COINTEGRATION AND UNIT ROOTS

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Abstract. This paper provides an updated survey of a burgeoning literature on testing, estimation and model specification in the presence of integrated variables. Integrated variables are a specific class of non-stationary variables which seem to characterise faithfully the properties of many macroeconomic time series. The analysis of cointegration develops out of the existence of unit roots and offers a generic route to test the validity of the equilibrium predictions of economic theories. Special emphasis is put on the empirical researcher's point of view.

Keywords. Unit root, cointegration, trends, error correction mechanisms.

1. Introduction

Economic theory generally deals with equilibrium relationships. Most empirical econometric studies are an attempt to evaluate such relationships by summarising economic time series using statistical analysis. To apply standard inference procedures in a dynamic time series model we need the various variables to be stationary, since the majority of econometric theory is built upon the assumption of stationarity. Until recently, this assumption was rarely questioned, and econometric analysis proceeded as if all the economic time series were stationary, at least around some deterministic trend function which could be appropriately removed. However, stationary series should at least have constant unconditional mean and variance over time, a condition which appears rarely to be satisfied in economics. The importance of the stationarity assumption had been recognised for many years, but the important papers by Granger and Newbold (1974), Nelson and Kang (1981) and Nelson and Plosser (1982) alerted many to the econometric implications of non-stationarity. Integrated variables are a specific class of non-stationary variables with important economic and statistical properties. These are derived from the presence of unit roots which give rise to stochastic trends, as opposed to pure deterministic trends, with innovations to an integrated process being permanent.
instead of transient. For example, the presence of a large permanent component in aggregate output conflicts with traditional formulations of both Keynesian and Classical macroeconomic theories in terms of countercyclical policies, implying, in addition, that the welfare costs and benefits of policy actions are far different than when output movements are seen as transitory deviations from a slowly growing natural level.

The presence of, at least, a unit root is implied by many economic models by the rational use of available information by economic agents. Standard applications include futures contracts, stock prices, yield curves, real interest rates, exchange rates, money velocity, hysteresis theories of unemployment, and, perhaps the most popular, the implications of the permanent income hypothesis for real consumption.

Statisticians have been aware for many years of the existence of integrated series and, in fact, Box and Jenkins (1970) argue that a non-stationary series can be transformed into a stationary one by successive differencing of the series. Therefore, from their point of view, the differencing operation seemed to be a pre-requisite for econometric modelling both from an univariate and a multivariate perspective. However Sargan (1964), Hendry and Mizon (1978) and Davidson et al. (1978), among others, have criticised on a number of grounds the specification of dynamic models in terms of differenced variables only, especially because it is then impossible to infer the long-run steady state solution from the estimated model.

Granger (1983) and Granger and Weiss (1983), resting upon the previous ideas, pointed out that a vector of variables, all of which achieve stationarity after differencing, may have linear combinations which are stationary without differencing. Engle and Granger (1987) formalise the idea of variables sharing an equilibrium relationship in terms of cointegration between time series, providing us with tests and an estimation procedure to evaluate the existence of equilibrium relationships, as implied by economic theory, within a dynamic specification framework. Standard examples include the relationship between real wages and productivity, nominal exchange rates and relative prices, consumption and disposable income, long and short-term interest rates, money velocity and interest rates, production and sales, etc.

In view of this epidemic of martingales in economics, a voluminous literature on testing, estimation, prediction, control and model specification in the presence of integrated variables has developed in the last few years. The purpose of this survey is to provide a useful guide through this increasingly technical literature, paying special attention to the point of view of the applied researcher with a good grounding in econometrics, who being a non-specialist in this particular subject wants to get a unified coverage of the main techniques available in this field.

The paper is organised as follows. The concepts of cointegration and unit roots are introduced in Section 2. In Section 3 we survey several alternative tests for the existence of unit roots, including cases where seasonality is present. Section 4 deals with alternative definitions of integration. Section 5 examines the application of some of the previous tests to determine the existence of cointegrating relationships. Section 6 contains a review of some new test procedures for cointegration. Finally, brief conclusions follow in Section 7.

2. Unit roots and cointegration

Wold's (1938) decomposition theorem states that a stationary time series process with no deterministic component has an infinite moving average (MA) representation. This, in turn, can be represented approximately by a finite autoregressive moving average (ARMA) process (see, e.g. Hannan, 1970).

However, as was mentioned in the Introduction, some time series need to be appropriately differenced in order to achieve stationarity. From this comes the definition of integration (as adopted by Engle and Granger, 1987): A variable $y_t$ is said to be integrated of order $d$ if $y_t - I(d)$ if it has stationary, invertible, non-deterministic ARMA representation after differencing $d$ times. Thus, a time series integrated of order zero is stationary in levels, while for a time series integrated of order one, the first difference is stationary. A white noise series and a stable first-order autoregressive [AR(1)] process are examples of $I(0)$ series, while a random walk process is an example of an $I(1)$ series.

Granger (1986) and Engle and Granger (1987) discuss the main differences between processes that are $I(0)$ and $I(1)$. They point out that an $I(0)$ series: (i) has finite variance which does not depend on time, (ii) has only a limited memory of its past behaviour (i.e. the effects of a particular random innovation are only transitory), (iii) tends to fluctuate around the mean (which may include a deterministic trend), and (iv) has autocorrelations that decline rapidly as the lag increases. For the case of an $I(1)$ series, the main features are: (i) the variance depends upon time and goes to infinity as time goes to infinity, (ii) the process has an infinitely long memory (i.e. an innovation will permanently affect the process), (iii) it wanders widely, and (iv) the autocorrelations tend to one in magnitude for all time separations.

Consider now two time series $y_t$ and $x_t$ which are both $I(d)$ (i.e. they have compatible long-run properties). In general, any linear combination of $y_t$ and $x_t$ will also be $I(d)$. If, however, there exists a vector $(1, -\beta)'$, such that the combination

$$z_t = y_t - \alpha - \beta x_t$$

(1)

is $I(d - b)$, $b > 0$, then Engle and Granger (1987) define $y_t$ and $x_t$ as cointegrated of order $(d, b)$ $[y_t, x_t]' - Ct(d, b)]$, with $(1, -\beta)'$ called the cointegrating vector. Note that a constant term has been included in (1) in order to allow for the possibility that $z_t$ may have a non-zero mean.

The concept of cointegration tries to mimic the existence of a long-run equilibrium to which an economic system converges over time. If, e.g., economic theory suggests the following long-run relationship between $y_t$ and $x_t$:

$$y_t = \alpha + \beta x_t$$

(2)
then $z_t$ can be interpreted as the equilibrium error (i.e., the distance that the system is away from the equilibrium at any point in time).

Engle and Granger also show that if $y_t$ and $x_t$ are cointegrated $CI(1,1)$, then there must exist an error correction model (ECM) representation of the following form:

$$\Delta y_t = \theta_0 + \theta_1 y_{t-1} + \Delta \Delta x_{t-1} + \Delta \Delta y_{t-1} + \epsilon_t$$  \hspace{1cm} (3)

where $\Delta$ denotes the first-order difference (i.e., $\Delta y_t = y_t - y_{t-1}$) and where $\epsilon_t$ is a sequence of independent and identically distributed random variables with mean zero and variance $\sigma^2$ (i.e., $\epsilon_t \sim iid(0, \sigma^2)$). Furthermore, they prove the converse result that an ECM generates cointegrated series.

