

2022-03

Working paper. Economics

ISSN 2340-5031

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Serie disponible en

<http://hdl.handle.net/10016/11>

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Competitive Search with Two-Sided Risk Aversion

Belén Jerez*

March, 2021

Abstract

We analyze a static competitive search model where risk-averse individuals with different wealth levels trade an indivisible good. The real estate market is a particularly relevant application. We show that the equilibrium is constrained efficient. Other properties of the equilibrium are derived, including the generalized version of the Hosios (1990) rule for this environment. Under risk aversion, buyers and sellers evaluate the trade-off between prices and trading probabilities differently as their wealth increases. As they become richer, buyers are relatively less concerned about paying higher prices and more concerned about increasing their trading probability. Conversely, richer sellers care less about increasing the probability of a sale than poorer sellers, and they care more about trading at a higher price. This results in positive sorting in equilibrium, that is, wealthier (poorer) buyers and sellers trading with each other. As transactions among wealthier agents involve higher prices, the equilibrium features frictional price dispersion. By contrast, with transferable utility, all individuals would trade at the same price, irrespectively of their wealth.

Journal of Economic Literature Classification Numbers: D50, D61, D83.

Key Words: competitive/directed search; risk aversion; wealth heterogeneity; constrained efficiency; sorting.

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1 Introduction

Models of competitive or directed search are widely used in economics to study markets where search frictions are prevalent.¹ Virtually all existing work assumes that either buyers or sellers (or both) are risk neutral. Indeed, in labor market applications, it is natural to think of workers as being risk averse, and of employers as maximizing expected profits (e.g. Shi, 2009; Menzio and Shi, 2010; Shi and Chaumont, 2020; Eeckhout and Sepahsalari, 2020). Yet the assumption of one-sided risk aversion, which is typically invoked for the sake of tractability, might be too strong to study other markets. An obvious example is real estate. Given that houses are indivisible big-ticket items, it seems more reasonable to assume that both home buyers and home sellers are risk averse.

To the best of our knowledge, the case of two-sided risk aversion has not been studied in the directed search literature. For instance, existing housing models with risk-averse households and directed search assume that home buyers and homeowners do not trade directly with each other (e.g. Hedlund, 2016; Garriga and Hedlund, 2020; Díaz et al., 2019). Instead, in these models, all real estate transactions are intermediated by risk-neutral agents who buy housing units from homeowners and then sell these units to buyers. Under this assumption, the model maps into one with one-sided risk aversion, a case that is well understood and particularly tractable (e.g. see Acemoglu and Shimer, 1999). However, this kind of intermediation is rare in reality, given the high transaction costs implied (e.g. taxes). In the real world, most homeowners sell their property directly to other households, and the role of real estate agents is to facilitate these transactions. That is, these intermediaries tend to be dealers rather than brokers

In this paper, we analyze a static competitive search model with full-information where agents on both sides of the market are risk averse and trade directly with each other. In doing so, we abstract away from intermediation. We use the price-taking approach in Jerez (2014) to define a competitive search equilibrium (see also Díaz et al., 2019). This allows us to use the tools and intuitions of general equilibrium theory to derive our main results. As shown in Jerez (2014), this price-taking equilibrium notion is equivalent to the (strategic) directed search equilibrium notion.

It is well-known that competitive search equilibria are constrained efficient in full-information environments with transferable utility (e.g. see Eeckhout and Kircher, 2010; Jerez, 2014) and

¹See Guerrieri et al. (2021) for a recent comprehensive survey.

one sided-risk aversion (see Acemoglu and Shimer, 1999). We extend this important result to our environment. Other properties of the equilibrium allocation are also derived, including the generalized version of the Hosios (1990) rule that applies to our setting. Under risk aversion, buyers and sellers evaluate the trade-off between prices and trading probabilities differently as their wealth increases. Specifically, as they become richer, buyers are relatively less concerned about paying higher prices and more concerned about increasing their trading probability (see also Díaz et al., 2019). On the other hand, richer sellers care less about increasing the probability of a sale than poorer sellers, and they care more about trading at a higher price (e.g. see also Shi and Chaumont, 2020; Eeckhout and Sepahsalari, 2020). This results in positive sorting in equilibrium, that is, wealthier (poorer) buyers and sellers trading with each other. Since transactions among wealthier agents involve higher prices, the equilibrium features frictional price dispersion. By contrast, with transferable utility, all individuals trade at the same price, irrespectively of their wealth.

The paper is organized as follows. Section 2 describes the environment. A competitive search equilibrium is defined in Section 3. The constrained efficiency result is established in Section 4, and our characterization results are presented in Section 5. Section 6 concludes. To ease the exposition, most of the proofs are relegated to the Appendix.

2 Environment

Consider a static exchange economy with a continuum of agents. Agents are either home buyers or home sellers, indexed by $i = b, s$, respectively. All agents have von Neumann Morgenstern preferences and derive utility both from housing services and from a divisible consumption good. Their common (Bernoulli) utility function is $u(c, h)$, where $c, h \in \mathfrak{R}_+$ are the amounts of the divisible good and housing services consumed. The utility function has standard properties; i.e. it is strictly increasing, concave and \mathcal{C}^2 , and satisfies $\lim_{c \rightarrow 0} u_c(c, h) = \infty$ for a given $h > 0$. The general case of interest is one where consumption and housing services are complements (so u_c increases with h). We take the consumption good as the numeraire. Each seller is endowed with an identical, indivisible housing unit that provides her with services $\underline{h} > 0$. Buyers have no housing endowment. On the other hand, agents on both sides of the housing market differ in their endowments of the consumption good. These endowments are represented by $w_i \in W \equiv [\underline{w}, \bar{w}] \subset \mathfrak{R}_{++}$ for $i = b, s$. We

refer to w_b as the buyer’s type and to w_s as the seller’s type. We assume that each agent consumes the services of at most one housing unit (for simplicity).

The subpopulations of buyers and sellers are described by a pair of (positive) Borel measures on W , $(\xi^b, \xi^s) \in M_+(W)^2$. That is, $\xi^s(F)$ is the mass of sellers whose endowment lies in F , for any Borel subset $F \subset W$ (and similarly for buyers). In particular, $\xi^s(W)$ is the total mass of homeowners—the economy’s housing stock.

The consumption good is traded in a frictionless competitive market. On the other hand, houses are illiquid assets and are traded in a competitive search market. Specifically, as in Moen (1997), home buyers and sellers can choose to trade in different submarkets where they meet bilaterally and at random (the matching process being one-to-one). The probability that a seller meets a buyer in a given submarket is $m(\theta)$, where $\theta \in \mathfrak{R}_+$ denotes the buyer-seller ratio (or tightness level) in that submarket, while the probability that a buyer meets a seller is $\alpha(\theta) = m(\theta)/\theta$. As is standard, $m(\theta)$ is strictly increasing, concave and \mathcal{C}^2 , with $m(0) = 0$ and $\lim_{\theta \rightarrow \infty} m(\theta) = 1$, and $\alpha(\theta)$ is strictly decreasing in θ , with $\lim_{\theta \rightarrow 0} \alpha(\theta) = 1$ and $\lim_{\theta \rightarrow \infty} \alpha(\theta) = 0$. So, in particular, the higher θ , the easier it is for sellers to meet buyers and the harder it is for buyers to meet sellers. Also, the elasticity $\eta(\theta) = [m'(\theta)\theta]/m(\theta)$ is non-increasing with $\lim_{\theta \rightarrow 0} \eta(\theta) = 1$ and $\lim_{\theta \rightarrow \infty} \eta(\theta) = 0$, and $\hat{m}(\alpha) \equiv m(\alpha^{-1}(\cdot))$ is concave.²

In our setting, not all meetings between a buyer and a seller in a given submarket lead to trade. Specifically, in a given random encounter between a buyer and a seller, there is a probability $q \in (0, 1)$ that the unit suits the buyer’s idiosyncratic needs. In this case, $h_b = \bar{h} > \underline{h}$ and the property is sold, so $h_s = 0$. With complementary probability, $h_b = 0$, and the buyer discards the unit, so $h_s = \underline{h}$.³ Therefore, buyers and sellers who search for a trading opportunity in a submarket

²The last assumption ensures that the agents’ preferences are convex on the relevant decision spaces (see Lemma 6). As noted by Menzio and Shi (2010), this assumption is satisfied by different specifications of the matching function which are commonly used in the competitive search literature. With transferable utility, the convexity requirement is always satisfied, so this assumption is not needed.

