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Rational Approximation and Sobolev-type Orthogonality

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Abstract

In this paper, we study the sequence of orthogonal polynomials $\{S_n\}_{n=0}^{\infty}$ with respect to the Sobolev-type inner product

$$\langle f,g\rangle = \int_{-1}^{1} f(x)g(x)\,d\mu(x) + \sum_{j=1}^{N} \eta_j f^{(d_j)}(c_j)g^{(d_j)}(c_j),$$

where μ is in the Nevai class $\mathbf{M}(0,1)$, $\eta_j > 0$, $N, d_j \in \mathbb{Z}_+$ and $\{c_1,\ldots,c_N\} \subset \mathbb{R}\setminus[-1,1]$. Under some restriction of order in the discrete part of $\langle\cdot,\cdot\rangle$, we prove that for sufficiently large n the zeros of S_n are real, simple, $n-N$ of them lie on $(-1,1)$ and each of the mass points c_j "attracts" one of the remaining N zeros.

The sequences of associated polynomials $\{S_n^{[k]}\}_{n=0}^{\infty}$ are defined for each $k \in \mathbb{Z}_+$. We prove an analogous of Markov's Theorem on rational approximation to a function of certain class of holomorphic functions and we give an estimate of the "speed" of convergence.

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1 Introduction

Let μ be a finite positive Borel measure whose support $\operatorname{supp}(\mu) \subset [-1,1]$ contains an infinite set of points, and $\{P_n\}_{n\geq 0}$ be the sequence of monic orthogonal polynomials with respect to μ, defined by the relations

$$\langle x^k, P_n\rangle_\mu = \int_{-1}^{1} x^k\, P_n(x)\, d\mu(x) = 0, \qquad k = 0, 1, \ldots, (n-1). \tag{1}$$

These polynomials satisfy the three-term recurrence relation

$$\begin{aligned} P_{n+1}(z) =& (z-b_n)P_n(z) - a_n^2 P_{n-1}(z), \quad n \geq 0, \\ P_{-1}(z) = 0 \quad & \text{and} \quad P_0(z) = 1; \end{aligned} \tag{2}$$

where $a_0 \neq 0$ is an arbitrary constant, $a_n = \|P_n\|_\mu / \|P_{n-1}\|_\mu$ for $n > 0$, $b_n = \langle P_n, x\, P_n\rangle_\mu / \|P_n\|_\mu^2$ and $\|\cdot\|_\mu = \sqrt{\langle \cdot,\cdot\rangle_\mu}$. Usually, an inner product is called standard if the multiplication operator is symmetric with respect to the inner product, i.e., $\langle xf, g\rangle_\mu = \langle f, xg\rangle_\mu$. Clearly, (1) is standard and (2) is an immediate consequence of (1) , which turns out to be an essential tool in the theory of standard orthogonal polynomials.

We say that a measure μ with support $[-1,1]$ is in the Nevai class $\mathbf{M}(0,1)$, $\mu \in \mathbf{M}(0,1)$, if the corresponding sequence of orthogonal polynomials $\{P_n\}_{n\geq 0}$ satisfies the recurrence relation (2), when $\lim_{n\to\infty} a_n = 1/2$ and $\lim_{n\to\infty} b_n = 0$. The condition $\mu' > 0$ a.e. on $[-1,1]$ is a sufficient condition for $\mu \in \mathbf{M}(0,1)$ (c.f. [14, 16]). The class $\mathbf{M}(0,1)$ has been thoroughly studied in [11], where it is proved that $\mu \in \mathbf{M}(0,1)$ is equivalent to

$$\frac{P_{n+1}(z)}{P_n(z)} \underset{n}{\rightrightarrows} \frac{\varphi(z)}{2}, \quad K \subset \Omega = \mathbb{C} \setminus [-1,1], \tag{3}$$

where $\varphi(z) = z + \sqrt{z^2-1}$ ($\sqrt{z^2-1} > 0$ for $z > 1$) is the function which maps the complement of $[-1,1]$ onto the exterior of the unit circle. Throughout this paper, we use the notation $f_n \underset{n}{\rightrightarrows} f$; $K \subset U$ when the sequence of functions f_n converges to f uniformly on every compact subset K of the region U.

Let us denote by $P_n^{[1]}$ the usually called *nth polynomial associated to* P_n, defined by the expression

$$P_n^{[1]}(z) = \int_{-1}^{1} \frac{P_{n+1}(z) - P_{n+1}(x)}{z-x} d\mu(x).$$

Note that $P_n^{[1]}$ is a polynomial of degree n with leading coefficient equal to $\mu([-1,1])$, which satisfies the three-term recurrence relation

$$\begin{aligned} P_{n+1}^{[1]}(z) =& (z-b_{n+1})P_n^{[1]}(z) - a_{n+1}^2 P_{n-1}^{[1]}(z), \quad n \geq 0, \\ P_{-1}^{[1]}(z) = 0 \quad & \text{and} \quad P_0^{[1]}(z) = \mu([-1,1]). \end{aligned} \tag{4}$$

As it is known, some particular families of orthogonal polynomials were studied in detail before a general theory existed. One of the starting points of this theory is closely related to the study of the convergence of certain sequences of rational functions, as can be seen in the first treatises on the subject [17, Ch. I-§4,] and [18, §3,5]. The analysis of the convergence of these sequences entails essential difficulties. One of the first, and most remarkable, general results in this sense is the following theorem established by A. A. Markov in 1895.

Markov's Theorem ([12, Th. 6.1]). *Let μ be a finite positive Borel measure supported in $[-1,1]$. Then*

$$\frac{P_n^{[1]}(z)}{P_{n+1}(z)} \underset{n}{\rightrightarrows} \hat{\mu}(z), \quad K \subset \Omega_\infty = \overline{\mathbb{C}} \setminus [-1,1],$$

where $\hat{\mu}(z) = \displaystyle\int_{-1}^{1} \frac{d\mu(x)}{z-x}$ is known as Markov's function of μ.

Note that $\hat{\mu}(z)$ is well defined and holomorphic in Ω_∞ ($\hat{\mu} \in \mathbb{H}(\Omega_\infty)$ for short). Some examples can be seen in [12, p. 64]. This classical theorem admits several generalizations, some of which are discussed in [1, 2, 3, 5] and references therein.

We define the *discrete Sobolev inner product* through the expression

$$\langle f,g \rangle = \int_{-1}^{1} f(x)g(x)\,d\mu(x) + \sum_{j=1}^{N}\sum_{i=0}^{d_j} \eta_{j,i} f^{(i)}(c_j)g^{(i)}(c_j); \tag{5}$$

where μ is as above, $N \geq 0$, $\eta_{j,i} \geq 0$, $\eta_{j,d_j} > 0$, $c_j \in \mathbb{R} \setminus [-1,1]$, $d_j \in \mathbb{Z}_+$ and $f^{(i)}$ denotes the ith derivative of a function f.

For $n \in \mathbb{Z}_+$ we denote by S_n the monic polynomial of lowest degree satisfying

$$\langle x^k, S_n \rangle = 0, \quad \text{for } k = 0,1,\ldots,n-1. \tag{6}$$

It is easy to see that for every $n \in \mathbb{Z}_+$, there exists a unique polynomial S_n of degree n. In fact, the existence of such polynomials is deduced by solving a homogeneous linear system with n equations and $n+1$ unknowns. Uniqueness follows from the minimality of the degree for the polynomial solution.

We refer the reader to [9, 10] for a review of this type of non-standard orthogonality. As is well known, most arguments for the standard theory of orthogonal polynomials fail in the Sobolev case. As shown in the next examples, it is no longer true that the zeros lie on the convex hull of the support of the measures involved in the inner product.

Examples.

1. *Set $\langle f,g \rangle = \displaystyle\int_{-1}^{1} f(x)g(x)\,dx + f'(3)g'(3) + f''(2)g''(2)$, then*

$$S_5(x) = x^5 + \frac{11282625}{1995289}x^4 + \frac{202236410}{1795760}x^3 + \frac{28506900}{1995289}x^2 - \frac{438413755}{41901069}x - \frac{11758825}{1995289},$$

whose zeros are approximately $\xi_1 \approx 0.4$, $\xi_2 \approx -0.7$, $\xi_3 \approx 1.1+2i$, $\xi_4 \approx 1.1-2i$ and $\xi_5 \approx 3.8$. Note that three of them are out of $[-1,1]$ and two are not real numbers.