Note that the term $z_{t-1}$ in equation (3) represents the extent of the disequilibrium between levels of $y$ and $x$ in the previous period. The ECM states that changes in $y_t$ depend not only on changes in $x_t$, but also on the extent of disequilibrium between the levels of $y$ and $x$. The appeal of the ECM formulation is that it combines flexibility in dynamic specification with desirable long-run properties: it could be seen as capturing the dynamics of the system whilst incorporating the equilibrium suggested by economic theory (see Hendry and Richard, 1983).³

Based upon the concept of cointegration (and on its closely related concept of ECM representation), Engle and Granger suggest a 2-step estimation procedure for dynamic modelling which has become very popular in applied research. Let us assume that $y_t$ and $x_t$ are both $I(1)$, then the procedure goes as follows:

(i) First, in order to test whether the series are cointegrated, the 'cointegrating regression'

$$y_t = \alpha + \beta x_t + z_t$$  \hspace{1cm} (4)

is estimated by ordinary least squares (OLS) and it is tested whether the 'cointegrating residuals' $z_t = y_t - \hat{\alpha} - \hat{\beta} x_t$ are $I(0)$. Stock (1987) has shown that if two $I(1)$ series are cointegrated, then the OLS estimates from equation (4) provide 'super-consistent' estimates of the cointegrating vector, in the sense that the estimates converge to the true parameter at a rate proportional to the inverse sample size, $T^{-1/2}$, and where

$$\Delta x_t = \epsilon_t; \hspace{0.5cm} (\epsilon_t = 0, \epsilon_t - iid(0, \sigma^2))$$  \hspace{1cm} (5)

Integrating (5) backwards we get

$$x_t = \sum_{i=1}^{T} \epsilon_i = S_t$$

and therefore $\text{Var}(x_t) = T\sigma^2$, exploding as $T \rightarrow \infty$. Nevertheless, $T^{-2} \Sigma x_t^2$ converges to a random variable. Similarly the cross-moment $T^{-1/2} \Sigma x_t \epsilon_t$ will explode, in contrast to the stationary case, where it is asymptotically normally distributed. In the $I(1)$ case $T^{-1} \Sigma x_t \epsilon_t$ converges also to a random variable. Both random variables are functionals of Brownian Motions or Wiener processes, which will be denoted henceforth, in general, as $f(W)$ (see Phillips (1987), Phillips and Perron (1988) and Park and Phillips (1988) for a general discussion on convergence of the above mentioned distributional limits). From the expression for the OLS estimator of $\beta$ we obtain

$$\hat{\beta} - \beta = \Sigma x_t \epsilon_t / \Sigma x_t^2$$

it follows from the previous discussion that $T(\hat{\beta} - \beta)$ is asymptotically the ratio of two non-degenerate random variables, and it is in general not normal. Thus, standard inference cannot be applied to $\hat{\delta}$, even if it is 'super-consistent', a question to which we will come back in Section 6.

(ii) Finally, the residuals $\hat{\epsilon}_t$ are entered into the ECM. Now, all the variables in equation (3) are $I(0)$ and conventional modelling strategies can be applied.

3. Testing the order of integration of the relevant variables

Once the relevant set of variables suggested by economic theory has been identified, the first stage in testing for cointegration between those variables is to determine the order of integration of the individual time series.

Several statistical tests for unit roots have been developed to test for stationarity in time series. Since many macroeconomic series have been found to be integrated of order one (see, e.g. Nelson and Plosser, 1982), we will only consider tests for a single unit root.

The previous tests can also be applied, with a slight change in their interpretation, for sequential testing of unit roots, i.e. when one wants to compare a null hypothesis of $k$ unit roots with an alternative of $k-1$ unit roots. In the sequential procedure, the investigator should start with the largest $k$ under consideration and work down; that is, decrease $d$ by one each time the null hypothesis is rejected.⁶

3.1. Tests of unit roots

(i) Dickey and Fuller (1979, 1981) present a class of test statistics, known as Dickey–Fuller (DF) statistics, generally used to test that a pure AR(1) process (with or without drift) has a unit root.

Let the time series $y_t$ satisfy the following data generating process (DGP)

$$y_t = \beta_0 + \beta_1 t + \rho y_{t-1} + \epsilon_t$$  \hspace{1cm} (6)

where $\epsilon_t \sim iid(0, \sigma^2)$, $t$ is a time trend and the initial condition, $y_0$, is assumed to be a known constant (zero, without loss of generality). Equation (6) can also be written as

$$y_t = \beta_0 + \sum_{j=1}^{T} \rho^{t-j} + \beta_1 \sum_{j=1}^{T} \rho^{t-j} + \sum_{j=1}^{T} \epsilon_t \rho^{t-j}$$  \hspace{1cm} (7)
while in the case that $\rho = 1$

$$y_t = \beta_0 t + \beta_1 t (t + 1)/2 + \epsilon_t$$

(8)

where $S_t = \Sigma_{t=1}^T \epsilon_t$.

Dickey and Fuller (1979) consider the problem of testing the null hypothesis

$H_0: \rho = 1$ versus $H_1: \rho < 1$, i.e. non-stationarity vs. stationarity around a deterministic trend, suggesting OLS estimation of a reparameterised version of (6), i.e.

$$\Delta y_t = \beta_0 + \beta_1 t + \gamma y_{t-1} + \epsilon_t$$

(9)

where $H_0: \gamma = 0$ is equivalent to $H_0: \gamma = \rho - 1$. The test is implemented though the usual $t$-statistic of $\hat{\gamma}$, denoted here as $\gamma$. In addition, Dickey and Fuller (1981) suggest two $F$-statistics for the joint null hypothesis $\beta_0 = \beta_1 = 0$ and $\beta_1 = 0$, denoted as $\Phi_2$ and $\Phi_1$ respectively. Note that under the null hypothesis $\tau$, $\Phi_2$ and $\Phi_1$ will not have the standard $t$ and $F$ distributions, instead they are functions of Brownian motion; we must use the asymptotic distributions tabulated in Fuller (1976, p. 373) and in Dickey and Fuller (1981, p. 1,063) respectively. If $\beta_1 = 0$ ($\beta_0 = 0$) in (9), the $t$ and $F$ statistics, corresponding to $H_0: \gamma = 0$ and $H_1: \beta_0 = 0$, are denoted $\tau$ ($\beta_1$) and $\Phi_1$ respectively and the corresponding critical values are also given in the previous references. In all cases the critical values given there crucially depend upon the sample size. It should also be noted that the critical values depend upon the 'nuisance' parameters contained in the model and in the DGP. To discuss this more formally, consider the sample variance of $y_t$ when it is generated by (8) (i.e. $\rho = 1$)

$$T^{-1} \Sigma y_t^2 = T^{-1} \Sigma (\beta_0 + \beta_1 t / 2)^2 + (\beta_1 t / 2)^2 t^2 + S_r^2 + (\beta_1 t / 2)^2 s_t t + 2(\beta_0 + \beta_1 t / 2) s_t + \epsilon_t^2$$