³We think of sellers as being “mismatched” with their property, and of buyers who purchase a suitable unit as becoming “matched” homeowners. Our model is easily extended to introduce a non-binary match quality distribution. In particular, Theorem 2 is a general result that is independent of the underlying preference structure. With a continuous match quality distribution, there would typically be a threshold value h_0 such that a house will be sold whenever the realized value of h_b exceeds h_0 . We assume a binary distribution because this allows us to focus on the agents’ participation thresholds (rather than the match quality threshold).

with tightness θ trade with probability

$$\pi_b(\theta) = q \alpha(\theta) = q m(\theta)/\theta, \tag{2.1}$$

$$\pi_s(\theta) = q m(\theta), \tag{2.2}$$

respectively. We assume that the Law of Large Numbers holds. Hence, $\pi_b(\theta)$ and $\pi_s(\theta)$ are also the fractions of buyers and sellers who complete a transaction in this submarket (whereas $1 - \pi_b(\theta)$ and $1 - \pi_s(\theta)$ are the fractions of buyers and sellers who are rationed).

3 Competitive Search Equilibrium

We use the price-taking approach in Jerez (2014) to define a competitive search equilibrium. The key idea is to think of houses traded in submarkets with different tightness levels as different commodities, which are characterized by different degrees of trading uncertainty. These commodities will be priced differently in equilibrium, even though all houses in our model have identical physical attributes. All agents behave as price-takers and have common rational expectations about the tightness level prevailing in each active submarket.

Let $p(\theta)$ denote the price of a house that is up for sale in a submarket with tightness θ . The function $p : \mathfrak{R}_+ \rightarrow \mathfrak{R}_+$ describes the price system in our environment. Following Mas-Colell's (1975) description of the price system with a continuum of differentiated commodities, we take p to be continuous. This means that submarkets with similar tightness levels have similar prices. It is useful to model the participation margin by introducing a fictitious submarket $\theta_0 \in \mathfrak{R}_{--}$, and extend the functions π_b , π_s and p to the set $\Theta \equiv \mathfrak{R}_+ \cup \{\theta_0\}$ by setting $\pi_b(\theta_0) = \pi_s(\theta_0) = p(\theta_0) = 0$. Joining θ_0 is equivalent to not participating in the housing market.

We assume for simplicity, that agents cannot borrow. Hence, only buyers with wealth $w_b \geq p(\theta)$ can trade in submarket θ . For these buyers, the (ex-ante) expected utility from participating in θ

is⁴

$$\begin{aligned}
U_b(w_b, \theta; p) &= \pi_b(\theta)u(w_b - p(\theta), \bar{h}) + (1 - \pi_b(\theta))u(w_b, 0) \\
&= u(w_b, 0) + \pi_b(\theta) [u(w_b - p(\theta), \bar{h}) - u(w_b, 0)].
\end{aligned} \tag{3.2}$$

These agents find a suitable home with probability $\pi_b(\theta)$, in which case they pay the price $p(\theta)$ and consume \bar{h} units of housing services and $w_b - p(\theta)$ units of consumption. With complementary probability, they do not trade. Their ex-post utility gain from a transaction in submarket θ is then $u(w_b - p(\theta), \bar{h}) - u(w_b, 0)$, and this gain is realized with probability $\pi_b(\theta)$.

Similarly, the expected utility of a type- w_s seller who joins submarket θ is

$$\begin{aligned}
U_s(w_s, \theta; p) &= \pi_s(\theta)u(w_s + p(\theta), 0) + (1 - \pi_s(\theta))u(w_s, \underline{h}) \\
&= u(w_s, \underline{h}) + \pi_s(\theta) [u(w_s + p(\theta), 0) - u(w_s, \underline{h})].
\end{aligned} \tag{3.3}$$

With probability $\pi_s(\theta)$, the unit is sold and the seller consumes $w_s + p(\theta)$ units of the numeraire. With complementary probability, there is no trade. The seller's ex-post gain from a transaction is then $u(w_s + p(\theta), 0) - u(w_s, \underline{h})$.

We use measures to describe allocations. Specifically, the buyers' choices as to which submarkets to join are described by a compactly supported measure Borel measure on $W \times \Theta$, $\mu^b \in M_{c+}(W \times \Theta)$. That is, $\mu^b(F \times \Xi)$ is the measure of buyers with wealth $w_b \in F$ who seek to trade in submarket $\theta \in \Xi$ (for arbitrary Borel sets $F \subset W$ and $\Xi \subset \Theta$). Likewise, the sellers' choices are described by $\mu^s \in M_{c+}(W \times \Theta)$, with a similar interpretation. We denote the support of a measure ν by $\text{supp}\nu$.

Two observations are in order. First, allocations assign (almost) all the agents in our economy to a submarket $\theta \in \Theta$. Hence, the marginals of μ^b and μ^s on W must coincide with the population measures ξ^b and ξ^s , respectively. We denote the marginal of μ^i on W by μ_W^i , for $i = b, s$.⁵ The second observation has to do with the marginal μ_Θ^i of μ^i on Θ . The measures μ_Θ^b and μ_Θ^s are

⁴The expected utility of a buyer is

$$U_b = \pi_b(\theta)u(\hat{c}_b, \bar{h}) + (1 - \pi_b(\theta))u(w_b, 0), \tag{3.1}$$

where $\hat{c}_b \geq 0$ is the amount of the numeraire consumed in the event a trading meeting takes place. The budget constraint of the buyer is $\hat{c}_b + p(\theta) \leq w_b$. Since u is strictly increasing in \hat{c}_b , this constraint binds (and similarly for sellers.) We could also use an (enlarged) description of a consumption bundle as a pair $(\theta_i, \hat{c}_i) \in \mathfrak{R}_+^2$ ($i = b, s$), but the notation would be more involved.

⁵ μ_W^b gives the total mass of buyers of each type who are assigned to some $\theta \in \Theta$, which should equal the measure of buyers of that type who live in the economy (and similarly for sellers).

important because they pin down the total mass of buyers and sellers (demand and supply) in each submarket $\theta \in \mathfrak{R}_+$. In particular, a submarket $\theta \neq \theta_0$ is active if it attracts both buyers and sellers; i.e. $\theta \in \text{supp}\mu_{\Theta}^b \cap \text{supp}\mu_{\Theta}^s$. Note that type- w_i agents participate in θ in equilibrium whenever $(w_i, \theta) \in \text{supp}\mu^{i*}$.

Definition 1. *A competitive (price-taking) search equilibrium in our economy is an allocation (μ^{b*}, μ^{s*}) and a price system p^* such that:*

(i) *(Individual optimization) Buyers and sellers choose a submarket $\theta \in \Theta$ so as to maximize their expected utility taking p^* as given, and (in doing so) buyers are restricted by the corresponding liquidity constraint:*

$$V_b(w_b, p^*) \equiv \sup_{\theta \in \Theta} \{U_b(w_b, \theta; p^*) \text{ s.t. } p^*(\theta) \leq w_b\} = U_b(w_b, \theta_{w_b}^*; p^*)$$

for a.a. $(w_b, \theta_{w_b}^*) \in \text{supp}\mu^{b*}$,

(3.4)

$$V_s(w_s, p^*) \equiv \sup_{\theta \in \Theta} U_s(w_s, \theta; p^*) = U_s(w_s, \theta_{w_s}^*; p^*) \text{ for a.a. } (w_s, \theta_{w_s}^*) \in \text{supp}\mu^{s*}.$$
(3.5)

(ii) *(Adding-up) The allocation (μ^{b*}, μ^{s*}) assigns almost all agents to a submarket $\theta \in \Theta$:*

$$\mu_W^{i*} = \xi^i, \quad i = b, s.$$
(3.6)

(iii) *(Generalized market clearing) All agents have rational beliefs about the tightness levels prevailing in active submarkets:*

$$\int_{\theta \in \Xi} d\mu_{\Theta}^{b*}(\theta) = \int_{\theta \in \Xi} \theta d\mu_{\Theta}^{s*}(\theta) \quad \text{for all Borel } \Xi \subseteq \mathfrak{R}_+.$$
(3.7)

Conditions (i) and (ii) are standard. Equation (3.4), together with the first “adding-up” condition in (ii), says that (almost) all buyers choose the markets they enter so as to maximize their expected utility subject to their liquidity constraint given the equilibrium prices. In this equation, $V_b(w_b, p^*)$ denotes the indirect utility of a type- w_b buyer at prices p^* . Equation (3.5) and the second condition in (ii) describe a similar optimization condition for sellers, where $V_s(w_s, p^*)$ represents the equilibrium indirect utility of a type- w_s seller.

The non-standard condition is the “generalized market clearing condition” in (iii).⁶ This condition says that the measures of buyers and sellers in each active submarket are consistent with the tightness levels that agents take as given when they make their optimal decisions. Suppose (μ^{b*}, μ^{s*}) is atomless. In this case, $dx_{\Theta}^{s*}(\theta)$ represents the density of sellers and $dx_{\Theta}^{b*}(\theta)$ represents the density of buyers in each active submarket $\theta \neq \theta_0$. If the traders’ conjectures about the buyer-seller ratio θ are correct then $dx_{\Theta}^{b*}(\theta)$ should be equal to $\theta dx_{\Theta}^{s*}(\theta)$. This is what (3.7) says.⁷ Note that, if a given submarket θ attracts no traders, (3.7) is vacuous. That is, (3.7) is a restriction on active submarkets $\theta \in \mathfrak{R}_+$ only. Formally, (3.7) says that the restriction of μ_{Θ}^{b*} to \mathfrak{R}_+ is absolutely continuous with respect to the restriction of μ_{Θ}^{s*} to the same set, with Radon-Nikodym derivative θ . So, in particular, both restrictions have the same support. This common support gives the set of submarkets that are active in equilibrium.

