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# Gromov Hyperbolicity in Directed Graphs

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**Abstract:** In this paper, we generalize the classical definition of Gromov hyperbolicity to the context of directed graphs and we extend one of the main results of the theory: the equivalence of the Gromov hyperbolicity and the geodesic stability. This theorem has potential applications to the development of solutions for secure data transfer on the internet.

**Keywords:** security; Gromov hyperbolicity; hyperbolic graph; directed graph

## 1. Introduction

A geodesic metric space is said to be (Gromov) hyperbolic if every geodesic triangle is uniformly “thin” (see Definition 2). The theory of hyperbolic spaces was introduced by M. Gromov in [1,2], and after that studied by many scientists, see, e.g., in [3–6]. It is especially remarkable that this “new” theory grasps the essence of negatively curved spaces, and the connections between graph theory and potential theory on Riemannian manifolds (see, e.g., in [7–11]).

In [12], a certain graph is constructed for each geodesic metric space and it is proved that one is hyperbolic if and only if the other is; furthermore, in [13–15] there are analogous results for Riemannian surfaces (and simple graphs). Therefore, the study of Gromov hyperbolic graphs can shed light on the hyperbolicity of more general geodesic metric spaces.

The research on mathematical properties of hyperbolic spaces and their applications is a subject of interest in graph theory; see, e.g., in [16–37] and the references therein.

Originally, the theory of hyperbolic spaces was developed to be applied to finitely generated groups and, in that field, it has proved relevant competence in practical applications. For example, they were historically applied to some computer science questions, as hyperbolic groups have a soluble word problem. In [38], it is proved that they are biautomatic and automatic; actually, they are strongly geodesically automatic, meaning that there is an automatic structure on the group, where the set of all geodesic words is the language accepted by the word acceptor. Many other applications are supported by the concept of Gromov hyperbolicity. For instance, research evidence suggests that the Internet is hyperbolic (see [39]); the outstanding growth/preferential attachment process as a mean to construct a scale-free graph produces a (scaled) hyperbolic graph (see [28,40]); the greedy geographical routing tries to embed the network graph in the (hyperbolic) Poincaré disk (see [41]). The spread of viruses through the Internet can be modeled by hyperbolic spaces (see [27,42]).

Today, and thanks to the geodesic stability of hyperbolic spaces (see Theorems 2 and 3), Gromov hyperbolicity is successfully applied to the development of solutions for secure data transfer on the internet. As Edmond Jonckheere states in [42], any electronic message is usually split into many different packets, which are sent separately to its destination. Most of them follow the same path, but, unfortunately, it facilitates the interception and subsequent reconstruction of the message by malicious listeners. A better strategy to bypass this type of electronic spying is to send packets along distinct routes [43]. However, this design raises a further difficulty: the reconstruction by the protocol of all those packets, which reach destination in a disorganized way (due to the intrinsic delays of the different routes chosen), is not trivial.

For all these reasons, a remarkable feature of hyperbolic networks is that in a (well-located by geodesic stability) neighborhood of any geodesic (optimal path) there are many other quasi-optimal paths, i.e., routes whose costs are close to the optimal one. Thus, hyperbolic networks facilitate the assembly of packets sent through different routes with similar delay times [44].

Nevertheless, when it comes to the application of Gromov spaces to this field, some problems arise due to the fact that modelization still not accurate of this process is not yet accurate. The communication between a client computer and its internet connection server is not “symmetric” in the following sense: data transferring from server to client is much faster than from client to server. Mathematically, it means that the function modeling the distance between client and server is not symmetric either, i.e., it is not a distance function.

We can first address the problem with a mathematical model that assumes that data transferring between server and client is symmetric. Although we know that this is not very accurate, it seems a very reasonable approach, as this makes it possible to work with a usual distance function (see the discussion in [42]).

However, it is desirable to pursue a more realistic model. For instance, Edmond Jonckheere cleverly decided to apply noncommutative geometry methods to overcome the difficulty introduced by the asymmetry (see [42]).

Our main aim in this paper is to show a new mathematical theory to provide an alternative approach to the problem, involving a less sophisticated mathematical foundation. In particular, we think that a more intuitive model can be found in “directed graphs”: each pair of adjacent vertices  $A$  and  $B$  is joined by two directed edges  $AB$  and  $BA$  of (possibly) different lengths. This point of view introduces asymmetry as something natural.

In real life asymmetric distances are natural; for instance, given a set  $X$  of mountain peaks connected by twisted roads, the usual driving times between these peaks form a quasi-metric because it takes longer to travel up hill than down hill; also, the taxi geometry with some one-way streets, where a path from location  $P$  to location  $Q$  contains a different set of streets than a path from  $Q$  to  $P$ .

Thus, in this paper we extend Gromov hyperbolic spaces to the new context of directed graphs. Specifically, we prove one of the most important results of the theory: a counterpart to the geodesic stability theorem, which happens to be the most utilized theorem both in theory and in applications of hyperbolic spaces in the usual context of geodesic metric spaces (see Theorem 5).

**Notation.** For each geodesic metric space  $X$ , we shall denote by  $d_X$  and  $L_X$ , the distance and the length, respectively.

## 2. Background on Hyperbolic Spaces

Although there are several definitions of hyperbolicity, in this paper we will use the one that involves thin triangles, as it is intuitively simple and has a clear geometric interpretation. For more background and results on hyperbolicity, see [2].