2. *Set* $\langle f,g\rangle = \int_{-1}^{1} f(x)g(x)(1-x)\,dx + f'(3)g'(3) + f''(2)g''(2)$, *then*

$$S_5(x) = x^5 + \frac{57943145}{27312164}x^4 - \frac{242237045}{13656082}x^3 - \frac{522277585}{20484123}x^2 - \frac{53214815}{40968246}x + \frac{220912645}{52141404},$$

whose zeros are approximately $\xi_1 \approx 0.3$, $\xi_2 \approx -0.6$, $\xi_3 \approx -1.1$, $\xi_4 \approx 3.9$ *and* $\xi_5 \approx -4.7$. *Note that three zeros are out of* $[-1,1]$ *and two of them, escape to the opposite side where the mass points are found.*

Definition 1. *Let* $\{(r_j,\nu_j)\}_{j=1}^{M} \subset \mathbb{R}\times\mathbb{Z}_+$ *be a finite sequence of* M *ordered pairs and* $A \subset \mathbb{R}$. *We say that* $\{(r_j,\nu_j)\}_{j=1}^{M}$ *is* sequentially-ordered with respect to A, *if*

1. $0 \le \nu_1 \le \nu_2 \le \cdots \le \nu_M$.
2. $r_k \notin \mathbf{C}_h(A\cup\{r_1,r_2,\ldots,r_{k-1}\})$ *for* $k=1,2,\ldots,M$; *where* $\mathbf{C}_h(B)$ *denotes the convex hull of an arbitrary set* $B\subset\mathbb{C}$.

If $A=\emptyset$, *we say that* $\{(r_j,\nu_j)\}_{j=1}^{M}$ *is* sequentially-ordered *for brevity.*

We say that the discrete Sobolev inner product (5) *is* sequentially-ordered, *if the set of ordered pairs* $\{(c_j,i) : 1\le j\le N, 0\le i\le d_j$ *and* $\eta_{j,i}>0\}$ *may be arranged to form a finite sequence of ordered pairs which is sequentially ordered with respect to* $(-1,1)$.

From the second condition of Definition 1, the coefficient η_{j,d_j} is the only coefficient $\eta_{j,i}$ $(i=0,1,\ldots,d_j)$ different from zero, for each $j=1,2,\ldots,N$. Hence, (5) takes the form

$$\langle f,g\rangle = \int_{-1}^{1} f(x)g(x)\,d\mu(x) + \sum_{j=1}^{N} \eta_{j,d_j} f^{(d_j)}(c_j)g^{(d_j)}(c_j). \tag{7}$$

Note that the inner products involved in the previous examples are not sequentially-ordered. In most of our work, we will restrict our attention to sequentially-ordered discrete Sobolev inner products. The following theorem shows our reasons for this assumption.

Theorem 1. *If* (7) *is a sequentially-ordered discrete Sobolev inner product, then* S_n *has at least* $n-N$ *changes of sign on* $(-1,1)$.

The previous Theorem is still true if $c_j=-1$ or $c_j=1$, for some j. Furthermore, if $N=1$ in (7), from Theorem 1 we get that all the zeros of S_n are real, simple, and at most one of them is outside of $(-1,1)$.

If $n\le N$, S_n can have changes of sign on $(-1,1)$ or not. For example, if $\sum_{j=1}^{N}\eta_{j,0}=0$, for all $n\ge1$, we have $\langle S_n,1\rangle = \langle S_n,1\rangle_\mu = 0$, which yields that S_n has at least one sign change on $(-1,1)$. On the other hand, if $\langle f,g\rangle = \int_{-1}^{1} f(x)g(x)\,dx + f(6)g(6)$, then $S_1(z)=z-2$, which is negative on $(-1,1)$.

As will be seen in Lemma 3.4, for sequentially-ordered discrete Sobolev inner products, the corresponding orthogonal polynomial S_n with degree n sufficiently large, has all its zeros real and simple, each sufficiently small neighborhood of c_j $(j=1,\ldots,N)$ contains exactly one zero of S_n, and from the Theorem 1 the remaining $n-N$ zeros lie on $(-1,1)$.

Let $\{Q_n\}_{n\geq 0}$ be the sequence of monic orthogonal polynomials with respect to the inner product

$$\langle f,g\rangle_\rho = \int_{-1}^{1} f(x)\, g(x)\, d\mu_\rho(x), \text{ where } \rho(z) = \prod_{c_j<-1} (z-c_j)^{d_j+1} \prod_{c_j>1} (c_j-z)^{d_j+1} \tag{8}$$
$$\text{and } d\mu_\rho(x) = \rho(x)d\mu(x).$$

Note that ρ is a polynomial of degree $d = N + \sum_{j=1}^{N} d_j$ and positive on $[-1,1]$.

Now, we associate to the sequence $\{S_n\}_{n=0}^{\infty}$ the next sequences of polynomials

$$S_n^{[k]}(z) = \int_{-1}^{1} \frac{S_{n+k}(z) - S_{n+k}(x)}{z-x}\, Q_{k-1}(x)\, d\mu_\rho(x), \tag{9}$$

for $k \in \mathbb{N}$ and $n \geq 0$. Additionally, we adopt the convention $S_n^{[0]} = S_n$. We call $\left\{S_n^{[k]}\right\}_{n=0}^{\infty}$ the sequence of *kth polynomials associated to* $\{S_n\}_{n=0}^{\infty}$.

As far as we know, the only extension of Markov's Theorem for Sobolev orthogonal polynomials appears in [8, Th. 5.5], when the inner product (5) is such that $N = 1$, $d_1 = 1$, $c_1 = 0$, $\eta_{1,0} = 0$, and $\eta_{1,1} > 0$. The main aim of the present paper is to prove the following theorem, which provides a natural extension of the Markov's Theorem for the Sobolev case.

Theorem 2 (Extended Markov's Theorem)**.** *Let* (7) *be a sequentially-ordered discrete Sobolev inner product with* $\mu \in \mathbf{M}(0,1)$. *Then, for* $k \in \mathbb{N}$,

$$R_n^{[k]} = \frac{S_n^{[k]}(z)}{S_{n+k}(z)} \underset{n}{\rightrightarrows} \widehat{\mu}_k(z) = \int_{-1}^{1} \frac{Q_{k-1}(x)}{z-x} d\mu_\rho(x), \quad K \subset \Omega_\infty^* = \Omega_\infty \setminus \{c_1, c_2, \ldots, c_N\}. \tag{10}$$

We call $\widehat{\mu}_k$ *the* kth Markov-type function associated with μ_ρ.

Also, in Corollary 2.1, we give the following estimate for the degree of convergence of the sequence of rational functions $\{R_n^{[k]}\}$ to the corresponding Markov-type function $\widehat{\mu}_k$.

$$\limsup_n \left\| \widehat{\mu}_k - R_n^{[k]} \right\|_K^{1/2n} \leq \|\varphi\|_K^{-1} < 1, \quad \text{where } \|f\|_K = \sup_{z\in K} |f(z)|.$$

The rest of the paper is organized as follows. The next section is devoted to the consequences of the quasi-orthogonality of S_n with respect to the measure μ. Sections 3 and 5 contain the proofs of Theorems 1 and 2 respectively, as well as some of their consequences. The Section 4 deals with the auxiliary results for the proof of the main result (Theorem 2).

2 Recurrence relations

Unlike the rest of the paper, the inner product (5) does not necessarily have to be sequentially-ordered in this section.

If $n > d$, from (6), we have that S_n satisfies the following quasi-orthogonality relations with respect to $d\mu_\rho$

$$\langle S_n, f\rangle_\rho = \langle S_n, \rho f\rangle_\mu = \int_{-1}^{1} S_n(x) f(x)\rho\, d\mu(x) = \langle S_n, \rho\, f\rangle = 0, \tag{11}$$

for all $f \in \mathbb{P}_{n-d-1}$, where $\mathbb{P}_n$ is the linear space of polynomials with real coefficients and degree at most $n \in \mathbb{Z}_+$. Hence, *the polynomial S_n is quasi-orthogonal of order d with respect to $d\mu_\rho$* and by this argument we get the next result.

