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Calmness of the optimal value in linear programming*

M.J. Gisbert[†] M.J. Cánovas[†] J. Parra[†] F.J. Toledo[†]

Abstract

The final goal of the present paper is computing/estimating the calmness moduli from below and above of the optimal value function restricted to the set of solvable linear problems. Roughly speaking, these moduli provide measures of the maximum rates of decrease and increase of the optimal value under perturbations of the data (provided that solvability is preserved). This research is developed in the framework of (finite) linear optimization problems under canonical perturbations; i.e., under simultaneous perturbations of the right-hand-side (RHS) of the constraints and the coefficients of the objective function. As a first step, part of the work is developed in the context of RHS pertubations only, where a specific formulation for the optimal value function is provided. This formulation constitutes the starting point in obtaining exact formulae/estimations for the corresponding calmness moduli from below and above. We point out the fact that all expressions for the aimed calmness moduli are conceptually tractable (implementable) as far as they are given exclusively in terms of the nominal data.

Keywords. Calmness, Optimal Value, Linear programming

Mathematics Subject Classification: 90C31, 49J53, 49K40, 90C05

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[†]Center of Operations Research, Miguel Hernández University of Elche, 03202 Elche (Alicante), Spain (mgisbert@umh.es, canovas@umh.es, parra@umh.es, javier.toledo@umh.es).

1 Introduction

We consider the parameterized linear optimization problem

$$\begin{array}{lll}\pi: & \text{minimize} & c'x \\ & \text{subject to} & \overline{a}_t'x \leq b_t, \;\; t \in T := \{1, 2, ..., m\},\end{array} \tag{1}$$

where $x \in \mathbb{R}^n$ is the vector of decision variables, $\overline{a} \equiv (\overline{a}_t)_{t \in T} \in (\mathbb{R}^n)^T$ is fixed, and $\pi \equiv (c, b) \in \mathbb{R}^n \times \mathbb{R}^T$ is considered as the parameter to be perturbed around a nominal element denoted by $\overline{\pi} \equiv (\overline{c}, \overline{b})$. Observe that, for the sake of simplicity in the notation, we are identifying our parameter (c, b) with the associated optimization problem π. This is the context of the so-called *canonical perturbations,* where the right-hand-side (RHS) of the constraints and the objective function coefficients are allowed to be perturbed simultaneously. All elements in $\mathbb{R}^n$ are regarded as column-vectors and y' denotes the transpose of $y \in \mathbb{R}^n$.

Associated with the previous parameterized problem, we consider the *feasible set mapping* $\mathcal{F} : \mathbb{R}^T \rightrightarrows \mathbb{R}^n$, given by

$$\mathcal{F}(b) := \left\{ x \in \mathbb{R}^n \mid \overline{a}_t'x \leq b_t, \;\; t \in T \right\}, \;\; b \in \mathbb{R}^T,$$

the *optimal value function* $\vartheta : \mathbb{R}^n \times \mathbb{R}^T \to [-\infty, +\infty]$, given by

$$\vartheta(\pi) := \inf\{c'x \mid x \in \mathcal{F}(b)\}, \;\; \pi \in \mathbb{R}^n \times \mathbb{R}^T,$$

(with the convention $\vartheta(\pi) := +\infty$ when $\mathcal{F}(b) = \emptyset$), and the *optimal set mapping* $\mathcal{F}^{op} : \mathbb{R}^n \times \mathbb{R}^T \rightrightarrows \mathbb{R}^n$, given by

$$\mathcal{F}^{op}(\pi) := \left\{ x \in \mathcal{F}(b) \mid c'x = \vartheta(\pi) \right\}, \;\; \pi \in \mathbb{R}^n \times \mathbb{R}^T.$$

The present paper is mainly focussed on the calmness of ϑ at a nominal parameter $\overline{\pi}$ such that $\vartheta(\overline{\pi})$ is finite. As a first stage, part of this work (Section 3) is developed in the setting of RHS perturbations; i.e., where c is assumed to be fixed, say $c = \overline{c}$.

The concept of calmness for a function $f : \mathbb{R}^p \to [-\infty, +\infty]$ $(p \in \mathbb{N})$ may be introduced through the simultaneous fulfilment of the so-called calmness from below and calmness from above (see, e.g., [37, Section 8.F]). Let $\overline{z} \in \mathbb{R}^p$ be such that $f(\overline{z})$ is finite; recall that f is *calm at* $\overline{z}$ *from below* if there exist a constant $\kappa_1 \geq 0$ and a neighborhood U_1 of $\overline{z}$ such that

$$f(\overline{z}) - f(z) \leq \kappa_1 \|z - \overline{z}\|, \text{ for all } z \in U_1. \tag{2}$$

Respectively, f is *calm at* $\overline{z}$ *from above* if

$$f(z) - f(\overline{z}) \leq \kappa_2 \|z - \overline{z}\|, \text{ for all } z \in U_2, \tag{3}$$

for some constant $\kappa_2 \geq 0$ and some neighborhood U_2 of $\overline{z}$. Along this paper, the infimum of those constants κ_1 and κ_2 for which (2) and (3), respectively, hold (for some associated neighborhoods) are called the *calmness modulus from below* and *above* of f at $\overline{z}$, and they are denoted by $\underline{\mathrm{clm}}f(\overline{z})$ and $\overline{\mathrm{clm}}f(\overline{z})$, respectively; these moduli can alternatively be expressed as

$$\underline{\mathrm{clm}}f(\overline{z}) = \limsup_{z \to \overline{z}} \frac{f(\overline{z}) - f(z)}{\|z - \overline{z}\|} \text{ and } \overline{\mathrm{clm}}f(\overline{z}) = \limsup_{z \to \overline{z}} \frac{f(z) - f(\overline{z})}{\|z - \overline{z}\|}. \tag{4}$$