**Definition 1.** If  $X$  is a metric space and  $I$  is a real interval, then a curve  $\gamma : I \rightarrow X$  is a geodesic if it is an isometry, i.e.,  $L_X(\gamma|_{[t,s]}) = d_X(\gamma(t), \gamma(s)) = |t - s|$  for each  $t, s$  in the interval  $I$ .  $X$  is said to be a geodesic

metric space if for each  $x_1, x_2 \in X$  there exists at least one geodesic joining  $x_1$  and  $x_2$ ; we denote by  $[x_1x_2]$  any of such geodesics (since uniqueness of geodesics is not required, this is an ambiguous but convenient notation).

**Definition 2.** If  $X$  is a geodesic metric space and  $Y = \{Y_1, Y_2, Y_3\}$ , with  $Y_j \subseteq X$ , we say that  $Y$  is  $\delta$ -thin if for each  $j \in \{1, 2, 3\}$  and  $y \in Y_j$  one has  $d(y, \cup_{i \neq j} Y_i) \leq \delta$ . If  $x, y, z \in X$ , then a geodesic triangle  $T = \{x, y, z\}$  is the union of the three geodesics  $[xy]$ ,  $[yz]$  and  $[zx]$ .  $X$  is said to be  $\delta$ -hyperbolic (or that it satisfies the Rips condition) if each geodesic triangle in  $X$  is  $\delta$ -thin. Denote by  $\delta(X)$  the sharp hyperbolicity constant of the space  $X$ , i.e.,  $\delta(X) = \inf\{\delta \geq 0 : X \text{ is } \delta\text{-hyperbolic}\}$ . The space  $X$  is said to be Gromov hyperbolic (or simply hyperbolic) if it is  $\delta$ -hyperbolic for some  $\delta \geq 0$ . If  $X$  is not hyperbolic, then define  $\delta(X) = \infty$ .

### Example 1.

- I Each bounded space  $S$  is  $(\text{diam } S)/2$ -hyperbolic [2] (p. 29).
- II Any complete simply connected manifold with sectional curvature above bounded by  $-k^2 < 0$  is hyperbolic (see, e.g., in [45] (p. 130) and [2] (p. 52)).
- III Every tree with edges of arbitrary length is 0-hyperbolic (see, e.g., in [2] (p. 29)).

In fact, the hyperbolicity constant  $\delta(X)$  of a geodesic metric space can be interpreted as a way to decide how far the space is from being a tree. Recall that the metric trees are the spaces with hyperbolicity constant equal to zero. In practice, the borderline between tractable and intractable problems can be the tree-like degree of the corresponding graph (see [46]).

Next, the class of maps which play a central role in the theory is presented.

**Definition 3.** Given constants  $a \geq 1$ ,  $b \geq 0$ , a function between two metric spaces  $f : X \rightarrow Y$  is said to be an  $(a, b)$ -quasi-isometry if

$$\frac{1}{a} d_X(x_1, x_2) - b \leq d_Y(f(x_1), f(x_2)) \leq a d_X(x_1, x_2) + b, \quad \text{for every } x_1, x_2 \in X.$$

An  $(a, b)$ -quasigeodesic in  $X$  is an  $(a, b)$ -quasi-isometry from a real interval to  $X$ .

Next result shows that quasi-isometries preserve hyperbolicity. That is why they play a central role in Gromov hyperbolic spaces.

**Theorem 1.** ([2] (p. 88)) Let us consider an  $(a, b)$ -quasi-isometry between two geodesic metric spaces  $f : X \rightarrow Y$ . If  $Y$  is hyperbolic, then  $X$  is hyperbolic. Furthermore, if  $f$  is onto, then  $X$  is hyperbolic if and only if  $Y$  is hyperbolic.

**Definition 4.** Let us consider  $H > 0$ , a metric space  $X$ , and subsets  $Y, Z \subseteq X$ . The set  $V_H(Y) := \{x \in X : d(x, Y) \leq H\}$  is called the  $H$ -neighborhood of  $Y$  in  $X$ . The Hausdorff distance between  $Y$  and  $Z$  is defined by  $\mathcal{H}(Y, Z) := \inf\{H > 0 : Y \subseteq V_H(Z), Z \subseteq V_H(Y)\}$ .

The following is a beautiful and useful result:

**Theorem 2.** ([2] (p. 87)) For each  $\delta \geq 0$ ,  $a \geq 1$  and  $b \geq 0$ , there exists a constant  $H = H(\delta, a, b)$ , which only depends on these three parameters, with the following property:

Let us consider any  $\delta$ -hyperbolic geodesic metric space  $X$ , any  $(a, b)$ -quasigeodesic  $g$  joining  $x$  and  $y$ . If  $\gamma$  is any geodesic joining  $x$  and  $y$ , then  $\mathcal{H}(g, \gamma) \leq H$ .

Mario Bonk proved that the above property, known as “geodesic stability”, is equivalent to hyperbolicity, as shown below.

**Theorem 3.** ([47] (p. 286)) Let us consider a geodesic metric space  $X$  with the following property:

For each  $a \geq 1$  there exists a constant  $H$  such that for any  $x, y \in X$ , any  $(a, 0)$ -quasigeodesic  $g$  in  $X$  joining  $x$  and  $y$ , and any geodesic  $\gamma$  joining  $x$  and  $y$ , the inequality  $\mathcal{H}(g, \gamma) \leq H$  holds. Then,  $X$  is hyperbolic.