Condition (iii) implies that the number of housing units purchased and sold in each active submarket is equal. Specifically, it is direct to check that (3.7) implies that

$$\int_{\theta \in \Xi} \pi_b(\theta) d\mu_{\Theta}^{b*}(\theta) = \int_{\theta \in \Xi} \pi_s(\theta) d\mu_{\Theta}^{s*}(\theta) \quad \text{for all Borel } \Xi \subseteq \mathfrak{R}_+, \quad (3.8)$$

since $\pi_s(\theta) = \theta \pi_b(\theta)$ and $\pi_b(\theta) > 0$ for all $\theta \in \mathfrak{R}_+$. Recall that μ_{Θ}^{b*} describes the total measure of buyers who participate in each submarket, and $\pi_b(\theta)$ is the fraction of buyers who trade in θ . The left-hand side of (3.8) is then the total number of houses purchased in an arbitrary Borel set of markets Ξ . Similarly, since μ_{Θ}^{s*} describes the total measure of sellers assigned to each θ and $\pi_s(\theta)$ is the fraction of sellers who complete a transaction in θ , the right-hand side of (3.8) represents total sales in Ξ .

The market clearing condition for the consumption good is not included in the definition of equilibrium because it is implied by (3.8). When (3.8) holds, the individual net trades of the consumption good—which describe the transfers of this good between home buyers and home sellers—necessarily cancel out when one aggregates all bilateral trades in the search market. Hence, (3.6) and (3.8) directly implies that the excess demand of the consumption good is zero. This is the corresponding version of Walras Law in our search environment.

⁶This term was coined by Peters (1997), who presents a related Walrasian equilibrium notion where prices (rather than rationing probabilities) are embedded in the objects of trade, and rationing probabilities adjust so that individual decisions are consistent at the aggregate level. See also Eeckhout and Kircher (2010).

⁷A similar interpretation applies if μ^{b*} and μ^{s*} have a mass point at θ .

4 Constrained efficiency

An allocation $(\hat{\mu}^b, \hat{\mu}^s)$ is feasible if it satisfies (3.6) and (3.8), while ensuring that the aggregate consumption of the numeraire does not exceed its aggregate endowment. The constrained efficiency result in Theorem 2 follows from a very similar argument to the one used to prove the First Welfare Theorem in a frictionless Arrow-Debreu economy.

Theorem 2. *Equilibrium allocations are constrained efficient. That is, there is no feasible allocation that almost all agents weakly prefer to (μ^{b*}, μ^{s*}) , the preference being strict for a non-negligible group of agents.*

Proof. Suppose not. Let $(\hat{\mu}^b, \hat{\mu}^s) \in M_{c^+}(W \times \Theta)^2$ denote the alternative Pareto superior allocation. By feasibility, $(\hat{\mu}^b, \hat{\mu}^s)$ satisfies (3.6) and (3.8). The alternative allocation also specifies the consumption levels $\hat{c}_i(w_i, \theta_i)$ of the divisible good associated to each element (w_i, θ_i) in the support of $\hat{\mu}^i$ for $i = b, s$ (i.e., the consumption of the divisible good by a type- w_i agent in the event that the agent is matched in submarket θ_i). These consumption levels cannot be budget feasible at the equilibrium prices for those agents who are strictly better off under $(\hat{\mu}^b, \hat{\mu}^s)$, and they must lie in the budget line in the case of the agents who are indifferent (since $u(c_i, h_i)$ is monotone in c_i); i.e.,

$$\begin{aligned} \hat{c}_b(w_b, \theta_{w_b}) + p^*(\theta_b) &\geq w_b && \text{for a.a. } (w_b, \theta_{w_b}) \in \text{supp} \hat{\mu}^b, \\ \hat{c}_s(w_s, \theta_{w_s}) - p^*(\theta_s) &\geq w_s && \text{for a.a. } (w_s, \theta_{w_s}) \in \text{supp} \hat{\mu}^s, \end{aligned}$$

with strict inequality for a non-negligible mass of agents who strictly prefer $(\hat{\mu}^b, \hat{\mu}^s)$ to (μ^{b*}, μ^{s*}) . (Otherwise, (μ^{b*}, μ^{s*}) would not satisfy condition (i) in Definition 1). Now, since $(\hat{\mu}^b, \hat{\mu}^s)$ satisfies (3.8), the transfers between buyers and sellers which are implied by the price system p^* cancel out within each active submarket under $(\hat{\mu}^b, \hat{\mu}^s)$, just as they cancel out under (μ^{b*}, μ^{s*}) . Hence, these transfers cancel out at the aggregate level:

$$\int_{W \times \Theta} p^*(\theta_b) d\hat{\mu}^b(w_b, \theta_b) - \int_{W \times \Theta} p^*(\theta_s) d\hat{\mu}^s(w_s, \theta_s) = 0.$$

But then the inequalities above imply that, under $(\hat{\mu}^b, \hat{\mu}^s)$, the aggregate consumption of the nu-

meraire exceeds its aggregate endowment:

$$\begin{aligned}
& \int_{W \times \Theta} \hat{c}_b(w_b, \theta_{w_b}) d\hat{\mu}^b(w_b, \theta_b) + \int_{W \times \Theta} \hat{c}_s(w_s, \theta_{w_s}) d\hat{\mu}^s(w_s, \theta_s) \\
& > \int_{W \times \Theta} w_b d\hat{\mu}^b(w_b, \theta_b) + \int_{W \times \Theta} w_s d\hat{\mu}^s(w_s, \theta_s) \\
& = \int_W w_b d\xi^b(w_b) + \int_W w_s d\xi^s(w_s),
\end{aligned}$$

where the last equality follows from (3.6). So $(\hat{\mu}^b, \hat{\mu}^s)$ is not feasible (a contradiction). \square

5 Equilibrium characterization

In this section, we derive some properties of the equilibrium. To ease the presentation, all the proofs are relegated to the Appendix.

5.1 Equilibrium prices

The first set of properties regards the equilibrium price system p^* . First of all, it is direct to see that prices in active submarkets are strictly positive and decreasing in θ . Intuitively, since buyers (sellers) who enter submarkets with higher tightness levels face higher (lower) rationing probabilities, they must also pay (receive) lower prices. This in turn implies that there will be price dispersion in the housing market whenever the set of active submarkets is not a singleton.

Lemma 3. *Suppose $\theta, \theta' \in \mathfrak{R}_+$ are active in equilibrium, and $\theta > \theta'$. Then $0 < p^*(\theta) < p^*(\theta')$.*

It is also easy to show that p^* is differentiable at $\theta \neq \theta_0$ for any active θ . To see why, note that, given the equilibrium prices, buyers and sellers choose the value of $\theta \in \Theta$ so as to maximize their respective (ex-ante) expected gains from trade:

$$S_b(w_b, \theta; p) = \pi_b(\theta) [u(w_b - p, \bar{h}) - u(w_b, 0)], \quad (5.1)$$

$$S_s(w_s, \theta; p) = \pi_s(\theta) [u(w_s + p, 0) - u(w_s, \underline{h})]. \quad (5.2)$$

Recall also that buyers must meet the liquidity constraint. Yet this constraint does not bind in equilibrium since it is never optimal for buyers to spend all their wealth in housing (as $\lim_{c \rightarrow 0} u_c(c, \bar{h}) =$

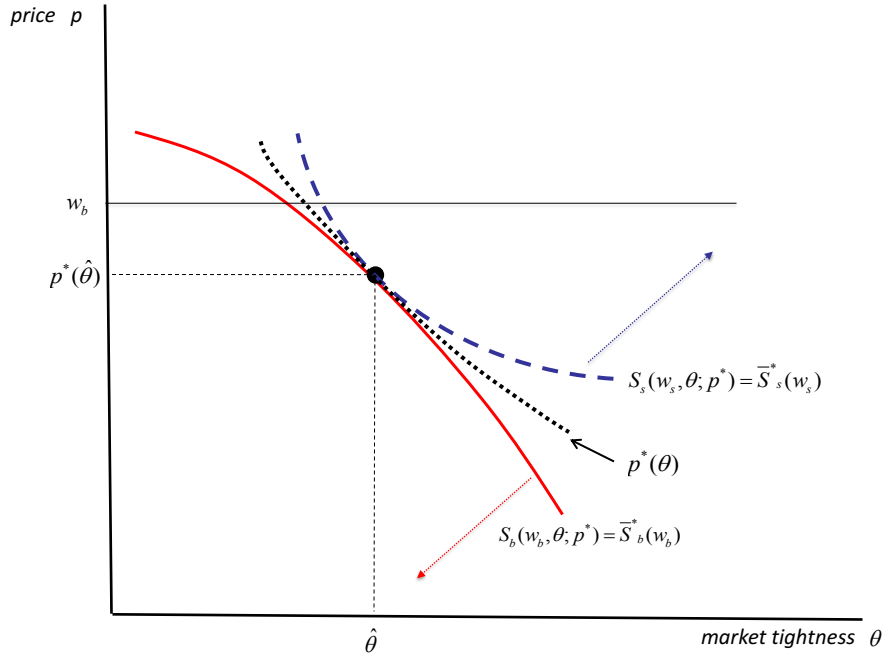


Figure 1: A submarket where type- \hat{w}_b buyers and type- \hat{w}_s sellers trade.