Proposition 2.1. *Let S_n be the n-th orthogonal polynomial with respect to* (5) *and $n > d$, then S_n has at least $(n-d)$ changes of sign on $(-1,1)$.*

Proposition 2.2. *Let $S_n^{[k]}$ be the kth associated polynomial defined by* (9)*. Then $S_n^{[k]}$ is a polynomial of degree n and leading coefficient equal to $\|Q_{k-1}\|^2_{\mu_\rho}$.*

Proof. Let $S_{n+k}(x) = \sum_{i=0}^{n+k} \theta_i\, x^i$ where $\theta_{n+k} = 1$, then

$$\begin{aligned}
S_n^{[k]}(z) &= \int_{-1}^{1} \frac{S_{n+k}(z) - S_{n+k}(x)}{z-x}\, Q_{k-1}(x)\, d\mu_\rho(x) = \sum_{i=1}^{n+k} \theta_i \int_{-1}^{1} \frac{z^i - x^i}{z-x}\, Q_{k-1}(x)\, d\mu_\rho(x) \\
&= \sum_{i=1}^{n+k} \theta_i \int_{-1}^{1} \left(\sum_{j=0}^{i-1} z^{i-1-j} x^j \right) Q_{k-1}(x)\, d\mu_\rho(x) \\
&= \sum_{i=1}^{n+k} \theta_i \sum_{j=0}^{i-1} z^{i-1-j} \left(\int_{-1}^{1} x^j\, Q_{k-1}(x)\, d\mu_\rho(x) \right) = \sum_{i=1}^{n+k} \theta_i \sum_{j=k-1}^{i-1} \langle x^j, Q_{k-1}\rangle_\rho\, z^{i-1-j} \\
&= \sum_{j=k-1}^{n+k-1} \langle x^j, Q_{k-1}\rangle_\rho\, z^{n+k-1-j} + \sum_{i=1}^{n+k-1} \theta_i \sum_{j=k-1}^{i-1} \langle x^j, Q_{k-1}\rangle_\rho\, z^{i-1-j} \\
&= \langle x^{k-1}, Q_{k-1}\rangle_\rho\, z^n + f_{n-1}(z) = \|Q_{k-1}\|^2_{\mu_\rho} z^n + f_{n-1}(z).
\end{aligned}$$

where f_{n-1} is a polynomial of degree at most $n-1$. $\square$

In the standard case of orthogonality, where the polynomials $\{P_n\}$ satisfy the three terms recurrence relation (2), the sequence of associated polynomials $\{P_n^{[1]}\}$ can be generated by the recurrence relation (4). The following proposition is an analogous result for the sequence of associated polynomials $\{S_n^{[k]}\}$.

Proposition 2.3 (Recurrence relation)**.** *For $n \geq 2d-1$, the sequences $\{S_n^{[k]}\}_{n=0}^{\infty}$ satisfy the following $2d+1$ term recurrence relation*

$$\rho(z) S_n^{[k]}(z) = \sum_{j=n-d}^{n+d} \mathfrak{a}_{n+k,j+k} S_j^{[k]}(z), \quad \textit{where} \quad \mathfrak{a}_{n+k,j+k} = \frac{\langle S_{n+k}, \rho S_{j+k}\rangle}{\langle S_{j+k}, S_{j+k}\rangle}. \tag{12}$$

Proof. It is straightforward to obtain (12) for $k=0$ as a consequence of (11), i.e.,

$$\rho(z)S_n(z)=\sum_{j=n-d}^{n+d}\mathfrak{a}_{n.j}\,S_j(z), \quad \text{where } \mathfrak{a}_{n.j}=\frac{\langle S_n,\rho S_j\rangle}{\langle S_j,S_j\rangle}. \tag{13}$$

Hence, if $k>0$

$$\frac{\rho(z)S_{n+k}(z)-\rho(x)S_{n+k}(x)}{z-x}\,Q_{k-1}(x)=\sum_{j=n-d}^{n+d}\mathfrak{a}_{n+k.j+k}\,\frac{S_{j+k}(z)-S_{j+k}(x)}{z-x}\,Q_{k-1}(x),$$

$$\int_{-1}^{1}\frac{\rho(z)S_{n+k}(z)-\rho(x)S_{n+k}(x)}{z-x}\,Q_{k-1}(x)d\mu_\rho(x)=\sum_{j=n-d}^{n+d}\mathfrak{a}_{n+k.j+k}\,S_j^{[k]}(z).$$

As $n\geq 2d-1$, from (11), we get $\displaystyle\int_{-1}^{1}S_{n+k}(x)\,\frac{\rho(z)-\rho(x)}{z-x}\,Q_{k-1}(x)d\mu_\rho(x)=0$. Hence,

$$\begin{aligned}\rho(z)S_n^{[k]}(z)=&\int_{-1}^{1}\frac{\rho(z)(S_{n+k}(z)-S_{n+k}(x))}{z-x}\,Q_{k-1}(x)d\mu_\rho(x)\\&+\int_{-1}^{1}S_{n+k}(x)\,\frac{\rho(z)-\rho(x)}{z-x}\,Q_{k-1}(x)d\mu_\rho(x)\\=&\int_{-1}^{1}\frac{\rho(z)S_{n+k}(z)-\rho(x)S_{n+k}(x)}{z-x}\,Q_{k-1}(x)d\mu_\rho(x),\end{aligned}$$

and we get (12). □

Remember that $\{Q_n\}_{n\geq 0}$ is the sequence of monic orthogonal polynomials with respect to $d\mu_\rho$, which was defined in (8). As it is known, this sequence satisfies the three-term recurrence relation

$$Q_{n+1}(z)=(z-\beta_n)Q_n(z)-\alpha_n^2\,Q_{n-1}(z), \quad n\geq 0, \tag{14}$$

where $Q_{-1}=0$, $Q_0=1$, $\|\cdot\|_{\mu_\rho}^2=\langle\cdot,\cdot\rangle_\rho$, $\beta_n=\langle Q_n,xQ_n\rangle_\rho/\|Q_n\|_{\mu_\rho}^2$, $\alpha_n=\|Q_n\|_{\mu_\rho}/\|Q_{n-1}\|_{\mu_\rho}$ and $\alpha_0^2=\displaystyle\int_{-1}^{1}d\mu_\rho(x)$.

Following [19], we define its kth sequence of associated polynomials $\{Q_n^{[k]}\}$ ($k\in\mathbb{Z}_+$) as

$$Q_n^{[k]}(z)=\int_{-1}^{1}\frac{Q_{n+k}(z)-Q_{n+k}(x)}{z-x}\,Q_{k-1}(x)d\mu_\rho(x), \tag{15}$$

where $Q_n^{[0]}=Q_n$. Note that $Q_n^{[k]}$ is a polynomial in z of degree n. From [19, (1.3) and (2.13)]

$$Q_{n+1}^{[k]}(x)=(x-\beta_{n+k})Q_n^{[k]}(x)-\alpha_{n+k}^2Q_{n-1}^{[k]}(x). \tag{16}$$

The next proposition is analogous to [19, (2.5)] for the Sobolev case.

Proposition 2.4. *For $n\geq d-1$, the sequences $\{S_n^{[k]}\}_{n=0}^{\infty}$, for $k\geq 2$, hold the following relation*

$$S_n^{[k]}(z)=(z-\beta_{k-2})S_{n+1}^{[k-1]}(z)-\alpha_{k-2}^2S_{n+2}^{[k-2]}(z). \tag{17}$$

Proof. From (14)-(15),

$$
\begin{aligned}
S_n^{[k]}(z) &= \int_{-1}^{1} \frac{S_{n+k}(z)-S_{n+k}(x)}{z-x} \left((x-\beta_{k-2})Q_{k-2}(x)-\alpha_{k-2}^2 Q_{k-3}(x)\right) \, d\mu_\rho(x) \\
&= \begin{cases} \displaystyle\int_{-1}^{1} \frac{S_{n+k}(z)-S_{n+k}(x)}{z-x} xQ_{k-2}(x)\, d\mu_\rho(x) - \beta_{k-2} S_{n+1}^{[k-1]}(z) - \alpha_{k-2}^2 S_{n+2}^{[k-2]}(z), & \text{if } k \geq 3, \\ \displaystyle\int_{-1}^{1} \frac{S_{n+2}(z)-S_{n+2}(x)}{z-x} x d\mu_\rho(x) - \beta_0 S_{n+1}^{[1]}(z), & \text{if } k = 2. \end{cases}
\end{aligned} \tag{18}
$$

From orthogonality,

$$
\int_{-1}^{1} \frac{S_{n+k}(z)-S_{n+k}(x)}{z-x} \,(z-x)\, Q_{k-2}(x)\, d\mu_\rho(x) = \begin{cases} 0, & \text{if } k \geq 3, \\ \alpha_0^2 S_{n+2}(z), & \text{if } k = 2. \end{cases}
$$

Therefore,

$$
\int_{-1}^{1} \frac{S_{n+k}(z)-S_{n+k}(x)}{z-x} xQ_{k-2}(x) d\mu_\rho(x) = \begin{cases} zS_{n+1}^{[k-1]}(z), & \text{if } k \geq 3, \\ zS_{n+1}^{[1]}(z) - \alpha_0^2 S_{n+2}(z), & \text{if } k = 2. \end{cases} \tag{19}
$$

Substituting (19) into (18), we get (17). □

3 Proof of Theorem 1

In the remainder of the paper, we assume that (5) is sequentially-ordered. Therefore, we can rewrite (5) as (7) with $0 \leq d_1 \leq d_2 \leq \cdots \leq d_N$. The next lemma is an extension of [7, Lemma 2.1].