In both expressions the possibility of approaching $\overline{z}$ by constant sequences is allowed under the convention $\frac{0}{0} := 0$; so, $\underline{\mathrm{clm}}f(\overline{z})$ and $\overline{\mathrm{clm}}f(\overline{z})$ are always non-negative. Alternatively, in order to ensure the nonnegativity of $\underline{\mathrm{clm}}f(\overline{z})$ and $\overline{\mathrm{clm}}f(\overline{z})$, we could define these moduli as the maximum between 0 and the corresponding 'lim sup' in (4) with $z \to \overline{z}$, $z \neq \overline{z}$. Observe that, for $f : \mathbb{R} \longrightarrow \mathbb{R}$ given by $f(z) := |z|$ and $\overline{z} := 0$, we have $\limsup_{z \to \overline{z}, z \neq \overline{z}} \frac{f(\overline{z}) - f(z)}{|z - \overline{z}|} = -1$, while, under our convention, $\limsup_{z \to \overline{z}} \frac{f(\overline{z}) - f(z)}{|z - \overline{z}|} = 0$.

Finally, f is said to be *calm at* $\overline{z}$ if it is calm from below and above at $\overline{z}$, and the *calmness modulus* of f at $\overline{z}$, $\mathrm{clm}f(\overline{z})$, is defined as

$$\mathrm{clm}f(\overline{z}) := \limsup_{z \to \overline{z}} \frac{|f(z) - f(\overline{z})|}{\|z - \overline{z}\|} = \max\left\{\underline{\mathrm{clm}}f(\overline{z}), \overline{\mathrm{clm}}f(\overline{z})\right\}.$$

Note that $\underline{\mathrm{clm}}f(\overline{z})$ is nothing else but the *strong slope* of f at $\overline{z}$, while $\overline{\mathrm{clm}}f(\overline{z})$ corresponds to that of $-f$ (see, e.g., [2]). Roughly speaking, they respectively provide measures of maximum rates of decrease and increase of f at $\overline{z}$.

Coming back to our optimal value function ϑ, it is well-known that $\vartheta(\pi)$ is finite if and only if $\mathcal{F}^{op}(\pi) \neq \emptyset$; i.e., if and only if $\pi \in \mathrm{dom}\mathcal{F}^{op}$ (the domain of $\mathcal{F}^{op}$). The following remark motivates the fact of considering the calmness property of ϑ restricted to $\mathrm{dom}\mathcal{F}^{op}$, denoted by ϑ^R (the notation is inspired by [10], where the feasible set mapping restricted to its domain is analyzed); so, $\vartheta^R : \mathrm{dom}\mathcal{F}^{op} \rightarrow]-\infty, +\infty[$ and the corresponding calmness modulus at $\overline{\pi} \in \mathrm{dom}\mathcal{F}^{op}$ is given by

$$\mathrm{clm}\vartheta^R(\overline{\pi}) = \limsup_{\pi \underset{\mathrm{dom}\mathcal{F}^{op}}{\longrightarrow} \overline{\pi}} \frac{|\vartheta(\pi) - \vartheta(\overline{\pi})|}{\|\pi - \overline{\pi}\|},$$

(the norm in question is introduced in Section 2) where this notation reflects that $\pi \to \overline{\pi}$ with $\pi \in \mathrm{dom}\mathcal{F}^{op}$. The calmness moduli of ϑ^R from below and above are analogously defined. In the following remark we also appeal to the *Slater constraint qualification* (SCQ, in brief) which is satisfied at $b \in \mathrm{dom}\mathcal{F}$ if there exists $\widehat{x} \in \mathbb{R}^n$ (called a *Slater point*) such that $\overline{a}_t'\widehat{x} < b_t$ for all $t \in T$.

Remark 1 It is clear from the definitions that if $\overline{\pi}$ belongs to the interior of $\mathrm{dom}\mathcal{F}^{op}$ (equivalently, SCQ holds at $\overline{b}$ and $\mathcal{F}^{op}(\overline{\pi})$ is nonempty and bounded; see e.g. [19, Th. 6.1 and Cor. 10.2]), then

$$\mathrm{clm}\vartheta^R(\overline{\pi}) = \mathrm{clm}\vartheta(\overline{\pi}).$$

Theorem 8 provides an exact formula for $\mathrm{clm}\vartheta(\overline{\pi})$ in such a case. Otherwise, if $\overline{\pi}$ is in the boundary of $\mathrm{dom}\mathcal{F}^{op}$, $\mathrm{clm}\vartheta(\overline{\pi}) = +\infty$ (ϑ is not calm at $\overline{\pi}$), while we will show that $\mathrm{clm}\vartheta^R(\overline{\pi})$ is always finite (see Section 6). In this way, $\mathrm{clm}\vartheta^R(\overline{\pi})$ still represents a certain measure of the stability of our problem $\overline{\pi}$ when either SCQ fails at $\overline{b}$ or $\mathcal{F}^{op}(\overline{\pi})$ is unbounded.

According to the previous notations, the main goal of this work consists in computing (or estimating) $\mathrm{clm}\vartheta^R(\overline{\pi})$ via the computation of the corresponding calmness moduli from below and above. At this moment we point out the fact that the present paper establishes point-based formulae for the aimed calmness moduli (sometimes estimations), i.e., formulae which only involve the nominal data $\overline{\pi}$ (not appealing to parameters in a neighborhood). In relation to this point, the main contributions of the paper are gathered in theorems 5, 6, and 7 (the last one under the boundedness of $\mathcal{F}^{op}(\overline{\pi})$); see also the announced Theorem 8 (stated for $\overline{\pi}$ in the interior of $\mathrm{dom}\mathcal{F}^{op}$). Our first step will be developed in the context of RHS perturbations, in which case the corresponding optimal value function is specially tractable; in fact, an explicit formula for computing the optimal values around $\overline{\pi}$ is provided (Corollary 1) and it is used as a starting point for deriving the results about the calmness modulus of the optimal value under RHS perturbations (Theorem 4 and Corollary 3).

