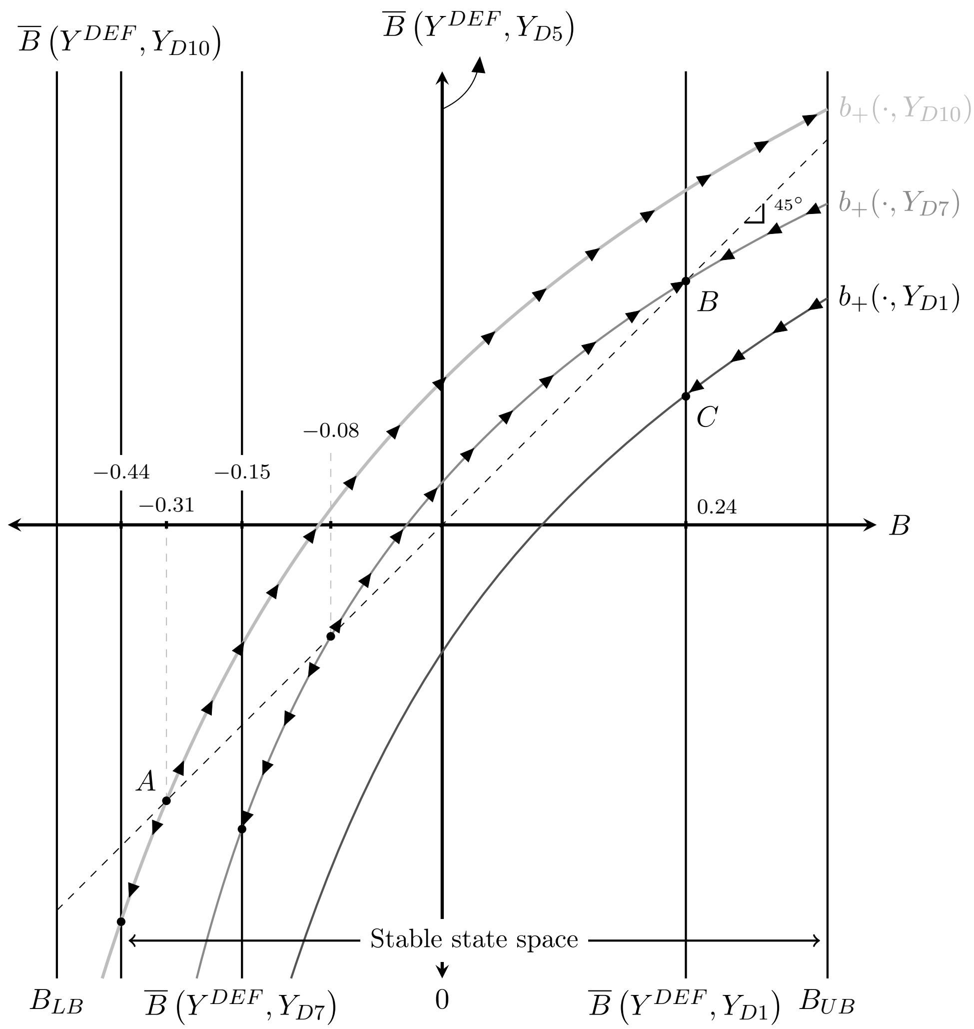

Figure 4: Calibrated phase diagram ($\mu = -2.0\%$, $STD = 0.08$) part A

Note: The phase diagram in this picture follows from the model calibrated to Argentina with the calibration described in Section 4.

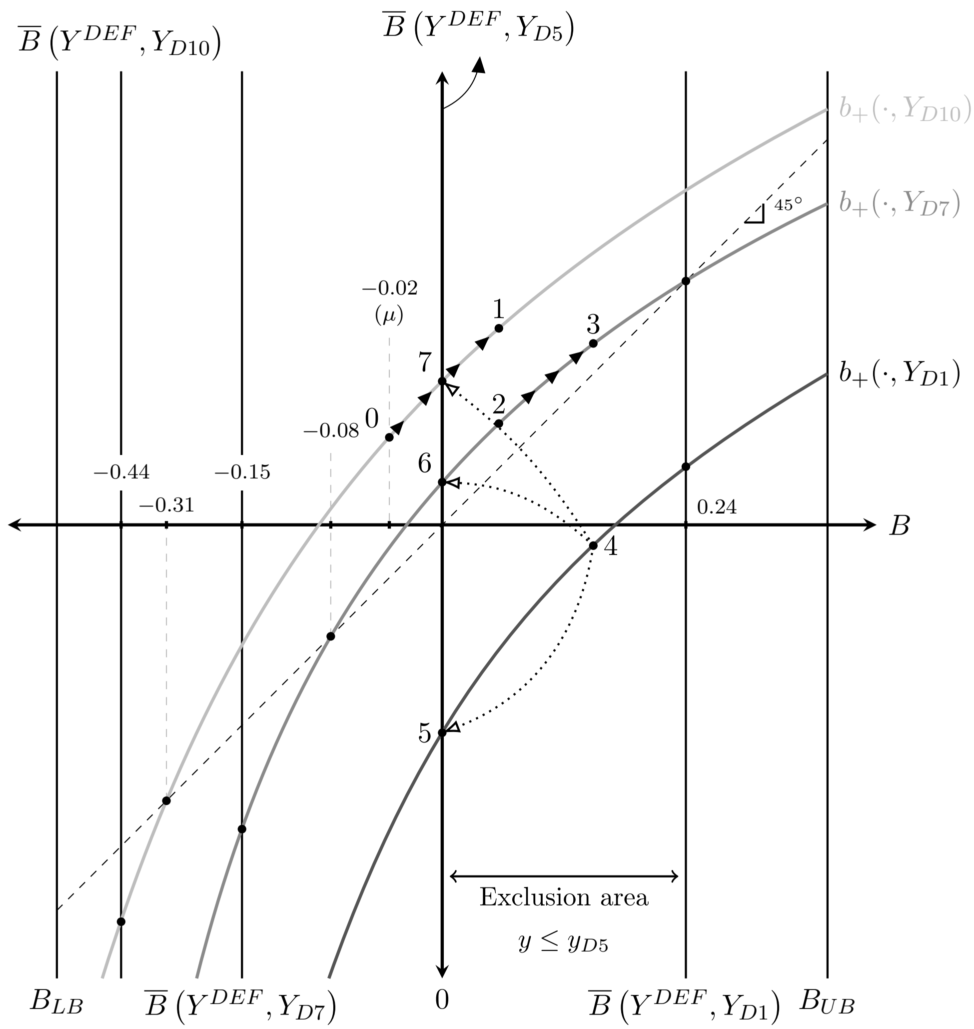

Figure 5: Calibrated phase diagram part B

Note: The phase diagram in this picture follows from the model calibrated to Argentina with the calibration described in Section 4. In this picture we consider the dynamics for various shock realizations.

We first focus on part A of the figure above. First, let's locate some preliminary elements in the figure. Demarcation curves are ordered based on different levels of output y, from the highest Y_{D10} to lowest Y_{D1}. We call these values deciles. As was discussed before, a higher decile implies that demarcation curves b_+ move to the north. We depict these curves only for $D1, D7$ and $D10$. However, as there is a monotonic increasing relationship, we know that "between" $b_+(., Y_{D7})$ and $b_+(., Y_{D10})$, we can find $b_+(., Y_{D9})$ which is above the former but below the latter. Moreover, contrarily to what we saw in figure 1, all depicted demarcation curves have at least 1 intersection with the 45° line. This result is due to the calibrated parameters. For instance, $b_+(., Y_{D7})$ has 2 intersections, one at $B = -0.08$ and the other at $B = 0.24$. We call these intersections "non-stochastic steady states" and satisfy $B_{NSS}(y) \equiv B = b_+(B, y)$ for some $y \in Y$. Finally, we will refer to a "high debt trap" as negative non-stochastic steady state $B_{NSS} < 0$. For example, for Y_{D7} and Y_{D10}, $B_{NSS}(Y_{D7}) = -0.08$ and $B_{NSS}(Y_{D10}) = -0.31$ respectively. It differs significantly with respect to figure 1 in at least 2 aspects.

i) We have 3 types of demarcation curves.

- For high shocks (Y_{D10} and Y_{D9}, where "D10" stands for decile 10) the intersection with the 45° line implies that the equilibrium is "unstable" (for instance point "A" in the figure). This also happens in deciles 6 to 8 but *only* when the economy has debt. Because private agents accumulate external assets at a fast pace, *except in the non-stochastic steady state*, in the absence of shocks the economy *would* converge to the boundary of the state space; outside the stable region. There are at least 2 things to be noted as regards these paths: a) the pace at which the economy accumulate assets is deeply connected with the *curvature* of the consumption function. For negative levels of net external assets, this function is *convex* and for high levels *concave*. Thus, the stability of the equilibrium without default depends on this last type of curvature. *At high and intermediate levels of GDP (i.e., deciles 6 to 10) households want to avoid a "high debt trap", as represented by the non-stochastic steady state, by accelerating the pace at which they accumulate assets; a fact that introduces instability into the private economy. The planner restores stability by introducing default into the decentralized equilibrium.* This is the next fact: b) the presence of default *stabilizes* the economy by returning paths inside the stable state space once the trajectory hits $\bar{B}(Y^{DEF}, Y_{D7})$.

- For low shocks (Y_{D1} to Y_{D5}) there is no intersections with the 45° line. We call these paths "non-sustainable": *beginning at every point of these demarcation curves*, in the absence of shocks, the economy will converge outside the stable state space. For these paths the planner also *stabilizes the economy by defaulting.* This happens when the economy hits, for instance, $\bar{B}(Y^{DEF}, Y_{D1})$. However, if the country has a positive external net asset position the government may choose to confiscate them. This is because, otherwise, households would get into a dynamic inefficient region. Behind the decision to confiscate assets there are at least 2 reasons, one technical and the other intuitive. As regards the latter, as in Aguiar and Amador (2019), to guarantee that default is only observed when the country is a net debtor, we would need to impose a restriction on endogenous variables, particularly the default penalty function. However, as we endogenously compute the default penalty, we cannot impose a restriction on primitives to rule out this case. There is also a powerful economic intuition behind the decision to confiscate assets: it only happen for those demarcation curves associated with unsustainable paths (i.e., those that do not intersect with the 45° line and thus do not have a non-stochastic steady state inside the stable state space). For instance, in point C in figure 4, the government chooses to default because the value of keep on honoring debt is affected by the presence of unsustainable paths. That is, in a not so distant future, the government will be forced to default with high probability as assets are in an explosive path.

