Online Appendices for “Who Acquires Information in Dealer Markets?”

Jesper Rüdiger and Adrien Vigier

Appendix C: Trading Game of Baseline Model (for online publication)

In this appendix we analyze the trading game induced by the baseline model. Specifically, throughout this appendix p_1, p_2 and q play the role of parameters: MMn is informed with probability p_n, and the speculator is informed with probability q. A strategy of MMn comprises cumulative distribution functions σ_n, $\underline{\sigma}_n$ and $\overline{\sigma}_n$ specifying respectively the distribution of the bid price b_n of MMnU, MMnL and MMnH. We assume in line with the baseline model that, conditional on MMnU, $1 - a_n$ is distributed like b_n. Similarly, we assume that the law of $1 - a_n$ conditional on MMnL (resp. MMnH) is the same as the law of b_n conditional on MMnH (resp. MMnL). A strategy of the speculator specifies her market order as a function of the information she possesses at that point.

The following notation will be used throughout:

- $\Pi_n(b|\text{sell})$ (respectively $\overline{\Pi}_n(b|\text{sell})$ and $\underline{\Pi}_n(b|\text{sell})$) for MM$n$U's (resp. MM$n$H's and MM$n$L's) expected trading profit conditional on a sell order, given $b_n = b$;
- $\sigma_n(b) := \mathbb{P}(b_n \leq b|\text{ MM}n\text{U})$, $\overline{\sigma}_n(b) := \mathbb{P}(b_n \leq b|\text{ MM}n\text{H})$ and $\underline{\sigma}_n(b) := \mathbb{P}(b_n \leq b|\text{ MM}n\text{L})$;
- $\Sigma_n := \text{supp}\,(\sigma_n)$ and $\overline{\Sigma}_n := \text{supp}\,(\overline{\sigma}_n)$;
- A_n (respectively $\overline{A}_n$) for the set of atoms in MMnU's (resp. MMnH's) strategy;
- $l_n := \sup \Sigma_n$;
- $\gamma := \mathbb{P}(V = 0|\text{sell})$.

Lemma C1. *If* $p_1 = p_2 = 1$ *then any trading equilibrium has* $a_1 = a_2 = b_1 = b_2 = V$. *If* $p_1 = p_2 = 0$ *then* $a_1 = a_2 = \frac{1-\pi(1-2q)}{2-2\pi(1-q)}$ *and* $b_1 = b_2 = \frac{1-\pi}{2-2\pi(1-q)}$. *Otherwise, any trading equilibrium satisfies the following properties:*

1. $\underline{\sigma}_1(0) = \underline{\sigma}_2(0) = 1$;
2. $\Sigma_1 \cup \overline{\Sigma}_1 = \Sigma_2 \cup \overline{\Sigma}_2 = [0, u]$, *where* $u \in (0, 1)$;
3. $A_1 \cup \overline{A}_1 \cup A_2 \cup \overline{A}_2 \subseteq \{0\}$;
4. *if* $p_m \in (0, 1)$ *then* $\Sigma_m \cap \overline{\Sigma}_m = \{l_m\}$ *and* $l_m < u$, *with* $l_m > 0$ *if and only if* $p_n < 1$;
5. *if* $1 > p_n \geq p_m > 0$ *then* $\mathbb{E}[V|\textit{sell}] > l_m \geq l_n > 0$, *with* $l_m > l_n$ *if* $p_n > p_m$;
6. *if* $p_n > p_m$ *then* $0 = \Pi_m < \Pi_n < \overline{\Pi}_n = \overline{\Pi}_m$.

Proof: The cases $p_1 = p_2 = 1$ and $p_1 = p_2 = 0$ are trivial. We prove below that, in any trading equilibrium, properties 1-6 hold in the case $\min\{p_1, p_2\} \in (0, 1)$, that is, when both market MMs acquire information with positive probability but neither of them becomes informed with probability 1; the proof for the case $p_m = 0 < p_n$ is similar.

Step 1: $\underline{\sigma}_1(0) = \underline{\sigma}_2(0) = 1$. Suppose by way of contradiction that $\underline{\sigma}_1(0) < 1$. Then we can find $b' > 0$ with $b' \in \arg\max_b \underline{\Pi}_1(b|\text{sell})$ and $\mathbb{P}(b_1 \geq b'|\text{ MM1L}) > 0$. The previous remarks imply $\mathbb{P}(b_1 = b' \text{ wins}|V = 0) = 0$, for otherwise $\underline{\Pi}_1(b'|V = 0) = -\mathbb{P}(b_1 = b' \text{ wins}|V = 0)b' < 0 = \underline{\Pi}_1(0|V = 0)$. Next, $\mathbb{P}(b_1 = b' \text{ wins}|V = 0) = 0$ implies the existence of $b'' \geq b'$ with $b'' \in \arg\max_b \underline{\Pi}_2(b|\text{sell})$ and $\mathbb{P}(b_2 = b'' \text{ wins}|V = 0) > 0$. We therefore obtain $\underline{\Pi}_2(b''|\text{sell}) < 0 = \underline{\Pi}_2(0|\text{sell})$, which cannot be.

Step 2: $p_n \in (0, 1) \Rightarrow l_n \leq \inf \overline{\Sigma}_n$. Suppose by way of contradiction that $l_n > \inf \overline{\Sigma}_n$. Then we can find $b'' > b'$ with $b'' \in \arg\max_b \Pi_n(b|\text{sell})$ and $b' \in \arg\max_b \overline{\Pi}_n(b|\text{sell})$. Next,

$$\begin{aligned}
\Pi_n(b''|\text{sell}) &= -\gamma\mathbb{P}(b_n = b'' \text{ wins}|V = 0)b'' + (1-\gamma)\mathbb{P}(b_n = b'' \text{ wins}|V = 1)(1 - b'') \\
&= -\gamma\mathbb{P}(b_n = b'' \text{ wins}|V = 0)b'' + (1-\gamma)\overline{\Pi}_n(b''|\text{sell}) \\
&< -\gamma\mathbb{P}(b_n = b'' \text{ wins}|V = 0)b' + (1-\gamma)\overline{\Pi}_n(b''|\text{sell}) \\
&\leq -\gamma\mathbb{P}(b_n = b' \text{ wins}|V = 0)b' + (1-\gamma)\overline{\Pi}_n(b'|\text{sell}) \\
&= \Pi_n(b'|\text{sell}).
\end{aligned}$$

The last inequality holds since $b' < b''$ and $\mathbb{P}(b_n \text{ wins}|V = 0)$ is non-decreasing in b_n. Thus $\Pi_n(b'|\text{sell}) > \Pi_n(b''|\text{sell}) = \max_b \Pi_n(b|\text{sell})$, which cannot be.

Step 3: $0 < \sup \overline{\Sigma}_1 = \sup \overline{\Sigma}_2 < 1$. We start by showing that $\sup \overline{\Sigma}_1 = \sup \overline{\Sigma}_2$. Suppose by way of contradiction that this is not the case, say $u_n > u_m$, where $u_n = \sup \overline{\Sigma}_n$ and $u_m = \sup \overline{\Sigma}_m$. Then, since increasing the bid beyond $u_m + \varepsilon$ does not increase the winning probability for n, $\exists\, \varepsilon > 0$ such that

$$\overline{\Pi}_n(u_m + \epsilon|\text{sell}) > \overline{\Pi}_n(u_n - x|\text{sell}), \quad \forall x \in [0, \varepsilon],$$

contradicting $u_n \in \overline{\Sigma}_n$. Hence, $u_n = u_m$. Next, let u denote the common supremum; we claim that $u \in (0, 1)$. Suppose by way of contradiction that $u = 0$. One of the two MMs does not win with probability 1 conditional on a tie at 0, say $\mathbb{P}(b_n = 0 \text{ wins}|b_n = b_m = 0) < 1$. Then bidding slightly above zero yields MMnH strictly larger expected profit than $b_n = 0$, $\overline{\Pi}_n(\epsilon|\text{sell}) > \overline{\Pi}_n(0|\text{sell})$, contradicting $u = 0$. Next, suppose by way of contradiction that $u = 1$. Then $\max_b \overline{\Pi}_n(b|\text{sell}) = 0$, for $n = 1, 2$. However, $\min\{p_1, p_2\} \in (0, 1)$. Say $p_m < 1$; then $l_m \leq \mathbb{E}[V] = \frac{1}{2}$. Therefore, $\overline{\Pi}_n(\frac{3}{4}|\text{sell}) = \frac{1}{4}(1 - p_m) > 0$, contradicting $\max_b \overline{\Pi}_n(b|\text{sell}) = 0$.

