Equivalently, $Q_n^*(\beta)$, the objective function of the maximum rank correlation estimator multiplied by $\{\binom{n}{2}\}/n^2$, can be written as

$$Q_n^*(\beta) = \int q_\beta(z,\eta;x,\varepsilon)\, dP_n(z,\eta)\, dP_n(x,\varepsilon)$$

where

$$q_\beta(z,\eta;x,\varepsilon) = 1_{\{\varepsilon\le x'\beta_0\}} 1_{\{\eta>z'\beta_0\}} 1_{\{x'\beta>z'\beta\}} + 1_{\{\varepsilon>x'\beta_0\}} 1_{\{\eta\le z'\beta_0\}} 1_{\{x'\beta<z'\beta\}}.$$

The difference between $R_n(\beta)$ and $Q_n^*(\beta)$ is given by

$$R_n(\beta) - Q_n^*(\beta) = \int 1_{\{\varepsilon\le x'\beta_0\}} 1_{\{\eta\le z'\beta_0\}} 1_{\{x'\beta\ge z'\beta\}} + 1_{\{\varepsilon>x'\beta_0\}} 1_{\{\eta>z'\beta_0\}} 1_{\{x'\beta<z'\beta\}}\, dP_n(z,\eta)\, dP_n(x,\varepsilon).$$

Define n_1 as the number of Y_i's equal to one and n_0 as the number of Y_i's equal to zero, then this difference is equal to

$$R_n(\beta) - Q_n^*(\beta) = \{n_1(n_1+1)/2 + n_0(n_0-1)/2\}/n^2,$$

that is, *independent* of β. So the difference remains constant for varying β, and consequently the maximum of both functions will be obtained for the same value of β.

93.2.2. *Deriving Restricted Least Squares without a Lagrangean*—Solutions.[1] The following solutions have been proposed independently by Farshid Vahid, Luis J. Alvarez and Juan J. Dolado, Paolo Paruolo (the poser of the problem), and John Xu Zheng. These solutions are based on different types of interesting arguments.

1. Solution—proposed by Farshid Vahid. The restricted OLS estimator should satisfy the r equations given by the restrictions $R'\beta = q$ together with $(k-r)$ normal equations arising from the fact that there are no restrictions in the $(k-r)$ directions orthogonal to the range of R.

We know that the orthogonal complement of the range of R is the nullspace of R', so if we let Q be a $k \times (k-r)$ matrix whose columns form a basis for the nullspace of R', i.e., $rank(Q) = k - r$ and $R'Q = 0$, then OLS in the unrestricted directions implies the following $(k-r)$ normal equation:

$$Q'X'\tilde{e} = 0,$$

where $\tilde{e}$ is the restricted OLS residuals. These normal equations together with the r restrictions form k equations in k unknowns and the result follows.

To obtain the specific formula asked in the question, it helps to rewrite the system of equations as follows ($\hat{e}$ denotes the unrestricted residuals and b_ϵ and b denote the restricted and unrestricted OLS estimators respectively):

$$Q'X'\tilde{e} = 0 \Rightarrow Q'X'\hat{e} - Q'X'\tilde{e} = 0 \Rightarrow Q'X'X(b_\epsilon - b) = 0$$

and

$$R'b_\epsilon = q \Rightarrow R'(b_\epsilon - b) = q - R'b$$

or

$$\begin{pmatrix} Q'X'X \\ R' \end{pmatrix}(b_\epsilon - b) = \begin{pmatrix} 0 \\ q - R'b \end{pmatrix}.$$

Matrix $P \doteq (Q'X'X \quad R')$ has full rank (remember Q is the orthogonal complement of R) and therefore is invertible. If we partition the inverse $P^{-1} = [M|N]$, where M is $k \times (k - r)$ and N is $k \times r$, then we will have $b_\epsilon - b = N(q - R'b)$. So we only need to determine N, which can be easily done using

$$\begin{aligned} P^{-1}P = I_k &\Rightarrow MQ'X'X + NR' = I_k \Rightarrow MQ' + NR'(X'X)^{-1} = (X'X)^{-1} \\ &\Rightarrow NR'(X'X)^{-1}R = (X'X)^{-1}R (\text{since } Q'R = 0) \\ &\Rightarrow N = (X'X)^{-1}R(R'(X'X)^{-1}R)^{-1}. \end{aligned}$$

Therefore, $b_\epsilon = b + (X'X)^{-1}R(R'(X'X)^{-1}R)^{-1}(q - R'b)$ ■

A similar derivation is in Amemiya [1], but it lacks the geometric intuition given here.