Note that Theorems 2 and 3 give that, in fact, the hypothesis in Theorem 3 is equivalent to the following: For each  $a \geq 1$  and  $b \geq 0$  there exists a constant  $H$  such that for any  $x, y \in X$ , any  $(a, b)$ -quasigeodesic  $g$  in  $X$  joining  $x$  and  $y$ , and any geodesic  $\gamma$  joining  $x$  and  $y$ , the inequality  $\mathcal{H}(g, \gamma) \leq H$  holds.

### 3. Directed Graphs

If  $G = (V(G), E(G))$  is a directed (finite or infinite) graph, with  $V(G)$  the set of vertices and  $E(G)$  the edges, we use the notation  $uv$  for an edge starting at a vertex  $u$  and ending at the vertex  $v$ . As we allow loops and multiple edges this notation is ambiguous, but it is convenient. We suppose that  $G$  is locally finite, i.e., each vertex belongs to a finite amount of edges. Also, assume that each edge  $uv$  has length  $L_G(uv)$ . We shall identify (by an isometry) each directed edge  $uv \in E(G)$  with the directed real interval  $[0, l]$  (where  $l := L_G(uv)$ ); therefore, any point in the interior of the edge  $uv$  is a point of  $G$ , and then  $G$  is a metric directed graph.

**Definition 5.** We say that a directed graph is admissible if it is connected (for every  $x, y \in G$  there exists a directed path from  $x$  to  $y$ ), locally finite (i.e., every vertex has finite degree) and we have  $uv \in E(G)$  if and only if  $vu \in E(G)$  (we allow  $L_G(uv) \neq L_G(vu)$ ). (See Figure 1a.)

Given an admissible graph  $G$  and two adjacent vertices  $u, v \in V(G)$ , let us define the parameters

$$C(u, v) := \frac{\max\{L_G(e) : e \text{ is an edge joining either } u \text{ and } v \text{ or } v \text{ and } u\}}{\min\{L_G(e) : e \text{ is an edge joining either } u \text{ and } v \text{ or } v \text{ and } u\}},$$

$$C(G) := \sup\{C(u, v)\},$$

$$D(G) := \sup\{L_G(e) : e \in E(G)\}.$$

Note that  $1 \leq C(u, v) \leq C(G) \leq \infty$  and  $0 < D(G) \leq \infty$ .

**Definition 6.** Given an admissible graph  $G$ , we define its canonical graph  $G_0$  as the simple (without loops and multiple edges) undirected graph with  $V(G_0) = V(G)$  obtained by replacing the directed edges joining two adjacent vertices  $u$  and  $v$  in  $G$  by a single undirected edge,  $uv \in E(G_0)$ , with

$$L_{G_0}(uv) := \max\{L_G(e) : e \text{ is an edge joining either } u \text{ and } v \text{ or } v \text{ and } u\}.$$

It can be checked that  $G_0$  is a geodesic metric space; thus, we also see  $G_0$  as a metric graph (see Figure 1b).

**Remark 1.** Note that if  $C(G) < \infty$ , then for any edge  $e \in E(G)$  joining  $u$  and  $v$  (or  $v$  and  $u$ ) it holds

$$L_G(e) \leq L_{G_0}(uv) \leq C(G) L_G(e). \tag{1}$$

Although there is no usual distance in an admissible directed graph  $G$ , we introduce now the following quasi-distance or quasi-metric  $D_G$ :

$$D_G(a, b) := \inf\{L_G(\eta) : \eta \text{ is a directed curve starting at } a \text{ and ending at } b\}.$$

Recall that a quasi-distance satisfies the properties of a distance except for the symmetry, e.g., it is possible to have  $D_G(a, b) \neq D_G(b, a)$ .

However, we have the following relationship between  $D_G(u, v)$  and  $D_G(v, u)$ : if  $C(G) < \infty$ , then

$$\frac{1}{C(G)} D_G(u, v) \leq D_G(v, u) \leq C(G) D_G(u, v), \quad \text{for every } u, v \in V(G).$$

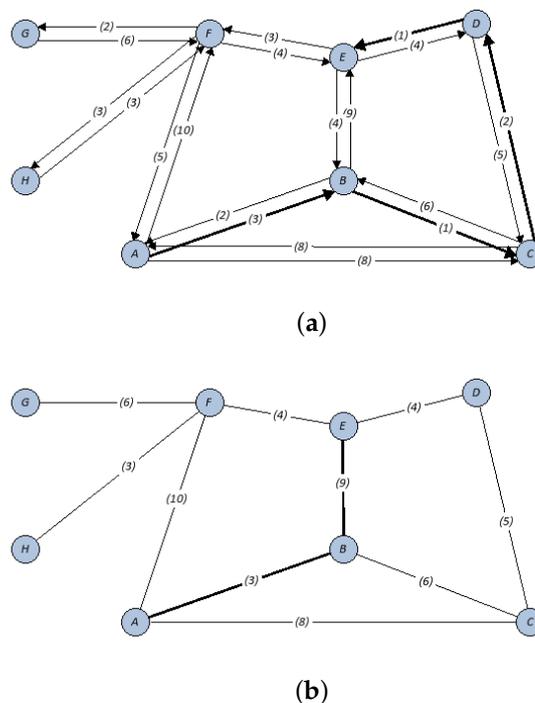
Quasi-metrics are used in many contexts, such as Computer Science and Approximation Theory. From a historical viewpoint, a kind of quasi-metric spaces were already considered for the first time by Hausdorff in his book on Set Theory [48]. This concept is extended later to introduce quasi-uniform topological spaces. Other authors also studied asymmetric distances. In [49], Künzi claims that there was much progress in quasi-uniform spaces between the years 1966 and 1982. This is just the period when different authors gave the first steps in asymmetric approximation. Indeed, to the intrinsic interest of this topic we might add applications in Approximation Theory and Computer Science. That is why there exists a vast amount of results both on abstract asymmetric normed spaces and on more general quasi-metric spaces.