Lemma 3.1. *Let L be a polynomial with real coefficients of degree $\geq m \in \mathbb{N}$, $\{\Delta_i\}_{i=0}^m$ be a set of intervals on the real line, and $I_k = \mathbf{C}_h\left(\cup_{i=0}^k \Delta_i\right)$ for $k = 0, 1, \ldots, m$. If*

$$
I_{k-1} \cap \Delta_k = \emptyset, \qquad k = 1, 2, \ldots, m; \tag{20}
$$

then

$$
\sum_{i=0}^{m} \mathscr{N}_0\left(L^{(i)}; \Delta_i\right) \leq \mathscr{N}_0\left(L^{(m)}; I_m\right) + m \leq \deg(L), \tag{21}
$$

where for a given non-null polynomial f and $A \subset \mathbb{R}$ the symbol $\mathscr{N}_0(f;A)$ denotes the total number of zeros (counting multiplicities) of f on A.

Proof. For $m = 0$, it is straightforward that $\mathscr{N}_0(L;\Delta_0) \leq \mathscr{N}_0(L;\Delta_0) + 0 \leq \deg(L)$. We now proceed by induction on m. Suppose that we have $\kappa + 1$ intervals $\{\Delta_i\}_{i=0}^{\kappa}$ that satisfy (20), and that (21) is true for the first $\kappa - 1$ intervals.

From Rolle's Theorem, $\mathscr{N}_0(f;A) \leq \mathscr{N}_0(f';A)+1$, where A is an interval of the real line and f' a non-null polynomial with real coefficients. Therefore,

$$\begin{aligned}
\sum_{i=0}^{\kappa} \mathscr{N}_0\left(L^{(i)};\Delta_i\right) &= \sum_{i=0}^{\kappa-1} \mathscr{N}_0(L^{(i)};\Delta_i) + \mathscr{N}_0\left(L^{(\kappa)};\Delta_\kappa\right) \\
&\leq \mathscr{N}_0\left(L^{(\kappa-1)};I_{\kappa-1}\right) + (\kappa-1) + \mathscr{N}_0\left(L^{(\kappa)};\Delta_\kappa\right) \\
&\leq \mathscr{N}_0\left(L^{(\kappa)};I_{\kappa-1}\right) + 1 + \mathscr{N}_0\left(L^{(\kappa)};\Delta_\kappa\right) + (\kappa-1) \\
&\leq \mathscr{N}_0\left(L^{(\kappa)};I_{\kappa-1}\cup\Delta_\kappa\right) + \kappa \leq \mathscr{N}_0\left(L^{(\kappa)};I_\kappa\right) + \kappa \leq \deg(L).
\end{aligned}$$

□

Lemma 3.2. *Let $\{(r_i,\nu_i)\}_{i=1}^{M}$ be a sequence of M ordered pairs which is sequentially-ordered. Then, there exists a unique monic polynomial U_M of minimal degree, such that*

$$U_M^{(\nu_i)}(r_i) = 0 \quad \textit{for } i = 1,2,\ldots,M. \tag{22}$$

Furthermore, the degree of U_M is $\kappa_M = \min \mathfrak{I}_M - 1$, where $\mathfrak{I}_M = \{i : 1 \leq i \leq M \text{ and } \nu_i \geq i\} \cup \{M+1\}$.

Proof. The existence of a not identically zero polynomial with degree $\leq M$ satisfying (22) reduces to solving a homogeneous linear system of M equations on $M+1$ unknowns (its coefficients). Thus, a non trivial solution always exists. In addition, if we suppose that there exist two different minimal monic polynomials U_M and $\widetilde{U}_M$, then the polynomial $\widehat{U}_M = U_M - \widetilde{U}_M$ is not identically zero, it satisfies (22), and $\deg(\widehat{U}_M) < \deg(U_M)$. So, if we divide $\widehat{U}_M$ by its leading coefficient, we reach a contradiction.

The rest of the proof runs by induction on the number of points M. For $M = 1$, the result follows taking

$$U_1(x) = \begin{cases} x - r_1 & , \text{ if } \nu_1 = 0, \\ 1 & , \text{ if } \nu_1 \geq 1. \end{cases}$$

Suppose that, for each sequentially-ordered sequence of M ordered pairs, the corresponding minimal polynomial U_M has degree κ_M.

Let $\{(r_i,\nu_i)\}_{i=1}^{M+1}$ be a sequentially-ordered sequence of $M+1$ ordered pairs. Obviously, $\{(r_i,\nu_i)\}_{i=1}^{M}$ is also sequentially-ordered, $\deg(U_{M+1}) \geq \deg(U_M)$, and from the induction hypothesis $\deg(U_M) = \kappa_M$. Now, we shall divide the proof in two cases:

1. If $\kappa_{M+1} = M+1$, then for all $1 \leq i \leq M+1$ we have $\nu_i < i$, which yields

   $$\deg(U_{M+1}) \geq \deg(U_M) = \kappa_M = M \geq \nu_{M+1}. \tag{23}$$

   Let $\Delta_k = \mathbf{C}_h(\{c_i : \nu_i = k\})$ for $k = 0,1,2,\ldots,\nu_{M+1}$. As $\{(r_i,\nu_i)\}_{i=1}^{M+1}$ is sequentially-ordered, the set of intervals $\{\Delta_k\}_{k=0}^{\nu_{M+1}}$ satisfy (20). Therefore, from (23) and Lemma 3.1 we get

   $$M+1 \leq \sum_{i=0}^{\nu_{M+1}} \mathscr{N}_0\left(U_{M+1}^{(i)};\Delta_i\right) \leq \deg(U_{M+1}),$$

   which implies that $\deg(U_{M+1}) = M+1 = \kappa_{M+1}$.

2. If $\kappa_{M+1} \leq M$, then there exists a minimal j $(1 \leq j \leq M+1)$, such that $\nu_j \geq j$, and $\nu_i < i$ for all $1 \leq i \leq j-1$. Therefore, $\kappa_{M+1} = j-1 = \kappa_M$. From the induction hypothesis

$$\deg(U_M) = \kappa_M = j-1 \leq \nu_j - 1 \leq \nu_{M+1} - 1,$$

which gives $U_M^{(\nu_{M+1})} \equiv 0$. Hence, $U_{M+1} \equiv U_M$ and $\deg(U_{M+1}) = \deg(U_M) = \kappa_M = \kappa_{M+1}$.

□

Observe that, in Lemma 3.2, the assumption of $\{(r_i,\nu_i)\}_{i=1}^{M}$ being sequentially-ordered is necessary for asserting that the polynomial U_M has degree κ_M. In fact, if we consider the non sequentially-ordered sequence $\{(-1,0),(1,0),(0,1)\}$, we get $U_3 = x^2-1$ and $\kappa_3 = 3 \neq \deg(U_3)$.

Proof of Theorem 1. From the sequentially-ordered conditions, the intervals

$$\Delta_0 = \mathbf{C}_h\left((-1,1)\cup\{c_i : d_i = 0\}\right) \quad , \quad \Delta_k = \mathbf{C}_h\left(\{c_i : d_i = k\}\right) \quad \text{for } \ k = 1,2,\ldots,N,$$

satisfy (20).

Let $\xi_1 < \xi_2 < \cdots < \xi_\ell$ be the points on $(-1,1)$ where S_n changes sign and suppose that $\ell < n-N$. Let $\{(r_i,\nu_i)\}_{i=1}^{N+\ell}$ be the sequentially-ordered sequence

$$(r_i,\nu_i) = \begin{cases} (\xi_i,0), & \text{if } i = 1,2,\ldots,\ell; \\ (c_{i-\ell},d_{i-\ell}), & \text{if } i = \ell+1,\ell+2,\ldots,\ell+N. \end{cases}$$

From Lemma 3.2, there exists a unique monic polynomial $U_{N+\ell}$ of minimal degree, such that

$$U_{N+\ell}^{(\nu_i)}(r_i) = 0; \qquad \text{for } \ i = 1,\ldots,N+\ell.$$

Furthermore,

$$\deg(U_{N+\ell}) = \min \mathfrak{I}_{N+\ell} - 1 \leq N+\ell, \tag{24}$$

where $\mathfrak{I}_{N+\ell} = \{i : 1 \leq i \leq N+\ell \ \text{ and } \ \nu_i \geq i\} \cup \{N+\ell+1\}$. Now, we need to consider the following two cases.