Step 4: $p_n = 1 \Rightarrow l_m = 0$; $\max\{p_1, p_2\} < 1 \Rightarrow \max\{l_1, l_2\} < \mathbb{E}[V|\text{sell}]$. The first part is trivial; we prove the second part. By Step 1, for both MMs and given any bid b, the probability of winning a sell order is maximized under $V = 0$. Hence, for all b,

$$\mathbb{E}[V|\text{sell}, b_n = b \text{ wins}] \leq \mathbb{E}[V|\text{sell}].$$

This implies, in turn, $l_n \leq \mathbb{E}[V|\text{sell}]$, otherwise MM$n$U could profitably deviate to $b_n = 0$. Now suppose $\max\{p_1, p_2\} < 1$ and, by way of contradiction, that $l_m = \mathbb{E}[V|\text{sell}]$. We consider two cases, $u = \mathbb{E}[V|\text{sell}]$ (Case 1) and $u > \mathbb{E}[V|\text{sell}]$ (Case 2). In Case 1, Step 2 gives $u = \mathbb{E}[V|\text{sell}] \in \overline{A}_m$. But then bidding slightly above u yields MMnH strictly larger expected profit than $b_n = u$: $\exists\, \varepsilon > 0$ such that

$$\overline{\Pi}_n(u + \epsilon|\text{sell}) > \overline{\Pi}_n(u - x|\text{sell}), \quad \forall x \in [0, \varepsilon],$$

contradicting $u = \sup \overline{\Sigma}_n$. Consider next Case 2. Note that in this case, by virtue of Steps 1 and 2, there exists $\delta > 0$ with

$$\mathbb{P}(b_m = b \text{ wins}|V = 1) < \mathbb{P}(b_m = b \text{ wins}|V = 0) - \delta, \quad \forall b \leq \mathbb{E}[V|\text{sell}].$$

Thus, $\exists\, \delta' > 0$ such that

$$\mathbb{E}[V|\text{sell}, b_m = b \text{ wins}] < \mathbb{E}[V|\text{sell}] - \delta', \quad \forall b \leq \mathbb{E}[V|\text{sell}].$$

We therefore obtain

$$\Pi_m(b|\text{sell}) = \mathbb{P}(b_m = b \text{ wins})\big(\mathbb{E}[V|\text{sell}, b_m = b \text{ wins}] - b\big) < 0, \quad \forall b \in \big[\mathbb{E}[V|\text{sell}] - \delta', \mathbb{E}[V|\text{sell}]\big],$$

giving $l_m \leq \mathbb{E}[V|\text{sell}] - \delta'$.

Step 5: $\big(A_1 \cup \overline{A}_1\big) \cap \big(A_2 \cup \overline{A}_2\big) = \emptyset$. That $\overline{A}_n \cap \big(A_m \cup \overline{A}_m\big) = \emptyset$ is trivial. Next, suppose by way of contradiction that we can find $b \in \big(A_1 \cap A_2\big) \setminus \big(\overline{A}_1 \cup \overline{A}_2\big)$. Then $b < \mathbb{E}[V|\text{sell}]$, by virtue of Step 4. Let $\Delta = \mathbb{E}[V|\text{sell}] - b$, and consider n such that $\mathbb{P}(\text{MM}n \text{ wins}|\text{tie at } b) < 1$. Notice that $\Pi_n(b + \varepsilon\Delta|\text{sell}, b_m = b) = \mathbb{E}[V|\text{sell}] - b - \varepsilon\Delta$ as (i) given $b_m = b$, $b_n = b + \varepsilon\Delta$ always wins, and (ii) conditional on $b_m = b$, MMm is uninformed with probability 1, from which $\mathbb{E}[V|\text{sell}, b_m = b] = \mathbb{E}[V|\text{sell}]$. Then,

$$\begin{aligned}
&\Pi_n(b + \varepsilon\Delta|\text{sell}) - \Pi_n(b|\text{sell}) \\
&= \mathbb{P}(b_m = b)\Big(\Pi_n(b + \varepsilon\Delta|\text{sell}, b_m = b) - \Pi_n(b|\text{sell}, b_m = b)\Big) \\
&+ (1 - \mathbb{P}(b_m = b))\Big(\Pi_n(b + \varepsilon\Delta|\text{sell}, b_m \neq b) - \Pi_n(b|\text{sell}, b_m \neq b)\Big) \\
&= \mathbb{P}(b_m = b)\Big(\mathbb{E}[V|\text{sell}] - b - \varepsilon\Delta - \mathbb{P}(\text{MM}n \text{ wins}|\text{tie at } b)\Delta\Big) \\
&+ (1 - \mathbb{P}(b_m = b))\Big(\Pi_n(b + \varepsilon\Delta|\text{sell}, b_m \neq b) - \Pi_n(b|\text{sell}, b_m \neq b)\Big) \\
&= \mathbb{P}(b_m = b)\Big((1 - \varepsilon)\Delta - \mathbb{P}(\text{MM}n \text{ wins}|\text{tie at } b)\Delta\Big) \\
&+ (1 - \mathbb{P}(b_m = b))\Big(\Pi_n(b + \varepsilon\Delta|\text{sell}, b_m \neq b) - \Pi_n(b|\text{sell}, b_m \neq b)\Big).
\end{aligned}$$

As $\lim_{\varepsilon\to 0}\Big(\Pi_n(b + \varepsilon\Delta|\text{sell}, b_m \neq b) - \Pi_n(b|\text{sell}, b_m \neq b)\Big) = 0$, we obtain

$$\lim_{\varepsilon\to 0}\Big(\Pi_n(b + \varepsilon\Delta|\text{sell}) - \Pi_n(b|\text{sell})\Big) = \Big(1 - \mathbb{P}(\text{MM}n \text{ wins}|\text{tie at } b)\Big)\Delta > 0,$$

contradicting $b \in A_n$.

Step 6: $\inf\left\{\Sigma_1 \cup \overline{\Sigma}_1\right\} = \inf\left\{\Sigma_2 \cup \overline{\Sigma}_2\right\}$. Assume $\max\{p_1, p_2\} < 1$ (other cases are similar), so that, by Step 2, $\inf\left\{\Sigma_1 \cup \overline{\Sigma}_1\right\} = \inf \Sigma_1$ and $\inf\left\{\Sigma_2 \cup \overline{\Sigma}_2\right\} = \inf \Sigma_2$. Suppose by way of contradiction that $b = \inf \Sigma_n > \inf \Sigma_m = b'$. Then $b \in \left(A_m \cup \overline{A}_m\right)$, otherwise we could find $\varepsilon > 0$ such that

$$\Pi_n(b - \varepsilon|\text{sell}) > \Pi_n(b + x|\text{sell}), \quad \forall x \in [0, \varepsilon],$$

contradicting $b = \inf \Sigma_n$, as then MMn could benefit from bidding below b. Applying Step 5 thus yields $b \notin \left(A_n \cup \overline{A}_n\right)$. Next, $b = \inf \Sigma_n$ together with $b \notin A_n$ implies $\sigma_n(b) = 0$. Therefore, using Steps 1 and 2, $b \in A_m$ would imply $\Pi_m(b|\text{sell}) = -\gamma p_n b < 0$, which cannot be. Similarly, $b \in \overline{A}_m$ would imply $\overline{\Pi}_m(b|\text{sell}) = 0$, which cannot be since, by virtue of Steps 2 and 3, $\max_b \overline{\Pi}_m(b|\text{sell}) > 0$.

Step 7: $0 \in \left(\Sigma_1 \cup \overline{\Sigma}_1\right) \cap \left(\Sigma_2 \cup \overline{\Sigma}_2\right)$. Assume $\max\{p_1, p_2\} < 1$ (other cases are similar). By Step 2, $\inf\left\{\Sigma_1 \cup \overline{\Sigma}_1\right\} = \inf \Sigma_1$ and $\inf\left\{\Sigma_2 \cup \overline{\Sigma}_2\right\} = \inf \Sigma_2$. Let b denote the common infinimum uncovered in Step 6, and suppose for a contradiction that $b > 0$. By Step 5, one of the MMs does not have an atom at b. Consider n such that $b \notin \left(A_n \cup \overline{A}_n\right)$. Then, by Step 1,

$$\lim_{\varepsilon \to 0} \Pi_m(b + \varepsilon|\text{sell}) = -\gamma p_n b < 0,$$

contradicting $b \in \Sigma_m$.

Step 8: $\Sigma_1 \cup \overline{\Sigma}_1 = \Sigma_2 \cup \overline{\Sigma}_2$. Suppose by way of contradiction that there exists $b' \in \left(\Sigma_n \cup \overline{\Sigma}_n\right) \setminus \left(\Sigma_m \cup \overline{\Sigma}_m\right)$, say $b' \in \Sigma_n \setminus \left(\Sigma_m \cup \overline{\Sigma}_m\right)$ (the other case is similar). Then $b' \in \arg\max_b \Pi_n(b|\text{sell})$. Moreover, by Step 7, $b' > 0$, and we can find $\delta > 0$ such that $[b' - \delta, b' + \delta] \cap \left(\Sigma_m \cup \overline{\Sigma}_m\right) = \emptyset$. Hence MM$n$ can lower his bid at b' without decreasing his winning probability, giving $\Pi_n(b' - \delta|\text{sell}) > \Pi_n(b'|\text{sell}) = \max_b \Pi_n(b|\text{sell})$, which cannot be.

Step 9: $\Sigma_1 \cup \overline{\Sigma}_1 = \Sigma_2 \cup \overline{\Sigma}_2 = [0, u]$. By Steps 2, 3, 7 and 8 all that remains to be shown is that the common support is an interval. Suppose by way of contradiction that this is not the case. Then we can find $b'' > b'$, both in the common support, and such that $(b', b'') \cap \left(\Sigma_1 \cup \overline{\Sigma}_1\right) = \emptyset$. By Step 5, there exists n such that $b'' \notin \left(A_n \cup \overline{A}_n\right)$. Hence, $\exists\, \varepsilon > 0$ such that, $\forall x \in [0, \varepsilon]$, $\Pi_m(b'' - \varepsilon|\text{sell}) > \Pi_m(b'' + x|\text{sell})$ and $\overline{\Pi}_m(b'' - \varepsilon|\text{sell}) > \overline{\Pi}_m(b'' + x|\text{sell})$, contradicting $b'' \in \left(\Sigma_m \cup \overline{\Sigma}_m\right)$.

Step 10: $p_n \in (0,1) \Rightarrow l_n < u$. Suppose by way of contradiction that $p_n \in (0,1)$ and $l_n = u$. Then, by Step 2, $u \in \overline{A}_n$. Assume $\mathbb{P}(\text{MM}m \text{ wins}|\text{tie at } u) < 1$ (the other case is similar). Then there exists $\varepsilon > 0$ such that

$$\overline{\Pi}_m(u + \epsilon|\text{sell}) > \overline{\Pi}_m(u - x|\text{sell}), \quad \forall x \in [0, \varepsilon],$$

contradicting $u \in \overline{\Sigma}_m$.