One of the main applications in Computer Science is the project to develop robust models for measuring the complexity distance between programs and algorithms, which has been considered in the last 40 years. A model in the context of quasi-metric spaces was introduced by M. Schellekens in [50] and further developed by different authors (see, e.g., [51] and the references therein).

**Definition 7.** Given constants  $a \geq 1, b \geq 0$ , we say that a function between an interval  $I$  of  $\mathbb{R}$  and a directed graph  $f : I \rightarrow G$  is an  $(a, b)$ -quasigeodesic if

$$\frac{1}{a} (t - s) - b \leq D_G(f(s), f(t)) \leq a (t - s) + b, \quad \text{for every } s, t \in I, \text{ with } s \leq t.$$

A geodesic in a directed graph  $(G, D_G)$  is a  $(1, 0)$ -quasigeodesic in  $(G, D_G)$ . We denote by  $[xy]$  a geodesic starting at  $x$  and ending at  $y$ . (See Figure 1).



**Figure 1.** (a) An admissible graph  $G$ . (In bold face, the directed geodesic joining  $A$  and  $E$ ); (b) Its canonical graph  $G_0$ . (In bold face, the geodesic joining  $A$  and  $E$ ).

Next, let us define the Hausdorff distance in any directed graph  $G$ .

**Definition 8.** Let us consider a directed graph  $G$ , and subsets  $A, B \subseteq G$ . We define the “Hausdorff distance”,  $\mathcal{H}_G(A, B)$ , between  $A$  and  $B$  in  $G$  as the infimum of the positive numbers  $\alpha$  satisfying the two following properties:

- I For every  $a \in A$  there exists  $b \in B$  such that  $D_G(a, b) \leq \alpha$  and  $D_G(b, a) \leq \alpha$ .
- II For every  $b \in B$  there exists  $a \in A$  such that  $D_G(b, a) \leq \alpha$  and  $D_G(a, b) \leq \alpha$ .

Note that  $\mathcal{H}_G$  is a symmetric function although  $D_G$  is, in general, non-symmetric.

**Definition 9.** Given an admissible directed graph  $G$  such that  $C(G), D(G)$  are finite numbers, we say that  $G$  is  $\delta$ -hyperbolic if and only if its canonical graph  $G_0$  is  $\delta$ -hyperbolic. We say that  $G$  is hyperbolic or Gromov hyperbolic if it is  $\delta$ -hyperbolic for some  $\delta \geq 0$ .

To prove the following theorem we need two last definitions.

**Definition 10.** If  $G$  is an admissible directed graph and  $x_1, x_2, x_3 \in G$ , a geodesic triangle  $T = \{x_1, x_2, x_3\}$  in  $G$  is the union of three directed geodesics  $J_1 = [x_1x_2]$  (or  $[x_2x_1]$ ),  $J_2 = [x_2x_3]$  (or  $[x_3x_2]$ ) and  $J_3 = [x_3x_1]$  (or  $[x_1x_3]$ ). We say that  $T$  is  $\delta$ -thin if for every  $x \in J_i$  there exists a point  $y \in \cup_{j \neq i} J_j$  such that  $D_G(x, y), D_G(y, x) \leq \delta$ .

**Definition 11.** If  $uv$  is an edge in the admissible directed graph  $G$  (or in the graph  $G_0$ ) and  $x \in uv$ , we denote by  $\overline{ux}$  (respectively,  $\overline{xv}$ ) the segment contained in  $uv$  starting at  $u$  and ending at  $x$  (respectively, starting at  $x$  and ending at  $v$ ). Let us define the canonical projection  $\Pi : G \rightarrow G_0$  as follows:  $\Pi(v) = v$  for every  $v \in V(G)$ ; if  $x \in uv \in E(G)$ , then  $\Pi(x)$  is the point in  $uv \in E(G_0)$  such that

$$L_{G_0}(\overline{u\Pi(x)}) = L_G(\overline{ux}) \frac{L_{G_0}(uv)}{L_G(uv)}.$$

The following result shows that we have a reasonable definition of Gromov hyperbolic directed graphs.

**Theorem 4.** Any geodesic triangle  $T$  in a  $\delta$ -hyperbolic admissible directed graph  $G$  with  $C(G), D(G)$  finite numbers is  $\delta'$ -thin, with  $\delta'$  a constant which only depends on  $\delta, C(G)$  and  $D(G)$ .