1. If $\deg(U_{N+\ell}) < N+\ell$, from (24), there exists $1 \leq j \leq N+\ell$ such that $\deg(U_{N+\ell}) = j-1$, $\nu_j \geq j$ and $\nu_i \leq i-1$ for $i = 1,2,\ldots,j-1$. Hence, $\nu_{j-1}+1 \leq j-1 = \deg(U_{N+\ell})$. Thus, from Lemma 3.1,

$$j-1 \leq \sum_{k=0}^{\nu_{j-1}} \mathcal{N}_0\left(U_{N+\ell}^{(k)};\Delta_k\right) \leq \deg(U_{N+\ell}) = j-1,$$

2. If $\deg(U_{N+\ell}) = N+\ell$, from (24), we get $\deg(U_{N+\ell}) = N+\ell \geq \nu_{\ell+N}+1 = d_N+1$ and from Lemma 3.1,

$$N+\ell \leq \sum_{k=0}^{d_N} \mathcal{N}_0\left(U_{N+\ell}^{(k)};\Delta_k\right) \leq \deg(U_{N+\ell}) = N+\ell,$$

In both cases, we obtain that $U_{N+\ell}$ has simple zeros on $(-1,1)\subset\Delta_0$ and has no other zeros than those given by construction. Now, since $\deg(U_{N+\ell})\leq \ell+N<n$, we arrive at the contradiction

$$
\begin{aligned}
0=\langle S_n,U_{N+\ell}\rangle &= \int_{-1}^{1} S_n(x)U_{N+\ell}(x)\,d\mu(x)+\sum_{j=1}^{N}\eta_{j,d_j}S_n^{(d_j)}(c_j)U_{N+\ell}^{(d_j)}(c_j)\\
&=\int_{-1}^{1} S_n(x)U_{N+\ell}(x)\,d\mu(x)\neq 0.
\end{aligned}
$$

□

The following Lemma is a direct consequence of [6, (1.10)], when instead of the inner product [6, (1.1)], we consider (7).

Lemma 3.3. *Consider the sequentially-ordered inner product* (7) *with* $\mu\in\mathbf{M}(0,1)$. *Then,*

$$
\frac{S_n(z)}{P_n(z)} \underset{n}{\rightrightarrows} \prod_{j=1}^{N}\frac{(\varphi(z)-\varphi(c_j))^2}{2\varphi(z)\,(z-c_j)},\quad K\subset\overline{\mathbb{C}}\setminus[-1,1], \tag{25}
$$

where φ *is as in* (3).

Now, combining Theorem 1 and Lemma 3.3, we get the following useful lemma.

Lemma 3.4. *If* (7) *is a sequentially-ordered Sobolev inner product such that* $\mu\in\mathbf{M}(0,1)$, *then:*

1. *For all n sufficiently large, each sufficiently small neighborhood of c_j; $j=1,\ldots,N$; contains exactly one zero of S_n, and the remaining $n-N$ zeros lie on $(-1,1)$.*
2. *For all n sufficiently large, the zeros of S_n are real and simple.*
3. *The set of zeros of $\{S_n\}_{n=1}^{\infty}$ is uniformly bounded.*

Proof. The first assertion of the lemma is a direct consequence of (25) and Rouché's Theorem (see [4, Th. 9.2.3]). Note that S_n is a polynomial with real coefficient. Therefore, the second and third sentences are consequences of the first assertion and Theorem 1. □

4 Auxiliary lemmas

Let S_n be the n-th orthogonal polynomial with respect to the sequentially-ordered inner product (7). Taking into consideration the Theorem 1, let $\{\xi_{n,i}\}_{i=1}^{n-N}$ be the $n-N$ simple zeros of S_n on $(-1,1)$ for all sufficiently large n and let $\{\xi_{n,n-N+i}\}_{i=1}^{N}$ be the remaining N zeros of S_n. Obviously, S_n admits the representation

$$
S_n(x)=S_{n,1}(x)\,S_{n,2}(x),\ \text{ where }\ S_{n,1}(x)=\prod_{i=1}^{n-N}(x-\xi_{n,i})\ \text{and}\ S_{n,2}(x)=\prod_{i=1}^{N}(x-\xi_{n,n-N+i}). \tag{26}
$$

From Lemma 3.4, for all sufficiently large n, the last N zeros of S_n are real and simple. Furthermore, the sign of $S_{n,2}$ is constant on $[-1,1]$ and equal to $(-1)^{\nu}$, where ν is the number of c_j greater than 1. Thus, the polynomial $S^+_{n,2}(x)=(-1)^{\nu}S_{n,2}(x)$ is positive on $[-1,1]$.

The following Lemma is an analogous of the Gauss-Jacobi quadrature formula for the sequentially-ordered Sobolev inner product, when n is sufficiently large.

Lemma 4.1. *Let S_n and $\{\xi_{n,i}\}_{i=1}^{n-N}$ as above. If n is sufficiently large, then for every polynomial T with* $\deg(T)\leq 2n-d-N-1$,

$$\int_{-1}^{1} T(x)S^+_{n,2}(x)d\mu_\rho(x)=\sum_{i=1}^{n-N}\lambda_{n,i}\,S^+_{n,2}(\xi_{n,i})\,T(\xi_{n,i}), \tag{27}$$

$$\textit{where}\ \ \lambda_{n,i}=\int_{-1}^{1}\frac{S_n(x)}{S'_n(\xi_{n,i})(x-\xi_{n,i})}d\mu_\rho(x).$$

Moreover, the number of positive coefficients $\lambda_{n,i}$ is greater than or equal to $\left(n-\frac{d+N}{2}\right)$. We call Christoffel-type coefficients *to the numbers $\{\lambda_{n,i}\}_{i=1}^{n}$.*

Proof. Let T be an arbitrary polynomial of degree at most $2n-d-N-1$ and denote by $\mathscr{L}$ the Lagrange polynomial interpolating T at the points $\xi_{n,1},\ldots,\xi_{n,n-N}$ ($\deg(\mathscr{L})<n-N$), i.e.,

$$\mathscr{L}(z)=\sum_{i=1}^{n-N}T(\xi_{n,i})\,\frac{S_{n,1}(z)}{S'_{n,1}(\xi_{n,i})(z-\xi_{n,i})}.$$

Then, $T-\mathscr{L}=f\,S_{n,1}$ where $\deg(f)\leq n-d-1$. From (11)

$$\int_{-1}^{1}(T-\mathscr{L})(x)\,S_{n,2}(x)\,d\mu_\rho(x)=\int_{-1}^{1}f(x)\,S_n(x)\,d\mu_\rho(x)=0.$$

Hence,

$$\begin{aligned}
\int_{-1}^{1}T(x)\,S_{n,2}(x)\,d\mu_\rho(x)&=\int_{-1}^{1}\mathscr{L}(x)\,S_{n,2}(x)\,d\mu_\rho(x),\\
&=\int_{-1}^{1}\left(\sum_{i=1}^{n-N}T(\xi_{n,i})\,\frac{S_{n,1}(x)}{S'_{n,1}(\xi_{n,i})(x-\xi_{n,i})}\right)S_{n,2}(x)d\mu_\rho(x),\\
&=\sum_{i=1}^{n-N}\left(\int_{-1}^{1}\frac{S_n(x)}{S'_{n,1}(\xi_{n,i})(x-\xi_{n,i})}d\mu_\rho(x)\right)\,T(\xi_{n,i}),
\end{aligned}$$

which establishes (27). Assume that n is fixed, let $I_+=\{1\leq i\leq n-N:\lambda_{n,i}>0\}$ and $\Lambda_+^2(x)=\prod_{i\in I_+}(x-\xi_{n,i})^2$. If $\deg(\Lambda_+^2)<2n-d-N$, from (27),

$$0<\int_{-1}^{1}\Lambda_+^2(x)\,S^+_{n,2}(x)d\mu_\rho(x)=\sum_{\substack{i=1\\ i\notin I_+}}^{n-N}\lambda_{n,i}\,\Lambda_+^2(\xi_{n,i})\,S^+_{n,2}(\xi_{n,i})\leq 0,$$

which is a contradiction and the second assertion is established. $\square$

Let us denote for $k \in \mathbb{N}$

$$R_{n,1}^{[k]}(z) = \frac{S_{n,1}^{[k]}(z)}{S_{n+k,1}(z)}, \text{ where } S_{n,1}^{[k]}(z) = \int_{-1}^{1} \frac{S_{n+k,1}(z) - S_{n+k,1}(x)}{z-x} Q_{k-1}(x)\, d\mu_{\rho,n}(x) \tag{28}$$
$$\text{and } \; d\mu_{\rho,n}(x) = S_{n+k,2}^{+}(x)\, \rho(x)\, d\mu(x).$$