- For intermediate shocks (deciles 6 to 8) there is a "stable" (when the economy has assets and the consumption function is convex) and an "unstable" (when the economy has debt and the consumption function is concave) region. Notice that this economy has a unique equilibrium as there is only 1 interest rate per element in the state space. Thus, *instability and multiplicity are not necessarily related to each other.*

ii) We now plot more than 1 shock. For the critical values of debt, $\bar{B}$, we depict those for the 2 extreme values in the support (Y_{D1}, Y_{D10}), the median (Y_{D5}),[20] and decile 7. Moreover, we identify the exact numerical values associated with each relevant point in the phase diagram. The region "exclusion", contains the subset of the stable state space at which the country remains in default even if the realization of the re-entry shock allows it to participate in asset markets.

We now focus on part B of the figure. One of the virtues of the framework derived in this paper, as we show the existence of a stable distribution, is that it allows us to *globally characterize stochastic paths with positive probability.* Thus, equipped with these tools, we can go beyond the implications suggested by an average across simulations with the same time spell, which is also well-defined because the equilibrium is stationary and ergodic. As the economy is on the stable state space, all

[20] We compute the empirical kernel for the GDP after removing the trend. We divide the support in 10 grid points. The accumulated probability between points 1 to 5 is 48.9% the the frequency of point 6 is 12.8%. Thus, decile 5 is the median.

the paths derived from the phase diagram are meaningful for the long run. Let's illustrate these points with an example. Suppose that the economy starts at point 0, around the long run mean, with the highest shock. Given that $\rho_e = 0.001$, $STD_e = 0.02$ and the discrete state space has 10 points any row of the transition matrix is given by the following vector: [0.0379 0.0537 0.0960 0.1415 0.1717 0.1716 0.1411 0.0956 0.0533 0.0376], where we are discretazing using standard techniques. Thus, the highest shock is observed with probability 0.0376. Remember that a recession is defined by 2 consecutive drops of the GDP. So, assume first that GDP moves from D10 to D7, which happens with probability 0.1411. Thus, it jumps from point 1 to point 2. Then, it keeps accumulating assets until the country is hit by the second negative shock, at point 3, that drags the economy to decile 1 with probability 0.0379; which in turn implies that the path jumps to point 4. Suppose that this happens at period τ. Then, we must have $b_\tau = b_+(b_{\tau-1}, Y_{D1}) < \bar{B}(Y^{def}, Y_{D1})$. Thus, the planner chooses to default and the economy jumps to the vertical axis and stays there until: a) we observe a low value for the re-entry distribution θ_i *and* b) GDP is above the median. That is, at point 5 (which happens with probability 0.0379 as the economy must be in D1 the economy remains in default due to the presence of the exclusion region.[21] In points 6 with probability 0.1411 as the economy jumps to D7 or 7 with probability 0.0376 associated with D10, the economy re-enter the international capital markets by accumulating net external assets. Thus, *a recession generates a default and the country will remain in autarky until GDP is above the median value.* Note that the planner defaults even if the country has positive net external assets. As we discussed above, when the economy is in an "non-sustainable" path, like in point 4, default occurs regardless of the level of assets.

The discussion above can be summarized by the following facts: i) *even if the country has a positive net private external position, if it is hit by a shock that takes GDP below the median, we will observe a default.* That is, despite the fact that we observe an appropriate draw from the re-entry probability (i.e. $\theta_i \in [0, \theta]$), it is possible to remain in default. This is the "exclusion" region and is characterized by the area between $\bar{B}(Y^{def}, Y_{D5})$ and $\bar{B}(Y^{def}, Y_{D1})$. This fact gives rise to point ii): *independently of the value of the exclusion parameter, the country will only re-enter to the international capital markets if the GDP is sufficiently high.* Facts i) and ii) have a factor

[21] Of course, this is a numerical result for this particular calibration. The exclusion region may be large or small depending on the calibration.

in common: if the economy is in a non-sustainable path, the Government will choose to default regardless of the stock of external assets and foreign investors will not purchase local bonds. iii) Contrarily to Arellano (2008), the country accumulates assets in an expansion: demarcation curves are increasing in Y while in Arellano (2008) they are decreasing. That is, *in our model, precautionary savings have an important role as the economy saves in good times.* This fact allows us to match the mean of yearly capital payments of net external debt, which is only 1.4% of the GDP. iv) *For deciles 6 to 10 of the GDP, which accumulates* 51.1% *of the mass in the empirical distribution of the GDP, even if the economy is near the long run mean of net external assets, debt destabilizes the economy.* This can be seen, for instance, in the intersection of $b_+(., Y_{D7})$ and the 45° degree line.

Even though the model is very stylized, as it is calibrated to Argentina there are two interesting features that are worth discussing. First, the model implies that output has to be relatively large such that the economy accepts participating in international markets after a default. This is something that we observed during the debt swap associated with the 2001 default in the first quarter of 2005, when the GDP were exactly at the median value. Second, the model implies that returning to private debt markets is hard for this calibration after default. In our calibration if the economy defaults and goes to autarky, when it comes back to international markets does it only for Y_{D6} to Y_{D10} and as a net lender. This is due to the exclusion zone.

4.2 Sensitivity analysis and Robustness

This section studies how two of the main distinctive features of the model affects its predictions. The first subsection deals with the effects of the endogenous penalty and how it varies with deep parameters; focusing on the effects of these changes on ergodic moments. The second subsection studies the implications of allowing for Markov shocks in the exogenous process driving the GDP. As ergodicity depends on i.i.d. shocks, in this subsection we focus on local moments and compare the results with the literature.

4.2.1 Sensitivity of Endogenous Penalty: implications on ergodic moments

The output cost of default is endogenous in this model. This endogeneity is critical to guarantee the stationarity and ergodicity of the equilibrium. Default costs change with the deep parameters of the economy (the discount factor, the probability of returning to the assets markets and the risk-free rate). This section studies those changes and how they affect the mean, the standard deviation and the accumulated mass in each quartile of the ergodic distribution of net external private assets. Table 6 presents the main results.

Table 6: Comparative statics of moments

Sim.	β	θ	$\bar{B}(Y_{D1})$	Y^{def}	$\mu(B/Y)$	$STD(B/Y)$	Def. freq	$E((B/Y)^2)$	$E(B/Y)^2$	r^*
BE	0.935	0.0725	0.24	1.00	-2.0	7.7	2.4	0.006	0.001	1.7
P1	0.935	0.150	-0.01	0.97	-8.8	15.6	5.1	0.032	0.008	1.7
P2	0.930	0.0725	-0.01	0.98	-6.1	13.7	2.1	0.02	0.004	1.7
P3	0.935	0.0725	-0.05	0.98	-7.0	14.7	4.1	0.027	0.005	2.7

Note: The first row contains the benchmark calibration (BE). Pn stands for policy $n = 1, 2, 3$. $\mu(B/Y)$ is the long run mean of the ratio of net external assets to GDP and is expressed in percentage points. Def. freq and $STD(B/Y)$ are also expressed in percentage points. $\bar{B}(Y_{D1}) \equiv \bar{B}(Y^{def}, Y_{D1})$ and Y^{def} are the threshold for debt for shocks at decile 1 and the value of GDP during default respectively. While R^* is the gross international risk free rate, r^* is the net expression for the same variable, expressed in percentage points. The remaining variables were already introduced or their interpretation is straightforward.

An increase in θ: a successful debt swap is more likely. We first compare row 1, the benchmark, with row 2, that only assumes a higher value for θ. Assumption 3 implies that a higher value for the re-entry parameter generates a lower Y^{def} through its effect on V_0^{def}. Consequently, the value in the first row 1.00 drops to 0.97. Moreover, the stationary continuation value, V_*^c, increases. In turn, this implies that the economy can sustain more debt: $\bar{B}(Y^{def}, Y_{D1})$ decreases from 0.24 to -0.01. Hence, the stable state space is now bigger. Consequently, $\mu(B/Y)$ falls from -2.0 to -8.8 while its standard deviation increases from 0.08 to 0.16. There are 3 simultaneous effects behind the change in the variance, 2 of them affects $E((B/Y)^2)$ and the other $E(B/Y)^2$. Remember that the variance VAR of a random variable X satisfies: $Var(X) = E((X)^2) - E(X)^2$. As the support of the distribution increases, $E((X)^2)$. However, as $\bar{B}(Y^{def}, Y_{D1}) < 0$, there is no exclusion region when $\theta = 0.15$. Thus, the mass allocated at zero, associated with the time that the process stays in autarky decreases; increasing $E((X)^2)$ even further. Finally, as the mean decreases, $E(X)^2$ increases, reducing the variance. The first 2 effect dominates which implies that *if we compare 2 countries, one with a higher probability of reaching a successful debt swap after default, both mean debt and variance in this country will be higher.* Hence, by the value of θ we can rationalize the differences in the debt swaps of Argentina and Ecuador in 2020. The later had an ongoing agreement with the IMF and the former decided to suspend the stand-by signed at 2018. Thus, with more institutional support, the likelihood of a successful swap was higher in Ecuador. The average private external debt in this country after default was 9% of the GDP while in Argentina it was 6%.

A change in β. The increase in impatience increases the support of the distribution, due to a smaller values for Y^{def} and $\bar{B}(Y^{def}, Y_{D1})$, there is also a simultaneous increase in the mean debt and variance. They accumulate more debt as demarcation curves rotates south-east. The frequency of slightly decreases, even though agents are more impatient, we can sustain more debt in equilibrium because of the changes in the default cost. The intuition is as follows: there are 2 channels working at the same time. As in a standard savings problem with precautionary savings, when the discount factor decreases, it increases indebtedness; lowering the value function associated with repayment V^c. At the same time, and this is entirely due to a specific characteristic of our model, Y^{def} decreases, lowering the value of default V^{def}. As a results, default probability could either increase or decrease. Thus, we only observed a mild change in it, decreasing slightly.

Hike in the international risk free rate, r^*. There is a decrease in Y^{def} (from assumption 3), the increase in the interest rate decreases V_0^c. As c_0 is the supremum of the of the space of function $\mathbb{C}$, it has the form: $y + (1 + r^*)B$. An increase in the interest rate affects negatively the low values of consumption and positively the high ones. As the instantaneous return function is strongly concave, the first effect dominates. Thus, Y^{def} falls. However, note that $\bar{B}(Y^{def}, Y_{D1})$ decreases. That is, even though Y^{def} is affected negatively, the planner tolerates higher debt levels. As the interest rate increases, demarcation curves rotate up, which implies more assets tomorrow for the same level of debt today. This is the typical Euler equation effect as we observe more savings. As during autarky, the country is not allowed to save, higher interest rates and more assets increase the value of continuation in the stationary equilibrium, V_*^c, rising the mean and variance of debt in the ergodic distribution. Thus, the model predicts that a worsening of international capital markets represented by *a 100 basis points increase in the risk free rate: more than triples net external debt and almost doubles the probability of default and the volatility of the economy, as measured by the standard deviation of net external assets.*

We now compare the different kernels. Table 7 contains the changes in the mass allocated to every quantile of the ergodic distribution with respect to the benchmark calibration (BE). Figure 6 supplements this information by plotting the kernel densities of debt to output ratio conditional that debt is not zero. For instance, the difference in mass between P1 and BE is given by $P1-BE$ at $[-0.80,-0.40)$[22] is 1.17 percentage points. Moreover, between -0.80 and the mean of the benchmark distribution there are 9.83 standard deviations of the same distribution.