Step 11: $A_1 \cup \overline{A}_1 \cup A_2 \cup \overline{A}_2 \subseteq \{0\}$. Suppose by way of contradiction that there exists $b \in \left(A_m \cup \overline{A}_m\right)$, with $b > 0$. Then, by virtue of Step 4, $\exists\, \varepsilon > 0$ such that, $\forall x \in (0, \varepsilon)$, $\overline{\Pi}_n(b + x|\text{sell}) > \overline{\Pi}_n(b - x|\text{sell})$ and $\Pi_n(b + x|\text{sell}) > \Pi_n(b - x|\text{sell})$. Thus $(b - \varepsilon, b) \cap \left(\Sigma_n \cup \overline{\Sigma}_n\right) = \emptyset$, contradicting Step 9.

Step 12: $\max_b \overline{\Pi}_1(b|\text{sell}) = \max_b \overline{\Pi}_2(b|\text{sell})$. The combination of Steps 2, 3 and 11 shows that $\max_b \overline{\Pi}_1(b|\text{sell}) = \overline{\Pi}_1(u|\text{sell}) = (1 - u) = \overline{\Pi}_2(u|\text{sell}) = \max_b \overline{\Pi}_2(b|\text{sell})$.

Step 13: $0 < p_m < p_n < 1 \Rightarrow 0 < l_n < l_m$. Let $0 < p_m < p_n < 1$ and suppose by way of contradiction that $l_n \geq l_m$. Note first that $l_n > 0$, for otherwise $\{0\} \in A_n \cap A_m$, which Step 5 ruled out. Hence, by Step 11, neither MM has an atom at l_n. Steps 2 and 9 therefore yield $\max_b \overline{\Pi}_n(b|\text{sell}) = \overline{\Pi}_n(l_n|\text{sell})$ and $\max_b \overline{\Pi}_m(b|\text{sell}) = \overline{\Pi}_m(l_n|\text{sell})$. On the other hand, $\overline{\Pi}_n(l_n|\text{sell}) \geq (1 - p_m)(1 - l_n)$ and $\overline{\Pi}_m(l_n|\text{sell}) = (1 - p_n)(1 - l_n)$. As $p_n > p_m$, combining the previous remarks yields

$$\max_b \overline{\Pi}_n(b|\text{sell}) > \max_b \overline{\Pi}_m(b|\text{sell}),$$

contradicting Step 12. Therefore, $l_n < l_m$. We next show that $l_n > 0$. Suppose by way of contradiction that $l_n = 0$. Then $0 \in A_n$, and, applying Step 5, $0 \notin A_m$. We therefore obtain

$$\Pi_n(0|\text{sell}) = \max_b \Pi_n(b|\text{sell}) = 0 < \max_b \Pi_m(b|\text{sell}). \tag{C1}$$

Yet, as $l_m > 0$, Steps 9, 11 and 12 give

$$\begin{aligned}\Pi_n(l_m|\text{sell}) &= -\gamma l_m + (1-\gamma)\overline{\Pi}_n(l_m|\text{sell}) \\ &= -\gamma l_m + (1-\gamma)\overline{\Pi}_m(l_m|\text{sell}) \\ &= \Pi_m(l_m|\text{sell}),\end{aligned}$$

contradicting (C1).

<u>*Step 14:* $p_n > p_m \Rightarrow 0 = \Pi_m < \Pi_n < \overline{\Pi}_n = \overline{\Pi}_m$.</u> Assume $0 < p_m < p_n < 1$ (other cases are similar). By Step 12, $\overline{\Pi}_n = \overline{\Pi}_m$. Moreover, Steps 1,2, 9, 11 and 13 give

$$\Pi_n(l_n|\text{sell}) = -\gamma\big(p_m + (1-p_m)\sigma_m(l_n)\big)l_n + (1-\gamma)\overline{\Pi}_n(l_n|\text{sell}) < \overline{\Pi}_n(l_n|\text{sell}).$$

Hence $\Pi_n < \overline{\Pi}_n$ (by symmetry of the bid and ask sides of the market). We next show that $\Pi_n > \Pi_m$. Reasoning like we did above, and using Step 12 together with $l_n < l_m$,

$$\begin{aligned}\Pi_n(l_n|\text{sell}) &= -\gamma\big(p_m + (1-p_m)\sigma_m(l_n)\big)l_n + (1-\gamma)\overline{\Pi}_n(l_n|\text{sell}) \\ &> -\gamma l_m + (1-\gamma)\overline{\Pi}_n(l_n|\text{sell}) \\ &= -\gamma l_m + (1-\gamma)\overline{\Pi}_m(l_m|\text{sell}) \\ &= \Pi_m(l_m|\text{sell}).\end{aligned}$$

Hence, $\Pi_n > \Pi_m$. Lastly, we show that $\Pi_m = 0$. Suppose by way of contradiction that $\Pi_m > 0$. Then Steps 2 and 7 imply $0 \in A_n$. It ensues, using Step 5, that $0 \notin \big(A_m \cup \overline{A}_m\big)$. We thus obtain $\Pi_n = 0 < \Pi_m$, contradicting $\Pi_n > \Pi_m$. ■

Proposition C1. *For all p_1, p_2 and q, a trading equilibrium exists. Moreover, except for $p_1 = p_2 = q = 0$ and $p_1 = p_2 = 1$, any two trading equilibria induce the same strategies and differ at most by the tie-breaking rules they induce.*[1] *For all p_1, p_2 and q, $\overline{\Pi}_n$, Π_n, $\overline{\Pi}_S$ and Π_S are independent of the trading equilibrium considered.*

Proof: The cases $p_1 = p_2 = 0$ and $p_1 = p_2 = 1$ are trivial. We prove below the existence of a trading equilibrium and the uniqueness of the strategies for $0 < p_m \leq p_n < 1$ and $q > 0$ (other cases are similar).

[1] If $p_1 = p_2 = q = 0$ then the uninformed speculator is indifferent between trading and abstaining. If $p_1 = p_2 = 1$ then any type of the speculator is indifferent between trading and abstaining.

We start by showing that in any trading equilibrium the speculator sells (resp. buys) with probability 1 when she is informed and $V = 0$ (resp. $V = 1$), and abstains when she is uninformed. First note that, by Lemma C1, $\hat{b} < 1$ with probability 1. So selling the asset is a strictly dominated strategy of the informed speculator when $V = 1$. Similarly, buying the asset is a strictly dominated strategy of the informed speculator when $V = 0$. Next, Suppose by way of contradiction that the speculator abstains with positive probability when she is informed and $V = 0$ (the other case is similar, by symmetry). Then $\mathbb{P}(\hat{b} = 0|V = 0) = 1$, otherwise the speculator would have a profitable deviation. But then $\sigma_1(0) = \sigma_2(0) = 1$, contradicting Step 5 in the proof of Lemma C1. We conclude that the speculator sells (resp. buys) with probability 1 when she is informed and $V = 0$ (resp. $V = 1$). We now prove that the speculator abstains when she is uninformed. Applying Lemma C1 gives $u < 1$ and $\max\{l_1, l_2\} < \mathbb{E}[V|\text{sell}]$. As we showed above that the speculator buys (resp. sells) with probability 1 when she is informed and $V = 1$ (resp. $V = 0$), we obtain $\mathbb{E}[V|\text{sell}] < \frac{1}{2}$. The uninformed speculator's expected profit from selling the asset is therefore bounded above by

$$\mathbb{P}(\text{sell order executed by an uninformed MM})\left(\max\{l_1, l_2\} - \frac{1}{2}\right)$$
$$+ \mathbb{P}(\text{sell order executed by an informed MM})(u - 1) < 0.$$

By symmetry, the uninformed speculator's expected profit from buying the asset is negative as well.

We next derive equilibrium strategies of the MMs. Since we saw above that in any trading equilibrium the speculator trades if and only if she is informed, we obtain $\gamma = \mathbb{P}(V = 0|\text{sell}) = (\frac{\pi q}{2} + \frac{1-\pi}{4})/(\frac{\pi q}{2} + \frac{1-\pi}{2})$ in any trading equilibrium. Now, by virtue of Lemma C1, if the pricing strategies σ_m, σ_n, $\underline{\sigma}_m$, $\underline{\sigma}_n$, $\overline{\sigma}_m$ and $\overline{\sigma}_n$ are in equilibrium then $\underline{\sigma}_m(0) = \underline{\sigma}_n(0) = 1$ and there exist $0 < l_n \leq l_m < u < 1$ such that:

$$\big[(1 - p_m) + p_m\overline{\sigma}_m(x)\big](1 - x) = 1 - u, \qquad \forall x \in [l_m, u]; \tag{C2}$$

$$\big[(1 - p_n) + p_n\overline{\sigma}_n(x)\big](1 - x) = 1 - u, \qquad \forall x \in [l_m, u]; \tag{C3}$$

$$\overline{\sigma}_m(l_m) = 0; \tag{C4}$$

$$-\gamma l_m + (1 - \gamma)(1 - u) = 0; \tag{C5}$$

$$-\gamma x + (1 - \gamma)\big[(1 - p_n) + p_n\overline{\sigma}_n(x)\big](1 - x) = 0, \qquad \forall x \in [l_n, l_m]; \tag{C6}$$

$$(1-p_m)\sigma_m(x)(1-x) = 1-u, \qquad \forall x \in [l_n, l_m]; \tag{C7}$$

$$\overline{\sigma}_n(l_n) = 0; \tag{C8}$$

$$\begin{aligned} &-\gamma\big[p_m + (1-p_m)\sigma_m(x)\big]x + (1-\gamma)(1-p_m)\sigma_m(x)(1-x) \\ &\quad = -\gamma\big[p_m + (1-p_m)\sigma_m(l_n)\big]l_n + (1-\gamma)(1-p_m)\sigma_m(l_n)(1-l_n), \qquad \forall x \in [0, l_n]; \end{aligned} \tag{C9}$$

$$-\gamma\big[p_n + (1-p_n)\sigma_n(x)\big]x + (1-\gamma)(1-p_n)\sigma_n(x)(1-x) = 0, \qquad \forall x \in [0, l_n]. \tag{C10}$$