(10)

From the distributional results in Park and Phillips (1988), it is known that $T^{-1} \Sigma y_t^2$, $T^{-3/2} \Sigma S_r^2$, and $T^{-7/2} \Sigma s_t^2$ tend to $t(W)$, hence, by taking probability limits in (10), we get

$$T^{-1} \Sigma y_t^2 = \beta_1^2 / 20 (O(T^0)) + (\beta_0 + \beta_1 / 2) / 4 (O(T^1)) + (\beta_0 + \beta_1 / 2) / 3 (O(T^3 / 2)) + \beta f(W)(O(T^{3 / 2})) + (\beta_0 + \beta_1 / 2)(f(W)(O(T^{3 / 2})) + f(W)(O(T^0)$$

whereby it is seen that

$$T^{-1} \Sigma y_t^2 = \beta_1^2 / 20$$

if $\beta_1 \neq 0$

$$T^{-1} \Sigma y_t^2 = \beta_1^2 / 3$$

if $\beta_0 \neq 0, \beta_1 = 0$

$$T^{-1} \Sigma y_t^2 = f(W)$$

if $\beta_0 = \beta_1 = 0$

That is, if the unit root process contains a linear trend or a drift, its variability will be dominated by a quadratic or a linear trend which, appropriately normalised, converges to constants. It is only when $\beta_0 = \beta_1 = 0$ that it converges to a non-standard distribution. This means that for example, if $\beta_1 \neq 0$ in (8) and the model is estimated as in (9), the quadratic term in (10) will dominate the integrated process and normality of the $\gamma$ will follow. Similarly if $\beta_0 \neq 0$ in (8) (with $\beta_1 = 0$) and only a constant appears in (9), the linear term will dominate and normality of $\gamma$ will also follow. It is only when $\beta_0 = \beta_1 = 0$, both in the DGP and in the model, that the non-standard distribution will dominate. This implies that in order to use the DF critical values if a linear trend is included in the maintained hypothesis (9), the relevant null hypothesis should be a random walk with drift if $\beta_0 \neq 0$ whilst if only a drift is included in (9), the relevant null hypothesis is a random walk without drift.

From the previous discussion we consider that the following testing strategy is most appropriate. First, start by the most unrestricted model (9), $(\beta_0 \neq 0, \beta_1 \neq 0)$ if it is suspected that the differenced series has a drift. Then use $\tau$ to test for the null hypothesis. If it is rejected there is no need to go further. If it is not rejected, test for the significance of the trend under the null. If it is significant, then test again for a unit root using the standardised normal. If the trend is not significant in the maintained model, estimate (9) without trend $(\beta_1 = 0)$. Test again for the unit root using $\tau$. If the null hypothesis is rejected, again there is no need to go further. If it is not rejected, test for the significance of the constant under the null hypothesis and so on.

(ii) In the analysis of the DF tests, we have assumed that the DGP is a pure AR(1) process. If instead, the DGP is AR($p$)

$$y_t = \beta_0 + \beta_1 t + \sum_{i=1}^{p} \rho_i y_{t-i} + \epsilon_t$$

(12)

let

$$\lambda^p - \sum_{i=1}^{p} \sigma_i \lambda^{p-i} = 0$$

(13)

be the characteristic equation of the time series, where $\lambda(i = 1, \ldots, p)$ are the eigenvalues of the process. Dickey and Fuller (1979, 1981) consider the problem of testing the null hypothesis $H_0: \lambda_1 = 1$ and $|\lambda_2| < 1$ for $i = 2, \ldots, p$, suggesting OLS estimation of the reparameterised regression model

$$\Delta y_t = \beta_0 + \beta_1 t + \gamma y_{t-1} + \sum_{i=1}^{p-1} \gamma_i \Delta y_{t-i} + \epsilon_t$$

(14)

where $p$ is large enough to ensure that the residual series $\epsilon_t$ is white noise. The tests are based on the $t$-ratio on $\hat{\gamma}_i$ and are known as 'Augmented Dickey Fuller' (ADF) statistics. The critical values are the same as those discussed for the DF statistics, since the $\hat{\gamma}_i$ ($i = 1, \ldots, p - 1$) estimates converge to their true values at a rate $O(T^{-1/2})$, being asymptotically dominated by the distribution of $\hat{\gamma}_1$ which, as we mentioned in (4), is $O(T^{-1})$. The same testing strategy discussed above applies in this case.

The sample distribution of the ADF statistics critically depend on the assumption that the time series $y_t$ is generated by a pure AR process. However,
since there is evidence that many macroeconomic series contain moving average (MA) components (see Schwert, 1987), we would want to consider also the possibility of an MA component in the DGP, so that the null hypothesis would be that the data are generated by a mixed autoregressive integrated moving average (ARIMA) process.

Said and Dickey (1984) extend the ADF test by exploiting the fact that an ARIMA \((p, 1, q)\) process can be adequately approximated by a high-order autoregressive process, \(AR(l)\), where \(l = O(T^{1/2})\) as \(T \to \infty\). In practice the test proceeds as before with \(p\) in (12) and (14) equal to \(l\). This approach permits one to test the null hypothesis of the presence of a unit root without knowing the orders of \(p\) and \(q\). However, it involves the estimation of additional nuisance parameters which reduces the effective number of observations due to the need for extra initial conditions.

When \(p\) and \(q\) are known, Said and Dickey (1985) present a test for the hypothesis that the process is ARIMA \((p, 1, q)\), i.e.

\[
H_0: \phi(L)\delta Y_t = 0(L)\epsilon_t
\]

where \(\phi(.)\) and \(\theta(.)\) are \(p\)th and \(q\)th order polynomials in the lag operator \(L\), versus the alternative hypothesis that it is ARIMA \((p, 0, q)\)

\[
H_1: \phi(L)(1 - \rho L)\delta y_t = \theta(L)\epsilon_t
\]

To perform the test of \(\rho = 1\), we specify initial estimates of the parameters that are consistent under the null and alternative hypothesis. We next perform a one-step of the Gauss–Newton numerical estimation procedure (see, e.g., Harvey, 1981 p. 17). The \(t\)-statistic associated with \(\rho\), after applying the iteration has the limiting distribution of a unit root in the AR polynomial and provides a wide range of possibilities of \(\rho\) and \(q\) in order to estimate the variance of the MA parameter close to one, the Said and Dickey test, which proceeds as before with \(p\) in (9), the corresponding statistic is

\[
Z(\tau) = (\delta s_{it})_T - (\delta s_{it} - \delta) T (\delta s_{it} - \delta^2) T (\delta s_{it} - \delta^2)^{-1},
\]

where \(\delta\) and \(\delta^2\) are defined as above, but with the residual \(\epsilon_t\) obtained from the estimation of (9) with \(\beta_l \neq 0\). \(\delta s_{it}\) is the determinant of the regressor cross-product matrix, given by

\[
D_{st} = [T^2(T - 1)/12] \Sigma y_{it}^2 - T(\Sigma y_{it}^2) + T(T + 1)(\Sigma y_{it}^2) - (T(1 + 2T + 1)/6)(\Sigma y_{it}^2)
\]

The Phillips and Perron statistics have the same limiting distributions as the corresponding DF and ADF statistics, provided that \(m \to \infty\) as \(T \to \infty\), such that \(mT^{1/4} \to 0\).