Table 7: Comparative statics of kernels (change in mass)

STDs	-9.83	-4.83	0.30	5.30
Bin	$[-0.80,-0.40)$	$[-0.40,0.01)$	$[0.01,0.41)$	$[0.41,0.81]$
P1-BE	1.17	16.53	-17.70	0
P2-BE	1.29	17.61	-19.64	0.74
P3-BE	1.13	11.51	-12.45	-0.19

$STDs$ stands for the number of standard deviations of the benchmark distribution between the mean of this distribution ($STD(B/Y)=0.08$, $\mu(B/Y)=-0.02$) and the left border of each bin. $Pn-BE$ with $n=1,2,3$ contains the difference in mass at each bin, expressed in percentage points, between the Pn distribution as characterized in table 6 and the benchmark BE.

A hike of 100 basis points in the international risk free rate (P3) generates an increase of 11.51 percentage points of the mass associated to the bin that is 4.83 standard deviations to the left of the mean. The mass allocated to the left increases, generating a reduction in mean external assets and a higher dispersion.

Finally, we characterize the concentration of the ergodic distribution for B/Y. Since we derive a stable process, we can study the fraction of time that the economy will spend in a given subset of the stable state space. We choose to construct these subsets using the standard deviation and the mean (i.e., $[\mu_*(B/Y)-STD(B/Y),\mu_*(B/Y)+STD(B/Y)]$. With this purpose we use the information in Table 6. For the case of the BE, the mass accumulated at $+/-1$ standard deviation, where the support takes values between $[-9.7\%,+5.6\%]$, is 89.0%; recall that in the case of the standard normal distribution $N(0,1)$ this value is 68.2%. Thus, the mean is a powerful attraction point of the process. In other words, *as there is 1 default per stable distribution, the fact that almost 90% of the time net external assets to output ratio will fluctuate at most at 1 standard deviation away from the mean (as against nearly 70% in the normal distribution), implies that this type of events have a drastic impact on the performance of the economy.* This is one side of the coin. The other is that the process is highly concentrated around the mean, which implies that *default stabilizes debt after the crises.* This result is robust: if we increase the international risk free rate 100 basis points, the

[22] The last bin is $[0.41,0.81]$.

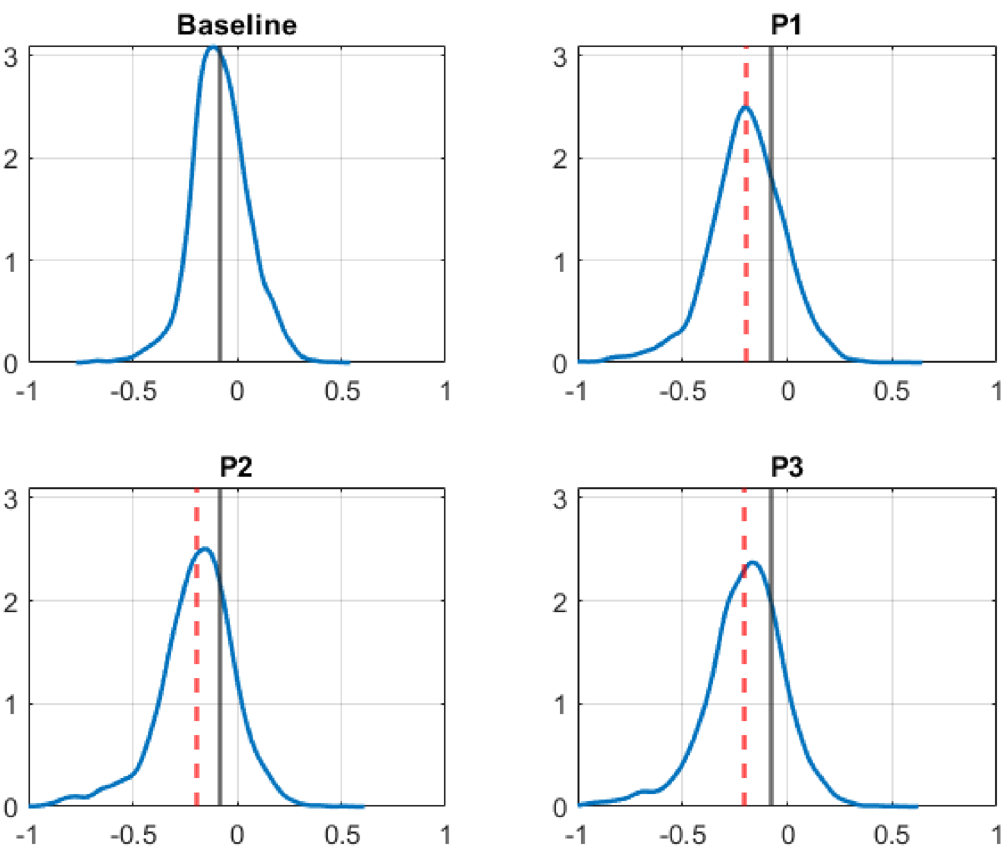

Figure 6: Debt to output ratio distributions

Note: Debt to output ratio distributions for each of the calibrations in table 6. The black vertical line is the mean of the debt to output ratio in the baseline economy. The dotted red line is the mean of P1, P2 and P3 calibrations in each of the figures, respectively. To compute each density we removed the zeros in the simulations, that is, we removed exclusion periods.

economy spends 89.7% of time $+/-1$ standard deviation away from the mean, taking values at $[-21.7\%, +7.1\%]$.

Recall that P1 represents a model with larger θ (i.e., higher probability of leaving autarky), P2 is associated with a lower β (.i.e., households are more impatient), and P3 represents an increase in the risk free rate.

In the model with a lower β, as households are more impatient, they have a higher incentive to front-load consumption. For this reason, the debt to output ratio distributions shifts to the left, as seen in Figure 6. In contrast to the case of Arellano (2008), the level of debt supported does not shrink, this is a consequence of the endogenous default costs that end up being larger in this economy compared to the baseline, as seen in Table 6. The households still have higher incentives to front-load consumption because it fails to internalize the impact of debt issuance on the spread. High θ will make the economy return faster to asset markets, which also implies more debt. However, when it comes to default probability, endogenous exclusion and endogenous default cost Y^{DEF} play a role: as this last variable goes down with respect to BE in P1 and P2, there is a simultaneous decrease in V^c and V^{def}. In the case of P2 this effect is so strong that it partially reverts the consequences of an increase in indebtedness, reducing the probability of default. Moreover, endogenous exclusion gives an additional value to default: it takes the economy out of explosive paths with high probability (i.e., in figure 4, after default, the economy returns to international capital markets only with a high value of the GDP and accumulate assets with high probability.). This explains why there is a significant increase in the probability of default in P1: endogenous exclusion "boosts" the effect of the increase in the re-entry parameter, lowering even more the cost of default. Behind endogenous exclusion we can find the value of default as a stabilization policy. As regards P3, a higher risk free rate increases the cost of debt; forcing households to borrow more to sustain the same level of consumption which in turn pushes for higher debt and higher default probability.

4.2.2 Endogenous Penalty and Markov shocks

In this subsection we test the implications of the endogenous penalty by comparing the predictions of the model with the literature. In particular, we use the parameters in Kim and Zhang (2012) and solve the model using the operator described in definition 2 under assumption 3.

The table below contains the results of solving and simulating the model using operator 2 with Markov shocks.

Table 8: Calibration with Markov shocks and an endogenous penalty

Results	Current account / GDP		Net external assets / GDP	
	01-83	16-83	01-83	16-83
Data	-2.4%	-0.8%	-36.5%	-34.0%
Stationary	-3.7%	-0.8%	-49.7%	-36.6%
Kim and Zhang (2012)	-1.2%	N/A	-22.5%	N/A

Note: We borrow the parameters from Kim and Zhang (2012). The Markov process for GDP has an auto-regressive coefficient (ρ) of 0.945 and the standard deviation of the residuals in this process (STD_e) equals 0.02. Then, $\theta = 0.1$ and $\beta = 0.97$. N/A stands for Not Available as Kim and Zhang (2012) only targets local moments.

Table 8 shows the effects of an endogenous punishment in a stationary model: simulations in our model over-estimate (i.e., more current account deficit and more debt than in data) local moments (i.e., between the defaults of 1982 and 2001) as the penalty does not truncate the high-end of the distribution of shocks, lowering the cost of default. Moreover, as shocks are Markov, they are persistent. Thus, the model with endogenous default stays a larger fraction of time in an expansion even under default when compared with an asymmetric penalty. This is not at odds with data as Argentina between 2002 and 2005, when the country was under default, grew at an average yearly rate of 8.7%, growing in every period.

5 Concluding remarks

This paper presents the conditions to characterize globally economies subject to sovereign default risk of private external debt. We show several properties connected with the stochastic stability of the equilibrium, a fact that is deeply connected with the ergodic behavior of endogenous variables.

We show that default is an instrument that can be used to derive a stable unconditional distribution, one for each possible default episode. In this way, we suggest a potential answer for the role of default in open economies: private external debt generates unstable and unsustainable debt paths, even for high levels of GDP and default can be used by a benevolent Government to stabilize the economy.

This is the first paper in the external default literature to present the conditions for stationarity and ergodicity. In this way, our model allows for a parametrization that targets unconditional data moments as well as local dynamics. We show that if we calibrate the model for unconditional (long run/global) Argentinean data, we can appropriately replicate the local (conditional) behavior around default too. For this purpose, we derive the notion of a stable state space and characterize the dynamics before the default using a phase diagram. Moreover, it is possible to use the theoretical structure in this paper to model countries with no default but with default risk. Based on the results in this paper, it is possible to recover the value of GDP that would be observed if the country decides to default, even if this event has never been observed recent in history. We leave this exercise for future research.

6 Bibliography

Aguiar, M. and M. Amador (2019): "A contraction for sovereign debt models," Journal of Economic Theory, 183, 842–875.

Aguiar, M. and G. Gopinath (2006): "Defaultable debt, interest rates and the current account," Journal of International Economics, 69, 64–83.

Arellano, C. (2008): "Default risk and income fluctuations in emerging economies," American Economic Review, 98, 690–712.

Arellano, C., A. Atkeson, and M. Wright (2016): "External and Public Debt Crises," NBER Macroeconomics Annual, 30, 191–244.

Auclert, A. and M. Rognlie (2016): "Unique equilibrium in the Eaton–Gersovitz model of sovereign debt," Journal of Monetary Economics, 84, 134–146.