Equations (C2) and (C3) are the equiprofit conditions of, respectively, MMnH and MMmH in the bid range $[l_m, u]$; (C4) is obtained by definition of l_m; equation (C5) captures $\Pi_m(l_m|\text{sell}) = 0$; equations (C6) and (C7) are the equiprofit conditions of, respectively, MMmU and MMnH in the bid range $[l_n, l_m]$; (C8) is obtained by definition of l_n; lastly, equations (C9) and (C10) are the equiprofit conditions of, respectively, MMnU and MMmU in the bid range $[0, l_n]$. That the system of equations (C2)-(C10) uniquely determines pricing strategies σ_m, σ_n, $\overline{\sigma}_m$ and $\overline{\sigma}_n$ is straightforward to check.[2]

By construction the strategies above are in equilibrium if no MM can profitably bid outside the support of their respective strategies. Observe to begin with that no MM can profitably bid outside $[0, u]$. So we only need to check the remaining cases. To see that MMmU has no profitable deviation to $b \in (l_m, u]$ note that

$$\Pi_m(b|\text{sell}) = -\gamma b + (1-\gamma)\big[(1-p_n) + p_n\overline{\sigma}_n(b)\big](1-b), \qquad \forall b \in [l_m, u].$$

Hence, by (C3),

$$\Pi_m(b|\text{sell}) = -\gamma b + (1-\gamma)(1-u), \qquad \forall b \in [l_m, u].$$

The last highlighted equation gives $\Pi_m(b|\text{sell}) < \Pi_m(l_m|\text{sell})$, for all $b \in (l_m, u]$. Similarly, to see that MMmH has no profitable deviation to $b \in [l_n, l_m)$ note that, by (C6),

$$\overline{\Pi}_m(b|\text{sell}) = \big[(1-p_n) + p_n\overline{\sigma}_n(b)\big](1-b) = \frac{\gamma b}{1-\gamma}, \qquad \forall b \in [l_n, l_m].$$

Hence $\overline{\Pi}_m(b|\text{sell}) < \overline{\Pi}_m(l_m|\text{sell})$ for all $b \in [l_n, l_m)$. MMmH has no profitable deviation to

[2]Combining (C2) and (C4) pins down l_m in terms of u; (C5) then gives u and, therefore, l_m as well. Applying (C2) and (C3) now gives $\overline{\sigma}_m$ and $\overline{\sigma}_n$ over the interval $[l_m, u]$. Next, Combining (C6) and (C8) pins down l_n, while (C6) and (C7) then give $\overline{\sigma}_n$ and σ_m over the interval $[l_n, l_m]$. Finally (C9) and (C10) give σ_m and σ_n over the interval $[l_n, l_m]$.

$b \in [0, l_n]$ either, since, by (C10),

$$\overline{\Pi}_m(b|\text{sell}) = (1-p_n)\sigma_n(b)(1-b) = \frac{\gamma\big[p_n + (1-p_n)\sigma_n(b)\big]b}{1-\gamma}, \quad \forall b \in [0, l_n]. \tag{C11}$$

Therefore, $\overline{\Pi}_m(b|\text{sell}) \leq \overline{\Pi}_m(l_n|\text{sell})$ for all $b \in [0, l_n]$, which, combined with the previous remark, gives $\overline{\Pi}_m(b|\text{sell}) < \overline{\Pi}_m(l_m|\text{sell})$ for all $b \in [0, l_n]$. This finishes to show that neither MMmU nor MMmH can profitably bid outside the support of their respective strategies. Similar arguments establish that neither MMnU nor MMnH can profitably bid outside the support of their respective strategies.

Lastly, Step 5 in the proof of Lemma C1 shows that for a tie to occur with positive probability requires both MMs to be informed, and either $V = 0$ and a sell order or $V = 1$ and a buy order. So MMs' profits are zero conditional on a tie, irrespective of the tie-breaking rule. It follows that uniqueness of the strategies implies uniqueness of $\overline{\Pi}_n$, Π_n, $\overline{\Pi}_S$ and Π_S. ■

The following technical lemma is used in the proof of Theorem 2.

Lemma C2. *Let $p^*(c, \pi)$ be given by (A16) and*

$$H(c; \pi) := \overline{\Pi}_S\big(p^*(c, \pi), 0\big) - c.$$

Then $H(c; \pi) = 0$ has exactly one solution in the interval $c \in \big(0, \frac{1-\pi}{4}\big)$.

Proof: Consider any equilibrium of the trading game with $q = 0$ and a given, arbitrary, p. Define $\beta := \mathbb{E}[b_n|\text{MM}n \text{ is uninformed}]$ and $\hat{\beta} := \mathbb{E}[b_n|\text{both MMs uninformed}, b_n \geq b_m]$. By symmetry of the bid and ask sides of the market, we can write

$$\overline{\Pi}_S(p, 0) = 2p(1-p)\beta + (1-p)^2\hat{\beta} \tag{C12}$$

and

$$\Pi_n(p, 0) = \left(\frac{1-\pi}{2}\right)\left[\frac{1}{2}\left(p(0-\beta) + (1-p)\frac{1}{2}(0-\hat{\beta})\right) + \frac{1}{2}(1-p)\frac{1}{2}(1-\hat{\beta})\right].$$

Rearranging the last highlighted equation gives

$$\begin{aligned}\Pi_n(p,0) &= \frac{1}{4}(1-p) - \frac{1}{2}\left[p\beta + (1-p)\hat{\beta}\right] \\ &= \frac{1}{4}(1-p) - \frac{1}{2}\frac{\overline{\Pi}_S(p,0)}{1-p} + \frac{1}{2}p\beta.\end{aligned}$$

Hence, as $\Pi_n(p,0) = 0$,

$$\overline{\Pi}_S(p,0) = \frac{1}{2}(1-p)^2 + p(1-p)\beta. \tag{C13}$$

Next, using (A9) gives

$$\beta = \int b\sigma'(b)db = \frac{p\left(\frac{1}{1-2b} + \ln(\frac{1}{2} - b)\right)}{4(1-p)}. \tag{C14}$$

Finally, combining (A16), (C13) and (C14) yields, for all $c \in \left(0, \frac{1-\pi}{4}\right)$,

$$H(c;\pi) = \frac{1}{(1-\pi-2c)^2}\left[\frac{(1-\pi-4c)^2}{4}\ln\left(1 - \frac{4c}{1-\pi}\right) + c(1-\pi-2c)(2c+\pi)\right].$$

Let $G(c;\pi)$ denote the expression inside the square bracket. One verifies that:

(i) $G(0;\pi) = 0$;

(ii) $G(c;\pi) \to \frac{(1-\pi)^2(1+\pi)}{16}$ as $c \to \frac{1-\pi}{4}$;

(iii) $G'(0;\pi) < 0 < G''(0;\pi)$;

(iv) $G'''(c;\pi) < 0$ for all $c \in \left(0, \frac{1-\pi}{4}\right)$.

Therefore, $G(c;\pi) = 0$ has exactly one solution in the interval $c \in \left(0, \frac{1-\pi}{4}\right)$.[3] ■

[3] On the interval $(0, \frac{1-\pi}{4})$, the function G is first convex, then concave. The function starts below the horizontal axis, and ends above it. Suppose it crossed the horizontal axis twice. Then at the second crossing, the function has to be decreasing and concave. But this contradicts G ending above the horizontal axis.

Appendix D: Trading Game with Observable Quotes (for online publication)

In this appendix we analyze the trading game induced by the observable quotes model, with $z > 0$ denoting the probability with which the speculator gets to observe the quotes before placing her market order. Specifically, throughout this appendix p and q play the role of parameters: each MM acquires information with probability p, while the speculator acquires information with probability q. A strategy of MMn comprises cumulative distribution functions σ_n, $\underline{\sigma}_n$ and $\overline{\sigma}_n$ specifying respectively the distribution of the bid price b_n of MMnU, MMnL and MMnH. As the bid and ask sides of the market are symmetric we assume as usual that, conditional on MMnU, $1 - a_n$ is distributed like b_n. Similarly, we assume that the law of $1 - a_n$ conditional on MMnL (resp. MMnH) is the same as the law of b_n conditional on MMnH (resp. MMnL). A strategy of the speculator specifies her market order as a function of the information she possesses at that point. A WELM trading equilibrium is a perfect Bayesian equilibrium such that

(i) $\sigma_1 = \sigma_2 = \sigma$, $\underline{\sigma}_1 = \underline{\sigma}_2 = \underline{\sigma}$ and $\overline{\sigma}_1 = \overline{\sigma}_2 = \overline{\sigma}$;

(ii) $\underline{\sigma}(0) = 1$;

(iii) either $p \in \{0, 1\}$ or σ and $\overline{\sigma}$ are atomless, with supp $(\sigma) = [0, l]$ and supp $(\overline{\sigma}) = [l, u]$.