**Proof.** We perform the proof in three steps. We first show that given any geodesic  $\gamma$  in  $G$ , then  $\Pi(\gamma)$  is a  $(C(G), 4D(G))$ -quasigeodesic in  $G_0$ . Then, we prove that every  $(a, b)$ -quasigeodesic triangle in  $G_0$  is  $(\delta + 2H(\delta, a, b))$ -thin, where  $H(\delta, a, b)$  is the constant in Theorem 2. These facts will allow us to conclude that every geodesic triangle in  $G$  is  $\delta'$ -thin, with  $\delta' = \delta + 2H(\delta, C(G), 4D(G)) + 4D(G)$ .

Step 1. From (1) we can deduce that

$$D_G(x, y) \leq d_{G_0}(\Pi(x), \Pi(y)) \leq C(G)D_G(x, y) \quad \text{for every } x, y \in V(G). \tag{2}$$

If  $u \in V(G)$  and  $x$  belongs to some edge starting or ending at  $u$ , then  $D_G(x, u), D_G(u, x) \leq 2D(G)$ ; this fact and (2) allows to deduce

$$D_G(x, y) - 4D(G) \leq d_{G_0}(\Pi(x), \Pi(y)) \leq C(G)D_G(x, y) \quad \text{for every } x, y \in G. \tag{3}$$

Therefore, if  $\gamma$  is a geodesic in  $G$  starting at  $x$  and ending at  $y$ , according to (3), we have that  $\Pi(\gamma)$  is a  $(C(G), 4D(G))$ -quasigeodesic in  $G_0$  joining  $\Pi(x)$  and  $\Pi(y)$ .

*Step 2.* Let us consider an  $(a, b)$ -quasigeodesic triangle  $T'$  in  $G_0$ , whose sides we will denote by  $g_1, g_2$  and  $g_3$ . Let  $\gamma_j$  be the geodesic in  $G_0$  joining the same endpoints of  $g_j$  for every  $j = 1, 2, 3$ , and let us denote by  $T_0$  the geodesic triangle in  $G_0$  with sides  $\gamma_1, \gamma_2, \gamma_3$ .

Let us consider any point  $p' \in T'$ ; without loss of generality we can assume that  $p' \in g_1$ . By Theorem 2 we know that there exist a constant  $H = H(\delta, a, b) \geq 0$  and a point  $p_0 \in \gamma_1$  with  $d_{G_0}(p', p_0) \leq H$ . As  $T_0$  is  $\delta$ -thin, there exists  $q_0 \in \gamma_2 \cup \gamma_3$  with  $d_{G_0}(p_0, q_0) \leq \delta$ . According to Theorem 2, there exists a point  $q' \in g_2 \cup g_3$  with  $d_{G_0}(q', q_0) \leq H$ .

Therefore,

$$d_{G_0}(p', g_2 \cup g_3) \leq d_{G_0}(p', q') \leq d_{G_0}(p', p_0) + d_{G_0}(p_0, q_0) + d_{G_0}(q_0, q') \leq H + \delta + H,$$

and  $T'$  is  $(\delta + 2H(\delta, a, b))$ -thin.

*Step 3.* Consequently, if  $T$  is a geodesic triangle in  $G$ , by Step 1 we have that  $T' := \Pi(T)$  is a  $(C(G), 4D(G))$ -quasigeodesic triangle in  $G_0$ , and by Step 2 we know that  $T'$  is  $(\delta + 2H)$ -thin with  $H = H(\delta, C(G), 4D(G))$ . Let us prove now that  $T$  is  $\delta'$ -thin in  $G$ .

Given any point  $p \in T$ , there exists a point  $q$  in a different side of  $T$  such that  $d_{G_0}(\Pi(p), \Pi(q)) \leq \delta + 2H$ . Consider a geodesic  $\gamma'$  joining  $\Pi(p)$  and  $\Pi(q)$  in  $G_0$ . Let us assume that  $p$  and  $q$  belong to the interior of the directed edges  $u_p v_p$  and  $u_q v_q$ , respectively (if  $p$  or  $q$  are vertices the argument is easier). Denote by  $w_p$  and  $w_q$  the vertices in  $\{u_p, v_p\}$  and  $\{u_q, v_q\}$ , respectively, such that  $\Pi(w_p), \Pi(w_q) \in \gamma'$ , and we define  $\gamma'_0$  as the closure of  $\gamma' \setminus \Pi(u_p v_p \cup u_q v_q)$ . Then, there exists a curve  $\sigma$  in  $G$  from  $w_p$  to  $w_q$  such that  $\Pi(\sigma) = \gamma'_0$  and  $L_G(\sigma) \leq L_{G_0}(\gamma'_0) < d_{G_0}(\Pi(p), \Pi(q)) \leq \delta + 2H$ . We can choose curves  $\sigma_p, \sigma_q$  in  $G$  from  $p$  to  $w_p$  and from  $w_q$  to  $q$ , respectively, with  $L_G(\sigma_p), L_G(\sigma_q) < 2D(G)$ . Then  $\sigma_p \cup \sigma \cup \sigma_q$  is a curve from  $p$  to  $q$ , and  $D_G(p, q) \leq L_G(\sigma_p) + L_G(\sigma) + L_G(\sigma_q) \leq \delta + 2H + 4D(G)$ . A similar argument gives  $D_G(q, p) \leq \delta + 2H + 4D(G)$ .