Ayres, J., G. Navarro, J. P. Nicolini, and P. Teles (2018): "Sovereign default: The role of expectations," Journal of Economic Theory, 175, 803–812.

Bianchi, J. (2011): "Overborrowing and systemic externalities in the business cycle," American Economic Review, 101, 3400–3426.

Braido, L. H. (2013): "Ergodic Markov equilibrium with incomplete markets and short sales," Theoretical Economics, 8, 41–57.

Coleman, W. (1991): "Equilibrium in a Production Economy with an Income Tax," Econometrica, 59, 1091–1104.

Duffie, D., J. Geanakopolos, A. Mas-Collel, and A. McLennan (1994): "Stationary Markov Equilibria," Econometrica, 62, 745–81.

Eaton, J. and M. Gersovitz (1981): "Debt with potential repudiation: Theoretical and empirical analysis," The Review of Economic Studies, 48, 289–309.

Feng, Z. and M. Santos (2021): "Markov Perfect Equilibrium in economies with sovereign default and distortionary taxation," Mimeo. https://sites.google.com/view/zhigangfeng/research?authuser=0.

Futia, C. A. (1982): "Invariant Distributions and the Limiting Behavior of Markovian Economic Models," Econometrica, 50, 377–408.

Kim, Y. J. and J. Zhang (2012): "Decentralized borrowing and centralized default," Journal of International Economics, 88, 121–133.

Ljungqvist, L. and T. Sargent (2012): "Recursive Macroeconomic Theory," .

Lucas, R., N. Stokey, and E. Prescott (1989): "Recursive Methods in Economic Dynamics," .

Meyn, S. and R. Tweedie (1993): "Markov Chains and Stochastic Stability," .

Mirman, L., K. Reffett, and O. Morand (2008): "A qualitative approach to Markovian equilibrium in infinite horizon economies with capital," Journal of Economic Theory, 139, 75–98.

Phelan, C. and E. Stacchetti (2001): "Sequential equilibria in a Ramsey tax model," Econometrica, 69, 1491–1518.

Pierri, D. R. and K. Reffett (2021): "Memory, multiple equilibria and emerging market crises," Universidad Carlos III de Madrid. Departamento de Economía. Working paper number 32871. https://ideas.repec.org/p/cte/werepe/32871.html.

Zame, W. R. (1993): "Efficiency and the role of default when security markets are incomplete," The American Economic Review, 1142–1164.

Appendix

We will show the results in each subsection separately.

Proofs for section 3.1

Before proving the results, we add some details as regards the Euler equation (3). Then, we prove lemmas 1 and 2.

Preliminary remarks

The purpose of this subsection is to carefully derive the Euler equation (3), which is the same used by Kim and Zhang (2012). As regards this equation, there are at least to major difference with respect to the centralized default literature (see, for instance, Arellano (2008)). First, the agent that issues debt / purchases assets does not internalize her portfolio decisions on market prices. This was already noted by Kim and Zhang (2012). Second, the Government chooses to default or repay but does not issue debt. Then, to compute the Euler equation, as we don't assume partial default, we must take the derivative with respect to the individual state b_+, not the aggregate state B_+, which only affects the value of repaying $V^c(b, B, y; h)$, not the option value $V(b, B, y; h) = max\{V^c(b, B, y; h), V^{def}(y)\}$. Thus, we can use the standard envelope theorem as we take the derivative with respect to b on $V^c(b, B, y; h)$ and then forward this expression 1 period. To take care of the interaction between default the the marginal value repaying, we use the fact that the interest rate depends only on the aggregate state. Thus, the discontinuities happened at equilibrium (or aggregate) level.

For exposition purposes, we first present the relevant value functions. The value for the household of the repaying option for the Government is given by:

$$V^c(b, B, y; h) = \max_{b_+ \geq -\bar{b}} u(F(b, B, y) - b_+) + \beta \mathbb{E}[V(b_+, h(B, y), y'; h)]. \tag{11}$$

Where $F(b, B, y) = y + bR(B)$ and h is the aggregate law of motion for assets B_+ in equation (9). Note that, due to the i.i.d structure in assumption 2, the interest rate is not a function of the exogenous shocks. This is clear from equation (1). However, even if we allow for Markov shocks or compute off (recursive) equilibrium paths using the operator presented in definition 2, we can factor the interest rate outside the expectation term in the Euler equation. As we assume the presence of incomplete markets, the return on net assets b_+ must be independent of y_+. Moreover, the interaction between default and private welfare is captured by the derivative of $V^c(b_+, B_+, y_+; h)$ with respect of b_+. The interest rate only enters into the Euler equation after replacing this derivative using the envelope theorem.

Formally, the general value (i.e., with Markov shocks and outside the recursive equilibrium) of the interest rate affected by default risk is given by:

$$
\begin{aligned}
&R(B', y) \equiv R^*(B, y; h^n) = \\
&R^* \left[\sum_{y' \in [Y_{LB}, Y_{UB}]} \pi(y', y) \mathbb{I} \left\{ V^c(b_+(b, B, y; h^n), h^n(B, y), y') > V^{def}(y') \right\} \right]^{-1},
\end{aligned}
\tag{12}
$$

where h^n is the n-th iteration using the operator presented in definition 2. Note that outside the recursive equilibrium, during iterations, we have $b_+(b, B, y; h^n) \neq h^n(B, y)$ but in a recursive equilibrium we get by definition $b_+(b, B, y; h^*) = h^*(B, y)$. As we are modeling a competitive equilibrium, individual states does affect the interest rate. Thus, marginal changes in b_+ must not affect equation (13). To enforce this fact even in a recursive equilibrium we must set a strict inequality inside the indicator function (i.e., when the Government is indifferent, defaults). Now, using this notation, we rewrite the right had side of equation (3):

$$
\begin{aligned}
&\mathbb{E}_y[V(b_+(b, B, y; h^n), h^n(B, y), y'; h^n)] = \sum_{y' \in [Y_{LB}, Y_{UB}]} \pi(y', y) \\
&[\mathbb{I} \left\{ V^c(b_+(b, B, y; h^n), h^n(B, y), y'; h^n) > V^{def}(y') \right\} V^c(b_+(b, B, y; h^n), h^n(B, y), y'; h^n) \\
&\mathbb{I} \left\{ V^c(b_+(b, B, y; h^n), h^n(B, y), y'; h^n) \leq V^{def}(y') \right\} V^{def}(y')].
\end{aligned}
\tag{13}
$$

Note that V^{def} and the first term in the second line do not change with b_+. The latter follows from the absence of partial default and the latter from the price taking assumption. Then, we can rewrite equation 3 as:

$$\begin{aligned}
&u'\left(F(b,B,y) - b_+(b,B,y;h^n)\right) \geq \\
&R^*(B,y;h^n) \sum_{y' \in [Y_{LB},Y_{UB}]} \pi(y',y)\mathbb{I}\left\{V^c(b_+(b,B,y;h^n),h^n(B,y),y') > V^{def}(y')\right\} \\
&u'\left(F(b_+(b,B,y;h^n),h^n(B,y),y') - b_+(b_+(b,B,y;h^n),h^n(B,y),y';h^n)\right),
\end{aligned} \tag{14}$$

where the second and the third lines capture the interaction between default and the marginal utility of consumption. That is, even though there are incomplete markets and the interest rate can be factored out the expectation in the Euler equation (i.e., does not depend on y' even outside the recursive equilibrium), the number of future exogenous states in which the Government repays affects consumption decisions today through the Euler equation.

Finally, note that the second line in equation (14) can be written as:

$$R^* \sum_{y' \in [Y_{LB},Y_{UB}]} \pi(y',y) \frac{\mathbb{I}\left\{V^c(b_+(b,B,y;h^n),h^n(B,y),y') > V^{def}(y')\right\}}{\sum_{y' \in [Y_{LB},Y_{UB}]} \pi(y',y)\mathbb{I}\left\{V^c(b_+(b,B,y;h^n),h^n(B,y),y') > V^{def}(y')\right\}}. \tag{15}$$

To derive the existence of a recursive equilibrium we must enforce, as in Coleman (1991), that $b_+ = B_+$. Thus, equation (15) must be restricted to satisfy:

$$\begin{aligned}
&R^* \sum_{y' \in [Y_{LB},Y_{UB}]} \pi(y',y) \\
&\frac{\mathbb{I}\left\{V^c(b_+(b,B,y;h^n),b_+(b,B,y;h^n),y') > V^{def}(y')\right\}}{\sum_{y' \in [Y_{LB},Y_{UB}]} \pi(y',y)\mathbb{I}\left\{V^c(b_+(b,B,y;h^n),b_+(b,B,y;h^n),y') > V^{def}(y')\right\}},
\end{aligned} \tag{16}$$

That is, we are imposing $B' = b_+(b,B,y;h^n)$. Using equation (16), we can define $R(B',y')$ when $b = B$:

$$R(B',y') \equiv R^*\pi(y',y) \frac{\mathbb{I}\left\{V^c(b_+(B,B,y;h^n),b_+(B,B,y;h^n),y') > V^{def}(y')\right\}}{\sum_{y'\in[Y_{LB},Y_{UB}]}\pi(y',y)\mathbb{I}\left\{V^c(b_+(B,B,y;h^n),b_+(B,B,y;h^n),y') > V^{def}(y')\right\}}. \tag{17}$$

Under the restriction in equation (16) and using the definition in equation (17), as V^c is increasing in b (which is a standard result due to Lucas et al. (1989)), $R(B',y')$ is decreasing in B' for any y'. This property will be essential to prove lemma 2. Using $R(B',y‘)$ we can write the Euler equation in (14) as:

$$u'\left(F(b,B,y) - b_+(b,B,y;h^n)\right) \geq \sum_{y'\in[Y_{LB},Y_{UB}]} \pi(y',y)R(B',y') \\ u'\left(F(b_+(b,B,y;h^n),B',y') - b_+(b_+(b,B,y;h^n),B',y';h^n)\right), \tag{18}$$

Both Euler equations (14) and (18) are equivalent to the one found in Kim and Zhang (2012) and to equation (3) in the body of the paper.

It is clear from the definition of $R(B',y')$ that, even though there are incomplete markets, the return on assets is affected by shocks y as noted by Zame (1993). In particular, in a recession y is low. Due to the Markov structure of π, it is possible that the probability of getting low shocks tomorrow y' would be high. Thus, the denominator in equation (17) goes down, raising the ex-ante return on net assets. Thus, default provides a natural hedge as in Zame (1993). In the i.i.d. case, we can't connect a recession today with an expected recession tomorrow, but we can still use the same argument due to the numerator of equation (17): ex-ante, when a recession occurs in the future, the return on net assets would be high.

Proofs

To prove lemma 1 we need an additional assumption that is standard in the literature: the Government will never default with assets. To enforce that result, we need to present a pure savings problem [23] that it is better placed in the context of the proof of theorem 1, which we present below.