REFERENCE

1. Amemiya, T. *Advanced Econometrics*. Cambridge: Harvard University Press, 1985.

2. Solution—proposed by Luis J. Alvarez and Juan J. Dolado. Consider the linear model

$$y = X\beta + \varepsilon, \quad (1)$$

where $E(\varepsilon\varepsilon') = I$ and the set of stochastic linear restrictions (extraneous information on β)

$$R'\beta + \nu = q, \quad (2)$$

where $E(\upsilon\upsilon') = V$. Note that the set of exact linear restrictions $R'\beta = q$ can be interpreted as the limiting case of (2) as $V \uparrow 0$ (each element of V tends to zero).

By combining sample and independent extraneous information in the compact form

$$\begin{pmatrix} y \\ q \end{pmatrix} = \begin{pmatrix} X \\ R' \end{pmatrix}\beta + \begin{pmatrix} \varepsilon \\ \nu \end{pmatrix}, \tag{3}$$

it is well known (see [2] and [3]) that the GLS estimator of β ($\tilde{b}$) in (3) is given by

$$\tilde{b} = [(X'X) + RV^{-1}R']^{-1}[X'y + RV^{-1}q\} \tag{4}$$

Now let us make use of the following well-known result on matrix inversion (see [1]).

$$(A + BDB')^{-1} = A^{-1} - A^{-1}B[B'A^{-1}B + D^{-1}]^{-1}B'A^{-1},$$

where in (4),

$$A = (X'X); \qquad B = R; \qquad D = V^{-1}.$$

Then,

$$\begin{aligned}\tilde{b} &= [(X'X)^{-1} - (X'X)^{-1}R(R'(X'X)^{-1}R + V)^{-1}R'(X'X)^{-1}] \\ &\quad \times [X'y + RV^{-1}q] \\ &= (a) + (b) + (c) + (d),\end{aligned}$$

where

(a) : $(X'X)^{-1}X'y = b$

(b) : $(X'X)^{-1}RV^{-1}q$

(c) : $-(X'X)^{-1}R[R'(X'X)^{-1}R + V]^{-1}R'b$

(d) : $-(X'X)^{-1}R[R'(X'X)^{-1}R + V]^{-1}R'(X'X)^{-1}RV^{-1}q.$

Taking $(X'X)^{-1}R[R'(X'X)^{-1}R + V]^{-1}$ as a common factor in (b) to (d), then $\tilde{b}$ can written as

$$\begin{aligned}\tilde{b} &= b - (X'X)^{-1}R[R'(X'X)^{-1}R + V]^{-1} \\ &\quad \times [R'b + R'(X'X)^{-1}RV^{-1}q - (R'(X'X)^{-1}R + V)V^{-1}q] \\ &= b - (X'X)^{-1}R[R'(X'X)^{-1}R + V]^{-1}[R'b - q]. \end{aligned} \tag{5}$$

Now let $V \uparrow 0$ and we get the well-known formula for the restricted OLS estimator without making use of the Lagrangean.

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1. Dhrymes, P.J. *Mathematics for Econometrics.* New York: Springer-Verlag, 1978.
2. Durbin, J. A note on regression when there is extraneous information about one of the coefficients. *Journal of the American Statistical Association* 48 (1953): 799–808.
3. Theil, H. & A.S. Goldberger. On pure and mixed statistical estimation in economics. *International Economic Review* 2 (1961): 65–78.

3. Solution – proposed by Paolo Paruolo. Two similar solutions are given, which revolve upon the property of invariance of least squares with respect to linear reparametrizations and orthogonal projections.

Solution A. Reparametrize the model using $\psi = A\beta$, where $A = (R,H)'$ and H is a $k \times (k-r)$ matrix whose columns are a basis of the orthogonal complement of the space spanned by the columns of R, $R'H = 0$. This ensures that A is invertible, and that $\beta = A^{-1}\psi$, so that ψ and β are equivalent parametrizations. Note that by virtue of the specific choice of $H, A^{-1} = (\bar{R},\bar{H})$, where an upper bar indicates $\bar{b} \equiv b(b'b)^{-1}$, for b of full column rank.