From Lemma 3.4, it is straightforward to see that:

1. If n is sufficiently large, $S_{n+k,2}^{+}(x)\, \rho(x) > 0$ for all $x \in [-1,1]$.
2. There exists a constant $\mathfrak{M}_\rho > 0$, such that for all $n \in \mathbb{Z}_+$

$$\mu_{\rho,n}([-1,1]) = \int_{-1}^{1} S_{n+k,2}^{+}(x)\, \rho(x)\, d\mu(x) \leq \mathfrak{M}_\rho. \tag{29}$$

Lemma 4.2 (Principal Lemma)**.** *Let $\{S_n\}_{n=0}^{\infty}$ be the monic orthogonal polynomial sequence with respect to a sequentially-ordered Sobolev inner product* (7)*. Then, for n sufficiently large*

$$R_{n,1}^{[k]}(z) = \sum_{j=1}^{n+k-N} \frac{S_{n+k,2}^{+}(\xi_{n+k,j})\, \lambda_{n+k,j}}{(z-\xi_{n+k,j})}. \tag{30}$$

Furthermore, $\{R_{n,1}^{[k]}\}$ is uniformly bounded on each compact subset $K \subset \mathbb{C} \setminus [-1,1]$.

Proof. Let n and k be fixed. For simplicity of notation, we write ξ_j instead of $\xi_{n+k,j}$. Then, $\{\xi_j\}_{j=1}^{n+k-N}$ is the set of zeros of S_{n+k} on $(-1,1)$.

From Theorem 1, for n sufficiently large, we have that the zeros of S_{n+k} are simple and $n+k-N$ of them lie on $(-1,1)$. Thus, $S'_{n+k}(\xi_j) \neq 0$ for $j = 1,\ldots,n+k-N$; and

$$R_{n,1}^{[k]}(z) = \sum_{j=1}^{n+k-N} \frac{b_j}{z-\xi_j},$$

where

$$\begin{aligned} b_j = \lim_{z\to\xi_j} (z-\xi_j) R_{n,1}^{[k]}(z) &= \lim_{z\to\xi_j} \frac{(z-\xi_j)}{S_{n+k,1}(z)} \lim_{z\to\xi_j} S_{n,1}^{[k]}(z) \\ =& S_{n+k,2}(\xi_j) \int_{-1}^{1} \frac{(-1)^{\nu} S_{n+k}(x)\, Q_{k-1}(x)\, d\mu_\rho(x)}{S'_{n+k}(\xi_j)(x-\xi_j)} = S_{n+k,2}^{+}(\xi_j)\, \lambda_{n+k,j}, \end{aligned}$$

and we get (30).

The second part of this proof, as [15, Lemma 1], is based on the second proof of Chebyshev-Markov-Stieltjes's Separation Theorem in [18, §3.41]. Through the proof, we use the following notations:

$$d\vartheta(x) = \sum_{j=1}^{n+k-N} \lambda_{n+k,j}\, S_{n+k,2}^{+}(\xi_j) \delta_{\xi_j}(x), \;\; \delta_{\xi_j}(x) = \begin{cases} 1, & x = \xi_j, \\ 0, & x \neq \xi_j. \end{cases},$$
$$\vartheta(x) = \int_{-1}^{x} d\vartheta(t), \; u_{\rho,n}(x) = \int_{-1}^{x} d\mu_{\rho,n}(t) \; \text{ and } \; \omega(x) = u_{\rho,n}(x) - \vartheta(x).$$

Let us recall that the function $u_{\rho,n}$ is monotone nondecreasing on $[-1,1]$. Set $\xi_0 = -1$ and $\xi_{n+k-N+1} = 1$. Then, ϑ is a step-function, which is constant on each interval (ξ_j, ξ_{j+1}) for $j = 0,1,\ldots,n+k-N$. Hence, ω is monotone nondecreasing on each of these open intervals.

With these notations, we can rewrite (27) as

$$\int_{-1}^{1} T(x)\, d\omega(x) = 0, \text{ for any polynomial } T \text{ of degree at most } (2(n+k)-d-N-1). \quad (31)$$

As $\omega(-1) = u_{\rho,n}(-1) - \vartheta(-1) = 0$ and

$$\omega(1) = u_{\rho,n}(1) - \vartheta(1) = \mu_{\rho,n}([-1,1]) - \mu_{\rho,n}([-1,1]) = 0,$$

integrating by parts in (31), we get

$$\int_{-1}^{1} \omega(x)\, T'(x)\, dx = 0. \quad (32)$$

We use the symbol $\mathscr{N}_1(q;I)$ to denote the number of points of sign change of the function q on the interval $I \subset \mathbb{R}$. Obviously, in (32), the polynomial T' can be replaced by any other polynomial of degree at most $(2(n+k)-d-N-2)$ and consequently, we can assert that $\mathscr{N}_1(\omega;(-1,1)) \geq 2(n+k)-d-N-1$.

Note that $\mathscr{N}_1(\omega;(\xi_0,\xi_1)) = \mathscr{N}_1(\omega;(\xi_{n+k-N},\xi_{n+k-N+1})) = 0$. Take into account that ω is monotone nondecreasing on each interval (ξ_j,ξ_{j+1}), $j = 1,\ldots,n+k-N-1$. Hence, it has at most one sign change on each of them. Therefore, we can conclude that the total number of sign changes of ω on $\bigcup_{j=1}^{n+k-N-1}(\xi_j,\xi_{j+1})$ is not greater than $(n+k-N-1)$. On the other hand, ω could change sign at each of the $n+k-N$ points ξ_j. In conclusion,

$$2(n+k-N)-(d-N)-1 \leq \mathscr{N}_1(\omega;(-1,1)) \leq 2(n+k-N)-1.$$

It thus follows that the number of intervals (ξ_j,ξ_{j+1}) where ω does not change sign is at most $(d-N)$. Indeed, if the number of intervals (ξ_j,ξ_{j+1}) where ω does not change sign is at least $(d-N+1)$, then $2(n+k)-d-N-1 \leq \mathscr{N}_1(\omega;(-1,1)) \leq 2(n+k)-1-d-N-2$, which is a contradiction.

We say that $\xi_j \in E_1$ if the function ω changes sign in each of the consecutive intervals (ξ_{j-1},ξ_j) and (ξ_j,ξ_{j+1}). In any other case, we say that $\xi_j \in E_2$.

Observe that if ω does not change sign on (ξ_j,ξ_{j+1}), then $\xi_j,\xi_{j+1} \in E_2$. From the previous considerations, the number of interval, where ω does not change sign is at most $(d-N)$. Therefore, E_2 cannot contain more than $2(d-N)$ elements.

Suppose that $\lambda_j \leq 0$. If $\xi_j \in E_1$, we know that ω changes sign in each of the consecutive intervals (ξ_{j-1},ξ_j) and (ξ_j,ξ_{j+1}). Let $x_1 \in (\xi_{j-1},\xi_j)$ such that $\omega(x_1) > 0$ and let $x_2 \in (\xi_j,\xi_{j+1})$ such that $\omega(x_2) < 0$. As $u_{\rho,n}(x)$ is monotone nondecreasing on $(-1,1)$, we get

$$0 < \omega(x_1) - \omega(x_2) = \big(u_{\rho,n}(x_1) - u_{\rho,n}(x_2)\big) + \lambda_j S^{+}_{n+k,2}(\xi_j) \leq 0.$$

This contradiction proves that $\xi_j \in E_1$ implies that $\lambda_j > 0$ (i.e., the Christoffel coefficients corresponding to the zeros $\xi_j \in E_1$ are positive).

Now, let $\xi_j \in E_1, x_1 \in (\xi_{j-1}, \xi_j)$ such that $\omega(x_1) \leq 0$ and $x_2 \in (\xi_j, \xi_{j+1})$ such that $\omega(x_2) \geq 0$. Recalling again that $u_{\rho,n}(x)$ is monotone nondecreasing on $(-1,1)$, then $0 \geq \omega(x_1) - \omega(x_2) = \big(u_{\rho,n}(x_1) - u_{\rho,n}(x_2)\big) + \lambda_j S^+_{n+k,2}(\xi_j)$ and $\lambda_j S^+_{n+k,2}(\xi_j) \leq u_{\rho,n}(x_2) - u_{\rho,n}(x_1) \leq \mu_{\rho,n}(\xi_{j+1}) - \mu_{\rho,n}(\xi_{j-1})$. From the last inequality, we get

$$\sum_{\xi_j \in E_1} |\lambda_j S^+_{n+k,2}(\xi_j)| = \sum_{\xi_j \in E_1} \lambda_j S^+_{n+k,2}(\xi_j) \leq 2\mu_{\rho,n}([-1,1]). \tag{33}$$

Set $K \subset \mathbb{C} \setminus [-1,1]$ compact and $\mathfrak{m} = \min\limits_{\substack{x\in[-1,1]\\ z\in K}} |x-z| > 0$, then

$$\sum_{\xi_j \in E_1} \left| \frac{S^+_{n+k,2}(\xi_j)\ \lambda_{n+k,j}}{(z-\xi_j)} \right| \leq \frac{2\mu_{\rho,n}([-1,1])}{\mathfrak{m}} \leq \frac{2\mathfrak{M}_\rho}{\mathfrak{m}}, \tag{34}$$

where $\mathfrak{M}_\rho$ was defined in (29).