[23] See equation (21) in section Proofs for section 3.2 that we describe below in this appendix.

In this subsection, we suppose that assumption 6 holds, which in turn prove lemma 3. This result, in turn, guarantees the absence of default with assets.

Proof of Lemma 1. Any sequence of consumption is valued by $U = \sum \beta^t u(c_t(\omega))\mu_{y_0}(\omega)$ and u bounded above and unbounded below. Under this assumptions, it is standard to show (see Duffie et al. (1994) page 765) that any utility maximizing sequence $\hat{c}_t(\omega) > \underline{c}$ with $\underline{c} > 0$ almost everywhere in Ω. This fact implies that marginal utility is uniformly bounded above. Under assumption 2 of this paper, Remark 1 together with Lemma 1 in Braido (2013) implies that there exists $\rho \in (0, 1)$ with: $b_+ \leq R^*/(1-\rho)$ almost everywhere in Ω, where R^* is the risk free gross rate. The lower bound on b follows from the restrictions on problem 2. Thus, $b_+ \in [B_{LB}, B_{UB}]$. Because there is a uniform lower bound in consumption, $c \geq C_{LB} \equiv \underline{c}$. Given these results, is is easy to show that $R(B)$ is bounded above. Suppose not. We will deal only with the case $b = B$ (i.e., only equilibrium candidates) and $R(B) = +\infty$ when $B = b < 0$. Under assumption 6, lemma 3 implies that $R(B) = +\infty$ when $B = b > 0$ is not possible as the Government would never choose to default with assets. Then, equations (13) and (14) implies that $u'(c(y, b, B; h)) \geq 0$, which satisfies $c, c_+ \geq \underline{c}$. This follows from equation (16): if $R(B)$ is unbounded, then the fraction in the second line of equation (16) is equal to 0. Then, as $c_+ \geq \underline{c}$ the right hand side of the Euler equation (14) is equal to 0. As we must have $c = F(b, B, y) - b_+ \geq \underline{c} > 0$ and $F(b, B, y) = bR(B) + y$, b_+ must be unbounded below, which contradicts the uniform lower bound on b_+ in equation (11) (i.e., $b_+ \geq -\bar{b}$). The lower bound on $R(B)$ is given by R^*, which is standard under risk neutral pricing. Thus, $R(B) \in [R_{LB}, R_{UB}]$. Finally, the upper bound on c is given by: $Y_{UB} + R_{UB}B_{UB} - B_{LB}$. Thus, $c \in [C_{LB}, C_{UB}]$. □

Proof of Lemma 2. We begin by defining a appropriate space of functions for c and h. Let C be space of candidate functions for h. As in in Coleman (1991), we require:

$$C(\mathbb{B} \times Y) = \left\{ \begin{array}{c} 0 \leq C(B, y) \leq F(B, y) \\ 0 \leq C(B', y) - C(B, y) \leq F(B', y) - F(B, y) \; if \; B' \geq B \end{array} \right\} \tag{19}$$

Where $F(B, y) = y + R(B)B$ with $B \in [B_{LB}, B_{UB}] \equiv \mathbb{B}$ from lemma 1. As in Aguiar and Amador (2019) we are proving the existence of a stationary equilibria using uniform bound on B

and then add the default state separately to construct h in equation (9)[24]. Further, equation (19) implies that both c and b_+ are (weakly) increasing in B for each $y \in Y$. We will now define an operator on $C(\mathbb{B} \times Y)$, A, and we will show that $Ac \in C(\mathbb{B} \times Y)$. We will show that any fixed point of this operator $Ac = c$ can be used to construct h as the optimization problem of the representative agent can be adjusted accordingly. Thus, for simplicity, in the proof of lemma 2 we will omit the dependence of the private policy function on the equilibrium law of motion h.

$$u'(Ac(B,y))) = \beta E\left[u'\left(c\left(F(B,y) - Ac(B,y), y'\right)\right) R\left(F(B,y) - Ac(B,y), y'\right)\right] \tag{20}$$

Where Ac defined the *Coleman-Reffett* operator and it may not be equal to c. Equation (20) simply defines the candidates for fixed point $Ac = c$.

We must show that A maps $C(\mathbb{B} \times Y)$ into itself. This will suffice to show the first part of lemma 2.

Take $c \in C(\mathbb{B} \times Y)$. Let $B'(B,y) = y + R(B)B - Ac(B,y)$. Thus, for any $\hat{c}, \tilde{c} \in C(\mathbb{B} \times Y)$, with $\hat{c} \leq \tilde{c}$, we must show that $A\hat{c} \leq A\tilde{c}$ and $\hat{B}' \leq \tilde{B}'$. In order to do so, notice that:

$$u'(A\hat{c}(B,y))) = \beta E\left[u'\left(\hat{c}\left(\hat{B}', y'\right)\right) R\left(\hat{B}', y'\right)\right] \geq \beta E\left[u'\left(\tilde{c}\left(\hat{B}', y'\right)\right) R\left(\hat{B}', y'\right)\right] \geq$$
$$\beta E\left[u'\left(\tilde{c}\left(\tilde{B}', y'\right)\right) R\left(\hat{B}', y'\right)\right] \geq \beta E\left[u'\left(\tilde{c}\left(\tilde{B}', y'\right)\right) R\left(\tilde{B}', y'\right)\right] = u'(A\tilde{c}(B,y)))$$

Where the first inequality follows from $\hat{c} \leq \tilde{c}$, the second from equation the optimality of consumption and the third because R is decreasing in B as any candidate function for consumption is increasing in B. Note that the last inequality implies $\hat{B}' \leq \tilde{B}'$ and the first and the last terms together imply $A\hat{c} \leq A\tilde{c}$ as desired. Thus, $AC(\mathbb{B} \times Y) \subseteq C(\mathbb{B} \times Y)$ which in turn implies that any fixed point of A is a good candidate for h.

It remains to show that either c or b_+ is strictly increasing. Suppose not. Then, for some $y \in Y$ and $\tilde{B}, B \in \mathbb{B}$, with $\tilde{B} > B$, we have $b_+(B,y) = b_+(\tilde{B},y)$ and $c(B,y) = c(\tilde{B},y)$. From equation (2) we know that:

[24] In Aguiar and Amador (2019) the value function for the default states in operator $\mathbb{T}$ can be selected arbitrarily from the feasible function set (see the proof of lemma 6 in page 865).

$$V(B,y) = u(c(B,y)) + \beta E[V(b_+(B,y),y')] \text{ with } c(B,y) + b_+(B,y) = y + R(B)B$$

Suppose that $B > 0$. This is without loss of generality as, from lemma 1, $B_{UB} > 0$. Thus, $c(\tilde{B},y) + b_+(\tilde{B},y) < y + R(\tilde{B})\tilde{B}$. This inequality implies that there is a basket $\tilde{c}(\tilde{B},y) > c(\tilde{B},y)$ which is also feasible and:

$$u(\tilde{c}(\tilde{B},y)) + \beta E[V(b_+(\tilde{B},y),y')] > V(\tilde{B},y)$$

The strict inequality implies a contradiction and it follows that $b_+(B,y) = b_+(\tilde{B},y)$ *or* $c(B,y) = c(\tilde{B},y)$ but not both. As y and $\tilde{B}, B$ are arbitrary, we can extend the result for any $y \in Y$ and any strictly ordered pair $\tilde{B}, B \in \mathbb{B}$. □

Proofs for section 3.2

We now turn to the proof of theorem 1. We will state the proof for the case with no re-entry (i.e., $\theta = 0$). Then, we show that we can extend the results for the general case. We will need an additional mild assumption on $y^{def}(y)$. This is only for the sake of clarity as we want the structure of the proof to be as close as possible to the ones in Aguiar and Amador (2019) and Coleman (1991). Under this assumption, we can show that $\hat{B}(y) < 0$ for all $y \in Y$, a fact that allow us to write the Aguiar-Amador operator in a tractable way. To state the assumption, we need a modified version of problem 2.

$$V(b,y;R^*) = Max_{b_+ \geq 0}\ u(y + R^*b - b_+) + \beta E[V(b_+,y';R^*)] \tag{21}$$

Problem 21 is a standard savings problem. In order to guarantee that it is well behaved, we need to assume that it does not generate extreme unstable path as defined in remark 1. We do this in the following assumption, which also contains the mentioned additional restriction on y^{def}.

Assumption 6 (Negative debt thresholds)**.** *Assume that $\beta R^* < 1$ and additionally:*

$$V(b,y;R^*) \geq u(y^{def}(y)) + E_1(\textstyle\sum_t \beta^t u(y^{def})) \text{ for all } y \in Y$$

Lemma 3 (Negative debt thresholds)**.** *Under assumptions 1, 2 and 6, $\bar{B} < \overrightarrow{0}$, where $\overrightarrow{0} \in \mathbb{R}^Y$.*

Proof. Follows immediately from lemma 1 (i) in Aguiar and Amador (2019). □

We are now in position to define formally the Aguiar-Amador operator.

Definition 3 (Utility maximization problem (UMP))**.**

$$V^c_{n+1,*}(B,y) = u(c_{n+1}(B,y)) + \beta E max \left\{ V^c_{n+1,*}(b_{+,n+1}(B,y),y'), V^{def}(y')) \right\}$$

Subject to

$$b_{+,n+1}(B,y) + c_{n+1}(B,y) = y + BR^* \left[\mathbb{I}\left(B > 0\right) + \mathbb{I}\left(B \leq 0\right) \sum_{y \in Y} \pi(y) \mathbb{I}\left(V^c_{n+1,*}(B,y) \geq V^{def}(y) \right) \right]$$

Where $c_{n+1} = Ac_n$ and defines the connection between the Coleman-Reffett operator in equation (20) and the Aguiar-Amador operator, to be defined. Note that we are using lemma 3 to write the equilibrium interest rate at iteration $n+1$. We now define the dual of the UMP, the expenditure minimization problem. In Aguiar and Amador (2019) $\nu = V^c_{n+1,*}(B_{n+1,*}(\nu,y),y)$ was stated without proof [25]. We proceed in the same way. However, we have to explicitly write the EMP in order to show the equivalence between it and the Aguiar-Amador operator $\mathbb{T}$.