Partition $\psi \equiv (\psi_1',\psi_2')'$ conformably with respect to R and H, i.e., $\psi_1 = R'\beta$, $\psi_2 = H'\beta$. Substituting $\beta = A^{-1}\psi$ into (1) one has

$$y = X^*\psi + \varepsilon = X_1^*\psi_1 + X_2^*\psi_2 + \varepsilon, \tag{3}$$

where $X^* \equiv (X_1^*,X_2^*) \equiv XA^{-1}$ and X_1^* and X_2^* are conformable with respect to ψ_1 and ψ_2, respectively. Under the restrictions of (2), $\psi_1 \equiv R'\beta = q$, so that (3) becomes $y = X_1^*q + X_2^*\psi_2 + \varepsilon$ or

$$y^* = X_2^*\psi_2 + \varepsilon, \tag{4}$$

where $y^* \equiv y - X_1^*q$. As the least squares are invariant with respect to linear reparametrizations, the analysis of (1) is equivalent to the analysis of (3); imposing the restrictions in the new parametrization is readily accomplished through (4), leading to

$$p_{2\text{OLS}} = (X_2^{*\prime}X_2^*)^{-1}X_2^{*\prime}y^* = (H'H)(H'(X'X)H)^{-1}H(X'y - X'X\bar{R}q).$$

Substituting in $b_{\text{RLS}} = (\bar{R},\bar{H})(q',p_{2\text{OLS}}')'$, one obtains

$$b_{\text{RLS}} = (I - H(H'SH)^{-1}H'S)\bar{R}q + H(H'SH)^{-1}H'Sb, \tag{5}$$

where $S \equiv X'X$. The first matrix on the right-hand side can be rewritten as follows.

$$\begin{aligned}(I - H(H'SH)^{-1}H'S) &= S^{-1}R(R'S^{-1}R)^{-1}R' \\ &= (X'X)^{-1}R(R'(X'X)^{-1}R)^{-1}R'\end{aligned}$$

Similarly,

$$\begin{aligned}H(H'SH)^{-1}H'S &= I - S^{-1}R(R'S^{-1}R)^{-1}R' \\ &= I - (X'X)^{-1}R(R'(X'X)^{-1}R)^{-1}R',\end{aligned}$$

(see e.g., Srivastava and Khatri [1, p. 19]). Hence, one obtains $b_{\text{RLS}} = b_*$, where $b_* = b - (X'X)^{-1}R(R'(X'X)^{-1}R)^{-1}(R'b - q)$ is the usual formulation for the constrained OLS estimator.

Solution B. The plan of this solution is to show (i) that (2) and

$$\beta = H\psi + h$$

are equivalent formulations; (ii) to substitute (6) into (1) and to apply least squares directly, and (iii) finally to verify that the obtained estimator is indeed $b_* = b - (X'X)^{-1}R(R'(X'X)^{-1}R)^{-1}(R'b - q)$.

(i) Indicate with $P_A \equiv A(A'A)^{-1}A' = \bar{A}A' = A\bar{A}'$ the projection matrix onto A, where an upper bar indicates $\bar{b} \equiv b(b'b)^{-1}$, for b of full column rank; then $P_H + P_R = I_k$, so that

$$\beta = (P_R + P_H)\beta = \bar{R}(R'\beta) + H(\bar{H}'\beta) = \bar{R}q + H\psi, \tag{7}$$

where $\psi \equiv \bar{H}'\beta$. Multiplying (7) by R', one obtains $R'\beta = q$. Note, however, that substituting ψ with $\gamma + h_0$ leads to the same result when multiplying by R'; in other words, adding a constant in the space spanned by the columns of H does not lead to changes in the linear restrictions. Note, however, that this corresponds to considering $\beta = H(\gamma + h_0) + \bar{R}q$, i.e., a reparametrization of ψ. Therefore, if one chooses $h = \bar{R}q$, ψ is identified, and (2) and (6) are the implicit and the explicit form of the same linear restrictions.

(ii) Substituting (6) into (1), one has $\tilde{y} = \tilde{X}\psi + \varepsilon$, where $\tilde{y} \equiv y - Xh$ and $\tilde{X} \equiv XH$, and applying least squares to the model leads to $p_{\text{RLS}} = (H'X'XH)^{-1}H'X'(y - Xh)$ and to

$$\begin{aligned} b_{\text{RLS}} = {} & (I - H(H'(X'X)H)^{-1}H'(X'X))h \\ & + H(H'(X'X)H)^{-1}H'(X'X)b. \end{aligned} \tag{8}$$

(iii) The solution now follows along the same lines as solution 1 from (5) onwards, since $h = \bar{R}q$.