In order to integrate the new contributions into the existent literature, first let us comment that exact formulae for the calmness moduli of multifunctions $\mathcal{F}$ and $\mathcal{F}^{op}$ have been already established, respectively, in [8] and [6] (see [24] and [28] in relation to the calmness of $\mathcal{F}$ and $\mathcal{F}^{op}$ in nonlinear contexts). In general, the concept of calmness for a function $f : \mathbb{R}^p \to [-\infty, +\infty]$ does not coincide with the corresponding one to the multifunction $z \longmapsto \{f(z)\}$. The latter does not entail the continuity of function f. Calmness property constitutes an important concept in the field of variational analysis; with respect to this point, the reader is addressed to the monographs [12, 27, 31, 37] and references therein.

The present research could also be integrated in the widely explored field of sensitivity analysis in linear programming (LP for short), where, from different approaches, one tries to answer the natural question of *what* happens *if* one modifies the nominal problem's data. Specifically, our focus is on a *local aspect* of sensitivity analysis in contrast to the classical theory of parametric linear optimization, which usually concerns the behavior of ϑ^R and $\mathcal{F}^{op}$ on $\text{dom}\mathcal{F}^{op}$ or some of its subsets.

The theory of parametric linear optimization goes back to the early time of LP (see, e.g., [16] and [38]). A systematic development of LP with canonical perturbations started in the 1970s. One direction of research was focussed on the behavior of ϑ^R. Specifically, the continuity of ϑ^R was proved through different approaches: via parametric analysis (see [32]), by a parametric approach using Berge's theory (see [3, 5]), and by a primal-dual approach (see [25] and [41]). A second direction of development of sensitivity analysis in LP starting in the late 1960s was the analysis of semicontinuity and Lipschitz semicontinuity properties, which was based on approaches of variational analysis like Berge's theory or Hoffman's error bounds (see [3, 11, 13, 29, 35, 36, 39, 41]). Along this paper the continuity in the Painlevé-Kuratowski sense of multifunctions $\mathcal{F}$ and $\mathcal{F}^{op}$ restricted to their domains plays a crucial role; see Section 2 for details and specific references on these results.

In the 1990s and continuing until today, both directions became of great interest; see the survey [40] on different approaches to sensitivity, and the monograph [15]; see also [17] for the study of regions in which ϑ is affine, and [1, 14, 22, 23] for an approach to the sensitivity analysis from an optimal partition perspective, related to support set invariancy. For extensions to linear semi-infinite optimization, where T is infinite, the reader is addressed to [18], [20], and [21]. In the context of conic linear systems (which includes our framework as a particular case), the pioneer works [33] and [34] provide a quantitative approach to the stability of optimization problems, by using as an ingredient the *distance to infeasibility.*

To the authors' knowledge, the contributions of this paper about the computation (or estimation) of calmness moduli, which are contained in theorems from 4 to 8 and corollaries 3 and 4, are new. As immediate antecedents we refer to as [33] (see also [34]) and [40]. Specifically, from [33, Theorem 1.1(5)] one immediately derives an upper bound for $\text{clm}\vartheta(\overline{\pi})$, provided that $\overline{\pi}$ belongs to the interior of $\text{dom}\mathcal{F}^{op}$, in terms of the distances to primal and dual infeasibility; the details are gathered in the last section at the end of the paper (in Theorem 9 and Corollary 5). In the same case, our Theorem 8 provides an exact formula for $\text{clm}\vartheta(\overline{\pi})$, which constitutes a refinement of Corollary 5 as far as the referred upper bound might be far from the exact value of $\text{clm}\vartheta(\overline{\pi})$ (see Remark 7). On the other hand, [40, Theorem 18], translated into our notation, provides a particular constant k_1 involved in the calmness of ϑ from below (2) in the context of RHS perturbations (vector $\overline{c}$ remains fixed).

The structure of the paper is as follows. Section 2 presents the necessary notation, key tools, and preliminary results on the continuity of $\mathcal{F}$, ϑ, and $\mathcal{F}^{op}$ restricted to their domains. Sections 3 and 4 contain the original contributions of this paper. Section 3 is concerned with the calmness modulus of the optimal value function under RHS perturbations only, while Section 4 considers canonical perturbations. We finish the paper with a section of conclusions, where all the new results about the calmness moduli are gathered in Table 1. By combining the results collected in this table, we compute (among others) the aimed $\text{clm}\vartheta(\overline{\pi})$, provided that $\overline{\pi}$ is in the interior of $\text{dom}\mathcal{F}^{op}$ (see Theorem 8). Finally, in order to better integrate the current work in the literature, a comparative analysis between Theorem 8 and a certain consequence of [33, Theorem 1.1(5).] is developed in Subsection 5.1.

2 Preliminaries

In this section we introduce some necessary notation and results which are used later on. Given $X \subset \mathbb{R}^p$, $p \in \mathbb{N}$, we denote by $\text{conv}X$, $\text{cone}X$, $\text{aff}X$, and $\text{span}X$, the *convex hull*, the *conical convex hull,* the *affine hull,* and the *linear hull* of X, respectively. Moreover, $X^{\perp}$ denotes the orthogonal complement of $\text{span}X$, and, provided that X is convex, $\text{extr}X$ stands for the set of extreme points of X. Recall that $x \in \text{extr}X$ means that it is impossible to express x as a convex combination of two points of $X \backslash \{x\}$. It is assumed that $\text{cone}X$ always contains the zero-vector 0_p, in particular $\text{cone}\,\emptyset = \{0_p\}$.