We focus throughout this appendix on $p \in (0, 1)$ and $q < 1$. The case $p = 0$ is almost identical. If $q = 1$, the observability of the quotes is inconsequential. The case $p = 1$ is straightforward: both MMs set prices equal to the realized asset value. Lastly, to shorten the exposition, we introduce the indicator variables I_S, I_n and Z respectively equal to 1 if and only if (a) the speculator acquires information, (b) MMn acquires information, (c) quotes are observable.

Proposition D1. *Assume $p \in (0, 1)$ and $q < 1$. In any WELM trading equilibrium:*

$$\overline{\Pi}_n(p, q) = \left(\frac{(1-\pi)(1-p)}{2}\right) \frac{1 - \pi(1-2q) + 2\pi p(1-q)z}{2 - p - 2\pi(1-q) + \pi p(1 + 2(1-q)z)}; \tag{D1}$$

$$\sigma(b) = \frac{(1 + \pi(2z-1) + 2\pi q(1-z))pb}{(1-p)(1-\pi-2b(1-\pi(1-q)))}, \qquad \forall b \in [0, l]; \tag{D2}$$

$$\overline{\sigma}(b) = \frac{2\overline{\Pi}_n(p, q) - (1-p)(1-\pi)(1-b)}{(1-b)(1-\pi)p}, \qquad \forall b \in [l, u]; \tag{D3}$$

$$l = \frac{(1-\pi)(1-p)}{2-p-2\pi(1-q)+\pi p(1+2(1-q)z)}; \tag{D4}$$

$$u = \frac{1-\pi-2\overline{\Pi}_n(p,q)}{1-\pi}. \tag{D5}$$

In particular, $\overline{\Pi}_n(p,q)$*,* l *and* u *given by, respectively,* (D1), (D4) *and* (D5) *satisfy* $\overline{\Pi}_n(p,q) > 0$ *and* $0 < l < u < 1$*.*

Proof: We start with a few preliminary remarks. Observe that WELM equilibria are separating equilibria. Hence, in any WELM equilibrium, $I_S \vee \Big((I_1 \vee I_2) \wedge Z\Big) = 1$ implies that, on the equilibrium path, the speculator learns the realization of V. In this case, by sequential rationality, the speculator buys if $V = 1$ and sells if $V = 0$.[4] On the equilibrium path sell orders are thus more likely conditional on $V = 0$ than they are conditional on $V = 1$, implying $l < \frac{1}{2}$.[5] Hence, on the equilibrium path, the speculator abstains whenever $I_S \vee \Big((I_1 \vee I_2) \wedge Z\Big) = 0$.[6]

Next, by definition of a WELM trading equilibrium, MMnU's expected profit on the bid side of the market has to be zero (the same being true of course on the ask side of the market).[7] As MMnU randomizes over $[0,l]$, we obtain

$$\begin{aligned} -\frac{1}{2}\Big[\pi\Big((1-p)q\sigma(b)+p(z+(1-z)q)\Big)+\left(\frac{1-\pi}{2}\right)\Big(p+(1-p)\sigma(b)\Big)\Big]b \\ +\frac{1}{2}\left(\frac{1-\pi}{2}\right)(1-p)\sigma(b)(1-b)=0, \qquad \forall b \in [0,l]. \end{aligned} \tag{D6}$$

The first term in equation (D6) can be decomposed as follows. With probability $\frac{1}{2}$ the asset value is $V = 0$, in which case a winning bid b induces a loss equal to b. With probability π the trader is a speculator. By the remarks made earlier in this proof the speculator sells if and only if one of the following 3 cases occurs: (i) $I_m = 0$ and $I_S = 1$, (ii) $I_m = 1$ and $Z = 1$, (iii) $I_m = 1$, $Z = 0$ and $I_S = 1$. In case (i) MMnU has the winning bid with probability $\sigma(b)$; in cases (ii) and (iii) MMnU has the winning bid with probability 1. With probability $\frac{1-\pi}{2}$ the

[4] We suppose here, without loss of generality, that the speculators always trades when she is indifferent between trading and abstaining.

[5] MMnU is subject to greater adverse selection than in the baseline case. As $l < \frac{1}{2}$ in the baseline model, $l < \frac{1}{2}$ with observable quotes as well.

[6] Observe that on the equilibrium path, if $I_S = I_1 = I_2 = 0$ and $Z = 1$ then the speculator's expected profit from trading the asset (either buying or selling) is at most $l - \frac{1}{2} < 0$. If instead $I_S = Z = 0$ then her expected profit from trading the asset is bounded above by $\mathbb{P}(\text{trade with an uninformed MM}|I_S \vee Z = 0)(l-\frac{1}{2}) + \mathbb{P}(\text{trade with an informed MM}|I_S \vee Z = 0)(u-1) < 0$.

[7] This must be since MMnU is indifferent between bids on the interval $[0,l]$, and the expected profit of $b_n = 0$ is zero due to the remark that, in any WELM trading equilibrium, $\mathbb{P}(b_n = 0 \text{ wins}|V = 1) = 0$.

trader is hit by the liquidity shock and sells the asset: either $I_m = 1$, in which case MMnU has the winning bid with probability 1, or $I_m = 0$, in which case MMnU has the winning bid with probability $\sigma(b)$. The second term in equation (D6) is decomposed as follows. With probability $\frac{1}{2}$ the asset value is $V = 1$, in which case a winning bid b induces a gain equal to $1 - b$. The probability of a sell order is the probability of a liquidity trader selling the asset, that is, $\frac{1-\pi}{2}$. Either $I_m = 1$, in which case MMnU has the losing bid, or $I_m = 0$, in which case MMnU has the winning bid with probability $\sigma(b)$.

As MMnH randomizes over $[l, u]$ we obtain in a similar way

$$\left(\frac{1-\pi}{2}\right)\left[p\overline{\sigma}(b) + (1-p)\right](1-b) = \left(\frac{1-\pi}{2}\right)(1-p)(1-l), \qquad \forall b \in [l, u]. \tag{D7}$$

We can now conclude the proof of the proposition. Rearranging (D6) yields (D2), from which solving $\sigma(l) = 1$ gives us (D4). Substituting (D4) into the right-hand side of (D7) and using the symmetry of the problem to write the resulting expression as $\overline{\Pi}_n(p, q)$ gives us (D1). Rearranging the terms in (D7) then yields (D3), from which solving $\overline{\sigma}(u) = 1$ yields (D5). To see that $l > 0$, substitute $b = l$ into (D6). Substituting $b = u$ into (D7) and using the fact that $l < \frac{1}{2}$ yields $u < 1$ and $\overline{\Pi}_n(p, q) > 0$. ■

Lemma D1. *Assume $p \in (0, 1)$ and $q < 1$. Let $\sigma(\cdot)$, $\overline{\sigma}(\cdot)$, l and u be defined by* (D2), (D3), (D4) *and* (D5), *respectively. Then*

$$\arg\max_{b\in[0,1]} \left(\frac{1-\pi}{2}\right)\big[p\overline{\sigma}(b) + (1-p)\sigma(b)\big](1-b) = [l, u], \tag{D8}$$

and

$$\begin{aligned}\arg\max_{b\in[0,1]} -\frac{1}{2}\Big[\pi\Big((1-p)q\sigma(b) + p(z + (1-z)q)\Big) + \left(\frac{1-\pi}{2}\right)\big(p + (1-p)\sigma(b)\big)\Big]b \\ + \frac{1}{2}\left(\frac{1-\pi}{2}\right)\big[p\overline{\sigma}(b) + (1-p)\sigma(b)\big](1-b) = [0, l].\end{aligned} \tag{D9}$$

The maximum values of (D8) *and* (D9) *are $\overline{\Pi}_n(p, q)$, given by* (D1), *and* 0, *respectively.*

Proof: By virtue of (D7),

$$\left(\frac{1-\pi}{2}\right)\big[p\overline{\sigma}(b) + (1-p)\sigma(b)\big](1-b) = \left(\frac{1-\pi}{2}\right)(1-u), \quad \forall b \in [l, u]. \tag{D10}$$

As $\overline{\sigma}(u) = \sigma(u) = 1$, notice that the left-hand side of (D10) is strictly decreasing in b for $b \geq u$. Next, rewriting (D6) as

$$-\frac{1}{2}\Big[\pi\Big((1-p)q\sigma(b) + p(z+(1-z)q)\Big) + \left(\frac{1-\pi}{2}\right)\big(p+(1-p)\sigma(b)\big)\Big]b$$
$$+\frac{1}{2}\left(\frac{1-\pi}{2}\right)\big[p\overline{\sigma}(b) + (1-p)\sigma(b)\big](1-b) = 0, \qquad \forall b \in [0,l],$$

gives

$$\left(\frac{1-\pi}{2}\right)\big[p\overline{\sigma}(b)+(1-p)\sigma(b)\big](1-b) = \Big[\pi\Big((1-p)q\sigma(b)+p(z+(1-z)q)\Big)+\frac{1-\pi}{2}\big(p+(1-p)\sigma(b)\big)\Big]b,$$

for all $b \in [0,l]$. The right-hand side of the last highlighted equation is strictly increasing in b. So combining the previous steps yields (D8). Finally, (D8) and the observation that the first term in the maximand of (D9) is a strictly decreasing function of b together yield (D9). ■

Proposition D2. *Assume $p \in (0,1)$ and $q < 1$. Let $\sigma(\cdot)$, $\overline{\sigma}(\cdot)$, l and u be defined by* (D2), (D3), (D4) *and* (D5), *respectively. Define*

$$h(b) := \frac{(1-u-b)(1-\pi) - 2b\pi q}{2b\pi(1-q)zp}, \tag{D11}$$

and suppose that

$$1 - \overline{\sigma}(b) \geq h(b), \qquad \forall b \in [l,u]. \tag{C}$$

Then a WELM trading equilibrium exists.

