95.4.2. *Ordering of Covariance Matrices*, proposed by Götz Trenkler. Let $S = (S_i)_{i=1}^n$ and $T = (T_j)_{j=1}^n$ be $\mathbf{R}^n$-valued random variables, where the covariance matrices $C(S,T) = [(\mathrm{Cov}(S_i, T_j)]_{i,j=1}^n$, $C(S) = C(S,S)$, $C(T) = C(T,T)$ exist. Show the generalized Cauchy–Schwarz inequality

$$C(T,S)\,[C(T)]^+C(S,T) \leq C(S),$$

where $[C(T)]^+$ is the Moore–Penrose inverse of $C(T)$ and $\leq$ denotes the Löwner ordering of matrices.

95.4.3. *The Moore–Penrose Generalized Inverse of a Symmetric Matrix*, proposed by R.W. Farebrother. Let Y be an $m \times m$ symmetric matrix of rank p and let X be an $m \times q$ matrix of rank $q = m - p$ such that $YX = 0$. Then we may select an $m \times q$ matrix S, say $S = X$, such that the augmented matrix

$$Q = Y + XS' + SX'$$

is nonsingular.

Further, let $M = I - XX^+$; then we may express Y in terms of Q and X as $Y = MQM$.

(a) Derive an explicit expression for the Moore-Penrose generalized inverse of Y in terms of Q and X.
(b) Establish conditions under which Y^+ may be expressed in the form $Y^+ = MQ^{-1}M$.

95.4.4. *Derivation of the Fully Modified Estimator*, proposed by Juan J. Dolado. Consider the model

$$y_t = \beta x_t + \varepsilon_t, \qquad t = 1 \ldots T, \tag{1}$$

$$\Delta x_t = u_t, \tag{2}$$

where x_t and y_t are scalar r.v.'s and $v_t = (\varepsilon_t, u_t)'$ is a stationary time series that satisfies the functional central limit theorem

$$T^{-1/2}\Sigma_1^{[T.]} v_t \rightarrow B(\cdot) \equiv BM(\Omega),$$

where $\Omega = \mathrm{lvar}(v_t)$, the long-run variance of v_t and $BM(\Omega)$, is bivariate Brownian motion with covariance matrix Ω given by

$$\Omega = \begin{pmatrix} \omega_{\varepsilon\varepsilon} & \omega_{\varepsilon u} \\ \cdot & \omega_{uu} \end{pmatrix} = \Sigma + \Lambda + \Lambda' = \Delta + \Lambda'$$

with

$$\Sigma = E(v_0 v_0'),\ \Omega = \sum_1^{\infty} E(v_0 v_k').$$

Phillips and Hansen (1990) proved that

$$T(\hat{\beta} - \beta) \to \left(\int B_u dB_\varepsilon + \Delta_{21} \right) \Big/ \int B_u^2$$

where $\hat{\beta}$ is the OLS estimate of β in (1).

Using the transformations

$$W_1 = h(B_\varepsilon - cB_u)$$

$$W_2 = bB_u,$$

where $h^{-2} = \omega_{\varepsilon\varepsilon} - \omega_{\varepsilon u}^2/\omega_{uu}$, $c = \omega_{\varepsilon u}/\omega_{uu}$, $b = \omega_{uu}^{-1/2}$, and $W(\cdot) = (W_1, W_2)'$ is a bivariate standardized Brownian motion, $W(\cdot) \equiv BM(I)$.

(i) Find the limiting distribution of $\hat{\beta}$ in terms on $W(\cdot)$.
(ii) From (i), derive the fully modified estimator suggested by Phillips and Hansen (1990).

REFERENCE

Phillips, P.C.B. & B.E. Hansen (1990) Statistical inference in instrumental variables regressions with I(1) processes. *Review of Economic Studies* 57, 99–125.

SOLUTIONS

94.1.1. *Efficient Estimation under Heteroskedasticity*—Solution, proposed by Jeffrey M. Wooldridge. It suffices to show that $[\text{Avar}\ \sqrt{n}(\beta^* - \beta)]^{-1} - [\text{Avar}\ \sqrt{n}(\hat{\beta} - \beta)]^{-1}$ is positive semidefinite (p.s.d.). Under (1) and (2), standard asymptotic analysis with independent and identically distributed (i.i.d.) data gives

$$\text{Avar}\ \sqrt{n}(\hat{\beta} - \beta) = \sigma^2[E(\mathbf{x}'\mathbf{x})]^{-1}. \tag{5}$$

To analyze β^*, it is useful to write

$$y = \mathbf{x}\beta + u,$$

$$E(u \mid \mathbf{x},\mathbf{z}) = 0,$$

$$E(u^2 \mid \mathbf{x},\mathbf{z}) = h^2(\mathbf{x},\mathbf{z}).$$

For notational ease, define $h^2 \equiv h^2(\mathbf{x},\mathbf{z})$. By standard asymptotics for weighted least squares with i.i.d. data,

$$\text{Avar}\ \sqrt{n}(\beta^* - \beta) = [E(\mathbf{x}'\mathbf{x}/h^2)]^{-1}. \tag{6}$$

Thus, from (5) and (6) it suffices to show that

$$E(\mathbf{x}'\mathbf{x}/h^2) - (1/\sigma^2)E(\mathbf{x}'\mathbf{x}) \tag{7}$$