From the topological side, if X is a subset of any topological space, $\text{int}X$, $\text{cl}X$ and $\text{bd}X$ stand, respectively, for the interior, the closure, and the boundary of X.

Throughout the paper, the *parameter spaces* $\mathbb{R}^T$ and $\mathbb{R}^n \times \mathbb{R}^T$ (associated with the contexts of RHS and canonical perturbations) are endowed, respectively, with the norms

$$\|b\|_\infty := \max_{t \in T} |b_t| \text{ and } \|\pi\| := \max\{\|c\|_*, \|b\|_\infty\}, \tag{5}$$

where $\mathbb{R}^n$ is equipped with an arbitrary norm, $\|\cdot\|$, with *dual norm* given by $\|u\|_* = \max_{\|x\|\leq 1} |u'x|$. Note that the choice of the dual norm $\|\cdot\|_*$ for measuring the perturbations of c comes from the fact that it is seen as the functional $x \mapsto c'x$. Moreover, the use of supremum (maximum indeed) norm for both b and π is a usual choice for measuring errors, and it is followed, for instance, in previous works on calmness of feasible and optimal solutions in the same parametric context, as [6] and [8].

Recall that the dual problem of $\pi \equiv (c, b)$ (introduced in (1)) is given by

$$\begin{array}{ll} \text{maximize} & -b'\lambda \\ \text{subject to} & \sum_{t\in T} \lambda_t \overline{a}_t = -c, \\ & \lambda = (\lambda_t)_{t\in T} \in \mathbb{R}_+^T. \end{array} \tag{6}$$

From now on, let us denote by $\Lambda : \mathbb{R}^n \rightrightarrows \mathbb{R}^T$ and $\Lambda^{op} : \mathbb{R}^n \rightrightarrows \mathbb{R}^T$ the feasible and optimal set mappings corresponding to the family of (dual) problems (6); i.e., $\Lambda(c)$ is the feasible set of (6), which does not depend on b, and $\Lambda^{op}(\pi)$ denotes the optimal set of (6).

2.1 Minimal KKT subsets of indices

Hereinafter, we use the following notation associated with any $\pi \equiv (c, b) \in \text{dom}\mathcal{F}^{op}$: The *set of active indices* at $x \in \mathcal{F}(b)$, for $b \in \mathbb{R}^T$, is denoted by $T_b(x)$; i.e.,

$$T_b(x) := \left\{ t \in T \mid \overline{a}_t' x = b_t \right\}.$$

We denote by $\mathcal{K}_\pi(x)$ the following family of subsets of indices involved in the Karush-Kuhn-Tucker (KKT in brief) conditions:

$$\mathcal{K}_\pi(x) := \{ D \subset T_b(x) \mid |D| \leq n, \ -c \in \text{cone}\{\overline{a}_t, \ t \in D\} \}.$$

(The condition '$|D| \leq n$' comes from Carathéodory's Theorem.) Moreover, we shall appeal to the family of *minimal KKT subsets of indices*

$$\mathcal{M}_\pi(x) := \{ D \in \mathcal{K}_\pi(x) \mid D \text{ is minimal for the inclusion order} \},$$

which constitutes a key ingredient in the formula of the calmness modulus of $\mathcal{F}^{op}$ established in [6]. Trivially, $\mathcal{M}_\pi(x) = \{\emptyset\}$ when $c = 0_n$.

Remark 2 Recall the stardard fact of LP theory: the dual optimal set $\Lambda^{op}(\pi)$ does coincide with the set of KKT multipliers associated with any primal solution $x \in \mathcal{F}^{op}(\pi)$. As a direct consequence, $\mathcal{M}_\pi(x)$ does not depend on point x. Formally,

$$\mathcal{M}_\pi(x) = \mathcal{M}_\pi(y), \text{ whenever } x, y \in \mathcal{F}^{op}(\pi),$$

and, accordingly, we may remove the optimal point in the notation of the minimal KKT subsets of indices. So, from now on, we just denote

$$\mathcal{M}_\pi := \mathcal{M}_\pi(x), \text{ for any } x \in \mathcal{F}^{op}(\pi),$$

provided that $\pi \in \mathrm{dom}\mathcal{F}^{op}$.

Remark 3 Observe that, a standard argument of linear algebra (in the line of Carathéodory's Theorem) yields the linear independence of $\{\overline{a}_t,\ t \in D\}$, whenever $D \in \mathcal{M}_\pi$. Specifically, arguing by contradiction, assume that $\sum_{t\in D} \mu_t \overline{a}_t = 0_n$ for some $(\mu_t)_{t\in D}$; without loss of generality, $\mu_t > 0$, for some $t \in D$. Write $\sum_{t\in D} \lambda_t \overline{a}_t = -c$, for certain $(\lambda_t)_{t\in D}$, and consider $t_0 \in D$ such that $\frac{\lambda_{t_0}}{\mu_{t_0}} = \min\left\{\frac{\lambda_t}{\mu_t} : \mu_t > 0, t \in D\right\}$. Then, $-c = \sum_{t\in D\setminus\{t_0\}} \left(\lambda_t - \frac{\lambda_{t_0}}{\mu_{t_0}}\mu_t\right) \overline{a}_t \in \mathrm{cone}\{\overline{a}_t,\ t \in D \setminus \{t_0\}\}$, which contradicts the minimality of D.