(iv) Simulation evidence in Molinas (1986) and Schwert (1986, 1989), shows that the tests proposed by Dickey and Fuller and by Phillips and Perron are affected by the process generating the data in large finite samples. In particular, when the underlying process is ARIMA \((0, 1, 1)\) with a MA parameter close to one, the Dickey–Fuller distributions i.e. these tests will lead to the conclusion that economic data are stationary too frequently). The intuition behind this result is that if the DGP of \(y_t\) is

\[
\Delta y_t = (1 - \theta L)\epsilon_t,
\]

if \(\theta\) is close to one, \((1 - L)\) will tend to cancel on both sides of (19), giving the impression that \(y_t\) behaves like a white noise. However, the Said and Dickey (1984) high-order autoregressive \(t\) test for the unit root, with a suitable choice of \(l\), has size close to its nominal level for all values of the MA parameter. Schwert suggests searching for the correct specification of the ARIMA process before testing for the presence of a unit root in the AR polynomial and provides the relevant critical values for the Said and Dickey (1984), Phillips (1987) and Phillips and Perron (1988) tests based on Monte Carlo experiments.

(v) Hall (1989) proposes a new approach to testing for a unit root in a time series with a moving average component based on an instrumental variable (IV) estimator.
Let \( y_t \) be generated by the DGP

\[
y_t = \beta_0 + \beta_1 t + \rho y_{t-1} + \theta(L)\epsilon_t
\]

(20)

where \( \theta(L) \) is again a \( q \)-th order lag polynomial. Then, the IV estimator for model (20) is defined as follows

\[
[\hat{\beta}_0, \hat{\beta}_1, \hat{\rho}]^{IV} = \left( \sum_{t=1}^{T} Z_{t}X_{t} \right)^{-1} \left( \sum_{t=1}^{T} Z_{t}Y_{t} \right)
\]

(21)

where \( Z_t = (1, t, y_{t-k}) \) and \( X_t = (1, t, y_{t-1}) \), \( k = (q + 1) \) (see Dolado (1989) for the choice of optimal IV in this framework). For the model (20) when \( \beta_1 = 0 \), the IV estimator is given by

\[
[\hat{\beta}_0, \hat{\beta}_1, \hat{\rho}]^{IV} = \left( \sum_{t=1}^{T} Z_{t}X_{t} \right)^{-1} \left( \sum_{t=1}^{T} Z_{t}Y_{t} \right)
\]

(22)

where now \( Z_t = (1, y_{t-k}) \), \( X_t = (1, y_{t-1}) \).

Let \( \hat{t}(\rho^{IV}) \) and \( \hat{t}(\rho^{III}) \) be the \( t \)-statistics associated with the null hypothesis \( \rho = 1 \) in (19) (with and without trend), then Hall proves that

\[
\hat{t}_{TV} = s_{\hat{\rho}}(\hat{\rho}^{IV})/\hat{\sigma}_{TV}
\]

(23)

and

\[
\hat{t}_{IV} = s_{\hat{\rho}}(\hat{\rho}^{IV})/\hat{\sigma}_{IV}
\]

(24)

where \( s_{\hat{\rho}}, \hat{\sigma}^2 \) and \( \hat{\theta}^2 \) are consistent estimators of the variances of \( \epsilon \) and the long-run variance \( \sigma^2 = \theta(L)\epsilon \) obtained as in (16) and (17).

(vi) As it might have been noticed, one important limitation of all the previous testing procedures is that they are not independent of the nuisance parameters contained in the deterministic component of the time-series process. This limitation has produced an alternative strand in the literature on testing. In this respect, Bhargava (1986) has developed most powerful invariant (MPI) tests for the null hypothesis corresponding to DGP (9) (with and without trend).

The corresponding critical values are given by Bhargava (1986, p. 378). The test is found to have slightly greater power than the tests proposed by Dickey and Fuller, when the data are generated by an AR(1) process.

(vii) Another limitation of all the previous testing procedures is that the distributions of the corresponding statistics are non-standard and hence a different set of critical values has to be used in each case. This problem has originated a new strand of research (see Phillips and Ouliaris, 1988), which exploits the fact that differencing a stationary series induces a unit root in the moving average representation. This fact provides a diagnostic for testing whether the series is \( I(0) \) or \( I(1) \), by using the long-run variance of the first difference of the time series \( y_t \). To clarify the interpretation of the test, let us assume that \( y_t \) is generated by

\[
Y_t = \theta(L)\epsilon_t; \theta(L) = (1 - \theta_1)\theta(L)
\]

Then the long-run variance of \( \Delta y_t \) is \( \sigma^2 = \sigma^2(\theta(1))^2 \). If \( \theta_1 \neq 1 \) and \( \theta(1) \neq 0 \), then \( \sigma^2 \) is finite, whilst if \( \theta_1 = 1 \), \( \sigma^2 \) is zero. In other words, if the time series \( y_t \) is \( I(0) \), \( \Delta y_t \) will have \( \sigma^2 = 0 \), whereas if it is \( I(1) \), \( \sigma^2 \neq 0 \). Therefore the null hypothesis is \( H_0: \sigma^2 = 0 \) or \( H_c: \sigma^2 \neq 0 \), getting rid of the units of measurement. Obtaining an estimate of \( \sigma^2 \) as in (16), Phillips and Ouliaris prove that

\[
m^{1/2}(\hat{\sigma}^2 - \sigma^2)/\sigma^2 \sim N(0, 1)
\]

(27)

Since only the alternative hypothesis is a simple hypothesis, i.e. \( H_1: \sigma^2 = 0 \), Phillips and Ouliaris propose a bounds procedure based upon the corresponding confidence interval in (27), yielding

\[
\hat{\sigma}^2[1 + (\omega_0m^{1/2})] \leq \hat{\sigma}^2 \leq \hat{\sigma}^2[1 - (\omega_0m^{1/2})]
\]

(28)

where \( \omega_0 \) is the (1 - \( \alpha \)) percentage point of the standard normal distribution. According to the bounds test, \( H_0 \) is rejected if the upper limit of \( \hat{\sigma}^2 \) in (28) is sufficiently small. Similarly \( H_0 \) is not rejected if the lower bound is sufficiently large. Phillips and Ouliaris recommend using 0.10 as the rejection point for the upper and lower bound. Simulation results show, however, that the suggested value can be very conservative in some instances. For example if the DGP is ARIMA \((1, 1, 1)\) with parameter values in the interval \((-0.6, 0.6)\), the average upper bound is 0.45 whereas the value of the lower bound is close to 0.10.