Definition 4 (Expenditure Minimization Problem (EMP))**.**

$$B_{n+1,*}(\nu,y) = \left[(b_{+,n+1} + c_{n+1})(\nu,y) - y\right] R^{-1}$$
$$\left[\mathbb{I}\left(B_{n+1,*}(\nu,y) > 0\right) + \mathbb{I}\left(B_{n+1,*}(\nu,y) \leq 0\right) \sum_{s \in Y} \pi(s) \mathbb{I}\left(\nu(s) \geq V^{def}(s) \right) \right]$$

Subject to

$$\nu = V^c_{n+1,*}(B_{n+1,*}(\nu,y),y), \quad \nu(s) = V^c_{n+1,*}(B_{n+1,*}(\nu,y),s) \quad for \quad s \in Y \tag{22}$$

[25]See page 850.

$$\nu = u(c_{n+1}(B_{n+1,*}(\nu,y),y)) + \beta Emax\left\{V^c_{n+1,*}(b_{+,n+1}(\nu,y),y'),V^{def}(y'))\right\} \tag{23}$$

The equivalence between the EMP and the UMP is automatic given the results in lemma 2. It turns out that EMP is not a contraction. However, we prove that there exist an equivalent representation to EMP, called *optimal contract (OC)*, which we will show that is well defined. This operator will allow us to iterate in j and find a fixed point for the pair $(B_{n+1,*}, V^c_{n+1,*})$ using equation (22).

Definition 5 (Optimal Contract (OC) and the Aguiar-Amador operator ($\mathbb{T}$))**.**

$$\mathbb{T}f_j(\nu,y) = SUP_{\{g_+(y')\}_{y'\in Y}}\left[(b_{+,n+1}+c_{n+1})(\nu,y)-y\right]R^{-1}$$
$$\left[\mathbb{I}\left(B_{n+1,*}(\nu,y)>0\right)+\mathbb{I}\left(B_{n+1,*}(\nu,y)\leq 0\right)\sum_{s\in Y}\pi(s)\mathbb{I}\left(V^c_{n+1,*}(B_{n+1,*}(\nu,y),s)\geq V^{def}(y)\right)\right]$$

Subject to

$$\nu = u(c_{n+1}(B_{n+1,*}(\nu,y),y)) + \beta Emax\left\{g_+(y'),V^{def}(y'))\right\} \tag{24}$$

$$b_{+,n+1}(\nu,y) = f_j(g_+(y'),y') \quad \textit{for all} \quad y'\in Y \quad \textit{such that} \quad g_+(y')\geq V^{def}(y') \tag{25}$$

A fix point $\mathbb{T}$, f, satisfies $f = B_{n+1,*}$ and by equation (23) we can recover $V^c_{n+1,*}$, which is given by the pre-image of $B_{n+1,*}(.,y)$ for each $y\in Y$. Intuitively, definition 5 gives the Government an additional instrument g_+ in order to enforce minimum expenditure f. In this sense, the maximal elements $\hat{g}_+(y)$ for all $y\in Y$ of a fixed point of $\mathbb{T}$, f, is a promised utility that sustain $(B_{n+1,*}, V^c_{n+1,*})$. Assuming that $\mathbb{T}$ has a fixed point, the next lemma shows that it is equivalent to $B_{n+1,*}$, which in turn has a unique value associated value function for the UMP, $V^c_{n+1,*}$. We later show that $\mathbb{T}$ has at least one non-trivial fixed point.

Lemma 4 (Optimal contract and expenditure minimization problem)**.** *Under assumptions 1, 2 and 6, any fixed point $\mathbb{T}f = f$, if it exists, satisfies: $f = B_{n+1,*}$.*

Proof. We will show this lemma in 2 steps.

We first show that EMP is a fixed point of $\mathbb{T}$. Let $\mathbb{V}$ the set of possible values of $V^c_{n+1,*}$ for all $B, y \in \mathbb{V} \times Y$. Because of lemma 1 and equation (6), $\mathbb{V}$ is compact. Take and arbitrary pair $\nu_0, y_0 \in \mathbb{V} \times Y$. This pair defines in turn a triple $b_{+,n+1}(\nu_0, y_0), c_{n+1}(\nu_0, y_0)$ and $B_{n+1,*}(\nu_0, y_0)$ from the EMP. Set $\hat{g}_+(y') = V^c_{n+1,*}(b_{+,n+1}, y')$ for all $y' \in Y$. We claim that setting, given that the objective function of EMP and OC are the same, $f = B_{n+1,*}$ suffices to show that $b_{+,n+1}$ and c_{n+1} satisfies equations (24) and (25). Equation (24) is satisfied by the definition of $V^c_{n+1,*}$ in equation (22). Equation (25) follows from the recursive structure given by private optimization in equation (3) and the equivalence between EMP and UMP [26]. As in Aguiar and Amador (2019), when $V^c_{n+1,*} < V^{def}$, $\hat{g}$ is any feasible function in the space $\mathbb{V}$. As the preceding argument can be done for any $(y_0, \nu_0) \in \mathbb{V} \times Y$, $b_{+,n+1}$ and c_{n+1} are feasible in OC which then implies $\mathbb{T}f \geq B_{n+1,*}$ or equivalently OC $\supseteq$ EMP.

We now show that a fixed point of $\mathbb{T}$ *is an EMP.* As $\mathbb{T}$ is assumed to have a fixed point we can use it as a candidate for $B_{n+1,*}$. Note then that the objective functions of EMP and OC are equal so, we must only verify equations (22) and (23). The objective function together with equation (22) form a system with $card(Y)$ unknowns for each ν given $\hat{g}_+(y')$ for some $y' \in Y$. As we are assuming that $\mathbb{T}$ has a fixed point, this system has at least 1 solution, so equation (22) is satisfied. Equations (24) and (25) together imply that (23) is satisfied. □

We now show that $\mathbb{T}$ has a fixed point which is an increasing function of ν, which in turn assures that: a) there is a well defined sequence of functions f_j generating a pair $(B_{n+1,j}, V^c_{n+1,j})$, b) $\mathbb{T}$ has a fixed point f which generates $(B_{n+1,*}, V^c_{n+1,*})$.

For that we need the following theorem.

Theorem 4 (Existence of a lower fixed point, Mirman, et. al. Proposition 5)**.** *Let* $\mathbb{F}$ *be a poset and* $h : \mathbb{F} \longrightarrow \mathbb{F}$ *be order continuous. Assume that there is an element* $a \in \mathbb{F}$ *such that i)* $a \leq h(a)$ *and ii) every countable chain in* $\mathbb{F}$ *has a supremum. Then,* h *has a fixed point and the sequence of elements in* $\mathbb{F}$ *generated iteratively using* h *and starting in* a*, converges to the infimum of the set of fixed points.*

[26] See Aguiar and Amador (2019) page 866.

Lemma 5 (Existence of a fixed point in the Aguiar-Amador operator)**.** *Under assumptions 1, 2 and 6,* $\mathbb{T}$ *has a fixed point* $\mathbb{T}f = f$*.*

Proof. As the monotonicity of $\mathbb{T}$ is straightforward and bounds are uniform, order continuity is rather immediate. The maximization clause is essential to guarantee that the operator maps a carefully selected initial condition up. We now prove this claim formally. To serve this purpose, we need the following iterative version of OC:

$$f_{j+1}(\nu, y) = SUP_{\{g_+(y')\}_{y' \in Y}}$$
$$[(b_{+,n+1} + c_{n+1})(\nu, y) - y]\, R^{-1}$$
$$\left[\mathbb{I}\left(f_j(\nu, y) > 0\right) + \mathbb{I}\left(f_j(\nu, y) \leq 0\right) \sum_{s \in Y} \pi(s) \mathbb{I}\left(V^c_{n+1,j}(f_j(\nu, y), s) \geq V^{def}(y)\right)\right]$$

Subject to

$$V^c_{n+1,j+1}(f_j(\nu, y), s) = u(c_{n+1}(f_j(\nu, y), s)) + \beta E max\left\{g_+(y'), V^{def}(y')\right\} \quad s \in Y \tag{26}$$

$$b_{+,n+1}(\nu, y) = f_j(g_+(y'), y') \quad \text{for all} \quad y' \in Y \quad \text{such that} \quad g_+(y') \geq V^{def}(y') \tag{27}$$

Let $\mathbb{F}$ be the space of real valued bounded measurable increasing functions mapping $\mathbb{V} \times Y$ to $\mathbb{R}$. This set is a poset and every countable chain in it has a supremum [27]. Take any $f_j \in \mathbb{F}$ with $f_0 = INF(\mathbb{F})$ and $V^c_{1,0}$ the initial condition in assumption 3. The results in Aguiar and Amador (2019) imply that $\mathbb{T}f_j$ is also increasing [28]. In order to show that $\mathbb{T}$ is order continuous note that the objective function in OC is bounded by lemma 1. Then, we have: $SUP \quad \mathbb{T}f_0 \leq SUP \quad \mathbb{T}f_1 = SUP \quad \mathbb{T}^2 f_0 \leq SUP \quad \mathbb{T}f_2, ..., \quad lim_n \quad SUP \quad Tf_n = lim_n \quad SUP \quad lim_n \mathbb{T}^n f_0 = SUP \quad lim_n \mathbb{T}^n f_0 = SUP \quad T(lim_n \mathbb{T}^n f_0) = SUP \quad T(lim_n f_n)$. Thus, $lim_n \quad SUP \quad Tf_n = SUP \quad T(lim_n f_n)$ which implies that the operator is order continuous. By setting $a = INF(\mathbb{F})$, by the definition of $\mathbb{T}$ we know that $a \leq \mathbb{T}a$

The desired result then follows. □

[27] This last property is easily achieved as long as shocks are finite. I would like to thank Kevin Reffett for pointing this out to me.

[28] See lemma 8.

We are now in position to prove theorem 1. We will use definition 1 and the iterative procedure in 2. To complete the proof, we need an additional result borrowed from Coleman (1991)

Theorem 5 (Existence of an upper fixed point, Coleman (1991), page 1098)**.** *An order continuous monotone operator A mapping a non-empty, partially ordered compact set $\mathbb{C}$ into itself, with an element c_0 such that $A(c_0) \leq c_o$, has a fixed point which can be computed by successive approximations $A^n(c_0)$ and converges to a maximal fixed point in the set $(c \leq c_o, c \in \mathbb{C})$.*

Note that theorems 4 and 5 can be used to find $c_*(\underline{c}_0)$ and $c_*(\overline{c}_0)$ in theorem 1. We will prove the result using lemmas 2, 4 and 5.

Proof of Theorem 1. Note that if $c_0 = SUP(\mathbb{C})$, then lemma 1 imply that $c_0(B, y) = Y_{UB} + R_{UB}B_{UB} - B_{LB}$ for all $B, y \in \mathbb{B} \times Y$. By assumption 3, $R_0 = R^*$ and thus equation (3) characterizes a standard savings problem. As $card(Y) > 1$, we know that $c_1 = A(c_0) \leq c_0$. Moreover, as problem 2 is a maximization problem, we know that, if $c_0 = INF(\mathbb{C})$, we have $c_1 = A(c_0) \geq c_0$. Note that, as consumption is uniformly bounded below and away from zero by lemma 1 and V^{def} is finite, $A(c_0)$ is well defined in this case. So we can set c_0 in either the supremum or the infimum of $\mathbb{C}$.