The aim of the last step of the proof is to show that the sum $G_2(z) = \sum\limits_{\xi_j \in E_2} \dfrac{S^+_{n+k,2}(\xi_j)\ \lambda_{n+k,j}}{(z-\xi_j)}$ is uniformly bounded on K. We renumber the zeros of $S_{n+k,1}$ in such a way that $E_2 = \{\xi_1, \dots, \xi_m\}$ and $E_1 = \{\xi_{m+1}, \dots, \xi_{n+k-N}\}$. From the previous result, $m \leq 2(d-N)$.

Firstly, we introduce several notations. Let σ_η be the ηth elementary symmetric polynomials evaluated in $(\xi_1, \dots, \xi_m)$ (see [13, (1.2.4)]), i.e.,

$$\begin{aligned}
\sigma_0 &= \sigma_0(\xi_1, \dots, \xi_m) = 1, \\
\sigma_\eta &= \sigma_\eta(\xi_1, \dots, \xi_m) = \sum_{1 \leq v_1 < \cdots < v_\eta \leq m} \prod_{l=1}^{\eta} \xi_{v_l}, \quad \text{for } \eta = 1, \dots, m.
\end{aligned}$$

The symbol $\sigma_{\eta,j} = \sigma_\eta(\xi_1, \dots, \xi_{j-1}, \xi_{j+1}, \dots, \xi_m)$ denotes the ηth elementary symmetric polynomial evaluated in $(\xi_1, \dots, \xi_{j-1}, \xi_{j+1}, \dots, \xi_m)$. It is straightforward to see that $\sigma_{\eta,j} = \sigma_\eta - \xi_j \sigma_{\eta-1,j}$ for $\eta = 1, \dots, m-1$, and iteratively applying this equality η times, we have

$$\sigma_{\eta,j} = \sum_{l=0}^{\eta} (-\xi_j)^l\ \sigma_{\eta-l}.$$

For simplicity of notation, we write $\rho_{n+k.j} = S^+_{n+k,2}(\xi_j)\ \lambda_{n+k,j}$. Hence, for $i = 1, \dots, m$,

$$\sum_{j=1}^{m} \rho_{n+k.j}\ \sigma_{i,j} = \sum_{j=1}^{m} \rho_{n+k.j} \left(\sum_{l=0}^{i} (-\xi_j)^l\ \sigma_{i-l} \right) = \sum_{l=0}^{i} \sigma_{i-l} \left(\sum_{j=1}^{m} \rho_{n+k.j}\ (-\xi_j)^l \right).$$

From Lemma 4.1 we have for $l \leq 2(d-N)$

$$\int_{-1}^{1} (-x)^l\, d\mu_{\rho,n}(x) = \sum_{j=1}^{n+k-N} \rho_{n+k.j}\ (-\xi_j)^l = \sum_{j=1}^{m} \rho_{n+k.j}\ (-\xi_j)^l + \sum_{j=m+1}^{n+k-N} \rho_{n+k.j}\ (-\xi_j)^l.$$

Thus, from (34)

$$\left|\sum_{j=1}^{m} \rho_{n+k.j}\,(-\xi_j)^l\right| \leq \left|\sum_{j=m+1}^{n+k-N} \rho_{n+k.j}\,(-\xi_j)^l\right| + \left|\int_{-1}^{1}(-x)^l\,d\mu_{\rho,n}(x)\right| \leq \frac{\mathfrak{m}+2}{\mathfrak{m}}\,\mu_{\rho,n}([-1,1]).$$

As $\{\xi_1,\ldots,\xi_m\} \subset [-1,1]$, it is straightforward to see that $|\sigma_\eta| \leq m$ for all $\eta = 0,\ldots,m$. Therefore, for $i = 1,\ldots,m$

$$\left|\sum_{j=1}^{m} \rho_{n+k.j}\,\sigma_{i,j}\right| \leq \sum_{l=0}^{i} |\sigma_{i-l}| \left|\sum_{j=1}^{m} \rho_{n+k.j}\,\left(-\xi_j\right)^l\right| \leq \frac{m^2\,(\mathfrak{m}+2)}{\mathfrak{m}}\,\mu_{\rho,n}([-1,1]). \tag{35}$$

Using the previous notation, we write

$$G_2(z) = \sum_{j=1}^{m} \frac{\rho_{n+k.j}}{(z-\xi_j)} = \frac{L_{m-1}(z)}{\prod_{j=1}^{m}(z-\xi_j)} \quad \text{where } L_{m-1}(z) = \sum_{j=1}^{m} \rho_{n+k.j} \prod_{\substack{i=1\\ i\neq j}}^{m}(z-\xi_i).$$

From the classical Formula of Viète, $\prod_{\substack{i=1\\ i\neq j}}^{m}(z-\xi_i) = \sum_{i=0}^{m-1}(-1)^i\,\sigma_{i,j}\,z^{m-1-i}$ (see [13, (1.2.2)]) and

$$L_{m-1}(z) = \sum_{j=1}^{m} \rho_{n+k.j}\,\left(\sum_{i=0}^{m-1}(-1)^i\,\sigma_{i,j}\,z^{m-1-i}\right) = \sum_{i=0}^{m-1}(-1)^i\,\left(\sum_{j=1}^{m} \rho_{n+k.j}\,\sigma_{i,j}\right)\,z^{m-1-i}.$$

Let $\mathfrak{M} = \max_{z\in K}|z|$. According to (35), for all $z \in K$,

$$\begin{aligned}
|L_{m-1}(z)| \leq& \sum_{i=0}^{m-1}\left|\sum_{j=1}^{m} \rho_{n+k.j}\,\sigma_{i,j}\right|\,|z|^{m-1-i} \leq \frac{m^2\,(\mathfrak{m}+2)\,\mu_{\rho,n}([-1,1])}{\mathfrak{m}} \sum_{i=0}^{m-1}|z|^{m-1-i}, \\
\leq& \frac{\mathfrak{m}+2}{\mathfrak{m}}\,m^3\,\max\{\mathfrak{M}^{m-1},1\}\,\mu_{\rho,n}([-1,1]) \leq \frac{\mathfrak{m}+2}{\mathfrak{m}}\,m^3\,\max\{\mathfrak{M}^{m-1},1\}\,\mathfrak{M}_\rho = \mathfrak{M}_1. \\
|G_2(z)| =& \frac{|L_{m-1}(z)|}{\prod_{j=1}^{m}|z-\xi_j|} \leq \frac{\mathfrak{M}_1}{\mathfrak{m}^m}.
\end{aligned} \tag{36}$$

Finally, (34) and (36) establish the second assertion. □

5 Proof of Theorem 2

Denote $R_n^{[k]} = \frac{S_n^{[k]}(z)}{S_{n+k}(z)}$ and let $\widehat{\mu}_k(z) = \int_{-1}^{1}\frac{Q_{k-1}(x)}{z-x}d\mu_\rho(x)$ be the kth Markov-type function associated to μ_ρ ($k \in \mathbb{N}$) as in (10). Note that $\widehat{\mu}_k(z)$ is well defined and holomorphic in Ω_∞ ($\widehat{\mu}_k \in \mathbb{H}(\Omega_\infty)$ for short) and $\widehat{\mu}_k(\infty) = 0$.

For the remainder $\left(\widehat{\mu}_k(z) - R_n^{[k]}(z)\right)$, the following formulas take place.

Lemma 5.1. *Let μ be a positive Borel measure supported on $[-1,1]$ and $S_n(z)$ and $S_n^{[k]}(z)$ defined as above. Then,*

$$\widehat{\mu}_{k,n}(z) - R_{n,1}^{[k]}(z) = S_{n+k,2}^{+}(z)\left(\widehat{\mu}_k(z) - R_n^{[k]}(z)\right) = \mathscr{O}\left(\frac{1}{z^{2(n+1)+k-d-N}}\right), \tag{37}$$

where $\widehat{\mu}_{k,n}(z) = \int_{-1}^{1} \frac{Q_{k-1}(x)}{z-x} d\mu_{\rho,n}(x)$.