Proof: The following notation will be used throughout the proof. Let the cdfs σ and $\overline{\sigma}$ be defined by (D2) and (D3), respectively. Define also the cdf $\underline{\sigma}$ such that $\underline{\sigma}(0) = 1$. Let Γ denote the set of bid-ask price pairs (b_n, a_n) consistent with the strategies σ, $\underline{\sigma}$, and $\overline{\sigma}$, that is,

$$\Gamma = \Big(\{0\} \times [1-u, 1-l]\Big) \cup \Big([0,l] \times [1-l, 1]\Big) \cup \Big([l,u] \times \{1\}\Big).$$

Similarly, let Γ^+ denote the set of tuples (b_1, a_1, b_2, a_2) consistent with the strategies σ, $\underline{\sigma}$, and

$\overline{\sigma}$, that is,

$$\Gamma^+ = \Big\{(b_1, a_1, b_2, a_2) : \ (b_1, a_1) \in \Gamma, (b_2, a_2) \in \Gamma,$$
$$(b_n, a_n) \in [l, u] \times \{1\} \Rightarrow (b_m, a_m) \notin \{0\} \times [1-u, 1-l],$$
$$(b_n, a_n) \in \{0\} \times [1-u, 1-l] \Rightarrow (b_m, a_m) \notin [l, u] \times \{1\}\Big\}.$$

Let $\beta : \Gamma^+ \to \{0, \frac{1}{2}, 1\}$ represent the mapping from consistent tuples (b_1, a_1, b_2, a_2) to posterior beliefs that $V = 1$, computed through Bayes' rule. Let μ_n denote the speculator's belief that $V = 1$ based only on the quotes of MMn, with $\mu_n = \emptyset$ in case $(b_n, a_n) \notin \Gamma$.[8] Let μ denote the speculator's belief that $V = 1$ at the time she chooses her market order.

Assume the condition (C) holds with equality (the other case is similar). We aim to show that the following strategies, beliefs and tie-breaking rule comprise a trading equilibrium:

(I) $\sigma_1 = \sigma_2 = \sigma$;

(II) $\underline{\sigma}_1 = \underline{\sigma}_2 = \underline{\sigma}$;

(III) $\overline{\sigma}_1 = \overline{\sigma}_2 = \overline{\sigma}$;

(IV) if $I_S = 1$ then $\mu = v$;

(V) if $I_S \vee Z = 0$ then $\mu = \frac{1}{2}$;

(VI) if $I_S = 0$ and $Z = 1$ then:

$$\mu = \begin{cases} \beta(b_1, a_1, b_2, a_2) & \text{if } (b_1, a_1, b_2, a_2) \in \Gamma^+; & \text{(D12a)} \\ I_{\{1-a_m > b_n\}} & \text{if } \mu_n = 1 \text{ and } \mu_m = 0; & \text{(D12b)} \\ \mu_m & \text{if } \mu_m \in \{0, 1\} \text{ and } \mu_n = \emptyset; & \text{(D12c)} \\ \dfrac{a_n + b_n}{2} & \text{if } \mu_m = \dfrac{1}{2}, \mu_n = \emptyset, \text{ and } b_n < a_n; & \text{(D12d)} \\ 1 & \text{if } \mu_m = \dfrac{1}{2}, a_n \le b_n, \text{ and } a_n \neq \hat{a}; & \text{(D12e)} \\ 0 & \text{if } \mu_m = \dfrac{1}{2}, a_n \le b_n, a_n = \hat{a} \text{ but } b_n \neq \hat{b}; & \text{(D12f)} \\ \dfrac{a_n + b_n}{2} & \text{if } \mu_m = \dfrac{1}{2}, a_n \le b_n, a_n = \hat{a} \text{ and } b_n = \hat{b}. & \text{(D12g)} \end{cases}$$

[8]We use the terminology "speculator's belief that $V = 1$" for the probability which the speculator attaches to the event $V = 1$.

(VII) ties are broken uniformly at random, except if $\mu_m = \frac{1}{2}, a_n \leq b_n, a_n = \hat{a}$ and $b_n = \hat{b}$, in which case any tie is broken in favor of MMn;

(VIII) the speculator's market order satisfies sequential rationality with the additional requirement that if $I_S = 0$, $Z = 1$, $\mu_m = \frac{1}{2}, a_n \leq b_n, a_n = \hat{a}$ and $b_n = \hat{b}$ (in which case, by (D12g), $\mu = \frac{a_n+b_n}{2}$) then the speculator buys with probability $\frac{1}{2}$ and sells with probability $\frac{1}{2}$.

The proposed equilibrium has the following features. If the speculator acquires information her beliefs concerning V are determined by the realized value v, that is, even if the quotes suggest otherwise (see (IV)). If the speculator does not acquire information and quotes are unobservable then μ is equal to the prior belief that $V = 1$, that is, $\mu = \frac{1}{2}$ (see (V)). The case in which the speculator does not acquire information but gets to observe the quotes is subdivided into 7 cases. If the quotes are consistent with the proposed equilibrium strategies, then μ is derived using Bayes' rule (see (D12a)). If MMn's quotes signals $V = 1$ while MMm's quotes signals $V = 0$, that is, $(b_n, a_n) \in [l, u] \times \{1\}$ and $(b_m, a_m) \in \{0\} \times [1-u, 1-l]$, then $\mu = 1$ if $1 - a_m > b_n$ and $\mu = 0$ otherwise (see (D12b)). If MMn's quotes are inconsistent with the proposed equilibrium strategies but MMm's quotes signal that MMm is informed then the speculator ignores MMn and bases her beliefs exclusively on the quotes of MMm (see (D12c)). The case in which MMn's quotes are inconsistent with the proposed equilibrium strategies and MMm's quotes signal that MMm is uninformed are further subdivided into 4 cases. If MMn's quotes satisfy $b_n < a_n$ then $\mu = \frac{a_n+b_n}{2}$ (see (D12d)), in which case sequential rationality precludes trading between the speculator and MMn. If $a_n \leq b_n$ and MMn does not offer the best ask price then $\mu = 1$, (see (D12e)), in which case sequential rationality precludes trading between the speculator and MMn.[9] If $a_n \leq b_n$, MMn offers the best ask price but not the best bid price then $\mu = 0$, (see (D12f)), in which case sequential rationality precludes trading between the speculator and MMn. Lastly, if $a_n \leq b_n$ and MMn offers the best bid and ask prices then $\mu = \frac{a_n+b_n}{2}$ (see (D12g)), in which case the tie-breaking rule ensures that, conditional on placing a market order, the speculator trades with MMn (see (VII)).

Note that the proposed equilibrium satisfies the requirements of a WELM equilibrium; thus, repeating arguments used to prove Proposition D1, on the equilibrium path, the specu-

[9]Observe that $a_n \neq \hat{a}$ implies $\hat{a} < 1$. So, for $\mu = 1$, the speculator's expected profit from buying the asset is strictly positive. On the other hand, the speculator's expected profit from selling is at most 0. Sequential rationality therefore requires the speculator to buy.

lator learns the realization of V if $I_S \vee \Big((I_1 \vee I_2) \wedge Z\Big) = 1$. Moreover, since $l < \frac{1}{2}$, sequential rationality requires the speculator to abstain if $I_S \vee \Big((I_1 \vee I_2) \wedge Z\Big) = 0$.[10] It ensues that, on the equilibrium path, MMnU's expected profit on the bid side of the market can be written as the left-hand side of (D6). Similarly, on the equilibrium path, MMnH's expected profit on the bid side of the market can be written as the left-hand side of (D7). These remarks, Lemma D1 and the symmetry of the bid and ask sides of the market together establish that, on the equilibrium path: MMnU's expected profit is equal to 0, while MMnH's expected profit equals $\overline{\Pi}_n(p,q)$ given by (D1). We establish in the rest of the proof that neither MMnU nor MMnH have a profitable deviation (which, by symmetry, implies that MMnL does not have a profitable deviation either).

Step 1: there exists no profitable deviation of MMnU to $(a_n, b_n) \notin \Gamma$, with $b_n < a_n$.

Suppose MMnU deviates to $(\tilde{a}_n, \tilde{b}_n) \notin \Gamma$, with $\tilde{b}_n < \tilde{a}_n$. Observe first that in this case, applying (IV), (V), (D12c) and (D12d), the speculator trades with MMnU if and only if $I_S \vee (I_m \wedge Z) = 1$ (notice that if $I_S = I_m = 0$ while $Z = 1$ then (D12d) yields $b_n < \mu < a_n$). So the "demand" facing MMnU is the same as it is on the equilibrium path. In consequence, MMnU's expected profit on the bid side of the market can be written like the maximand of (D9), with $b = \tilde{b}_n$. Yet, by virtue of Lemma D1, the maximand of (D9) is maximized when MMnU sticks to the proposed equilibrium strategy. The symmetry between the bid and ask sides of the market finishes to establish that $(\tilde{a}_n, \tilde{b}_n)$ is not a profitable deviation of MMnU.

Step 2: there exists no profitable deviation of MMnU to $(a_n, b_n) \notin \Gamma$, with $a_n \leq b_n$.