A very nice implication of this type of tests is that, given their asymptotic normality, they can be applied to deal with very general trend-cycle models (e.g. piecewise linear functions of time, any type of impulse or step dummy). All that is needed is to perform the previous test on the differenced residuals of the regression of \( y_t \) on the general trend function.
3.2. Integration and seasonality

Due to the fact that many economic time series contain important seasonal components, there have been several developments in the concept of seasonal integration.

Osborn et al. (1988) amend the Engle and Granger (1987) definition of integration to account for seasonality: a variable $y_t$ is said to be integrated of order $(d, D)$ or $y_t - I(d, D)$, if it has a stationary, invertible, non-deterministic ARMA representation after one-period differencing $d$ times and seasonally differencing $D$ times.

Following Pierce (1976), let us assume that seasonality has both deterministic and stochastic components, then a seasonal observed series $y_t$ can be seen as the sum of a purely stochastic process $x_t$ and a purely deterministic seasonal component $\mu_t$

$$y_t = x_t + \mu_t$$  \hspace{1cm} (29)

where

$$\mu_t = \beta_0 + \beta_1 t + \sum_{j=1}^{q-1} \beta_j S_{jt}$$  \hspace{1cm} (30)

where $S_{jt}$ are zero/one seasonal dummies, and $q = 12$ for monthly data, $q = 4$ for quarterly data and so on.

By regressing $y_t$ on $\mu_t$, we can remove the deterministic seasonality, using the residuals from that regression as if they were the true $x_t$. Then, the following tests can be applied for testing $I(d, D)$ integration, where we present the case for $q = 4$ (i.e. we are dealing with quarterly data).

(i) Dickey et al. (1984) present a test for the presence of a single unit root at a seasonal lag. The null hypothesis is $H_0$: $I(0,1)$ and the alternative is $H_1$: $I(0,0)$. The test is a 3-step procedure as follows:

1. The regression equation

$$\Delta x_t = \theta_0 + \sum_{i=1}^{p} \theta_i \Delta x_{t-i} + \varepsilon_t$$

is estimated by OLS, where $\Delta x_t = x_t - x_{t-4}$

2. Using the estimates $\hat{\theta}_1, \hat{\theta}_2, \ldots, \hat{\theta}_p$ define

$$z_t = \hat{\theta}(L)x_t = (1 - \hat{\theta}_1 L - \cdots - \hat{\theta}_p L^p)x_t$$

3. Run the regression

$$\Delta x_t = \hat{\phi}_0 + \sum_{i=1}^{p} \hat{\phi}_i \Delta x_{t-i} + \varepsilon_t$$

and compute the $t$-ratio on $\hat{\phi}_0$. This sample statistic, denoted $\tau_d$, is compared to the tabulated critical values given in Table 7 of Dickey, Hasza and Fuller (1984), p. 362).

(ii) Dickey et al. (1986, Appendix B) show that the limiting distribution of the unit root statistics is not affected by removal of seasonal means from autoregressive series. Therefore, we can use the ADF statistic from the regression equation

$$\Delta x_t = \gamma_1 \Delta x_{t-1} + \sum_{i=1}^{p} \gamma_{1i} \Delta x_{t-i} + \varepsilon_t$$  \hspace{1cm} (31)

to test the null hypothesis $H_0$: $x_t - I(1,0)$ versus the alternative $H_1$: $x_t - I(0,0)$. The relevant critical values are given by Fuller (1976, p. 373) for $\tau_d$.

(iii) Engel et al. (1987) present the following 3-step procedure to test for seasonal unit roots in the possible presence of a zero frequency unit root:

1. Compute $\hat{\theta}(L)$ as for the Dickey et al. (1984) statistic

2. Compute

$$z_{11} = \hat{\theta}(L)(1 + L + L^2 + L^3) x_t$$

$$z_{21} = -\hat{\theta}(L)(1 - L + L^2 - L^3) x_t$$

$$z_{31} = -\hat{\theta}(L)(1 - L^2) x_t$$

3. Run the regression

$$\Delta x_t = \pi_1 z_{1t-1} + \pi_2 z_{2t-1} + \pi_3 z_{3t-2} + \sum_{i=1}^{p} \pi_4 \Delta x_{t-i} + \varepsilon_t$$

and compute the values of the $t$-ratios on $\hat{\pi}_1, \hat{\pi}_2$ and $\hat{\pi}_3$. The critical values are given in Table 2.1 of Engle et al. (1987, p. 14). If $x_t - I(0,0)$, then all three of these statistics should be significant. If the test statistic for $\pi_1 = 0$ is not significant, then $x_t - I(1,0)$. If either of the test statistics for $\pi_2 = 0$ or $\pi_3 = 0$ is not significant, then $x_t - I(0,1)$.

(iv) Osborn et al. (1988) present an alternative 3-step test procedure:

1. Run the regression

$$\Delta x_t = \psi_0 + \sum_{i=1}^{p} \psi_i \Delta x_{t-i} + \varepsilon_t$$

and compute $\hat{\psi}_1, \hat{\psi}_2, \ldots, \hat{\psi}_p$

2. Compute $z_{10} = \hat{\psi}(L) \Delta x_t$ and $z_{20} = \hat{\psi}(L) \Delta x_t$, where

$$\hat{\psi}(L) = (1 - \hat{\psi}_1 L - \cdots - \hat{\psi}_p L^p)$$

3. Run the regression

$$\Delta x_t = \phi_0 z_{10} + \phi_2 z_{20} + \sum_{i=1}^{p} \phi_3 \Delta x_{t-i} + \varepsilon_t$$

and compute the $F$-statistics for the null hypothesis $H_0$: $\phi_0 = \phi_2 = 0$, and the $t$-ratio on $\phi_1$ and $\phi_2$. The null hypothesis for both type of statistic is $H_0$: $x_t - I(1,1)$ with alternative hypothesis $H_1$: $x_t - I(0,0)$ or $H_2$: $x_t - I(0,1)$. The
critical values of these statistics are given in Table A.1 of Osborn et al. (1988, p. 376).

4. Other forms of integration
In this section, we review alternative forms of integration based upon the possibility that the model parameters are allowed to vary (periodic integration) or the possibility of using non-integer differencing orders to achieve stationarity in the data (fractional integration). Both ideas have received recent attention in the literature.

4.1. Periodic Integration
Osborn et al. (1988), building upon the framework developed by Tiao and Grupe (1980), investigate the use of a periodic model (whose parameters are allowed to vary according to the time at which observations are made) as an alternative to the conventional approaches to modelling for seasonal data.

The non-deterministic periodic AR(1) process is given by the following expression
\[ y_t = \sum_{j=1}^{q} \omega_j S_j y_{t-j} + \epsilon_t \] (32)
or
\[ y_t = \omega_2 y_{t-1} + \epsilon_t \] (33)
when \( t \) falls in season \( j \). As in equation (30), \( S_j \) are seasonal dummy variables corresponding to season \( j (j = 1, \ldots, q) \). Equation (33) states that \( y_t \) is seasonal, seasonality arising not from any direct dependence of \( y_t \) on \( y_{t-q} \), but from the annual variation in the autoregressive coefficients \( \omega_j \). This dependence can arise, for example, if the allocation of expenditure over the year reflects seasonal tastes and hence seasonality in the underlying utility function (see Osborn, 1988).