Therefore, any geodesic triangle  $T$  in  $G$  is  $\delta'$ -thin, with  $\delta' = \delta + 2H(\delta, C(G), 4D(G)) + 4D(G)$  and  $H$  the constant in Theorem 2.  $\square$

We prove now one of the main results both of the theory and the applications: the geodesic stability is equivalent to the Gromov hyperbolicity for directed graphs.

**Theorem 5.** *The following statements hold:*

I For each  $\delta, b, D \geq 0$  and  $a, C \geq 1$ , there exists a constant  $H = H(\delta, a, b, C, D)$ , which only depends on these five parameters, with the following property:

Let us consider any admissible  $\delta$ -hyperbolic directed graph  $G$  with  $C(G) \leq C$  and  $D(G) \leq D$ , any  $x, y \in G$ , and any  $(a, b)$ -quasigeodesic  $g$  starting at  $x$  and ending at  $y$ . If  $\gamma$  is any geodesic starting at  $x$  and ending at  $y$  (or starting at  $y$  and ending at  $x$ ), then  $\mathcal{H}_G(g, \gamma) \leq H$ .

II Let us consider any admissible directed graph  $G$  with  $C(G), D(G)$  finite numbers, satisfying the following property.

For each  $a \geq 1$  and  $b \geq 0$  there exists a constant  $H$  such that for any  $x, y \in G$ , any  $(a, b)$ -quasigeodesic  $g$  in  $G$  starting at  $x$  and ending at  $y$ , and any geodesic  $\gamma$  starting at  $x$  and ending at  $y$ , the inequality  $\mathcal{H}_G(g, \gamma) \leq H$  holds.

Then,  $G$  is hyperbolic.

**Proof.** First of all, note that from (3) we deduce for every  $A, B \subseteq G$ ,

$$\mathcal{H}_G(A, B) - 4D(G) \leq \mathcal{H}_{G_0}(\Pi(A), \Pi(B)) \leq C(G) \mathcal{H}_G(A, B). \tag{4}$$

Let us prove now the first statement. Consider  $x, y \in G$ , an  $(a, b)$ -quasigeodesic  $g$  starting at  $x$  and ending at  $y$ , and a geodesic  $\gamma$  starting at  $x$  and ending at  $y$  (or starting at  $y$  and ending at  $x$ ). According

to Step 1 in the proof of Theorem 4 we have that  $\Pi(\gamma)$  is a  $(C, 4D)$ -quasigeodesic in  $G_0$  joining  $\Pi(x)$  and  $\Pi(y)$ .

We prove now that  $\Pi(g)$  is a  $(Ca, Cb + 4D)$ -quasigeodesic in  $G_0$  joining  $\Pi(x)$  and  $\Pi(y)$ . If  $g : [\alpha, \beta] \rightarrow G$  is a parametrization of  $g$  with  $\frac{1}{a}(t - s) - b \leq D_G(g(s), g(t)) \leq a(t - s) + b$  for every  $\alpha \leq s \leq t \leq \beta$ , then we obtain by using (3)

$$\frac{1}{a}(t - s) - b - 4D \leq d_{G_0}(\Pi(g(s)), \Pi(g(t))) \leq Ca(t - s) + Cb \quad \text{for every } \alpha \leq s \leq t \leq \beta. \quad (5)$$

Therefore, we conclude that  $\Pi(g)$  is a  $(Ca, Cb + 4D)$ -quasigeodesic in  $G_0$ , since  $C \geq 1$ .

Now, we can finish the proof of the first statement from Definition 9 and Equation (4):

If the directed graph  $G$  is  $\delta$ -hyperbolic, then  $G_0$  is  $\delta$ -hyperbolic by Definition 9. According to Theorem 2 there exists a constant  $\tilde{H} = \tilde{H}(\delta, Ca, Cb + 4D) \geq 0$  such that if  $\gamma_0$  is a geodesic in  $G_0$  joining  $\Pi(x)$  and  $\Pi(y)$ , then  $\mathcal{H}_{G_0}(\Pi(\gamma), \gamma_0) \leq \tilde{H}$  and  $\mathcal{H}_{G_0}(\Pi(g), \gamma_0) \leq \tilde{H}$ ; consequently,  $\mathcal{H}_{G_0}(\Pi(\gamma), \Pi(g)) \leq 2\tilde{H}$ . Applying now (4) we deduce  $\mathcal{H}_G(\gamma, g) \leq H$  with  $H := 2\tilde{H} + 4D$ .

Let us prove now the second statement. Consider  $x_0, y_0 \in G_0$ , an  $(a, 0)$ -quasigeodesic  $g_0$  joining  $x_0$  with  $y_0$ , and a geodesic  $\gamma_0$  joining  $x_0$  with  $y_0$ . Fix  $x, y \in G$ , with  $\Pi(x) = x_0$  and  $\Pi(y) = y_0$ . Let us assume that  $x$  and  $y$  belong to the interior of the directed edges  $e_x$  and  $e_y$ , respectively, with  $\Pi(e_x) \neq \Pi(e_y)$  (if  $x$  or  $y$  are vertices or  $\Pi(e_x) = \Pi(e_y)$ , the argument is easier).