∞). Figure 1 depicts the indifference curves of buyers and sellers on the space (θ, p) . The indifference curves of a type- w_i agent are given by $S_i(w_i, \theta; p) = \bar{S}_i(w_i)$ for some fixed level $\bar{S}_i(w_i) \geq 0$. Sellers prefer high prices and high tightness levels, and the opposite is true for buyers. As shown in Figure 1, an agent's optimal choice of θ attains the highest value of $\bar{S}_i(w_i)$ along the price function p^* . This maximal value, denoted by $\bar{S}_i^*(w_i)$, is non-negative since θ_0 is a feasible choice for all agents.

By definition, an active submarket is the optimal choice of some buyers and some sellers. Say $\hat{\theta} \in \mathfrak{R}_+$ is optimal for buyers of type \hat{w}_b and sellers of type \hat{w}_s . As depicted in Figure 1, in equilibrium, the indifference curves of type- \hat{w}_b buyers and type- \hat{w}_s sellers must be tangent to p^* at $(\hat{\theta}, p^*(\hat{\theta}))$. Hence, these curves must be tangent to each other. The differentiability of p^* at $\hat{\theta}$ follows from the fact that the agents' indifference curves are smooth.

Lemma 4. *If $\theta \in \mathfrak{R}_+$ is active in equilibrium then p^* is differentiable at θ .*

5.2 The generalized Hosios condition

We now turn to characterize the agents' optimal decisions. Since the function $p^*(\theta)$ may be non-linear, the problem of solved by buyers and sellers need not be convex. So, in general, first-order conditions are not sufficient for optimality. Neither need the optimal choice of an agent be unique. Whether it is or not will depend on the shape of $p^*(\theta)$. Figure 1 depicts a situation where the optimal choices of the buyer and the seller are unique. However, in general, agents of a given type may trade in more than one submarket in equilibrium. In this case, the agent's optimal choice can be interpreted as a lottery that randomizes over the set of submarkets that are optimal at prices p^* .

Take a seller of type w_s . Using (2.2) and (3.3), the first-order necessary condition of the seller's problem in (3.5) can be written as

$$m'(\theta) [u(w_s + p^*(\theta), 0) - u(w_s, \underline{h})] + m(\theta) u_c(w_s + p^*(\theta), 0) p^{*'}(\theta) = 0. \quad (5.3)$$

Rearranging this expression yields

$$\frac{m'(\theta)}{m(\theta)} \left(\frac{u(w_s + p^*(\theta), 0) - u(w_s, \underline{h})}{u_c(w_s + p^*(\theta), 0)} \right) = -p^{*'}(\theta). \quad (5.4)$$

Similarly, using (2.1), (3.2) and (3.4), the first-order condition for a type- w_b buyer can be written as

$$\left(\frac{1}{\theta} - \frac{m'(\theta)}{m(\theta)} \right) \left(\frac{u(w_b - p^*(\theta), \bar{h}) - u(w_b, 0)}{u_c(w_b - p^*(\theta), \bar{h})} \right) = -p^{*'}(\theta). \quad (5.5)$$

The left-hand side of (5.4) is the seller's marginal rate of substitution of p for θ , whereas the left-hand side of (5.5) is that of the buyer. These equations then describe the tangency between the agents' indifference curves on the space (θ, p) and the equilibrium price function.

Combining (5.4) and (5.5) and rearranging, we obtain the tangency condition between the indifference curves of the buyer and the seller:

$$\left(\frac{u(\hat{w}_b - p^*(\theta), \bar{h}) - u(\hat{w}_b, 0)}{u_c(\hat{w}_b - p^*(\theta), \bar{h})} \right) = \frac{\eta(\theta)}{1 - \eta(\theta)} \left(\frac{u(\hat{w}_s + p^*(\theta), 0) - u(\hat{w}_s, \underline{h})}{u_c(\hat{w}_s + p^*(\theta), 0)} \right). \quad (5.6)$$

To interpret this expression, consider first the well-known case of transferable utility. When the

utility function takes the quasilinear form $u(c, h) = v(h) + c$, (5.6) simplifies to

$$\frac{v(\bar{h}) - p^*(\theta)}{p^*(\theta) - v(\underline{h})} = \frac{\eta(\theta)}{1 - \eta(\theta)}. \quad (5.7)$$

Here, $v(\bar{h}) - p^*(\theta)$ represents the buyer's (ex-post) surplus in a bilateral transaction, whereas $p^*(\theta) - v(\underline{h})$ is the seller's surplus. Equation (5.7) says that, in equilibrium buyers get a share $\eta(\theta)$ of the total bilateral trading surplus, $v(\bar{h}) - v(\underline{h})$. The rest of the surplus is appropriated by sellers. This is the familiar Hosios (1990) condition, which characterizes the division of the bilateral surplus in competitive search models with transferable utility.

Equation (5.6) is a generalization of the Hosios condition for the case of two-sided risk aversion. The term in the left-hand side of (5.6) is the ratio of the (ex-post) utility gain the buyer obtains from a transaction in submarket θ and the associated marginal utility of wealth (after the purchase). This ratio then represents the buyer's (ex-post) gains from trade in submarket θ measured *in units of consumption*. Similarly, the sellers' gains from trade in submarket θ in units of consumption are given by the second term in the right-hand side of (5.6). The sum of the two terms then represents the total gains in a bilateral meeting between a type- \hat{w}_b buyer and a type- \hat{w}_s seller when the meeting takes place in submarket θ . According to (5.6), in equilibrium, a fraction $\eta(\theta)$ of these gains are appropriated by the buyer and the rest go to the seller. This is why we refer to this equation as the generalized Hosios rule.

5.2.1 Market segmentation and price dispersion

It is well-known that, in transferable utility environments with two-sided heterogeneity, the Hosios condition is necessary but not sufficient for constrained optimality (e.g. see Shi, 2001; Eeckhout and Kircher, 2010; Garibaldi and Moen, 2010). The same is true here. Specifically, efficiency has an “extra layer”, as one needs to characterize the optimal matching pattern: which types of buyers and sellers join each submarket and in what proportions. What the division of the surplus in (5.6) ensures is that, conditional on the assignment of buyers and sellers across submarkets, the private benefits of buyers and sellers in each bilateral match are equal to their respective social contribution to the match (by Theorem 2).

Consider first the case of transferable utility. In this case, the buyers' marginal rates of substitution, and thus their optimal choices, do not depend on their non-housing wealth. The same

is true for sellers. It is also direct to check that the indifference curves of buyers and sellers have the standard convex shape depicted in Figure 1. This means that the indifference curves of buyers and sellers are tangent at a single point. It thus follows that a single submarket θ^* is active in equilibrium, so there is no price dispersion. Note that equation (5.7) has at most one solution, θ^* , because $\eta(\theta)$ is non-increasing and $p(\theta)$ is decreasing.

With transferable utility, sellers always participate in the housing market, since they get a positive fraction $1 - \eta(\theta^*)$ of the bilateral surplus. This need not be the case for buyers though, due to the liquidity constraint. One possibility is that all buyers participate in equilibrium. In this case, θ^* is equal to economy's aggregate buyer-seller ratio, $\theta^* = \xi^b(W)/\xi^s(W)$, and $p^*(\theta^*)$ is calculated by substituting θ^* in (5.7). The other possibility is that some of the poorer buyers stay out of the market because they cannot afford to trade in submarket θ^* . This happens whenever the price that solves (5.7) when $\theta = \xi^b(W)/\xi^s(W)$ exceeds the endowment of the poorest buyer type in the economy.

Proposition 5. *When utility is transferable, a single submarket θ^* is active in equilibrium, so there is no price dispersion. All agents participate in submarket θ^* in equilibrium except for those buyers whose type w_b is lower than $p^*(\theta^*)$.*

We now show that, when agents –on one or both sides of the market– are risk averse, price dispersion arises endogenously from the existing wealth heterogeneity. We also derive some properties of the matching pattern that emerges in equilibrium.