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1. Srivastava, M.S. & C.G. Khatri. *An Introduction to Multivariate Statistics.* New York: North Holland, 1979.

4. Solution—proposed by John Xu Zheng. By OLS principle, b_ϵ is the solution to the constrained minimization problem

$$\min(y - X\beta)'(y - X\beta) \qquad \text{subject to} \quad R'\beta = q. \tag{4}$$

Since $(y - Xb)'X = y'(I - X(X'X)^{-1}X')X = 0$, we have

$$\begin{aligned} (y - X\beta)'(y - X\beta) &= [(y - Xb) + (Xb - X\beta)]'[(y - Xb) + (Xb - X\beta)] \\ &= (y - Xb)'(y - Xb) + (\beta - b)'X'X(\beta - b). \end{aligned}$$

The term $(y - Xb)'(y - Xb)$ does not depend on β; thus the constrained minimization problem (4) is equivalent to

$$\min(\beta - b)'X'X(\beta - b) \qquad \text{subject to} \quad R'\beta = q. \tag{5}$$

Let $\beta = b + (X'X)^{-1/2}\alpha$, where $(X'X)^{-1/2}$ is the square root of the positive definite matrix $(X'X)^{-1}$. Then the problem (5) can be further simplified as

$$\min \alpha'\alpha \qquad \text{subject to} \quad R'(X'X)^{-1/2}\alpha = q - R'b. \tag{6}$$

Multiplying both sides of the constraint equation in (6) by $(q - R'b)'$ and applying the Schwarz inequality $a'b \leq (a'a)^{1/2}(b'b)^{1/2}$ for any vectors a, b where the equality holds if and only if a is proportional to b, we have

$$\begin{aligned}(q - R'b)'(q - R'b) &= (q - R'b)'R'(X'X)^{-1/2}\alpha \\ &\leq [(q - R'b)'R'(X'X)^{-1}R(q - R'b)]^{1/2} \\ &\quad \times [\alpha'\alpha]^{1/2},\end{aligned} \tag{7}$$

or

$$\alpha'\alpha \geq \frac{[(q - R'b)'(q - R'b)]^2}{(q - R'b)'R'(X'X)^{-1}R(q - R'b)}. \tag{8}$$

Thus, subject to the constraint in (6), $\alpha'\alpha$ is minimized at $\hat{\alpha}$ where

$$\hat{\alpha} = \lambda(X'X)^{-1/2}R(q - R'b) \qquad \text{for some scalar } \lambda. \tag{9}$$

Substituting $\hat{\alpha}$ in (9) into the constraint equation (6), we have

$$q - R'b = R'(X'X)^{-1/2}\hat{\alpha} = \lambda R'(X'X)^{-1}R(q - R'b).$$

Solving $\lambda(q - R'b)$, we get $\lambda(q - R'b) = (R'(X'X)^{-1}R)^{-1}(q - R'b)$. Substituting the formula for $\lambda(q - R'b)$ into (9), we get

$$\hat{\alpha} = (X'X)^{-1/2}R(R'(X'X)^{-1}R)^{-1}(q - R'b). \tag{10}$$

Therefore, the constrained OLS estimator of β is

$$\begin{aligned}b_\epsilon &= b + (X'X)^{-1/2}\hat{\alpha} = b + (X'X)^{-1}R(R'(X'X)^{-1}R)^{-1}(q - R'b) \\ &= b - (X'X)^{-1}R(R'(X'X)^{-1}R)^{-1}(R'b - q).\end{aligned}$$

NOTE

1. R.W. Farebrother has drawn the Editor's attention to the fact that this problem has been solved by Farebrother [1, pp. 258–259] using an argument based on that used by Gauss [2] for the case $X = I$.

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1. Farebrother, R.W. *Linear Least Squares Computations.* New York: Marcel Dekker, 1988.
2. Gauss, C.F. Supplementum theoriae combinationis erroribus minimis obnoxiae. *Commentationes societatis Regiae Scienti—arum Gottengensis Recentiores* 6 (1828). Reprinted in his *Werke* 4 (1880): 57–93.