In this way, for any $D \in \mathcal{M}_\pi$, we define $\lambda^D := \left(\lambda_t^D\right)_{t\in T}$ as the unique element in $\mathbb{R}_+^T$ verifying

$$\sum_{t\in D} \lambda_t^D \overline{a}_t = -c, \ \textit{and}\ \lambda_t^D = 0, \ \textit{whenever}\ t \in T \setminus D. \tag{7}$$

Observe that the minimality of D entails $\lambda_t > 0$ for all $t \in D$. In the case $c = 0_n$, we have $\lambda^\emptyset = 0_T$.

Lemma 1 *Let $\pi \in \mathrm{dom}\mathcal{F}^{op}$. We have*

$$\left\{\lambda^D,\ D \in \mathcal{M}_\pi\right\} = \mathrm{extr}\Lambda^{op}(\pi).$$

Proof. Consider the nontrivial case $\mathcal{M}_{\overline{\pi}} \neq \{\emptyset\}$ (otherwise, $c = 0$ and it is clear that $\mathrm{extr}\Lambda^{op}(\overline{b}) = \{0_T\}$). One easily sees (according to the previous remark) that $D \in \mathcal{M}_\pi$ if and only if $D \subset T_b(x)$, for any $x \in \mathcal{F}^{op}(\overline{\pi})$, the set of vectors $\{\overline{a}_t,\ t \in D\}$ is linearly independent, and

$$\sum_{t\in D} \lambda_t \overline{a}_t = -c,$$

for some $\lambda_t > 0$, $t \in D$. The latter condition (with the componets λ_t there) is equivalent to

$$\lambda^D \in \mathrm{extr}\Lambda^{op}(\pi),$$

for $\lambda_t^D := \lambda_t > 0, t \in D$, $\lambda_s^D := 0$, $s \in T \setminus D$. ■

2.2 On the continuity of $\mathcal{F}$, ϑ, and $\mathcal{F}^{op}$ restricted to their domains

Recall that a multifunction between metric spaces, $\mathcal{G} : Y \rightrightarrows X$, is said to be lower semicontinuous in the sense of Berge at $\overline{y} \in \mathrm{dom}\mathcal{G}$ (*lsc*, in brief) if for any open set $V \subset X$ such that $\mathcal{G}(\overline{y}) \cap V \neq \emptyset$, there exists $\varepsilon > 0$ such that

$$\mathcal{G}(y) \cap V \neq \emptyset, \text{ whenever } d(y, \overline{y}) < \varepsilon.$$

It is well-known that the lower semicontinuity of $\mathcal{G}$ at $\overline{y}$ can be characterized in terms of the Painlevé-Kuratowski lower/inner limit as follows: $\mathcal{G}$ is *lsc* at $\overline{y} \in \mathrm{dom}\mathcal{G}$ if and only if

$$\mathcal{G}(\overline{y}) \subset \operatorname{Lim\,inf}_r \mathcal{G}(y^r), \tag{8}$$

for any $\{y^r\} \subset Y$ converging to $\overline{y}$. Since we are restricting our mappings $\mathcal{F}$ and $\mathcal{F}^{op}$ to their domains, we may confine ourselves to the case $\mathcal{G}(y^r) \neq \emptyset$ for all r. In such a case, recall that the Painlevé-Kuratowski lower/inner limit, $\operatorname{Lim\,inf}_r \mathcal{G}(y^r)$, is formed by all possible limits of sequences $\{x^r\}$, with $x^r \in \mathcal{G}(y^r)$, for all r. Recall also that the Painlevé-Kuratowski upper/outer limit, $\operatorname{Lim\,sup}_r \mathcal{G}(y^r)$, consists of all the cluster points (limits of subsequences) of sequences $\{x^r\}$, with $x^r \in \mathcal{G}(y^r)$, for all r. It is clear that

$$\operatorname{Lim\,inf}_r \mathcal{G}(y^r) \subset \operatorname{Lim\,sup}_r \mathcal{G}(y^r).$$

When these two sets coincide, we say that there exists the limit of $\{\mathcal{G}(y^r)\}_{r\in\mathbb{N}}$ in the Painlevé-Kuratowski sense, and we write

$$\operatorname{Lim}_r \mathcal{G}(y^r) = \operatorname{Lim\,inf}_r \mathcal{G}(y^r) = \operatorname{Lim\,sup}_r \mathcal{G}(y^r).$$

Remark 4 In this paper we opt for stating Painlevé-Kuratowski convergence results in a sequential form, following [37, p. 109]. Functional expressions of the type $\operatorname{Lim\,inf}_{y\to\overline{y}} \mathcal{G}(y)$ can be found, for instance, in [12, p. 142] or [31, p. 13]. In the latter, the notation Lim inf, with capital L, is used for multifunctions in order to distinguish this concept from its counterpart for real-valued functions.

The following theorem, which can be traced out from the literature, establishes the Painlevé-Kuratowski continuity of $\mathcal{F}$ restricted to $\mathrm{dom}\mathcal{F}$ (recall that it is closed in $\mathbb{R}^T$). Indeed, it can be found under different approaches. It comes from [11, Corollary II.3.1] (dealing with the continuity of $\mathcal{F}$ in the Hausdorff metric); see also [3, Theorem 3.4.1] for a proof of this result in terms of the representation of $\mathcal{F}(b)$ as a compact polyhedron plus its recession cone, similar to [30, Lemma 3.3].