We proceed in a series of steps. The first is to show that the indifference curves of a given buyer type and a given seller type are tangent at most one point. This implies that, if type- w_b buyers trade with type- w_s sellers in equilibrium, all these transactions occur in the same submarket. Unlike in the case of transferable utility, this result is not direct, for the following reason. Whereas the seller's indifference curves in the space (θ, p) still have a convex shape, this need be the case for risk-averse buyers. However, this issue is easily circumvented by assuming that agents choose the buyer's matching probability α , rather than θ . This is without loss of generality since there is a one-to-one mapping between both variables. It is easy to see that the indifference curves of the buyer and the seller are convex in the space (α, p) , and so they are tangent at most one point (as in Figure 2).

Lemma 6. *Suppose type- w_b buyers and type- w_s sellers trade with each other in equilibrium. Then all the transactions between these types take place in the same submarket.*

Consider now the agents' participation decisions. Lemma 7 shows that the ex-post utility gain that buyers obtain when they trade at a given price p increases with their wealth, while the opposite is true for sellers. If the lowest buyer type is sufficiently poor, there will then be a threshold value $\hat{w}_b^* \in W$ such that buyer types *below* \hat{w}_b^* do not participate in the housing market. The reverse result holds for sellers.⁸ If the highest seller type is sufficiently rich, there may be a threshold value $\hat{w}_s^* \in W$ such that sellers types *above* \hat{w}_s^* do not participate in equilibrium. Characterizing the equilibrium allocation entails (among other things) characterizing these participation thresholds when relevant.

Lemma 7. *If agents are risk averse, the buyer's ex-post utility gain from trading at a given price p is increasing in their type, whereas the opposite is true for sellers. Hence, if θ_0 is optimal for type- w_b buyers, θ_0 also optimal for all types $w'_b < w_b$. Similarly, if θ_0 is optimal for type- w_s sellers, it is also optimal for all types $w'_s > w_s$.*

Our next result says that, when buyers are risk averse, different buyer types always participate in different submarkets, so the equilibrium displays frictional price dispersion. This result follows from the fact that higher buyer types have steeper indifference curves on the space (α, p) (see Figure 2). This single crossing property is intuitive. *The wealthier the buyer, the larger the price increase he is willing to accept in order to increase his trading probability α by a given amount while remaining indifferent.* This is because richer buyers have lower marginal utilities of wealth, and their ex-post utility gain when they buy a house at a given price is also larger (by Lemma 7).⁹ Because the buyers' optimal choice is characterized by a tangency between their indifference curves and p^* , the fact that different types have different marginal rates of substitution implies that, at a given θ , p^* cannot be simultaneously tangent to the indifference curves of two or more buyer types.

Lemma 8. *Suppose that buyers are risk averse. For buyers who prefer (α, p) to non-participation, the marginal rate of substitution of p for α increases with w_b . Hence, in equilibrium, different buyer types necessarily trade in different submarkets.*

⁸The assumption that $\underline{h} > 0$ is key for this result. If $\underline{h} = 0$, the seller's gain from joining any active market is positive (since $p^*(\theta) > 0$), so θ_0 can never be an optimal choice for a seller.

⁹See also the stationary dynamic model in Díaz et al. (2019).

Let us now focus on the sellers' indifference curves. Again, the marginal utility of wealth is lower for richer agents. Yet so is the sellers' ex-post utility gain of trading at a given price p (by Lemma 7). Hence, the sellers' marginal rate of substitution need not be monotone with respect to w_s . This is, again, intuitive. On the one hand, since the marginal utility of wealth is decreasing, the utility loss from a given price reduction is lower for richer sellers. But, on the other hand, the utility gain from an increase in their trading probability (a decrease in α) associated with such a price reduction is also lower. Lemma 9 shows that, when the coefficient of relative risk aversion is non-increasing, the second effect outweighs the first one, and the sellers' marginal rate of substitution of p for α decreases with w_s (as depicted in Figure 2). That is, under these conditions, *the poorer the seller, the larger the price reduction he is willing to accept in order to increase the probability of completing a transaction* by a given amount while remaining indifferent. In this case, different seller types participate in different submarkets.

Lemma 9. *Suppose that sellers are risk averse and the coefficient of relative risk aversion (e.g. conditional on $h = 0$) is non-increasing. For sellers who prefer who prefer (α, p) to non-participation, the marginal rate of substitution of p for α decreases with w_s for any $p > 0$. Hence, under these assumptions, different seller types trade in different submarkets.*

Loosely speaking, as their wealth increases, buyers and sellers evaluate the trade-off between prices and trading probabilities differently. Specifically, richer buyers are more concerned about increasing the probability that a purchase is completed and less concerned about paying a higher price in return. By contrast, richer sellers are relatively less concerned about increasing the probability of a sale than poorer sellers, and more concerned about selling at a higher price.

To recapitulate, we have shown that risk-averse buyers with different wealth levels participate in different submarkets and that this is also the case for sellers provided the coefficient of relative risk aversion is non-increasing. We stick to the latter assumption throughout the rest of the section. Under this assumption, each active submarket is then joined by a single buyer type and a single seller type.

Proposition 10. *All the buyers who participate in a given active submarket θ are of the same type, and, under the conditions in Lemma 9, the same is true for sellers. The equilibrium price in this submarket, $p^*(\theta)$, satisfies the tangency condition (5.6).*

5.2.2 Sorting

We next turn to inspect the equilibrium sorting pattern. Consider first a situation in which the solution of the agents' optimization problem is unique. In this case, all traders of a given type participate in the same submarket in equilibrium—so the equilibrium allocation is “pure”. Our previous results imply that a pure equilibrium allocation features positive assortative matching, with the wealthier (poorer) buyer and seller types trading with each other (see Figure 2). This is intuitive. Since wealthier buyers care relatively more about trading delays and relatively less about prices, and the opposite is true for wealthier sellers, the greater gains from trade arise when the wealthier (poorer) buyer and seller types trade with each other.

Proposition 11. *(Positive sorting by wealth in pure equilibrium allocations)*

Suppose that all buyers/sellers of a given type participate in the same submarket in equilibrium, and that the assumptions in Lemma 9 hold. Take any two seller types w_s and w'_s which participate in the housing market in equilibrium, with $w'_s < w_s$. In equilibrium type- w_s sellers trade with type- w_b buyers and type- w'_s sellers trade with type- w'_b buyers, where $w'_b < w_b$. Moreover, the submarket where type- w_s sellers and type- w_b buyers trade is characterized by higher prices and higher (lower) trading probabilities for buyers (sellers).

In general, agents who are ex-ante alike may end up trading in different submarkets (i.e., at different prices). For instance, consider a setting with a finite number of buyer and seller types, denoted by N_b and N_s respectively. In this economy, the measures ξ^b and ξ^s have a finite support. For simplicity, suppose that all buyers participate in the market in equilibrium (e.g. the lowest buyer type in the economy is sufficiently rich), and so do all sellers (e.g. $\underline{h} = 0$). By Proposition 10, the number of active submarkets is at least $\max\{N_b, N_s\}$. But then, if $N_b \neq N_s$, the allocation cannot be pure. For instance, if $N_b > N_s$, some seller types must be trading in more than one submarket. Suppose that type- w_s sellers participate in both θ and θ' , being indifferent between the two submarkets. The buyers who participate in θ and θ' are of different types (by Lemma 6), and prices are lower in the submarket with higher tightness (by Lemma 3). Now, in equilibrium, the indifference curve of a type- w_s seller must then be tangent to the indifference curves of the two buyer types at $(\theta, p^*(\theta))$ and $(\theta', p^*(\theta'))$, respectively. This, combined with Lemma 8, implies that the higher buyer type trades in the submarket where prices are higher and where he is more likely to trade (see Figure 3).

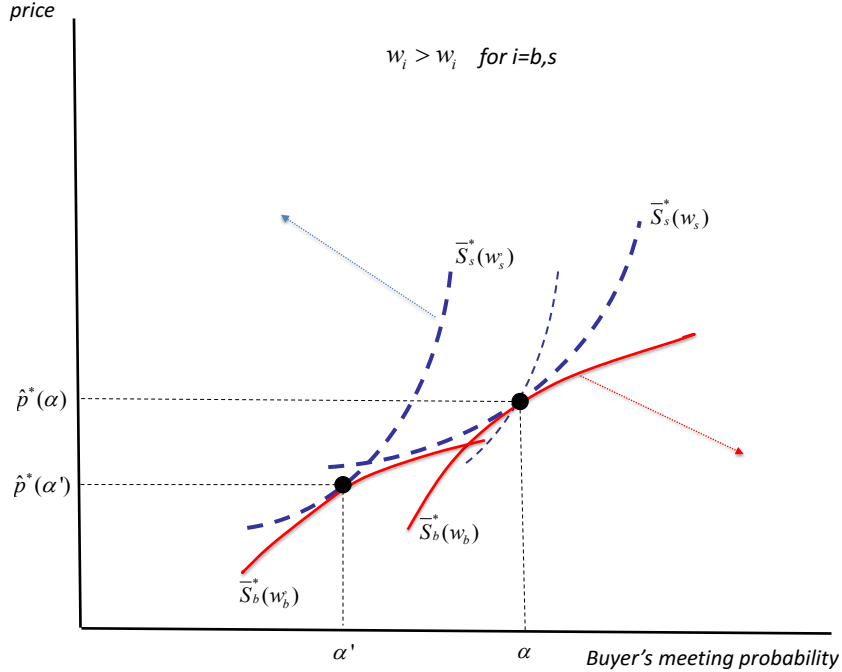


Figure 2: With non-increasing relative risk aversion, a pure equilibrium allocation displays positive assortative matching

Lemma 12. *Suppose that type- w_s sellers participate in submarkets θ and θ' in equilibrium, where $\theta < \theta'$. Then buyers who trade in θ are wealthier than those who trade in θ' .*

A similar result holds for buyer types who participate in more than one submarket (by Lemmas 3, 6 and 9). This case is depicted in Figure 4.