If  $g_0 : [\alpha, \beta] \rightarrow G_0$  is a parametrization of  $g_0$  with  $\frac{1}{a}(t - s) \leq d_{G_0}(g_0(s), g_0(t)) \leq a(t - s)$  for every  $\alpha \leq s \leq t \leq \beta$ , then we define

$$\begin{aligned} \alpha_1 &:= \sup\{t \in [\alpha, \beta] / g_0(t) \in \Pi(e_x)\}, \\ \beta_1 &:= \inf\{t \in [\alpha, \beta] / g_0(t) \in \Pi(e_y)\}. \end{aligned}$$

As  $g_0$  is a continuous and injective map, we have  $g_0(t) \in \Pi(e_x)$  for every  $t \in [\alpha, \alpha_1]$ , and  $g_0(t) \in \Pi(e_y)$  for every  $t \in [\beta_1, \beta]$ .

For each  $e \in E(G_0) \setminus \{\Pi(e_x), \Pi(e_y)\}$  such that  $e \cap g_0$  has more than one point, we define

$$\begin{aligned} t_1^e &:= \min\{t \in [\alpha, \beta] / g_0(t) \in e\}, \\ t_2^e &:= \max\{t \in [\alpha, \beta] / g_0(t) \in e\}. \end{aligned}$$

As  $g_0$  is a continuous and injective map, we have  $g_0(t_1^e) \neq g_0(t_2^e)$  and  $g_0(t) \in e$  if and only if  $t \in [t_1^e, t_2^e]$ .

If  $t \in [\alpha_1, \beta_1]$ , then there exists  $e \in E(G_0) \setminus \{\Pi(e_x), \Pi(e_y)\}$  such that  $e \cap g_0$  has more than one point, with  $t \in [t_1^e, t_2^e]$ ; let us choose a directed edge  $g_0(t_1^e)g_0(t_2^e) \in E(G)$  and define a curve  $g_2 : [\alpha_1, \beta_1] \rightarrow G$  as follows;  $g_2(t)$  is the unique point in  $g_0(t_1^e)g_0(t_2^e)$  with  $\Pi(g_2(t)) = g_0(t)$ .

If  $e_x$  ends at  $g_0(\alpha_1)$ , then we define a curve  $g_1 : [\alpha', \alpha_1] \rightarrow G$  as follows:  $g_1(t)$  is the unique point in  $e_x$  with  $\Pi(g_1(t)) = g_0(t)$ , and  $\alpha' = \alpha$ . If  $e_x$  starts at  $g_0(\alpha_1)$  and ends, say, at  $u$ , then let us choose an oriented edge  $ug_0(\alpha_1) \in E(G)$ ; we define a curve  $g_1 : [\alpha', \alpha_1] \rightarrow G$  such that  $g_1$  is an arclength parametrization of  $\bar{xu} \cup ug_0(\alpha_1)$  (obviously,  $\alpha' = \alpha_1 - L_G(\bar{xu} \cup ug_0(\alpha_1))$ ). Note that  $\alpha_1 - \alpha' = L_G(\bar{xu} \cup ug_0(\alpha_1)) < 2D(G)$ .

If  $e_y$  starts at  $g_0(\beta_1)$ , then we define a curve  $g_3 : [\beta_1, \beta'] \rightarrow G$  as follows:  $g_3(t)$  is the unique point in  $e_y$  with  $\Pi(g_3(t)) = g_0(t)$ , and  $\beta' = \beta$ . If  $e_y$  ends at  $g_0(\beta_1)$  and starts, say, at  $v$ , then let us choose an oriented edge  $g_0(\beta_1)v \in E(G)$ ; we define a curve  $g_3 : [\beta_1, \beta'] \rightarrow G$  such that  $g_3$  is an arclength parametrization of  $g_0(\beta_1)v \cup \bar{v}y$  (obviously,  $\beta' = \beta_1 + L_G(g_0(\beta_1)v \cup \bar{v}y)$ ). Note that  $\beta' - \beta_1 = L_G(g_0(\beta_1)v \cup \bar{v}y) < 2D(G)$ .

We prove now that  $g := g_1 \cup g_2 \cup g_3$  is an  $(aC(G), 8D(G))$ -quasigeodesic in  $G$  starting at  $x$  and ending at  $y$ . We deal just with the case  $\alpha' = \alpha_1 - L_G(\bar{xu} \cup ug_0(\alpha_1))$  and  $\beta' = \beta_1 + L_G(g_0(\beta_1)v \cup \bar{v}y)$ , as the cases  $\alpha' = \alpha$  and  $\beta' = \beta$ ,  $\alpha' = \alpha$  and  $\beta' = \beta_1 + L_G(g_0(\beta_1)v \cup \bar{v}y)$ , and  $\alpha' = \alpha_1 - L_G(\bar{xu} \cup ug_0(\alpha_1))$  and  $\beta' = \beta$  are simpler.