Theorem 1 *Let* $\{b^r\} \subset \mathrm{dom}\mathcal{F}$ *be a sequence converging to* $\overline{b}$*. Then*

$$\mathcal{F}\left(\overline{b}\right) = \mathrm{Lim}_r\, \mathcal{F}\left(b^r\right).$$

Remark 5 The situation is different when dealing with $\mathcal{F}^{op}$. Specifically, one have

$$\mathrm{Lim\,sup}_r\, \mathcal{F}^{op}\left(\pi^r\right) \subset \mathcal{F}^{op}\left(\overline{\pi}\right), \tag{9}$$

for any $\{\pi^r\} \subset \mathrm{dom}\mathcal{F}^{op}$ converging to $\overline{\pi}$, as it follows from the Berge's theory (see [3, Theorem 5.5.1]) or from the upper Lipschitz property for polyhedral multifunctions, which is the case of $\mathcal{F}^{op}$ (see [36]). However $\mathcal{F}^{op}\left(\overline{\pi}\right)$ may not be included in $\mathrm{Lim\,inf}_r\, \mathcal{F}^{op}\left(\pi^r\right)$. Just consider the counterexample, in $\mathbb{R}$, *minimize* cx s.t. $x \in [-1, 1]$ around $\overline{c} = 0$.

The following lemma will be used later in the computation of our aimed calmness modulus of the optimal value function. In it we use the notation

$$\begin{aligned} \mathcal{E}\left(b\right) &:= \mathrm{extr}\left(\mathcal{F}\left(b\right) \cap \mathrm{span}\left\{\overline{a}_t,\ t \in T\right\}\right),\ b \in \mathrm{dom}\mathcal{F}, \\ \mathcal{E}^{op}\left(\pi\right) &:= \mathrm{extr}\left(\mathcal{F}^{op}\left(\pi\right) \cap \mathrm{span}\left\{\overline{a}_t,\ t \in T\right\}\right),\ \pi \in \mathrm{dom}\mathcal{F}^{op}. \end{aligned} \tag{10}$$

In order to motivate the use of mappings $\mathcal{E}$ and $\mathcal{E}^{op}$ from a geometrical point of view, it is easy to see that $\left\{\overline{a}_t,\ t \in T\right\}^{\perp}$ is the lineality space of both $\mathcal{F}\left(b\right)$ and $\mathcal{F}^{op}\left(\pi\right)$, provided that $\pi \equiv (c, b) \in \mathrm{dom}\mathcal{F}^{op}$; i.e., $\left\{\overline{a}_t,\ t \in T\right\}^{\perp}$ consists of those 'directions' $d \in \mathbb{R}^n$ such that $x + \mu d \in \mathcal{F}\left(b\right)$ (resp. $x + \mu d \in \mathcal{F}^{op}\left(\pi\right)$) for all $x \in \mathcal{F}\left(b\right)$ (resp. $x \in \mathcal{F}^{op}\left(\pi\right)$) and all $\mu \in \mathbb{R}$. It is easy to see that, for $\pi \in \mathrm{dom}\mathcal{F}^{op}$, we have $\mathrm{extr}\mathcal{F}^{op}\left(\pi\right) = \emptyset$, equivalently $\mathrm{extr}\mathcal{F}\left(b\right) = \emptyset$, if and only if $\left\{\overline{a}_t,\ t \in T\right\}^{\perp} \neq \{0_n\}$. In such a case, a way to ensure the existence of extreme points is intersecting $\mathcal{F}\left(b\right)$, and $\mathcal{F}^{op}\left(\pi\right)$, with $\left(\left\{\overline{a}_t,\ t \in T\right\}^{\perp}\right)^{\perp} = \mathrm{span}\left\{\overline{a}_t,\ t \in T\right\}$. This construction is inspired by the definition of multifunction F_0 considered in [30, p. 142].

In fact, in the case when $\mathrm{span}\left\{\overline{a}_t,\ t \in T\right\} \subsetneqq \mathbb{R}^n$, we can take a basis of $\left\{\overline{a}_t,\ t \in T\right\}^{\perp}$, $\{u_1, ..., u_p\}$, and form the matrix Q whose rows are u_i', $i = 1, ..., p$; then, in order to apply the results of [30] we consider the following convenient representation of $\mathcal{E}\left(b\right)$ and $\mathcal{E}^{op}\left(\pi\right)$, for $\pi = (c, b) \in \mathrm{dom}\mathcal{F}^{op}$: take any $D \in \mathcal{M}_\pi$, and write

$$\mathcal{E}\left(b\right) = \mathrm{extr}\left\{x \in \mathbb{R}^n \mid \overline{a}_t' x \leq b_t,\ t \in T;\ Qx = 0\right\}, \tag{11}$$

and

$$\mathcal{E}^{op}\left(\pi\right) = \mathrm{extr}\left\{x \in \mathbb{R}^n \mid \overline{a}_t' x \leq b_t,\ t \in T \setminus D;\ \overline{a}_t' x = b_t,\ t \in D;\ Qx = 0\right\}. \tag{12}$$

(In the case when $\operatorname{span}\{\overline{a}_t,\ t \in T\} = \mathbb{R}^n$, we just omit equation '$Qx = 0$'.) Then, as a consequence of [30, Lemma 3.3] we derive the following lemma. In it, and throughout the paper, π^r is identified with $(c^r, b^r) \in \mathbb{R}^n \times \mathbb{R}^T$ for all $r \in \mathbb{N}$ and the nominal problem $\overline{\pi}$ is with parameter $(\overline{c}, \overline{b})$.

Lemma 2 *Let* $\{\pi^r\} \subset \operatorname{dom}\mathcal{F}^{op}$ *converge to* $\overline{\pi}$. *We have:*
(*i*) $\{\mathcal{E}(b^r)\}_{r\in\mathbb{N}}$ *is uniformly bounded and* $\emptyset \neq \operatorname{Lim}\mathcal{E}(b^r) = \mathcal{E}(\overline{b})$,
(*ii*) $\emptyset \neq \operatorname{Lim\,sup}_r \mathcal{E}^{op}(\pi^r) \subset \mathcal{E}^{op}(\overline{\pi})$.