Lemma 13. *Suppose that type- w_b buyers participate in submarkets θ and θ' in equilibrium, where $\theta < \theta'$. Under the conditions in Lemma 9, sellers who trade in θ are wealthier than those who trade in θ' .*

Lemmas 12 and 13 imply that there is positive sorting in equilibrium (whether or not the allocation is pure) in the sense that, *on average*, wealthier agent types trade with each other in more expensive submarkets. To make this point, consider an environment with a finite number of types, where we shall use the following lemma.

Lemma 14. *With a finite number of buyer and seller types, the number of active submarkets is finite.*

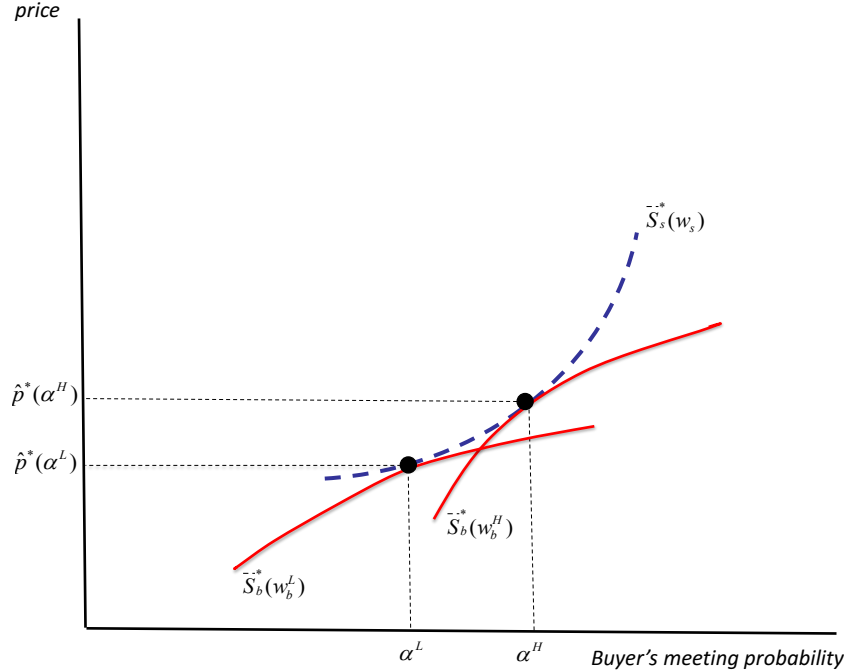


Figure 3: An equilibrium where type- w_s sellers trade with type- w_b^H and type- w_b^L buyers.

Proposition 15 shows that, as we shift to active submarkets with higher congestion θ , the types of the buyers and sellers who join these submarkets either increase or remain constant. Also, if the buyer type remains constant, the seller type necessarily increases, and vice versa. This implies that there is positive sorting *on average*. That is, on average, wealthier sellers receive a higher price when their property is sold but are less likely to trade than poorer sellers. On the other hand, on average, wealthier buyers pay higher prices and are more likely to trade than poorer ones.

Proposition 15. *Suppose that there is a finite number of buyer and seller types, denoted by $N_b, N_s > 1$, and that the conditions in Proposition 9 hold. Take two arbitrary active submarkets θ, θ' with $\theta < \theta'$, and let (w_b, w_s) and (w'_b, w'_s) be the types of the traders who participate in these submarkets in equilibrium, respectively. Then $w'_i \leq w_i$ for $i = b, s$, where the equality holds for at most one i .*

The former result implies that, if an agent type participates in two different submarkets with tightness θ and θ' , that type must also participate in any active submarket where the tightness level lies between θ and θ' .

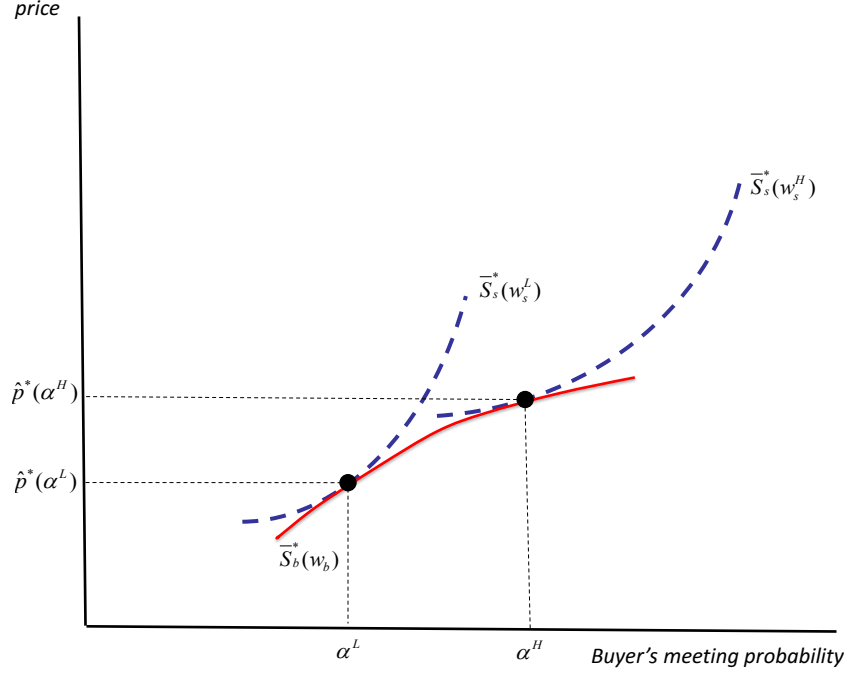


Figure 4: An equilibrium where type- w_b buyers trade with type- w_s^H and type- w_s^L sellers.

Corollary 16. *Suppose the conditions in Proposition 15 hold. If two different active submarkets θ and θ' are joined by agents of type w_i , then so is any active submarket $\hat{\theta} \in (\theta, \theta')$.*

5.3 Indeterminacy of prices in inactive submarkets and price selection rule

A standard feature of general equilibrium models with a continuum of differentiated commodities is that the prices of the commodities that are not traded in equilibrium are indeterminate. A related issue arises in directed search models where out-of-equilibrium beliefs are indeterminate and refinements are imposed to pin down these beliefs (e.g. Peters, 1997; Eeckhout and Kircher, 2010).

Let us describe a submarket by the associated level of α , and the equilibrium price system by $\hat{p}^*(\alpha) = p^*(\alpha^1(\cdot))$. Note that \hat{p}^* is a function mapping $[0, 1]$ into \mathfrak{R}_+ , with the following properties. First, \hat{p}^* is continuous since $p^*(\theta)$ and $\alpha(\theta)$ are continuous. Second, \hat{p}^* lies *between* the indifference curves that all buyer types attain in equilibrium and the indifference curves that all seller types attain in equilibrium. In other words, it lies between the lower envelope of the sellers' indifference