Note that (3) gives

$$\frac{1}{C(G)} d_{G_0}(\Pi(z), \Pi(w)) \leq D_G(z, w) \leq d_{G_0}(\Pi(z), \Pi(w)) + 4D(G), \quad \text{for every } z, w \in G. \quad (6)$$

If  $\alpha_1 \leq s \leq t \leq \beta_1$ , then (6) gives

$$\frac{1}{aC(G)} (t - s) \leq D_G(g(s), g(t)) \leq a(t - s) + 4D(G).$$

If  $\alpha' \leq s \leq t \leq \alpha_1$  or  $\beta_1 \leq s \leq t \leq \beta'$ , then

$$D_G(g(s), g(t)) = t - s.$$

If  $\alpha' \leq s \leq \alpha_1 \leq t \leq \beta_1$ , then as  $a, C(G) \geq 1$ , we deduce

$$\begin{aligned} D_G(g(s), g(t)) &\leq D_G(g_1(s), g(\alpha_1)) + D_G(g(\alpha_1), g_2(t)) \leq 2D(G) + d_{G_0}(g_0(\alpha_1), g_0(t)) + 4D(G) \\ &\leq a(t - \alpha_1) + 6D(G) \leq a(t - s) + 6D(G), \\ D_G(g(s), g(t)) &\geq D_G(g(\alpha_1), g_2(t)) - D_G(g(\alpha_1), g_1(s)) \\ &\geq \frac{1}{C(G)} d_{G_0}(g_0(\alpha_1), g_0(t)) - 2D(G) \geq \frac{1}{aC(G)} (t - \alpha_1) - 2D(G) \\ &\geq \frac{1}{aC(G)} (t - s) - \frac{2D(G)}{aC(G)} - 2D(G) \geq \frac{1}{aC(G)} (t - s) - 4D(G). \end{aligned}$$

If  $\alpha_1 \leq s \leq \beta_1 \leq t \leq \beta'$ , then a similar argument also gives

$$\frac{1}{aC(G)} (t - s) - 4D(G) \leq D_G(g(s), g(t)) \leq a(t - s) + 6D(G).$$

If  $\alpha' \leq s \leq \alpha_1$  and  $\beta_1 \leq t \leq \beta'$ , then

$$\begin{aligned} D_G(g(s), g(t)) &\leq D_G(g_1(s), g(\alpha_1)) + D_G(g(\alpha_1), g(\beta_1)) + D_G(g(\beta_1), g_3(t)) \\ &\leq 2D(G) + d_{G_0}(g_0(\alpha_1), g_0(\beta_1)) + 4D(G) + 2D(G) \\ &\leq a(\beta_1 - \alpha_1) + 8D(G) \leq a(t - s) + 8D(G), \\ D_G(g(s), g(t)) &\geq D_G(g(\alpha_1), g(\beta_1)) - D_G(g(\alpha_1), g_1(s)) - D_G(g_3(t), g(\beta_1)) \\ &\geq \frac{1}{C(G)} d_{G_0}(g_0(\alpha_1), g_0(\beta_1)) - 4D(G) \geq \frac{1}{aC(G)} (\beta_1 - \alpha_1) - 4D(G) \\ &\geq \frac{1}{aC(G)} (t - s) - \frac{4D(G)}{aC(G)} - 4D(G) \geq \frac{1}{aC(G)} (t - s) - 8D(G). \end{aligned}$$

Therefore, we conclude

$$\frac{1}{aC(G)} (t - s) - 8D(G) \leq d_{G_0}(\Pi(g(s)), \Pi(g(t))) \leq a(t - s) + 8D(G), \quad \text{for every } \alpha' \leq s \leq t \leq \beta',$$

and  $g$  is an  $(aC(G), 8D(G))$ -quasigeodesic in  $G$  starting at  $x$  and ending at  $y$ , as  $C(G) \geq 1$ . Furthermore,  $g_0 \subseteq \Pi(g)$  and  $L_{G_0}(\Pi(g) \setminus g_0) \leq D(G)$ .

If we replace  $g_0$  by  $\gamma_0$ , then a similar process also gives an  $(aC(G), 8D(G))$ -quasigeodesic  $\gamma$  in  $G$  starting at  $x$  and ending at  $y$ , with  $\gamma_0 \subseteq \Pi(\gamma)$  and  $L_{G_0}(\Pi(\gamma) \setminus \gamma_0) \leq D(G)$ .

Then, by hypothesis, there exists a constant  $H$ , which depends neither on  $g$  or  $\gamma$  ( $C(G)$  and  $D(G)$  are constants which only depends on  $G$ ) such that  $\mathcal{H}_G(\gamma, g) \leq H$ . By (4), we deduce that  $\mathcal{H}_{G_0}(\Pi(\gamma), \Pi(g)) \leq C(G)\mathcal{H}_G(\gamma, g) \leq C(G)H$ . As  $g_0 \subseteq \Pi(g)$ ,  $\gamma_0 \subseteq \Pi(\gamma)$ ,  $L_{G_0}(\Pi(g) \setminus g_0) \leq D(G)$

and  $L_{G_0}(\Pi(\gamma) \setminus \gamma_0) \leq D(G)$ , we conclude that  $\mathcal{H}_{G_0}(\gamma_0, g_0) \leq C(G)H + 2D(G)$ . Finally, by Theorem 3,  $G_0$  is hyperbolic and, by Definition 9,  $G$  is hyperbolic.  $\square$

#### 4. Conclusions

The theory of hyperbolic metric spaces is a well-studied subject. A natural question is how to develop this theory in the context of spaces with “asymmetric distances”. This problem was faced by Edmond Jonckheere in [42] by using noncommutative geometry. Our main aim in this paper is to show a new mathematical theory to provide an alternative and simpler approach to the problem: directed graphs. This point of view introduces asymmetry as something natural. Therefore, in this work, we extend the theory of Gromov hyperbolicity to the new context of directed graphs. In particular, we prove one of the most important results of the theory: a counterpart to the geodesic stability theorem, which happens to be the most utilized theorem both in theory and in applications of hyperbolic spaces in the usual context of geodesic metric spaces.

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