Proof. (*i*) According to (11), $\mathcal{E}(b)$ is nothing else but $\operatorname{extr}F_0(b)$ in [30, Lemma 3.3] (here we omit d since we have no equations). So, the current statement is a direct consequence of [30, Lemma 3.3] where the Lipschitz contituity of $\mathcal{E}$ in the Hausdorff metric is established.

(*ii*) First, for each r, take any $D^r \in \mathcal{M}_{\pi^r}$ and write (by (12))

$$\mathcal{E}^{op}(\pi^r) = \operatorname{extr}\left\{x \in \mathbb{R}^n \mid \overline{a}_t' x \leq b_t^r,\ t \in T \setminus D^r;\ \overline{a}_t' x = b_t^r,\ t \in D^r;\ Qx = 0\ \right\}.$$

The finiteness of T entails the existence of a constant subsequence $\{D^{r_k}\}_{k\in\mathbb{N}}$; say $D^{r_k} = D$ for $r_1 < r_2 < \ldots$, so $\mathcal{E}^{op}(\pi^r)$ coincides with the set of extreme points of feasible sets corresponding to the same feasible set mapping. Then, we can apply [30, Lemma 3.3] for deriving, in particular,

$$\begin{aligned} \emptyset &\neq \operatorname{Lim}_k \mathcal{E}^{op}(\pi^{r_k}) \\ &= \operatorname{extr}\left\{x \in \mathcal{F}(\overline{b}) \mid \overline{a}_t' x = \overline{b}_t,\ t \in D;\ Qx = 0\right\} \subset \mathcal{E}^{op}(\overline{\pi}), \end{aligned}$$

where the last inclusion comes from the fact that $-\overline{c} \in \operatorname{cone}\{\overline{a}_t,\ t \in D\}$, which follows from $-c^r \in \operatorname{cone}\{\overline{a}_t,\ t \in D\}$, for all r (although the minimality of D in relation to $-c^r$ does not entail the minimality of D for $-\overline{c}$). Since obviously $\operatorname{Lim}_k \mathcal{E}^{op}(\pi^{r_k}) \subset \operatorname{Lim\,sup}_r \mathcal{E}^{op}(\pi^r)$, the latter turns out to be nonempty.

Now, if we consider any element of $\operatorname{Lim\,sup}_r \mathcal{E}^{op}(\pi^r)$, say $z = \lim_s z^{r_s}$ with $z^{r_s} \in \mathcal{E}^{op}(\pi^{r_s})$, then, by repeating the previous argument for an appropriate subsequence $\{\pi^{r_{s_k}}\}$ of $\{\pi^{r_s}\}$, we obtain $z \in \operatorname{Lim}_k \mathcal{E}^{op}(\pi^{r_{s_k}}) \subset \mathcal{E}^{op}(\overline{\pi})$.
■

The next result is well-known in the literature. The reader is addressed to [3, Theorem 4.5.2] for a proof based on the Berge's theory, or to [25, Satz 2.7] and [41, Theorem 14] for a primal-dual approach to the continuity of ϑ^R (see also [32] for a parametric analysis). Indeed, one can find stronger versions: ϑ^R is Lipschitz on bounded subsets of $\operatorname{dom}\mathcal{F}^{op}$; see [36, p. 214] in the context of canonically perturbed convex quadratic problems (see also [41, p. 25]). On the other hand, [29] proved the continuity of the optimal value function for a (generally non-convex) quadratic program under canonical perturbations.

Nevertheless, for the reader's convenience, we include here a direct proof based on the previous lemma.

Theorem 2 ϑ^R *is continuous on* $\mathrm{dom}\mathcal{F}^{op}$*.*

Proof. Let $\{\pi^r\} \subset \mathrm{dom}\mathcal{F}^{op}$ be convergent to $\overline{\pi}$ (belonging to $\mathrm{dom}\mathcal{F}^{op}$ because of the closedness of this set) and let us see that

$$\lim_r \vartheta(\pi^r) = \vartheta(\overline{\pi}).$$

Take any $\overline{x} \in \mathcal{F}^{op}(\overline{\pi})$ and, appealing to Theorem 1, take any sequence $\{x^r\}$ converging to $\overline{x}$, with $x^r \in \mathcal{F}(b^r)$ for all r. Then,

$$\vartheta(\overline{\pi}) = \overline{c}'\overline{x} = \lim_r (c^r)' x^r \geq \limsup_r \vartheta(\pi^r).$$

Now, reasoning by contradiction, assume that $\liminf_r \vartheta(\pi^r) < \vartheta(\overline{\pi})$. Write $\liminf_r \vartheta(\pi^r)$ as $\lim_k \vartheta(\pi^{r_k})$ for an appropriate subsequence. By Lemma 2(ii), without loss of generality (taking a subsequence if necessary), we can assume the existence of $x^k \in \mathcal{E}^{op}(\pi^{r_k})$, for all k, such that $\{x^k\}$ converges to some $\overline{x} \in \mathcal{E}^{op}(\overline{\pi})$. Therefore, we attain the contradiction

$$\overline{c}'\overline{x} = \lim_k (c^{r_k})' x^k = \lim_k \vartheta(\pi^{r_k}) < \vartheta(\overline{\pi}).$$

■

Finally, recall that the restriction of $\mathcal{F}^{op}$ to its domain is not continuous (in the Painlevé-Kuratowski sense) as shown in Remark 5. However, it is if we only perturb b, as the following theorem asserts. In fact, it is a well-known result of stability theory in LP. Specifically, it can be derived from the fact that $\mathcal{F}^{op}(\overline{c}, \cdot)$ is Lipschitzian on $\mathrm{dom}\mathcal{F}$, provided that

$$-\overline{c} \in \mathrm{cone}\{\overline{a}_t,\ t \in T\} \tag{13}$$

(in which case $(\overline{c}, b) \in \mathrm{dom}\mathcal{F}^{op}$ if and only if $b \in \mathrm{dom}\mathcal{F}$); see, e.g. [26, p. 232] or [13, Chapter IX (Sec. 7)].