Take V_0^c, V_0^{def} from assumption 3. As $c \in \mathbb{C}$, under standard results equations 5 and 13 imply that R_1 is monotone. Them under lemma 2, the Coleman-Reffett operator in equation 20 implies that $c_1, b_{+,1}$ are monotone. Then, using equations 6 and 7 and lemmas 4 and 5, $B_{1,*}, V_{1,*}^c$ are well defined. Moreover, as $B_{1,*}$ is a fixed point of $\mathbb{T}$, it is increasing. Thus, as $\nu = V_{1,*}^c(y, B_{1,*}(\nu, y)) = V_{1,*}^c(y, f_{1,*}(\nu, y))$, by equations 5 and 13 R_2 is also monotone. Continuing with this logic, we can construct a sequence of ordered functions $SUP\ Ac_0 \leq SUP\ Ac_1 = SUP\ A^2c_0 \leq SUP\ Ac_2, ..., $. As $\mathbb{C}$ is compact by lemma 1, we can use the same argument as in lemma 5 to show that A is order continuous. As $\mathbb{C}$ is compact, we know that it is countable chain complete. Thus, under theorems 4 and 5, A has 2 ordered fixed points, depending on the initial condition c_0.

Until now V^{def} was assumed to have the form: $V^{def}(y) = u(y^{def}(y)) + \beta E(V^{def}(y))$. That is, there is no re-entry (i.e., $\theta = 0$). However, equation (7) assumes that $\theta \in (0, 1)$. We now extend the argument for a model with re-entry. The outside option with and without re-entry are connected as follows [29]:

$$\tilde{V}^{def}(y) = V^{def}(y) + \gamma v_0, \quad \text{where } \gamma \equiv \frac{\theta\beta}{1-\beta(1-\theta)}$$

Where $v_0 \equiv E(V^c_{n+1,*}(0,y) - V^{def}(y))$. Then, the UMP has the form:

$$V^c_{n+1,*}(B,y;v_0) = u(c_{n+1}(B,y)) - (1-\beta)\gamma v_0 + \beta Emax\left\{V^c_{n+1,*}(b_{+,n+1}(B,y),y';v_0), V^{def}(y')\right\}$$

Subject to

$$b_{+,n+1}(B,y) + c_{n+1}(B,y) = y + BR^* \left[\mathbb{I}(B>0) + \mathbb{I}(B\leq 0)\sum_{y\in Y}\pi(y)\mathbb{I}\left(V^c_{n+1,*}(B,y;v_0) \geq V^{def}(y)\right)\right]$$

The following argument based on a modified version of $\mathbb{T}$ shows that there is a unique $v_{0,*}$ which satisfies: $v_{0,*} = E(V^c_{n+1,*}(B,y;v_{0,*}) - V^{def}(y))$. Let f_0 be the adequate initial condition based on theorem 4. Let $a < b$ be 2 possible values for v_0. Let $\mathbb{T}(. \mid v_0)$ be given by:

$$\mathbb{T}(f_j(\nu,y) \mid v_0) = SUP_{\{g_+(y')\}_{y'\in Y}} \left[(b_{+,n+1} + c_{n+1})(\nu,y) - y\right] R^{-1}$$
$$\left[\mathbb{I}(f_j(\nu,y)>0) + \mathbb{I}(f_j(\nu,y)\leq 0)\sum_{s\in Y}\pi(s)\mathbb{I}\left(V^c_{n+1,j}(f_j(\nu,y),s) \geq V^{def}(s) + \gamma v_0\right)\right]$$

Subject to

$$V^c_{n+1,j+1}(f_j(\nu,y),s) = u(c_{n+1}(f_j(\nu,y),s)) - (1-\beta)\gamma v_0 + \beta Emax\left\{g_+(y'), V^{def}(y')\right\} \quad s\in Y \quad (28)$$

$$b_{+,n+1}(\nu,y) = f_j(g_+(y'),y') \quad \text{for all} \quad y'\in Y \quad \text{such that} \quad g_+(y') \geq V^{def}(y') \quad (29)$$

[29] A detailed computation of the steps required to connect both equations is available under request.

Note that $\mathbb{T}(. \mid a) \geq \mathbb{T}(. \mid b)$. Then, $f_{1,a} = \mathbb{T}(f_0 \mid a) \geq \mathbb{T}(f_0 \mid b) = f_{1,b}$. Then applying $\mathbb{T}(. \mid a)$ to both sides, we get: $f_{2,a} = \mathbb{T}^2(f_0 \mid a) = \mathbb{T}(f_{1,a} \mid a) \geq \mathbb{T}(f_{1,b} \mid a) \geq \mathbb{T}(f_{1,b} \mid b) = \mathbb{T}^2(f_0 \mid b) = f_{2,b}$, where the first inequality follows from the monotonicity of $\mathbb{T}(. \mid a)$ and the second one by the fact that $a < b$. Continuing with this logic, we obtain: $f_{*,a} \geq f_{*,b}$, which shows that any fixed point of $\mathbb{T}(. \mid v_0)$ is decreasing in v_0. Then, using the equivalence between EMP and UMP, the arguments in Aguiar and Amador (2019) [30] shows that $v_{0,*} = E(V^c_{n+1,*}(B, y; v_{0,*}) - V^{def}(y))$ as desired.

Now it remains to be shown that any fixed point can be used to construct a candidate policy h. Let $b_{+,*}(B, y) = y + R_*(B)B - c_*(B, y)$, where R_* is the interest rate using $V^c_{*,*}, V^{def}_{*,*}, c_*$ for the model with re-entry. Let θ_i a realization from a uniform $[0, 1]$ distribution. Then, we have:

$$h(B, y) =$$
$$\mathbb{I}\left\{b_{+,*}(B, y) < \overline{B}(y)\right\}(\mathbb{I}\left\{\theta_i \leq \theta\right\} b_{+,*}(0, y) + \mathbb{I}\left\{\theta_i > \theta\right\} 0) + \mathbb{I}\left\{b_{+,*}(B, y) \geq \overline{B}(y)\right\} b_{+,*}(B, y)$$

$$c(B, y) = \mathbb{I}\left\{b_{+,*}(B, y) < \overline{B}(y)\right\}\left(y^{def}(y)\right) + \mathbb{I}\left\{b_{+,*}(B, y) \geq \overline{B}(y)\right\}(y + R_*(B)B - h(B, y))$$

□

Proofs for section 3.3

Proof of Theorem 2. Let $\underline{c}_* \leq \overline{c}_*$ be the 2 candidate fixed points in theorem 1. Take α such that: $\underline{c}_*(B, y) \geq \alpha\overline{c}_*(B, y)$ for all $B, y \in \mathbb{B} \times Y$ and $\underline{c}_*(B, y) = \alpha\overline{c}_*(B, y)$ for some B, y. Note that this equality is possible as consumption is bounded below and away from zero. Then, as u is pseudo-concave, theorem 11 in Coleman (1991) implies: $\underline{c}_*(B, y) = A(\underline{c}_*(B, y)) \geq A(\alpha\overline{c}_*(B, y)) > \alpha A(\overline{c}_*(B, y)) = \alpha\overline{c}_*(B, y)$. Note that the last equality implies $A(\alpha\overline{c}_*(B, y)) > \alpha\overline{c}_*(B, y)$ which is a contradiction as $\overline{c}_*(B, y)$ is assumed to be a fixed point. □

To show ergodicy, We first must define an equilibrium state space of the markov process. We begin by the minimal state space Z_1:

[30] See page 861.

Definition 6 (Equilibrium state space Z)**.** *Let $\overline{B}(y)$ be the upper bounds for debt implied by h. The minimal state space for the equilibrium process generated by the markov kernel P_φ is given by: $Z_1 \equiv \left[\overline{B}(y_{LB}), B_{UB}\right] \times Y$ Then, there exist a function φ mapping (B, y, c, R) to (B_+, y_+, c_+, R_+) and these elements belong to a SCE. The state space Z is composed by all possible (B, y, c, R) spanned by Z_1 using φ. That is, for all possible SCE candidates according to definition 1.*

Equation (10) implies that we can construct Z using φ. Note that theorem 2 guarantees the uniqueness of the SCE. Thus, given an element in Z_1 and $y_+ \in Y$ we can find at most 1 vector $(c, R, B_+, y_+, c_+, R_+)$ associated with it. That is, iterating this procedure, it is possible to construct a *finite time path from the SCE* using φ. Using these paths we will show that a unique SCE is also ergodic, although it is discontinuous.

Let us start by formally defining an "accessible atom", which can be thought as a point that is non-negligible from a probabilistic perspective and gets "hit" frequently. Let $P^n_\varphi(z, A)$ be the probability that the Markov chain goes from z to any point in A in n steps with A being measurable, let ψ be some measure, and $B(Z)$ be the Borel sigma algebra generated by Z. Then the set $A \in B(Z)$ is *non-negligible* if $\psi(A) > 0$. A chain is called *irreducible* if, starting from any initial condition, the chain hits all non-negligible sets with positive probability in finite time (i.e. $\psi(A) > 0 \rightarrow P^n_\varphi(z, A) > 0$.) Intuitively, irreducibility is a notion of connectedness for the Markov process as it implies non-negligible sets are visited with positive probability in finite time.

We are now in position to define an atom and state an important intermediate result.

Definition 7 (Accessible Atom)**.** *A set $\alpha \in B(Z)$ is an atom for (Z, P_φ) if there exists a probability measure μ such that $P_\varphi(z, A) = \mu(A)$ with $z \in \alpha$ for all $A \in B(Z)$. The atom is accessible if $\psi(\alpha) > 0$.*

Intuitively an atom is a set containing points in which the chain behave like an i.i.d. process. Any singleton $\{\alpha\}$ is an atom. Note that there is a trade off: if the atom is a singleton, the i.i.d. requirement is trivial but, taking into account that the state space is uncountable, the accessibility clause becomes an issue as it is not clear how to choose ψ. The same happens with irreducibility: when the state space is finite, it suffices to ask for a transition matrix with positive values in all its positions. In the general case, we need to define carefully what is a meaningful set. Fortunately, when the state space Z is a product space between a finite set (Y) and an uncountable subset of ^{3}, containing (B, c, R), there is a well know results that help us find an accessible atom in an irreducible chain (for proof, see Proposition 5.1.1 in Meyn and Tweedie (1993).)

Lemma 6 (Irreducibility and accesible atoms)**.** *Suppose that $P_\varphi^n(z, \alpha) > 0$ for all $z \in Z$. Then α is an accessible atom and (Z, P_φ) is a $P_\varphi(\alpha, .)-$irreducible.*

Proposition 6 follows directly from standard results in Meyn and Tweedie (1993) [31]. Note the relevance of the atom, $\alpha = z_* = (0, y_{LB}, y^{def}, R^*)$ for the stochastic stability of the process: we define a meaningful set to be the one that can be hit by the chain starting from it. In this sense, it is similar to a saddle path point in a phase diagram in non-stochastic models where endogenous variables can only take 1 initial condition that leads to convergence to the steady state state.