Proof. From the definition of $S_n^{[k]}$, we get

$$\begin{aligned}
S_n^{[k]}(z) = & \int_{-1}^{1} \frac{S_{n+k}(z) - S_{n+k}(x)}{z-x} Q_{k-1}(x) d\mu_\rho(x) \\
= & S_{n+k}(z) \int_{-1}^{1} \frac{Q_{k-1}(x)}{z-x} d\mu_\rho(x) - \int_{-1}^{1} \frac{S_{n+k}(x)\, Q_{k-1}(x)}{z-x} d\mu_\rho(x) \\
= & S_{n+k}(z)\widehat{\mu}_k(z) - \int_{-1}^{1} \frac{S_{n+k}(x)\, Q_{k-1}(x)}{z-x} d\mu_\rho(x).
\end{aligned}$$

Then, we have

$$\widehat{\mu}_k(z) - R_n^{[k]}(z) = \int_{-1}^{1} \frac{S_{n+k}(x)\, Q_{k-1}(x)}{S_{n+k}(z)\,(z-x)} d\mu_\rho(x). \tag{38}$$

On the other hand, from the orthogonality condition (6)

$$\begin{aligned}
0 = & \left\langle S_{n+k}(x),\ \frac{(S_{n-d+1}(z) - S_{n-d+1}(x))\, Q_{k-1}(x)\rho(x)}{z-x} \right\rangle \\
= & \int_{-1}^{1} S_{n+k}(x)\ \frac{S_{n-d+1}(z) - S_{n-d+1}(x)}{z-x}\ Q_{k-1}(x) d\mu_\rho(x).
\end{aligned}$$

Hence, it follows that

$$\int_{-1}^{1} \frac{S_{n+k}(x) S_{n-d+1}(z)}{z-x}\ Q_{k-1}(x)\, d\mu_\rho(x) = \int_{-1}^{1} \frac{S_{n+k}(x) S_{n-d+1}(x)}{z-x}\ Q_{k-1}(x)\, d\mu_\rho(x),$$

and from (38), we obtain

$$\begin{aligned}
\widehat{\mu}_k(z) - R_n^{[k]}(z) = & \int_{-1}^{1} \frac{S_{n+k}(x)}{S_{n+k}(z)} \frac{Q_{k-1}(x)}{z-x}\ d\mu_\rho(x) = \int_{-1}^{1} \frac{S_{n+k}(x) S_{n-d+1}(x)}{S_{n+k}(z) S_{n-d+1}(z)} \frac{Q_{k-1}(x)}{z-x}\ d\mu_\rho(x) \\
= & \mathscr{O}\left(\frac{1}{z^{2(n+1)+k-d}}\right)
\end{aligned}$$

The second equality in (37) is a direct consequence of the above equality. Lastly, we compute

$$\begin{aligned}
S_{n+k,2}^{+}(z)\widehat{\mu}_k(z) = & \int_{-1}^{1} \frac{S_{n+k,2}^{+}(z)\, Q_{k-1}(x)}{z-x} d\mu_\rho(x) \\
= & \int_{-1}^{1} \frac{S_{n+k,2}^{+}(z) - S_{n+k,2}^{+}(x)}{z-x}\ Q_{k-1}(x) d\mu_\rho(x) + \widehat{\mu}_{k,n}(z),
\end{aligned} \tag{39}$$

and

$$
\begin{aligned}
S^{+}_{n+k,2}(z)R^{[k]}_{n}(z) =& \frac{(-1)^{\nu}S^{[k]}_{n}(z)}{S_{n+k,1}(z)} = \frac{(-1)^{\nu}}{S_{n+k,1}(z)}\int_{-1}^{1}\frac{S_{n+k}(z)-S_{n+k}(x)}{z-x}\,Q_{k-1}(x)\,d\mu_{\rho}(x),\\
=&\int_{-1}^{1}\frac{S_{n+k,1}(z)S^{+}_{n+k,2}(z)-S_{n+k,1}(x)S^{+}_{n+k,2}(x)}{S_{n+k,1}(z)(z-x)}\,Q_{k-1}(x)\,d\mu_{\rho}(x),\\
=&R^{[k]}_{n,1}(z)+\int_{-1}^{1}\frac{S^{+}_{n+k,2}(z)-S^{+}_{n+k,2}(x)}{z-x}\,Q_{k-1}(x)\,d\mu_{\rho}(x).
\end{aligned}
\tag{40}
$$

The first equality now follows by subtracting (40) from (39). □

Proof of Theorem 2. Let K be any compact set on Ω_∞ and consider the level curve ℓ_τ defined by

$$\ell_\tau = \{z\in\mathbb{C} : |\varphi(z)| = \tau\}, \qquad \text{where } \tau>1 \text{ and } \varphi \text{ as in (3).}$$

Since $\varphi(K)$ is a compact set, we can take τ sufficiently close to 1 such that $1<\tau<\min|\varphi(K)|$ (remember that φ is the conformal map of the exterior of $[-1,1]$ onto the exterior of the unit circle). From Lemma 4.2 and (29), the sequences $\{\widehat{\mu}_{k,n}\}$ and $\{R^{[k]}_{n,1}\}$ are uniformly bounded over ℓ_τ. Then, there exists a constant $\mathfrak{L}_\tau$, independent of n, such that for all $z\in\ell_\tau$

$$\left|\left(\widehat{\mu}_{k,n}(z)-R^{[k]}_{n,1}(z)\right)\varphi^{2(n+1)+k-d-N}(z)\right| \le \mathfrak{L}_\tau\,\tau^{2(n+1)+k-d-N}. \tag{41}$$

Taking into account that φ has a simple pole at ∞, from (37), we have

$$\left(\left(\widehat{\mu}_{k,n}-R^{[k]}_{n,1}\right)\varphi^{2(n+1)+k-d-N}\right)\in\mathbb{H}(\Omega_\infty).$$

Now, from the maximum modulus principle the bound (41) also holds on K. Consequently, we have

$$\left|\widehat{\mu}_{k,n}(z)-R^{[k]}_{n,1}(z)\right| \le \mathfrak{L}_\tau\left(\frac{\tau}{|\varphi(z)|}\right)^{2(n+1)+k-d-N} \qquad z\in K.$$

Hence

$$\sup_{z\in K}\left|\widehat{\mu}_{k,n}(z)-R^{[k]}_{n,1}(z)\right| \le \mathfrak{L}_\tau\left(\frac{\tau}{\min|\varphi(K)|}\right)^{2(n+1)+k-d-N} \underset{n\to\infty}{\longrightarrow} 0, \tag{42}$$

which is equivalent to say that $R^{[k]}_{n,1}(z) \underset{n}{\rightrightarrows} \widehat{\mu}_{k,n}(z) \quad K\subset\Omega_\infty$.

As before, $\Omega^*_\infty = \Omega_\infty\setminus\{c_1,c_2,\dots,c_N\}$. For the rest of the proof we assume that the compact set K is a subset of Ω^*_∞. From Lemma 3.4, there exists a constant $\mathfrak{L}_2>0$, independent of n, such that $\mathfrak{L}_2\le|S_{n+k,2}(z)|$ for all $z\in K$. Therefore, taking into account (37), we get

$$\sup_{z\in K}\left|\widehat{\mu}_{k}(z)-R^{[k]}_{n}(z)\right| \le \frac{\mathfrak{L}_\tau}{\mathfrak{L}_2}\left(\frac{\tau}{\min|\varphi(K)|}\right)^{2(n+1)+k-d-N} \underset{n\to\infty}{\longrightarrow} 0, \tag{43}$$

□

As a complement of Theorem 2, we have the following estimate for the degree of convergence ("speed") of $R_n^{[k]}$ to $\widehat{\mu}_k$.

Corollary 2.1. *Under the same hypotheses of Theorem 2, we have*

$$\limsup_n \left\| \widehat{\mu}_k - R_n^{[k]} \right\|_K^{1/2n} \leq \|\varphi^{-1}\|_K < 1. \tag{44}$$

Proof. Taking the 2nth root in (43), we get

$$\left\| \widehat{\mu}_k - R_n^{[k]} \right\|_K^{1/2n} \leq \left(\frac{\mathfrak{L}_\tau}{\mathfrak{L}_2} \right)^{1/2n} \left(\frac{\tau}{\min|\varphi(K)|} \right)^{(2(n+1)+k-d-N)/(2n)}. \tag{45}$$

Since $\tau < \min|\varphi(K)|$, (44) follows from (45). □

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