Suppose MMnU deviates to $(\tilde{a}_n, \tilde{b}_n) \notin \Gamma$, with $\tilde{a}_n \leq \tilde{b}_n$. Now in this case, applying (IV), (V), (D12c), (D12e), (D12f), (D12g), (VII) and (VIII) the speculator trades with MMn if and only if either (a) $I_S \vee (I_m \wedge Z) = 1$ or (b) $I_S \vee I_m = 0$, $Z = 1$, $\tilde{a}_n = \hat{a}$ and $\tilde{b}_n = \hat{b}$. Moreover, in the latter event, (VIII) assures that the speculator buys with probability $\frac{1}{2}$ and sells with probability $\frac{1}{2}$. These remarks enable us to write the expected profit of MMnU as

[10] See the third footnote in the proof of Proposition D1.

$$\Big\{-\frac{1}{2}\Big[\pi\Big((1-p)q\sigma(\tilde b_n)+p(z{+}(1-z)q)\Big)+\frac{1-\pi}{2}\big(p+(1-p)\sigma(\tilde b_n)\big)\Big]\tilde b_n$$
$$+\frac{1}{2}\left(\frac{1-\pi}{2}\right)(1-p)\sigma(\tilde b_n)(1-\tilde b_n)\Big\}$$

$$+\Big\{-\frac{1}{2}\Big[\pi\Big((1-p)q\sigma(1-\tilde a_n)+p(z{+}(1-z)q)\Big)+\frac{1-\pi}{2}\big(p+(1-p)\sigma(1-\tilde a_n)\big)\Big](1-\tilde a_n)$$
$$+\frac{1}{2}\left(\frac{1-\pi}{2}\right)(1-p)\sigma(1-\tilde a_n)\tilde a_n\Big\}$$

$$+\left\{\pi(1-q)(1-p)z\sigma(\tilde b_n)\sigma(1-\tilde a_n)\left[\frac{1}{2}\left(-\frac{\tilde b_n}{2}+\frac{\tilde a_n}{2}\right)+\frac{1}{2}\left(\frac{1-\tilde b_n}{2}+\frac{\tilde a_n-1}{2}\right)\right]\right\},$$

where the first two curly brackets capture case (a) in the previous paragraph, and the last curly bracket captures case (b). Now, using Lemma D1, the term inside the first curly bracket is at most 0. By symmetry, the same remark applies to the second curly bracket. Finally, the third curly bracket is equal to $\pi(1-q)(1-p)z\sigma(\tilde b_n)\sigma(1-\tilde a_n)\left(\frac{\tilde a_n-\tilde b_n}{2}\right)$, which, since $\tilde a_n\le\tilde b_n$, is at most 0. So $(\tilde a_n,\tilde b_n)$ is not a profitable deviation of MMnU.

Step 3: there exists no profitable deviation of MMnU to $(a_n,b_n)\in\Gamma$.

Suppose MMnU deviates to $(\tilde a_n,\tilde b_n)\in\Gamma$, say $\tilde a_n=1$ and $\tilde b_n\in[l,u]$ (the other case is analogous, by symmetry). Consider first the ask side of the market: either $V=1$ or $\tilde a_n\neq\hat a$ with probability 1. So the expected profit of MMnU on the ask side of the market is at most 0. Next, consider the bid side of the market. By virtue of (IV), (V), (D12a) and (D12b) the speculator sells and trades with MMnU if and only if $V=0$ and:

- either $I_S=1$;
- or $I_m\wedge Z=1$ and $\tilde b_n\le 1-a_m$.

Thus MMnU's expected profit on the bid side of the market may be written as

$$-\frac{1}{2}\left[\pi\Big(q+(1-q)zp\mathbb{P}(\tilde b_n\le 1-a_m|\text{MM}m\text{L})\Big)+\frac{1-\pi}{2}\right]\tilde b_n$$
$$+\frac{1}{2}\left(\frac{1-\pi}{2}\right)[(1-p)+p\overline{\sigma}(\tilde b_n)](1-\tilde b_n).$$

Conditional on MMmL, $1 - a_m$ is distributed according to the cdf $\overline{\sigma}$. Hence, using condition (C), $\mathbb{P}(\tilde{b}_n \leq 1 - a_m | \text{MM}m\text{L}) = 1 - \overline{\sigma}(\tilde{b}_n) \geq h(\tilde{b}_n)$. Substituting this inequality into the last highlighted expression shows that MMnU's expected profit on the bid side of the market is bounded above by

$$-\frac{1}{2}\left[\pi\big(q + (1-q)zph(\tilde{b}_n)\big) + \frac{1-\pi}{2}\right]\tilde{b}_n + \frac{1}{2}\left(\frac{1-\pi}{2}\right)[(1-p) + p\overline{\sigma}(\tilde{b}_n)](1-\tilde{b}_n). \quad \text{(D13)}$$

By (D8), we can rewrite (D13) as

$$-\frac{1}{2}\left[\pi\big(q + (1-q)zph(\tilde{b}_n)\big) + \frac{1-\pi}{2}\right]\tilde{b}_n + \frac{(1-\pi)(1-u)}{4},$$

which, by definition of $h(\tilde{b}_n)$, is equal to 0. So $(\tilde{a}_n, \tilde{b}_n)$ is not a profitable deviation of MMnU.

Step 4: there exists no profitable deviation of MMnH to $(a_n, b_n) \notin \Gamma$, with $b_n < a_n$.

Suppose MMnH deviates to $(\tilde{a}_n, \tilde{b}_n) \notin \Gamma$, with $\tilde{b}_n < \tilde{a}_n$. Note to start with that MMnH's expected profit on the ask side of the market has to be non-positive. Consider next the bid side of the market. Observe that by (IV), (V), (D12c) and (D12d), the speculator never sells to MMnH. Hence, the "demand" facing MMnU is the same as it is on the equilibrium path. In consequence, MMnH's expected profit on the bid side of the market can be written like the maximand of (D8), with $b = \tilde{b}_n$. Yet, by virtue of Lemma D1, the maximand of (D8) is maximized when MMnH sticks to the proposed equilibrium strategy. So $(\tilde{a}_n, \tilde{b}_n)$ is not a profitable deviation of MMnH.

Step 5: there exists no profitable deviation of MMnH to $(a_n, b_n) \notin \Gamma$, with $a_n \leq b_n$.

Suppose MMnH deviates to $(\tilde{a}_n, \tilde{b}_n) \notin \Gamma$, with $\tilde{a}_n \leq \tilde{b}_n$. We start by showing that MMnH cannot make positive expected profit against the speculator. First, by virtue of (IV) and (D12c), if $I_S \vee (I_m \wedge Z) = 1$ then the speculator never sells. Furthermore, it is impossible to make profit against the speculator if she buys, since $V = 1$ and $\tilde{a}_n \leq 1$. Hence, conditional on $I_S \vee (I_m \wedge Z) = 1$, MM$n$H makes at most zero profit against the speculator. Next, by virtue of (V), (D12e), (D12f) and (D12g), if $I_S \vee I_m = 0$ then the only case in which MMnH trades with the speculator is if $Z = 1$, $\tilde{a}_n = \hat{a}$ and $\tilde{b}_n = \hat{a}$. Furthermore, in that case, by (VIII) the speculator buys and sells the asset with probabilities $\frac{1}{2}$ each. Applying (VII), the expected profit made by MMnH against the speculator is then

$$\frac{1}{2}(1-\tilde{b}_n)+\frac{1}{2}(\tilde{a}_n-1)=\frac{\tilde{a}_n-\tilde{b}_n}{2}.$$

Yet $\tilde{a}_n \leq \tilde{b}_n$. Thus MM$n$H makes at most zero expected profit against the speculator. The expected profit of MMnH is then bounded above by the expected profit made against the liquidity trader, which we can write as $\frac{1-\pi}{2}\left[p\overline{\sigma}(\tilde{b}_n)+(1-p)\sigma(\tilde{b}_n)\right](1-\tilde{b}_n)+\frac{1-\pi}{2}\mathbb{P}(\tilde{a}_n=\hat{a})(\tilde{a}_n-1)$. Since the second term is non-positive, the former expression is at most equal to $\frac{1-\pi}{2}\left[p\overline{\sigma}(\tilde{b}_n)+(1-p)\sigma(\tilde{b}_n)\right](1-\tilde{b}_n)$, which by (D8) is at most equal to MMnH's expected profit in the proposed equilibrium. So $(\tilde{a}_n,\tilde{b}_n)$ is not a profitable deviation of MMnH.

Step 6: there exists no profitable deviation of MMnH to $(a_n,b_n)\in\Gamma$.

There are two possible cases. MMnH could deviate to masquerade as MMnL or MMnH could deviate to masquerade as MMnU. Suppose MMnH deviates to masquerade as MMnL. Then $b_n=0<\hat{b}$ with probability 1. So the expected profit of MMnH on the bid side of the market is 0. On the other hand, since $V=1$, the profit of MMnH on the ask side of the market is bounded above by 0. Since sticking to his proposed equilibrium strategy yields MMnH an expected profit of $\overline{\Pi}(p,q)>0$, deviating to masquerade as MMnL is therefore not a profitable deviation. Next, suppose MMnH deviates to masquerade as MMnU. Reasoning as above, the expected profit of MMnH on the ask side of the market is bounded above by 0. Consider now the bid side of the market, with $b_n=\tilde{b}_n\in[0,l]$. Since $V=1$, we deduce from (IV), (V) and (D12a) that the speculator never sells. MMnH's expected profit on the bid side of the market can thus be written as $\left(\frac{1-\pi}{2}\right)\left[p\overline{\sigma}(\tilde{b}_n)+(1-p)\sigma(\tilde{b}_n)\right](1-\tilde{b}_n)$, which, applying Lemma D1, is bounded above by MMnH's expected profit on the bid side of the market in the proposed equilibrium. So deviating to masquerade as MMnU is not a profitable deviation either. ∎

Lemma D2. *Assume $p\in(0,1)$ and $q<1$. Let $\overline{\Pi}_n(p,q)$, l, u and $h(\cdot)$ be defined by* (D1)*,* (D4)*,* (D5)*, and* (D11) *respectively. Then:*

(i) for all $\varepsilon>0$, $p>1-\frac{2\varepsilon}{1-\pi}$ implies $\overline{\Pi}_n(p,q)<\varepsilon$;

(ii) for all $\delta>0$, $p>1-\delta$ implies $l<\delta$ and $h(b)<0$ for all $b\in[\delta,u]$;

(iii) $1-u>\frac{1-p}{2}$.