To apply Proposition 6, the finiteness of Y in assumption 1 and the definition of the Markov kernel P_φ in equation (10) are essential. As we are considering a point, in order to show that $P_\varphi^\tau(z, \{z_*\}) > 0$, it suffices to find a finite sequence $\{y_0, ..., y_\tau\}$ such that the economy defaults when $y_\tau = y_{LB}$.

The effect of an atom in the recurrence structure of the chain is essential to define an invariant measure (i.e. a measure μ which satisfy $\mu = \int P_\varphi(z, A)\mu(dz)$). Suppose that the atom is hit for the first time with positive probability in period $\tau_{z_*} < \infty$ starting from z_0. Then, it is possible to define a (not necessarily probability) measure μ which gives the expected number of visits to a particular set in $B(Z)$, called it A, before τ_{z_*}. Then $\mu(A)$ gives the sum of the probabilities of hitting A *avoiding* the atom. In period $\tau_{z_*} - 1$ when "forward" μ 1 period (i.e. by applying the Markov operator to it, $\int P_\varphi(z, A)\mu(dz)$) the expected number of visits to A avoiding the atom is the same as the chain will hit z_* in period $t = \tau_\alpha$. Thus, μ must not change or equivalently $\mu = \int P_\varphi(z, A)\mu(dz)$. That is, μ is an invariant measure. Provided that $\tau_{z_*} < \infty$, it is possible to normalize μ to be a probability measure. Further, as the accessibility of the atom comes together with the irreducibility of the chain (see lemma 6), the invariant measure is unique as the chain does

[31] If in In proposition 5.1.1 we assume that the atom is a singleton, we still have to deal with the reference measure ψ. Typically, ψ is set to be the "maximal" measure. Fortunately, if the chain is irreducible with respect to some measure, say $P_\varphi(\alpha, .)$, then it can be "expanded" to ψ (e.g, see Meyn and Tweedie (1993), Proposition 4.2.2)

not break into different "unconnected islands". Finally, the Krein-Milman theorem guarantees the ergodicity of the chain provided its uniqueness (see Futia (1982)).

We first show that (Z, P_φ) satisfy the conditions of proposition 6.

Lemma 7 (Accessible atom in the default model)**.** *Let the atom be $z_* = (0, y_{LB}, y^{def}, R^*)$. Then, under assumptions 1, 4 and 5, for any $(B_0, y_0) \in Z_1$, $P_\varphi^{\tau(B_0,y_0)}(z, \{z_*\}) > 0$ and $\tau(B_0, y_0) < \infty$.*

Proof. To show that $B^N(y_{LB}) < \bar{B}(y^{def}) < 0$ note that under assumption 5, $B^N(y_{LB}) < 0$. Then, equation 8 and lemma 2, implies that there is at least 1 y^{def} with such a property.

We now show that starting from any initial condition, the chain hits the atom. We will first show that for any $y \neq y_{LB}$ and any B_0, there is a positive probability path $\{y_0, y_1, ..., y_\tau\} = \{y_0, y_{LB}..., y_{LB}\}$, and an associated sequence $z(y^t)$ for which the economy defaults when $y_\tau = y_{LB}$.

If $b_{+,*}(B_0, y_0) < B_0$ using lemma 2 we know that: $b_{+,*}(B_0, y_{LB}) < b_{+,*}(B_0, y_0) = B_1$. Then, $B_2 = b_{+,*}(B_1, y_{LB}) < b_{+,*}(B_0, y_{LB}) < B_1$. Then, $B_3 = b_{+,*}(B_2, y_{LB}) < b_{+,*}(B_1, y_{LB})$. Continuing with this logic, as $\bar{B}(y^{def})$ is finite, $B_{\tau+1} < \bar{B}(y^{def})$ as the chain would have converged to $B^N(y_{LB})$ in the absence of default. By the definition of h, then the planner chooses to default at period τ, which in turn implies that the economy hits the atom in this time period.

If $b_{+,*}(B_0, y_0) \geq B_0$, choose $y_t = y_0$ until $B^{y_0} = b_{+,*}(B^{y_0}, y_0)$. Because of remark 4, we know that this point exist for every $y \in Y$ and is finite. Thus, the chain hits B^{y_0} in finite time. Call this period $s = t$ and thus $B^{y_0} = B_{s+1}$. Choose $y_{s+1} = y_{LB}$. Then, $B_{s+2} = b_{+,*}(B_{s+1}, y_{LB}) < B_{s+1}$ and $B_{s+3} = b_{+,*}(B_{s+2}, y_{LB}) < b_{+,*}(B_{s+1}, y_{LB}) = B_{s+2}$. Continuing with this logic, the chain will hit $\bar{B}(y^{def})$ and thus the atom in finite time.

If $y_0 = y_{LB}$, choose $y_1 = y$ with $y \neq y_{LB}$ and repeat the previous reasoning. □

Now using lemma 7, we show that the chain has a unique invariant measure.

Proof of theorem 3. Note that lemma 7 imply that $P_\varphi^\tau(z_*, \{z_*\}) > 0$ with $\tau < \infty$. The results in Remark 4.2.1, proposition 4.2.2, theorem 8.2.1 and theorem 10.2.1 in Meyn and Tweedie (1993) imply that (Z, P_φ) has an unique invariant measure. As $\tau < \infty$ for any initial condition in Z, theorem 10.2.2 in Meyn and Tweedie (1993) implies that the invariant measure is a probability measure. As it is unique, the Krein-Milman theorem (See Futia (1982)) implies that this measure is ergodic.

□

Supplementary Appendix

Supplementary Appendix for section 3.1

Lemma 8 (Unstable paths)**.** *Under assumptions 1 and 2, if $R(B)$ is decreasing in B, then there exist some $\hat{B} < 0$ such that for any $y \in Y$ with $-\hat{B} > y$ and $\beta R(\hat{B}) > 1$, $B \leq \hat{B}$ implies that $b_+(B, y)$ converges to $\bar{B}$ for any weakly decreasing path $[y \downarrow, ..., y_T]$.*

Proof of Lemma 8. Under the assumptions of these lemma, (3) implies $u(c) > E[u'(c_+)]$, where the dependence on $b, B, y; h$ was omitted for simplicity. Then, as $-\hat{B} > y$ and consumption is uniformly bounded away from zero because of lemma 1, $b_+ - \hat{B} < y + (R(\hat{B}) - 1)\hat{B} < 0$. Equivalently, $b_+(\hat{B}, y) < \hat{B}$. Because of 2, b_+ is increasing in B and by assumption R is decreasing in B. Thus, $\beta R(b_+(b_+(\hat{B}, y), y)) > \beta R(b_+(\hat{B}, y)) > 1$. As problem (2) is a standard savings problem and y follows a weakly decreasing path, we know that $u(c_+) > E[u'(c_{++})]$. Taking expectations on the second inequality and using the first we get $u(c) > E[u'(c_{++})]$. Continue with these logic and noting that u' is bounded below by zero, we get $lim_{T\rightarrow\infty} E\left[u'(c_T)\right] = 0$ (A1). Because of lemma 1, we know that $c_T + b_{T+1} \leq Y_{UB} + R_{UB}B_{UB}$, which then implies that $b_{T+1} \rightarrow \bar{B}$. As y and $\hat{B}$ were arbitrary and A1 was obtained after taking expectations for every period, the convergence is in finite time, which in turn implies that the weakly decreasing path has positive probability.

□

The intuition for lemma 8 follows from the conditions $-\hat{B} > y$, a debt to GDP ratio bigger than 100%, and $\beta R(\hat{B}) > 1$, a sufficiently high interest rate. By noting that a weakly decreasing path represents a persistent recession, we can say that sufficiently high debt levels coupled with a poor growth prospect lead to a default. Depending on the level of GDP during default, $y^{def}(y)$, its welfare effects can vary significantly. We will see the values of $y^{def}(y)$ are critical to show the existence of equilibrium and to generate an ergodic representation. Notice that there is a clear connection between $\hat{B}$, $\bar{B}$ and the type of recession required to induce default. Clearly, $\bar{B}(y)/y$ for any $y \in Y$ defines an upper bound for the debt to GDP ratio. If in any period t, $\hat{B}/y_t$ is close to $\bar{B}(y_t)/y_t$, then it takes a short and mild recession to cause a default.

Supplementary Appendix for section 3.4

VI) Severity of the crisis and permanent effects on the stable distribution. If we measure the severity of the crises as the difference between the average detrended GDP (i.e., $\mathbb{E}(y)$) and the level of activity after default and during exclusion (i.e., y^{def}), the figure above can be used to illustrate the effects of default on the long run distribution of the model, and consequently on key observed unconditional moments. Let us compare 2 economies, i, j, that only differ in the severity of the crises with $[\mathbb{E}(y_i) - y_i^{def}]/\mathbb{E}(y_i) > [\mathbb{E}(y_j) - y_j^{def}]/\mathbb{E}(y_j) > 0$. In economy i, the most affected one, point 3 will be closer to A in figure 1. Note that this last point is the same in both economies as definition 2, which is used to construct the non-stochastic steady state, is independent of the default decision. As the process is ergodic, all the points in the state space, characterized by $[\bar{B}(y_{LB}), B_{UB}]$, are hit with positive probability starting from any initial condition. If the crises is more severe, the support of the stable distribution increases, i.e. $[\bar{B}(y_{i,LB}), B_{i,UB}] \supset [\bar{B}(y_{j,LB}), B_{j,UB}]$. If this is case it is likely that the most affected country:

- has a smaller level of assets on average (i.e. $\mathbb{E}(B; \mu_{i,*}^{def}) < \mathbb{E}(B; \mu_{j,*}^{def})$). That is, a more serious crises is associated with a higher level of net external private debt, and,
- as the support is bigger, the variance of the distribution of debt increases. As the interest rate spread $R(B) - R^*$ is monotonic in B, we will observe a higher and more volatile spread *even after the default occurs.*

We study whether this intuition is right for a relevant calibration in the next section. As y_i^{def}, y_j^{def} are endogenous as well as point 3, we need to solve the model for different values of the deep parameters and compute the effect on the threshold $\bar{B}(y^{def})$, the contours $b_+(., y)$ and the ergodic statistics $\mathbb{E}(B; \mu_{i,*}^{def})$, $\mathbb{E}(B; \mu_{j,*}^{def})$), among others.

VII) Modelling countries with no default but with default risk. As we show that μ_*^{def} is ergodic, we know that $\sum f(z)/N \to \mathbb{E}(z; \mu_*^{def})$ for any value of y^{def} even if it has never been observed. Thus, we can design a calibration or estimation procedure to recover the value of GDP that would be observed if the country decides to default, even if this event has never been observed recent in history. Thus, it is possible to use a default model to explain the risk premium during 2008 for countries like Spain or Portugal, which experienced a hike in this variable without actually defaulting.