Online Appendix to "Dynamic Persuasion with Outside Information" by Jacopo Bizzotto, Jesper Rüdiger and Adrien Vigier

Proof of Lemma 5: If $a_{T-1}=b_{T-1}=b_T$, the claim of the lemma is straightforward.[2] Assume now $b_{T-1}>b_T$. At $q_{T-1}=b_{T-1}$, in equilibrium the agent is indifferent between waiting and accepting. The agent's expected payoff from accepting is $b_{T-1}V_G+(1-b_{T-1})V_B$. On the other hand, using Lemmata 1 and B.1, the agent's expected payoff from waiting can be written as $\delta[b_{T-1}V_G+(1-b_{T-1})(\gamma V_R+(1-\gamma)V_B)]$. So b_{T-1} is the unique solution of

$$xV_G+(1-x)V_B=\delta[xV_G+(1-x)(\gamma V_R+(1-\gamma)V_B)]. \tag{B.1}$$

Next, consider $t<T-1$ such that $b_{t+1}=b_{T-1}$. Suppose $q_t=b_t$, so that, by definition, in equilibrium the agent is indifferent between waiting and accepting. The agent's expected payoff from accepting is $b_tV_G+(1-b_t)V_B$. On the other hand, using Lemma B.1, $q_t=b_t\geq b_{T-1}=b_{t+1}$. Hence, conditional on $s_t=g$, the agent optimally accepts in the next period. It ensues that b_t solves (B.1) and, therefore, that $b_t=b_{T-1}$. A recursive argument then yields $b_t=b_{T-1}$ for all $t<T$. ■

Proof of Lemma 6: Suppose that in equilibrium the principal is aggressive in period $1<t+1<T$. If $a_t=b_t$ the statement of the lemma is straightforward. Assume therefore that $a_t<b_t$. By virtue of Lemma B.2, in order to establish that the principal is also aggressive in period t it is enough to show that, at $p_t=a_t$, the principal strictly prefers the experiment $M_t=\{0,b_t\}$ over the uninformative experiment. On one hand, the principal's expected payoff from designing $M_t=\{0,b_t\}$ is $\frac{a_t}{b_t}$. On the other hand, her expected payoff from designing the uninformative experiment is given by $\delta\mathbb{E}_{s_t}[\hat{f}_{t+1}(p_{t+1})\,|\,q_t=a_t]$. The next sequence of inequalities therefore concludes the proof:

$$\delta\mathbb{E}_{s_t}[\hat{f}_{t+1}(p_{t+1})\,|\,q_t=a_t]\leq\delta\hat{f}_{t+1}(a_t)=\delta\frac{a_t}{b_{t+1}}<\frac{a_t}{b_t}.$$

The first inequality follows from noting that $\hat{f}_{t+1}$ is concave; the equality follows from the assumption that the principal is aggressive in period $t+1$, and the second inequality is due to Lemma 5. ■

[2] $a_{T\ 1}=b_{T\ 1}=b_T=\underline{b}$ implies $a_{t\ 1}=b_{t\ 1}=b_t$ whenever $a_t=b_t=\underline{b}$. Hence, a recursive argument yields the result in this case.

period $T-1$. In particular, since $\tilde{\gamma}(\delta_A) > 1$ for δ_A sufficiently small, we find that for δ_A small enough the principal is aggressive in period $T-1$ irrespective of γ and of δ_P.

Suppose next that $\gamma > \tilde{\gamma}(\delta_A)$. Then for $q_{T-1} = a_{T-1}$ in equilibrium the agent is indifferent between waiting and rejection. The agent's expected payoff from rejection is given by V_R. His expected payoff from waiting is on the other hand given by $\delta_A[a_{T-1}V_G + (1-a_{T-1})(\gamma V_R + (1-\gamma)V_B)]$, where we deduced from Lemma E.1 that $s_{T-1} = g$ induces $p_T > b_T = \underline{b}$. We therefore obtain $V_R = \delta_A[a_{T-1}V_G + (1-a_{T-1})(\gamma V_R + (1-\gamma)V_B)]$, giving

$$a_{T-1} = \frac{V_R - \delta_A(\gamma V_R + (1-\gamma)V_B)}{\delta_A(V_G - \gamma V_R - (1-\gamma)V_B)}. \tag{E.6}$$

Now, using Lemma E.5, the necessary and sufficient condition for the principal not to be aggressive in period $T-1$ in equilibrium is $f_{T-1}(a_{T-1}) \geq \frac{a_{T-1}}{b_{T-1}}$.[4] Noting that $f_{T-1}(a_{T-1}) = \delta_P\left[a_{T-1} + (1-\gamma)(1-a_{T-1})\right]$ and substituting for a_{T-1} and b_{T-1} using (E.5) and (E.6), the former inequality becomes

$$\begin{aligned} &V_B\delta_A(\gamma-1) + \delta_A V_G(1-\gamma) + V_R\gamma(1-\delta_A) \\ &\qquad \geq \frac{[V_B\delta_A(\gamma-1) + V_R(1-\delta_A\gamma)]\,[\delta_A\gamma(V_B - V_R) + (V_B - V_G)(1-\delta_A)]}{\delta_P\left[(V_B - V_R)\delta_A\gamma + V_B(1-\delta_A)\right]}. \end{aligned} \tag{E.7}$$

One checks that if (E.7) holds for some δ'_P, it must hold for $\delta''_P > \delta'_P$: either the right-hand side is positive, and therefore decreasing in δ_P, or it is negative, but the left-hand side is always positive,[5] so in this case the inequality is always satisfied. Moreover, for $\delta_A = 1$ the quadratic equation in γ obtained from (E.7) has roots $\gamma = 0$ and $\gamma = 1$. On the other hand, for $\delta_A < 1$, (E.7) is violated whenever either $\gamma = 1$, or $\gamma = \tilde{\gamma}(\delta_A)$. So (E.7) holds for all values of γ in between the roots of the corresponding quadratic equation. Letting $\underline{\gamma}(\delta_A, \delta_P)$ and $\overline{\gamma}(\delta_A, \delta_P)$ denote the real roots, the previous remarks yield $\tilde{\gamma}(\delta_A, \delta_P) < \underline{\gamma}(\delta_A, \delta_P) \leq \overline{\gamma}(\delta_A, \delta_P) < 1$ and show that these roots only exist for $\delta_A > \underline{\delta}_A$ and $\delta_P > \underline{\delta}_P(\delta_A)$, where (i) $\underline{\delta}_A$ is defined implicitly by $\underline{\gamma}(\underline{\delta}_A, 1) = \overline{\gamma}(\underline{\delta}_A, 1)$ and (ii) $\underline{\delta}_P(\delta_A)$ is defined implicitly for $\delta_A > \underline{\delta}_A$ by $\underline{\gamma}(\underline{\delta}_P(\delta_A), \delta_A) = \overline{\gamma}(\underline{\delta}_P(\delta_A), \delta_A)$. Noting that, by Lemma E.11, whenever the principal is not aggressive she is conservative concludes the proof. ■

Lemmata E.1 to E.12 together with Proposition E.1 now prove Proposition 3 following the same steps as the proof of Theorem 1.

[4]That is, at $p_{T\ 1} = a_{T\ 1}$ the principal must prefer the uninformative experiment over $M_{T\ 1} = \{0, b_{T\ 1}\}$.

[5]Since $V_G > V_G$ and $V_R > 0$ imply $V_B\delta_A(\gamma-1) + \delta_A V_G(1-\gamma) + V_R\gamma(1-\delta_A) > \delta_A(1-\gamma)(V_G - V_B) > 0$.