Proof: By Lemma D1,

$$\overline{\Pi}_n(p,q) = \left(\frac{1-\pi}{2}\right)(1-u) = \left(\frac{1-\pi}{2}\right)(1-p)(1-l).$$

Hence, $1-p < \frac{2\varepsilon}{1-\pi}$ implies $\overline{\Pi}_n(p,q) < \varepsilon$, giving part (i) of the lemma. Part (iii) follows from the remark that $l < \frac{1}{2}$.

We now show part (ii) of the lemma. The denominator on the right-hand side of (D4) is minimized at $q = 0$ and $z = 0$, with minimum value $(2-p)(1-\pi) > 1-\pi$. Hence,

$$l \leq \frac{(1-\pi)(1-p)}{1-\pi} = 1-p.$$

Pick a $\delta > 0$. Then, $p > 1-\delta$ implies $l < \delta$. We next show that choosing $p > 1-\delta$ also implies $h(b) < 0$ for all $b \in [\delta, u]$. First, rearranging (D11) gives

$$-\frac{1}{2}\left[\pi\big(q + (1-q)zph(b)\big) + \frac{1-\pi}{2}\right] b + \frac{(1-\pi)(1-u)}{4} = 0,$$

which, by Lemma D1, we can rewrite as

$$\left[\pi\big(q + (1-q)zph(b)\big) + \frac{1-\pi}{2}\right] b = \overline{\Pi}_n(p,q).$$

Solving for $h(b)$ gives

$$h(b) = \frac{2\overline{\Pi}_n(p,q) - b(1-\pi) - 2b\pi q}{2bpz\pi(1-q)}.$$

In particular,

$$h(b) \leq \frac{1}{2bpz\pi(1-q)}\left[2\overline{\Pi}_n(p,q) - b\,(1-\pi)\right], \qquad \forall b \in [l,u]. \tag{D14}$$

Now let $\varepsilon := \frac{\delta(1-\pi)}{2}$. By part (i) of the lemma, $p > 1 - \frac{2\varepsilon}{1-\pi}$ implies $\overline{\Pi}_n(p,q) < \varepsilon$, so, $p > 1-\delta$ implies $\overline{\Pi}_n(p,q) < \varepsilon$. Finally, using (D14), $p > 1-\delta$ implies

$$h(b) < \frac{1}{2bpz\pi(1-q)}\left[2\varepsilon - \delta\,(1-\pi)\right] = 0, \qquad \forall b \in [\delta, u].$$

$\blacksquare$

Proposition D3. *There exists a function $\overline{z}(\cdot) > 0$, independent of q, such that a WELM trading equilibrium exists whenever $z \leq \overline{z}(p)$. Moreover, $\overline{z}(p) = 1$ for $p = 0$ and all $p \geq \frac{\sqrt{2\pi}}{\sqrt{2\pi}+\sqrt{1-\pi}}$. If $q = 1$, a WELM trading equilibrium exists for all values of p and z.*

Proof: We remarked at the beginning of this appendix that if $q = 1$ or $p = 1$ (or both) the existence of a WELM trading equilibrium then follows from the existence of a trading equilibrium in the baseline model. That $\overline{z}(p) = 1$ for $p = 0$ is easy to show. We assume in the rest of the proof that $p \in (0, 1)$ and $q < 1$.

Step 1: there exists $\overline{z}(p) > 0$, independent of q, such that $z \leq \overline{z}(p)$ implies that a WELM trading equilibrium exists.

Define, for all $b \in [l, u]$, $D(b) := 1 - \overline{\sigma}(b) - h(b)$, where $\overline{\sigma}(\cdot)$, l, u and $h(\cdot)$ are defined respectively by (D3), (D4), (D5) and (D11). Thus,

$$D(b) = \frac{(1-\pi)(1-b) - 2\overline{\Pi}_n(p,q)}{(1-\pi)(1-b)p} - \frac{2\overline{\Pi}_n(p,q) - b(1-\pi) - 2b\pi q}{2bpz\pi(1-q)}, \qquad \forall b \in [l, u], \tag{D15}$$

with $\overline{\Pi}_n(p, q)$ given by (D1). By Proposition D2, it suffices for our purpose to show the existence of $\overline{z}(p) > 0$, independent of q, such that $z \leq \overline{z}(p)$ implies $D(b) \geq 0$ for all $b \in [l, u]$. First, straightforward algebra establishes that $h(l) = 1$ and $\overline{\sigma}(l) = 0$. Hence,

$$D(l) = 0. \tag{D16}$$

Next, differentiating (D15) gives

$$D'(b) = \frac{\overline{\Pi}_n(p,q)}{p}\left(\frac{1}{b^2\pi(1-q)z} - \frac{2}{(1-b)^2(1-\pi)}\right), \qquad \forall b \in [l, u]. \tag{D17}$$

The bracketed expression on the right-hand side of (D17) is decreasing in b and increasing in q, so

$$D'(b) \geq \frac{\overline{\Pi}_n(p,q)}{p}\left[\frac{1}{u^2\pi z} - \frac{2}{(1-u)^2(1-\pi)}\right], \qquad \forall b \in [l, u].$$

We showed in Lemma D2 that $1-u > \frac{1-p}{2}$, so the last inequality implies

$$D'(b) \geq \frac{\overline{\Pi}_n(p,q)}{p}\left[\frac{1}{\pi z} - \frac{8}{(1-p)^2(1-\pi)}\right], \quad \forall b \in [l,u]. \tag{D18}$$

The expression inside the square bracket is independent of q, and tends to $+\infty$ as z tends to 0. Hence, there exists $\overline{z}(p) > 0$, independent of q, such that $z \leq \overline{z}(p)$ implies $D'(b) \geq 0$ for all $b \in [l,u]$. Since $D(l) = 0$, we obtain $D(b) \geq 0$ for all $b \in [l,u]$ whenever $z \leq \overline{z}(p)$.

Step 2: $\overline{z}(p) = 1$ for all $p \geq \frac{\sqrt{2}}{\sqrt{2}+\sqrt{1-\pi}}$

By virtue of (D17),

$$D'(b) \geq \frac{\overline{\Pi}_n(p,q)}{p}\left[\frac{1}{b^2\pi} - \frac{2}{(1-b)^2(1-\pi)}\right], \quad \forall b \in [l,u]. \tag{D19}$$

Define

$$\delta := \frac{\sqrt{1-\pi}}{\sqrt{2\pi}+\sqrt{1-\pi}}.$$

Thus,

$$\frac{1}{\delta^2\pi} = \frac{2}{(1-\delta)^2(1-\pi)},$$

and, using (D19), $D'(b) \geq 0$ for all $b \leq \delta$. By Lemma D2, $p > 1-\delta$ implies $l < \delta$. So $p > 1-\delta$ implies $D'(b) \geq 0$ for all $b \in [l,\delta]$. By (D16), $p > 1-\delta$ therefore implies $D(b) \geq 0$ for all $b \in [l,\delta]$. Yet, by Lemma D2, $p > 1-\delta$ also implies $h(b) < 0$ for all $b \in [\delta,u]$. So $p > 1-\delta$ implies $D(b) \geq 0$ for all $b \in [l,u]$. ∎

Proposition D4. *A WELM trading equilibrium exists for all values of p and q if $z \leq \frac{(1-\pi)^2}{8\pi(\sqrt{2\pi}+\sqrt{1-\pi})^2}$. In particular, for $(1-\pi)^2 \geq 8\pi\left(\sqrt{2\pi}+\sqrt{1-\pi}\right)^2$, a WELM trading equilibrium exists for all values of p, q and z.*

Proof: Recall: if $q = 1$ or $p \in \{0,1\}$ (or both) the existence of a WELM trading equilibrium then follows from the existence of a trading equilibrium in the baseline model. We therefore assume in the rest of the proof that $p \in (0,1)$ and $q < 1$.

Next, assume $z \leq \frac{(1-\pi)^2}{8\pi(\sqrt{2\pi}+\sqrt{1-\pi})^2}$. By Proposition D3, a WELM trading equilibrium exists whenever $p \geq \tilde{p} := \frac{\sqrt{2\pi}}{\sqrt{2\pi}+\sqrt{1-\pi}}$. Next, let, as in the proof of Proposition D3, $D(b) := 1 - \overline{\sigma}(b) -$

$h(b)$. By (D18), $p < \tilde{p}$ implies

$$D'(b) \geq \frac{\overline{\Pi}_n(p,q)}{p}\left[\frac{1}{\pi z} - \frac{8}{(1-\tilde{p})^2(1-\pi)}\right], \qquad \forall b \in [l,u].$$

Yet,

$$\frac{1}{\pi z} - \frac{8}{(1-\tilde{p})^2(1-\pi)} \geq 0 \Longleftrightarrow z \leq \frac{(1-\pi)^2}{8\pi(\sqrt{2\pi}+\sqrt{1-\pi})^2}.$$

Thus, $p < \tilde{p}$ implies $D'(b) \geq 0$ for all $b \in [l,u]$. Since $D(l) = 0$, we obtain $D(b) \geq 0$ for all $b \in [l,u]$ whenever $p < \tilde{p}$. By Proposition D2, a WELM trading equilibrium therefore exists for all $p < \tilde{p}$. ■