Osborn et al. (1988) define periodic integration as follows: A variable \( y_t \) is periodically integrated of order one [or \( y_t \sim PI(1) \)] if \( y_t \) is non-stationary and \( \delta y_t \) is stationary, where the generalised difference operator \( \delta \) is defined as
\[ \delta y_t = y_t - \omega_2 y_{t-1} \] (34)
the product \( \omega_1, \omega_2, \ldots, \omega_2 \) being equal to one.

Osborn et al. (1988) propose two ways of testing for periodic integration:
(i) After regressin g \( y_t \) on \( \mu_t \) (as defined in (30)) to remove conventional deterministic seasonality, a non-deterministic periodic AR(1) process (as defined in (33)) is fitted to the residuals \( \epsilon_t \). This case is referred to as the removed deterministic seasonality case.

(ii) The case of included deterministic seasonality is given by fitting the following periodic AR(1) process to the original observations \( y_t \)
\[ y_t = \delta_t + \omega_2 y_{t-1} + \epsilon_t \] (j = 1, \ldots, q)

To allow for the possibility of a periodic disturbance variance, they suggest a 2-step estimation procedure for both cases. In the first step, the appropriate equation is estimated by OLS applied to observations on each of the \( q \) seasonal realisations, (i.e. four for quarterly data); then the equation is transformed by dividing each variable by the appropriate seasonal residual standard deviation estimated in this first stage regression. Using the transformed data, in the second step the periodic AR(1) model is estimated in its two versions (i.e. removed and included deterministic seasonality), with imposition of the restriction \( \omega_1, \omega_2, \ldots, \omega_2 = 1 \).

Finally, the tests (i) to (iii) in Section 3.2 are applied to the residuals of the periodic AR(1) model.

4.2. Fractional Integration
As was seen in Section 2, one of the main characteristics of the existence of unit roots in the Wold representation of a time series is that they have ‘long memory’ (i.e. shocks have a permanent effect on the level of the series). In general it is known that the coefficient on \( \epsilon_{t-j} \) in the MA representation of any \( I(d) \) process has a leading term \( j^{-d} \) (for example, the coefficient in a random walk is unity, since \( d = 1 \)). This implies that the variance of the original series is \( O(t^{2d-1}) \). So, all that is needed to have ‘long-memory’, in the sense that the variance explodes as \( t \to \infty \), is a degree of differencing \( |d| > 0.5 \). Thus, it is clear that a wide range of dynamic behaviour is ruled out a priori if \( d \) is restricted to integer values.

Granger and Joyceux (1980) and more recently Diebold and Rudebusch (1989) have proposed a new family of “long-memory” processes, denoted by ARFIMA (autoregressive fractionally integrated moving-average processes), of which the ARIMA processes are particular cases: A variable \( y_t \) is fractionally integrated of order \( d \) if \( y_t \sim \text{FIMA}(d) \) if \( y_t \) is non-stationary and \( \Delta^d \) is stationary, where the operation \( \Delta^d \), using a binomial expansion, is as follows
\[ (1 - L)^d = 1 - dL + \frac{d(d-1)}{2!} L^2 - \frac{d(d-1)(d-2)}{3!} L^3 + \cdots \] (35)
where \( d \) belongs to the rational set of numbers and \( d > 0.5 \).

Note that these processes can always be constrained to belong to the open interval \((0.5, 1.0)\) by subtracting the integer part of the differencing order. So if the degree of differencing is, for example, 1.7, we can always redefine the degree of differencing as \( d = 1 \) (0.7 in this case).

Diebold and Rudebusch (1989) propose the following method of testing and estimation for fractional integration:
(i) First difference the relevant series denoted \( \delta y_t = (1 - L) y_t \). As \( d \) of the level series equals \( 1 + d \), a value of \( d \) equal to zero corresponds to a unit root in \( y_t \).
Thus, we wish to estimate $\hat{d}$ in the model

$$
(1 - L)^d \hat{y}_t = \theta(L)e_t
$$

(36)

(ii) Estimate by OLS the following regression

$$
\ln(|I(\lambda_j)|) = \beta_0 - \beta_1 \ln(4 \sin^2(\lambda_j/2)) + \eta_j, \quad j = 1, \ldots, T^{1/2}
$$

(37)

where $\lambda_j = 2\pi j/T (j = 0, \ldots, T - 1)$ denote the harmonic ordinates of the sample and $I(\lambda_j)$ denote the periodogram at ordinate $j$ (see Harvey, 1981, p. 66). Geweke and Porter-Hudak (1983) prove that $\beta_1$ is a consistent and asymptotically normal estimate of $d$. Furthermore, the variance of the estimate of $\beta_1$ is given by the usual OLS estimator, which can be used to test the null hypothesis $H_0: d = 0$ (i.e., $\eta_j = I(1)$). Moreover, they show that the variance of the disturbance $\eta_j$ is known to be equal to $\pi^2/6$, which can be imposed to increase efficiency.

(iii) Given an estimate of $d$ we transform the series $\hat{y}_t$ by the 'long-memory' filter (35), truncated at each point to the available sample. The transformed series is then modelled as in (36) (or in the ARMA representation) following the traditional Box and Jenkins (1970) procedure.

5. Testing for stationarity in the cointegrating residuals

In the two previous sections we have discussed procedures to test for the order of integration of individual time series. This is, as we mentioned in Section 2, a first stage in the estimation and testing of cointegrating relationships. The reason is a matter of 'integration or growth accounting' in the words of Pagan and Wickens (1989) (i.e., the left and right hand sides of an equation, such as (4) must be of the same order of integration, otherwise, the residual will not be stationary). If, for example, the dependent variable is $I(1)$, the independent variables need to be $I(1)$ variable, or, perhaps, be $I(2)$ and cointegrate among themselves to an $I(1)$ variable.

In order to illustrate testing for cointegration, we will consider a bivariate case where, say, $y_t$ and $x_t$ have been found to contain a unit root at the regular frequency (i.e., both are $I(1)$). Then, the following part of the cointegration test is to estimate the cointegrating regression (4) and test whether the 'cointegrating residuals' ($\tilde{z}_t = y_t - \hat{\alpha} - \hat{\beta} x_t$) are $I(0)$.

Engle and Granger (1987) suggest seven alternative tests for determining if $z_t$ is stationary. Here we will consider only two of their suggested tests, namely the Durbin–Watson statistic for the cointegration equation (CRDW) and the ADF statistic for the cointegrating residuals (CRADF).