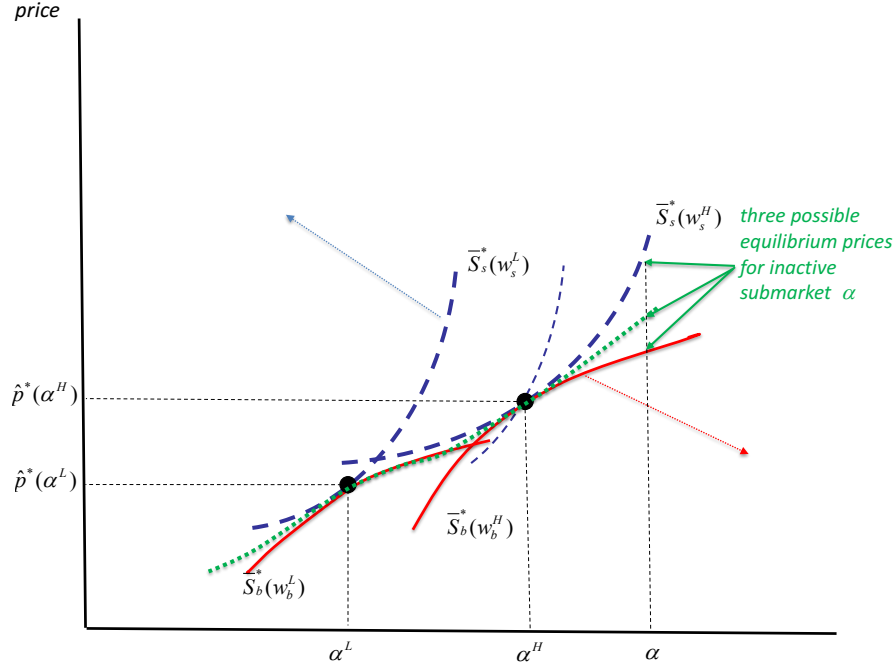


Figure 5: Prices in inactive submarkets are indeterminate

curves and the upper envelope of the buyers' indifference curves. Third, \hat{p}^* passes through all the tangency points between the indifference curves of the buyer type and the seller type which participate in each *active* submarket. So, if α is active, $\hat{p}^*(\alpha)$ is equal to the marginal rates of substitution of the buyer and the seller types who participate in α evaluated at $(\alpha, \hat{p}^*(\alpha))$.

To illustrate, take the example of the pure equilibrium allocation depicted in Figure 2. In this example, there are two buyer and two seller types, and two active submarkets, α^L and α^H . Values of $\alpha \in [0, 1]$ different from α^L and α^H correspond to inactive submarkets. These submarkets are inactive because, given the agents' equilibrium payoffs, the highest price a buyer is willing to pay in α is lower than the minimum price a seller is willing to accept. The value of $\hat{p}^*(\alpha)$, in this case, is indeterminate. It could be as high as the minimum price sellers are willing to accept, as low as the maximum price buyers are willing to pay, or anything in between. In other words, $\hat{p}^*(\alpha)$ could lie on the indifference curve of the seller type who is willing to accept a lower price in submarket α , on the indifference curve of the buyer type who is willing to pay the highest price in α , or anywhere in between. All these prices are consistent with α being inactive. A possible selection criterion is to take the infimum over the set of supporting prices —the “cheapest” supporting prices. The selected

price system $\hat{p}^*(\alpha)$ corresponds to the upper envelope of the indifference curves that the buyer types who participate in the housing market attain in equilibrium, which is a strictly increasing function.¹⁰ Alternatively, we could select the highest supporting prices, which correspond to the lower envelope of the indifference curves that the seller types who participate attain in equilibrium, so again $\hat{p}^*(\alpha)$ is strictly increasing.

6 Conclusion

We analyze a static competitive search model with full information where individual traders are risk averse and have different wealth levels. To the best of our knowledge, the case of two-sided risk aversion has not been studied in the directed search literature. This paper is a first step in filling this gap. The real estate market is a particularly relevant application of our model since houses are indivisible big-ticket items.

The key departure from the directed search literature is that we use the price-taking approach in Jerez (2014) to define a competitive search equilibrium. This allows us to use the tools and intuitions of general equilibrium theory to derive the First Welfare Theorem in our environment. Other properties of the equilibrium are also derived—including the generalized version of the Hosios (1990) rule in our environment. Under risk aversion, buyers and sellers evaluate the trade-off between prices and trading probabilities differently as their wealth increases. Richer buyers care more about increasing their trading probability than poorer buyers and less about paying a higher price in return. On the other hand, richer sellers care less about increasing the probability of a sale than poorer sellers, and they care more about trading at a higher price. This results in positive sorting in equilibrium, that is, wealthier (poorer) buyers and sellers trading with each other at higher (lower) prices. By contrast, with transferable utility, all individuals trade at the same price, irrespectively of their wealth.

We have assumed, for simplicity, that all homes are identical. Yet the argument that is used to establish the First Welfare Theorem is general and applies to an environment where houses differ in their hedonic attributes. Indeed, in the real world, wealthier buyers typically purchase higher quality homes, which in turn tend to be owned by wealthier sellers. Still, our characterization

¹⁰With this price selection criterion, $p^*(\theta)$ is strictly decreasing on its entire domain because α is a decreasing function of θ .

results should still hold within each market segment: conditional on the hedonic attributes of the property, there is also positive sorting by wealth.

Díaz et al. (2019) study a related dynamic model of the real estate market which allows for credit. There, home buyers are risk averse, and houses are sold by risk-neutral intermediaries. So their competitive search model is effectively one with one-sided risk aversion. The advantage of their simpler structure is that the equilibrium is “block recursive”, meaning that the households’ policy and value function do not depend on the endogenous wealth distribution (see also Hedlund, 2016). The focus in Díaz et al. (2019) is also different and consists of studying the effect of credit conditions on house prices. Still, in their stationary equilibrium, wealthier buyers also pay higher prices for identical homes than poorer buyers, and they face shorter trading delays. Establishing this result in a dynamic setting is more involved, in the sense that needs to show first that the households’ value functions have standard properties (mainly, continuity, monotonicity, and concavity), and this is not trivial. In any case, the results in Díaz et al. (2019) and in the current paper suggest that increases in wealth inequality may amplify into higher house price dispersion.

Equilibrium existence is a very important topic that is, unfortunately, outside the scope of this paper. Existence of a competitive search equilibrium has been established in environments with transferable utility (e.g. Eeckhout and Kircher, 2010; Jerez, 2014) and one-sided risk aversion (e.g. Acemoglu and Shimer, 1999), but again there are no results for the case of two-sided risk aversion. From our general equilibrium perspective, this is not surprising. First of all, it is well-known that establishing existence of equilibrium is more involved than establishing the First Welfare Theorem, in particular, because additional (convexity) assumptions ought to be imposed. More importantly, the crucial difference between our price-taking model and the standard (frictionless) Arrow-Debreu model is that markets do not clear. Instead, the relevant aggregate feasibility condition is a consistency condition that depends on the exogenous random matching process that brings buyers and sellers together in the market. The key issue is then whether standard approaches to existence in the general equilibrium literature can be adapted to study competitive search equilibria with two-sided risk aversion. We leave this issue for future work. Our conjecture is that the general equilibrium approach in this paper is likely to prove useful to study existence, and well as to derive algorithms that allow to compute the equilibrium in applications.

7 Acknowledgements

I am grateful to Antonia Díaz, Emma Moreno, and Juan Pablo Rincón for their comments. This work was supported by the Spanish Ministerio Economía y Competitividad, grants ECO2016-76818, PID2019-107161GB-C31 and MDM 2014-0431, and Comunidad de Madrid, MadEco-CM (S2015/HUM-3444). Any errors are mine. Declarations of interest: none.

A Appendix

Proof of Lemma 3. Suppose not. Given the strict monotonicity of π_b and $u(\cdot, \bar{h})$, buyers strictly prefer θ' to θ irrespectively of their type. Hence, condition (i) in Definition 1 implies that $\theta \notin \text{supp}\mu_{\Theta}^{b*}$ (a contradiction). Also, for any active submarket θ , $U_s(w_s, \theta; p) \geq 0$. Since $u(w_s, 0) < u(w_s, \underline{h})$, this implies $p^*(\theta) > 0$. \square

Proof of Lemma 6. The indifference curves of the buyer and the seller in the space (α, p) are described by

$$\bar{S}_b(w_b) = q\alpha [u(w_b - p, \bar{h}) - u(w_b, 0)], \quad (\text{A.1})$$

$$\bar{S}_s(w_s) = q\hat{m}(\alpha) [u(w_s + p, 0) - u(w_s, \underline{h})]. \quad (\text{A.2})$$

Buyers prefer low values of p and high values of α , and the opposite is true for sellers. The buyer's marginal rate of substitution of p for α is

$$MRS_b(\alpha, p) = \left(\frac{u(w_b - p, \bar{h}) - u(w_b, 0)}{\alpha u_c(w_b - p, \bar{h})} \right). \quad (\text{A.3})$$

This rate decreases monotonically as both p and α increase along the buyer's indifference curve because $u(\cdot, \bar{h})$ is concave. The marginal rate of substitution of the seller is

$$MRS_s(\alpha, p) = \left(\frac{-m'(\alpha^{-1})}{m(\alpha^{-1})} \right) \left(\frac{u(w_s + p, 0) - u(w_s, \underline{h})}{u_c(w_s + p, 0)} \right). \quad (\text{A.4})$$

This rate increases monotonically when both p and α increase along the seller's indifference curve, since $u(\cdot, 0)$ is concave and $m(\alpha^{-1})$ is decreasing and concave in α . \square