The DW statistic for equation (4) will approach zero if the cointegrating residuals contain an autoregressive unit root, and thus the test rejects the null hypothesis of non-cointegration if the CRDW is significantly greater than zero. The intuition underlying this test can be understood by means of a simple example. Suppose that $z_t$ is assumed to follow an AR(1) process with coefficient $\rho$. Then the null hypothesis of non-cointegration is $H_0: \rho = 1$. Since it can be shown that the DW statistic is such that $DW = 2(1 - \rho)$ (see, e.g., Harvey, 1981, p. 20), the previous null hypothesis can be translated into $H_0: DW = 0$ versus the alternative $H_1: DW > 0$. Engle and Granger (1987, p. 269) present the critical values of this test for 100 observations.

The CRADF statistic is based upon the OLS estimation of

$$
\Delta z_t = \gamma_1 z_{t-1} + \sum_{i=1}^{p} \gamma_i \Delta z_{t-i} + \epsilon_t
$$

(38)

where again $p$ is selected on the basis of being sufficiently large to ensure that $\epsilon_t$ is a close approximation to white noise. The $t$-ratio statistic on $\gamma_1$ is the CRADF statistic. We cannot use the critical values tabulated by Fuller (1976) to test for a unit root in the cointegrating residuals. Intuitively, since OLS estimation of the cointegrating regression equation chooses $\alpha$ and $\beta$ to minimise the residual variance, we might expect to reject the null hypothesis $H_0: z_t \sim I(1)$ rather more often than suggested by the nominal test size, so that the critical values have to be raised in order to correct the test bias. Engle and Granger (1987, p. 269) present the critical values for the CRADF statistic generated from Monte Carlo simulations of 100 observations.

Note that the critical values for both CRDW and CRADF statistics are for the bivariate case (i.e., for one dependent and one independent variable in the cointegrating regression), and for 100 observations. Engle and Yoo (1987) produce expanded critical values for CRDW and CRADF statistics for 50, 100 and 200 observations, and for systems of up to five variables.

6. Some new developments in cointegration

In this section we survey some new test procedures for cointegration that have recently been proposed in the literature. Most of these procedures extend the testing and estimation approach introduced in Section 2 to a multivariate context where there may exist more than a single cointegrating relationship among a set of $n$ variables. For example, among nominal wages, prices, employment and productivity, there may exist two relationships, one determining employment and another determining wages (see, inter alia, Hall, 1986, and Jenkinson, 1986).

In general, if $X_t$ represents a vector of $n$ $I(1)$ variables whose Wold representation is

$$
\Delta X_t = C(L)e_t
$$

(39)

where now $\alpha \sim \text{nid}(0, \Sigma)$, $\Sigma$ being the covariance matrix of $e_t$ and $C(L)$ an invertible matrix of polynomial lags. If there exists a cointegrating vector $\alpha$, then premultiplying (39) by $\alpha'$, we obtain

$$
\alpha' \Delta X_t = \alpha' [C(L) + C^*(L)(1 - L)]e_t
$$

(40)

where $C(L)$ has been expanded around $L = 1$ and $C^*(L)$ can be shown to be invertible (see Engle and Granger, 1987). If the linear combination $\alpha' X_t$ is
stationary, then \( \alpha' C(1) = 0 \) and then \((1 - L)\) would cancel out on both sides of (40). If (39) is represented in AR form, we have that
\[
A(L)C(L) = (1 - L)I
\]
where \( I \) is an identity matrix, and hence
\[
A(1)C(1) = 0
\]
This implies that \( A(1) \) can be written as \( A(1) = \gamma L \). If there were \( r \) cointegrating vectors \((r \leq n - 1)\), then \( A(1) = B' \), where \( B \) and \( \Gamma \) are \((n \times r)\) matrices which collect the \( r \) different \( \gamma \) and \( \alpha \) vectors. Testing the rank of \( A(1) \) or \( C(1) \) constitutes the basis of the following procedures:

(i) Johansen (1988) and Johansen and Juselius (1988) develop a maximum likelihood estimation procedure that has several advantages on the 2-step regression procedure suggested by Engle and Granger. It relaxes the assumption that the cointegrating vector is unique and it takes into account the error structure of the underlying process.

Johansen considers the \( p \)-th order autoregressive representation of \( X_t \)
\[
X_t = \Pi_1 X_{t-1} + \Pi_2 X_{t-2} + \cdots + \Pi_p X_{t-p} + \epsilon_t
\]
which, following a similar procedure to the ADF test, can be reparameterised as
\[
\Delta X_t = \Pi_1 \Delta X_{t-1} + \cdots + \Pi_p \Delta X_{t-p} + \epsilon_t
\]
where \( \Pi_p = -\Pi(1) = (-\Pi_1 + \cdots + \Pi_p) \). To estimate \( \Pi_p \) by maximum-likelihood, we estimate by OLS the following regressions
\[
\Delta X_t = \Gamma_0 \Delta X_{t-1} + \cdots + \Gamma_{p-1} \Delta X_{t-p} + \eta_t
\]
and
\[
X_{t-p} = \Gamma_1 \Delta X_{t-1} + \cdots + \Gamma_{p-1} \Delta X_{t-k+1} + \epsilon_t
\]
and compute the product moment matrices of the residuals
\[
S_{ij} = T^{-1} \sum_{t=1}^{T} \epsilon_t \epsilon'_t; \quad i, j = 0, 1
\]
The likelihood ratio test statistic of the null hypothesis \( H_0: \Pi'_p = B' \), i.e., there are at most \( r \) cointegrating vectors, is
\[
-2 \ln(Q) = -T \sum_{i=r+1}^{p} (1 - \hat{\lambda}_i)
\]
where \( \hat{\lambda}_{r+1}, \ldots, \hat{\lambda}_p \) are the \( n-r \) smallest eigenvalues of \( S_{10}S_{00}S_{01} \) with respect to \( S_{11} \), obtained from the determinant
\[
| \hat{\lambda}_{S_{11}} - S_{10}S_{00}S_{01} | = 0
\]
Under the hypothesis that there are at most \( r \) cointegrating vectors, Johansen (1988) shows that the likelihood ratio test (45) is asymptotically distributed as a functional \( f(W) \). Johansen (1988, p. 239) provides a table with various quantiles of the distribution of the likelihood ratio test for \( r = 1, 2, \ldots, 5 \). He also shows that these quantiles can be obtained by approximating the distribution by \( c \chi^2(f) \) where \( c = 0.85 - 0.85/f \), and \( \chi^2(f) \) is a central chi-square distribution with \( f = 2(n - r)^2 \) degrees of freedom.

(ii) Stock and Watson (1988) focus on testing for the rank of \( C(1) \) in (40) and denote their approach as a ‘common trends’ approach, by noticing that if there exist \( r \) cointegrating vectors in (40), then there exists a representation such that
\[
x_t = C(1) \Phi \tau_t + C^*(L)x_t
\]
where \( \Phi \) is an \( r \times (n - r) \) matrix and \( \tau_t \) is an \( n - r \) vector random walk. In other words, \( x_t \) can be written as the sum of \( n-r \) common trends and an \( I(0) \) component. Estimating (39) as a multivariate ARMA \((1, q)\) model, the null hypothesis that there are \( r \) cointegrating vectors is equivalent to the null hypothesis that there are \( n-r \) ‘common trends’. This implies that, under the null hypothesis, the first \((n-r)\) eigenvalues of the autoregressive matrix should be unity and the remaining eigenvalues should be smaller than one. The test is based on \( T(\hat{\lambda}_{n-r+1} - 1) \) and the critical values can be found in Stock and Watson (1988, p. 1104).