Proof of Lemma 7. The buyer's ex-post utility gain from trading price $p > 0$ is $u(w_b - p, \bar{h}) - u(w_b, 0)$. The derivative of this term with respect to w_b is $u_c(w_b - p, \bar{h}) - u_c(w_b, 0)$. This derivative is strictly positive. This is because $u_c(w_b - p, \bar{h}) \geq u_c(w_b - p, 0) > u_c(w_b, 0)$, since u_c is non-decreasing in h and strictly decreasing in c . Similarly, the sellers' ex-post utility gain, $u(w_s + p, 0) - u(w_s, \underline{h})$, is decreasing in w_s . That is, $u_c(w_s + p, 0) - u_c(w_s, \underline{h}) < 0$, since $u_c(w_s + p, 0) < u_c(w_s, 0) \leq u_c(w_s, \underline{h})$. \square

Proof of Lemma 8. The term $[u(w_b - p, \bar{h}) - u(w_b, 0)]/u_c(w_b - p, \bar{h})$ in (A.3) is positive provided the numerator (i.e., the buyer's gains from trading at price p) is positive. The denominator is strictly decreasing in w_b because u_c is strictly decreasing, and the numerator is increasing by Lemma 7. \square

Proof of Lemma 9. The derivative of the sellers' gain in (A.4) with respect to w_s is

$$\frac{[u_c(w_s + p, 0) - u_c(w_s, \underline{h})]u_c(w_s + p, 0) - [u(w_s + p, 0) - u(w_s, \underline{h})]u_{cc}(w_s + p, 0)}{(u_c(w_s + p, 0))^2}. \quad (\text{A.5})$$

The sign of this derivative is that of the numerator in (A.5). The first term of the numerator is negative since $u_c(w_s, \underline{h}) \geq u_c(w_s, 0) > u_c(w_s + p, 0)$. The second term is positive because u_c is decreasing. So, in principle, the sign of the numerator is indeterminate.

Define the coefficient of relative risk aversion associated to $u(\cdot, 0)$:

$$R(a, 0) = \frac{-u_{cc}(a, 0)a}{u_c(a, 0)}.$$

The numerator in (A.5) can be written as

$$\begin{aligned} & [u_c(w_s + p, 0) - u_c(w_s, \underline{h})]u_c(w_s + p, 0) + [u(w_s + p, 0) - u(w_s, \underline{h})] \left(\frac{R(w_s + p, 0)u_c(w_s + p, 0)}{w_s + p} \right) \\ &= \frac{u_c(w_s + p, 0)}{w_s + p} ([u_c(w_s + p, 0) - u_c(w_s, \underline{h})](w_s + p) + [u(w_s + p, 0) - u(w_s, \underline{h})]R(w_s + p, 0)). \end{aligned}$$

The first term in this equation is positive. Take the second term. Recall that $u_c(w_s, \underline{h}) \geq u_c(w_s, 0) > u_c(w_s + p, 0)$ and $u(w_s, 0) > u(w_s, \underline{h})$. Hence, when $p = 0$, the term in brackets is negative. We now show that the derivative of this term with respect to p is negative, which in turn implies that the expression in (A.6) is negative for any $p > 0$. This derivative is equal to

$$u_{cc}(w_s + p, 0)(w_s + p) + u_c(w_s + p, 0)[1 + R(w_s + p, 0)] - u_c(w_s, \underline{h}) + [u(w_s + p, 0) - u(w_s, \underline{h})]R_c(w_s + p, 0).$$

Using the definition of $R(a, 0)$, the above equation can be expressed as

$$u_c(w_s + p, 0) - u_c(w_s, \underline{h}) + [u(w_s + p, 0) - u(w_s, \underline{h})]R_c(w_s + p, 0). \quad (\text{A.6})$$

Since $R_c(\cdot, 0) \leq 0$, the derivative is negative, as claimed. \square

Proof of Proposition 11. Let a submarket be described by the associated level of α (rather than θ). Let $\hat{p}^*(\alpha)$ denote the equilibrium price in submarket α ; i.e., $\hat{p}^*(\alpha) = p^*(\alpha^1(\cdot))$. Suppose that α is joined by buyers of type w_b and sellers of type w_s . Consider sellers of type $w'_s < w_s$. By Lemma 9, the indifference curve of this seller type at $(\alpha, \hat{p}^*(\alpha))$ is steeper than that of w_s . Also, any two indifference curves corresponding to types w_s and w'_s cross at most once (i.e., the single crossing property holds). But then, since $(\alpha, \hat{p}^*(\alpha))$ is not optimal for them, type- w'_s sellers must necessarily choose a submarket $\alpha' > \alpha$ where prices are lower. (The single crossing property implies that, if they chose a submarket $\alpha'' < \alpha$ where prices were higher, sellers of type w_s would also prefer α'' to α ; a contradiction). Similarly, take an arbitrary buyer type $w'_b < w_b$. Lemma 8 implies that the single crossing property holds, and that the indifference curve of this buyer type at $(\alpha, \hat{p}^*(\alpha))$ is flatter than that of w_b . But then, since $(\alpha, \hat{p}^*(\alpha))$ is not optimal for them, type- w'_b buyers must necessarily choose a submarket $\alpha' > \alpha$ where prices are lower. \square

Proof of Lemma 14. This is straightforward. A given seller type can trade with a finite number of buyer types (and vice versa). By Lemma 6, there is a unique submarket where the bilateral trade with each such buyer would take place. This implies that the set of submarkets where each seller type trades is finite. Since there is a finite number of sellers, the set of active submarkets is also finite. \square

Proof of Proposition 15. If ξ^b and ξ^s have finite support, the set of active submarkets is finite (by Lemma 14). Order this set as $\{\theta_1, \dots, \theta_N\}$, so $\theta_n < \theta_{n+1}$ for all $n = 1, \dots, N - 1$. Take two consecutive elements θ_n and θ_{n+1} . Let (w_b^n, w_s^n) and (w_b^{n+1}, w_s^{n+1}) be the types of the traders who participate in these submarkets in equilibrium, respectively. There are two possibilities. The first is that $w_i^{n+1} \neq w_i^n$ for both $i = b, s$, in which case the arguments in the proof of Proposition 11 imply that $w_i^{n+1} > w_i^n$ for $i = b, s$. The second is that $w_i^{n+1} = w_i^n$ for at most one i (recall Lemma 6). Say, $w_s^{n+1} = w_s^n$. In this case, Lemma 12 implies that $w_b^{n+1} < w_b^n$. Otherwise, $w_b^{n+1} = w_b^n$, and Lemma 13 implies that $w_s^{n+1} < w_s^n$. \square

References

- Acemoglu, D. and Shimer, R. (1999). Holdups and efficiency with search frictions. *International Economic Review*, 40(4):827–49.
- Díaz, A., Jerez, B., and Rincón-Zapatero, J. P. (2019). Housing prices and credit constraints in competitive search. Economics working papers, UC3M.
- Eeckhout, J. and Kircher, P. (2010). Sorting and decentralized price competition. *Econometrica*, 78(2):539–574.
- Eeckhout, J. and Sepahsalari, A. (2020). The Effect of Wealth on Worker Productivity. Economics discussion papers, University of Bristol.
- Garibaldi, P. and Moen, E. R. (2010). Job to job movements in a simple search model. *American Economic Review*, 100(2):343–47.
- Garriga, C. and Hedlund, A. (2020). Mortgage Debt, Consumption, and Illiquid Housing Markets in the Great Recession. *American Economic Review*, 110(6):1603–1634.
- Guerrieri, V., Julien, B., Kircher, P., and Wright, R. (2021). Directed Search and Competitive Search Equilibrium: A Guided Tour. *Journal of Economic Literature*, 59(1):90–148.
- Hedlund, A. (2016). Illiquidity and its discontents: Trading delays and foreclosures in the housing market. *Journal of Monetary Economics*, 83:1–13.
- Hosios, A. J. (1990). On the efficiency of matching and related models of search and unemployment. *Review of Economic Studies*, 57(2):279–98.
- Jerez, B. (2014). Competitive equilibrium with search frictions: A general equilibrium approach. *Journal of Economic Theory*, 153:252–286.
- Mas-Colell, A. (1975). A model of equilibrium with differentiated commodities. *Journal of Mathematical Economics*, 2(2):263–295.
- Menzio, G. and Shi, S. (2010). Block recursive equilibria for stochastic models of search on the job. *Journal of Economic Theory*, 145(4):1453–1494.
- Moen, E. R. (1997). Competitive search equilibrium. *Journal of Political Economy*, 105:385–411.

Peters, M. (1997). On the equivalence of Walrasian and Non-Walrasian equilibria in contract markets: The case of complete contracts. *Review of Economic Studies*, 64(2):241–64.

Shi, S. (2001). Frictional assignment. i. efficiency. *Journal of Economic Theory*, 98(2):232–260.

Shi, S. (2009). Directed search for equilibrium wage-tenure contracts. *Econometrica*, 77(2):561–584.

Shi, S. and Chaumont, G. (2020). Wealth Accumulation, On the Job Search and Inequality. Mimeo.