Phillips and Ouliaris (1988) have also proposed a multivariate extension of their unit root test, as discussed in Section 3, based upon the eigenvalues of the long-run variance of the differenced multivariate series.

(iii) As discussed in Section 2, when concentrating on a single equation estimator in the case of a single cointegrating \((1, 1)\) relationship, the OLS estimator of the slope in the static regression (4) is ‘super-consistent’ but its estimator in the case of a single cointegrating vector is biased and yields asymptotic normality in the case where such correlation exists. This bias and non-normality stem from the non-parametric correction by which the error term \( z_t \) is conditioned on the process followed by \( \Delta x_t \) and, hence, orthogonality between regressors and disturbance is achieved by construction. The FME estimators of \( \alpha \) and \( \beta \) in (4) are given by
\[
(\hat{\alpha}, \hat{\beta}) = \left( \sum_{t=1}^{T} X_t X'_t \right)^{-1} \left[ \sum_{t=1}^{T} X_t (y_t - \hat{\beta} \Delta x_t - \epsilon_t) \left( \Delta \hat{x}_t - \hat{\beta} \hat{\Delta} x_t \right) \right]
\]
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The existence of such long-run relationship can, and should, be tested, using the battery of tests for unit roots discussed in this paper.

The considerable gap between the economic theorist, who has much to say about equilibrium but relatively little to say about dynamics, and the econometrician, whose models concentrate on dynamic adjustment process, has, to some extent, been bridged by the concept of cointegration. In addition to allowing the data to determine the dynamics of the model, cointegration suggests that models can be significantly improved by introducing, and allowing the data to parameterise, equilibrium conditions suggested by economic theory. Furthermore, the generic existence of such long-run relationship can, and should, be tested, using the battery of tests for unit roots discussed in this paper.

7. Brief conclusion

The considerable gap between the economic theorist, who has much to say about equilibrium but relatively little to say about dynamics, and the econometrician, whose models concentrate on dynamic adjustment process, has, to some extent, been bridged by the concept of cointegration. In addition to allowing the data to determine the dynamics of the model, cointegration suggests that models can be significantly improved by introducing, and allowing the data to parameterise, equilibrium conditions suggested by economic theory. Furthermore, the generic existence of such long-run relationship can, and should, be tested, using the battery of tests for unit roots discussed in this paper.

Notes

1. There is an early survey by two of us (see Dolado and Jenkinson, 1987) and a more recent one by one of us (see Sosvilla-Rivero, 1989), and some excellent overviews by Granger (1986), Hendry (1986), Stock and Watson (1987), Diebold and Nerlove (1988), Pagan and Wickens (1989) and Haldrup and Hylleberg (1989).

2. Even though cointegration implies at least one causal direction, it does not imply any explicit causal relationship. Here we have assumed that the causal relation suggested by the theory (i.e. $x$ causes $y$) is the correct one. See Granger (1988) for a study of cointegration and causality.

3. Nickell (1985) shows that the ECM is also consistent with optimising behaviour on the part of economic agents.

4. Alternatively we would say that a 'super-consistent' estimator is such that $\hat{\beta} - \beta$ has probabilistic order of magnitude $O(T^{-1})$.

5. The explosivity of the variance characterises the 'integration in variance'. Integration can also be applied to other higher moments (see Escrivano (1987) and Hansen (1988)).

6. See the Appendix for a description for the sequential test procedure $k (k \geq 2)$ unit roots. Note that the alternative sequence of testing for the presence of a unit root in the series levels and if it is not rejected, then test for a second unit root, i.e. a unit root in the differences, and so on is not well founded on statistical grounds since the unit root tests described in Section 3 are based on the assumption of stationarity under the alternative hypothesis.

7. This result has been noticed by West (1988) and it is applicable also to regression models like (2) where $x_t$ has a unit root with drift. However, Hylleberg and Mizon (1989) have noted in simulation studies that the drift has to be quite large for the deterministic trend to dominate the integrated component. If there are two $I(1)$ regressors with drift in the model, a trend should also be included to avoid asymptotic perfect collinearity.

8. Ouliaris et al. (1988) compute critical values when in the maintained hypothesis there is up to a quintic trend. Similarly, Perron (1987) computes critical values when there is a piecewise linear trend under the maintained hypothesis.

9. Sims et al. (1986) and Banerjee and Dolado (1988), have shown that the estimates of coefficients on $I(0)$ variables in regression models with $I(1)$ variables are $O(T^{-1/2})$ and asymptotically normally distributed.

10. In the frequency domain notation, the long-run variance is equal to $2 \pi f_0$, where $f_0$ is given by Fuller (1976), p. 373).

11. The size of $C(1)$ in a univariate context, has been called the 'size the unit root', giving rise to a literature (see Cochrane (1988) and references therein) which deals with the relative importance of the trend and cyclical components in the decomposition of a time-series.

Appendix: testing for $k$ unit roots

Dickey and Pantula (1987) suggest a sequence of tests for unit roots, starting with the largest number of roots under consideration ($k$) and decreasing by one each time the null hypothesis is rejected, stopping the procedure when the null hypothesis is accepted.

They illustrate their sequential procedure for the case $k = 3$. It is as follows:

1. Run the regression

$$\Delta^3 y_t = \delta_0 + \xi_1 \Delta^2 y_{t-1} + \epsilon_t$$

(where $\Delta^k$ denotes third difference, and compute the 'pseudo t-statistic' $t^*_c(3)$ (i.e. the $t$-statistic on $\xi_1$). Reject the null hypothesis $H_3$ of three unit roots and go to step 2 if $t^*_c(3) < \tau_4$ (or if $t^*_c(3) > \tau_3$ if absolute values are considered) where $\tau_3$ is given by Fuller ([1976], p. 373).

2. Run the regression

$$\Delta^3 y_t = \delta_0 + \xi_1 \Delta^2 y_{t-1} + \xi_2 \Delta y_{t-1} + \epsilon_t$$

and compute $t^*_c(3)$ and $t^*_c(3)$. Reject the null hypothesis $H_2$ of exactly two unit
roots and go to step 3 if in addition to $t^* \sim (3) < \tau$, it is also found that $t^*_u \sim (3) < \tau$.

3. Run the regression

$$
\Delta^3 y_t = \epsilon_t + \epsilon_t^2 \Delta y_{t-1} + \epsilon_t \Delta y_{t-2} + \epsilon_t \Delta y_{t-3} + \epsilon_t
$$

and compute $t^*_u \sim (3)$, $t^*_u \sim (3)$ and $t^*_u \sim (3)$. Reject the null hypothesis $H_0$ if $t^*_u \sim (3) < \tau (i = 1, 2, 3)$.

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**Notes**