Theorem 3 *Let* $\overline{c} \in \mathbb{R}^n$ *verify* (13)*. For any* $\{b^r\} \subset \mathrm{dom}\mathcal{F}$ *converging to* $\overline{b}$ *we have*

$$\mathcal{F}^{op}(\overline{\pi}) = \mathrm{Lim}_r\, \mathcal{F}^{op}(\overline{c}, b^r).$$

3 Calmness modulus under RHS perturbations

Along this section we deal with linear optimization problems with a fixed c, say $\overline{c}$, which is assumed to verify (13). So, the only parameter to be considered here is $b \in \mathrm{dom}\mathcal{F}$ (equivalently $(\overline{c}, b) \in \mathrm{dom}\mathcal{F}^{op}$). Formally, we consider the new optimal value function $\vartheta_{\overline{c}}^{R} : \mathrm{dom}\mathcal{F} \to \left]-\infty, +\infty\right[$ defined as

$$\vartheta_{\overline{c}}^{R}(b) := \vartheta(\overline{c}, b), \text{ for all } b \in \mathrm{dom}\mathcal{F}.$$

The final goal of this section is to compute/estimate the calmness modulus of $\vartheta_{\overline{c}}^{R}$ at $\overline{b} \in \mathrm{dom}\mathcal{F}$, which is given by

$$\mathrm{clm}\vartheta_{\overline{c}}^{R}(\overline{b}) = \limsup_{b \underset{\mathrm{dom}\mathcal{F}}{\to} \overline{b}} \frac{|\vartheta(\overline{c}, b) - \vartheta(\overline{\pi})|}{\|b - \overline{b}\|_{\infty}},$$

(recall $\overline{\pi} \equiv (\overline{c}, \overline{b})$) and the corresponding calmness moduli from below and above (which are analogously defined). Observe that we use indistinctly the notation $\vartheta_{\overline{c}}^{R}(b)$ and $\vartheta(\overline{c}, b)$ whenever $b \in \mathrm{dom}\mathcal{F}$. In fact, for clarity, we usually write $\vartheta_{\overline{c}}^{R}$ when talking about the function itself and write $\vartheta(\overline{c}, b)$ for the image of $\vartheta_{\overline{c}}^{R}$ at $b \in \mathrm{dom}\mathcal{F}$.

To start with, we have the well-known expression for $\vartheta_{\overline{c}}^{R}$ as a piecewise linear function (see, e.g., [4, p. 214]) given by

$$\vartheta(\overline{c}, b) = \max_{\lambda \in \mathrm{extr}\Lambda(\overline{c})} -b'\lambda, \text{ for all } b \in \mathrm{dom}\mathcal{F}.$$

The following results are devoted to refine the previous expression in a neighborhood of $\overline{b}$ by appealing to the family of minimal KKT subsets of indices, $\mathcal{M}_{\overline{\pi}}$; specifically, to replace $\mathrm{extr}\Lambda(\overline{c})$ with a smaller set written in terms of $\mathcal{M}_{\overline{\pi}}$.

The following result is standard (the finiteness of $\mathrm{extr}\Lambda(\overline{c})$ is a key fact).

Lemma 3 *Let $\overline{b} \in \mathrm{dom}\mathcal{F}$. There exists a neighborhood $U_{\overline{b}} \subset \mathbb{R}^T$ of $\overline{b}$ such that*

$$\mathrm{extr}\Lambda^{op}(\overline{c}, b) \subset \mathrm{extr}\Lambda^{op}(\overline{\pi}), \text{ whenever } b \in \mathrm{dom}\mathcal{F} \cap U_{\overline{b}}.$$

As a consequence of the previous lemma, and taking into account the obvious fact that

$$\vartheta(\overline{c}, b) = \max_{\lambda \in \mathrm{extr}\Lambda^{op}(\overline{c}, b)} -b'\lambda, \text{ for all } b \in \mathrm{dom}\mathcal{F},$$

we derive the following corollary.

Corollary 1 *Let $\overline{b} \in \mathrm{dom}\mathcal{F}$ and let $U_{\overline{b}}$ be as in Lemma 3. Then*

$$\vartheta(\overline{c}, b) = \max_{D \in \mathcal{M}_{\overline{\pi}}} -b'\lambda^D, \text{ for all } b \in \mathrm{dom}\mathcal{F} \cap U_{\overline{b}}.$$

Now, applying the previous corollary, and taking the fact that $\vartheta(\overline{\pi}) = -\overline{b}'\lambda^D$ for all $D \in \mathcal{M}_{\overline{\pi}}$ into account, we deduce

$$\vartheta(\overline{c}, b) - \vartheta(\overline{\pi}) = \left(\max_{D \in \mathcal{M}_{\overline{\pi}}} -b'\lambda^D \right) - \vartheta(\overline{\pi}) = \max_{D \in \mathcal{M}_{\overline{\pi}}} -\left(b - \overline{b}\right)' \lambda^D, \tag{14}$$

while

$$\vartheta(\overline{\pi}) - \vartheta(\overline{c}, b) = \vartheta(\overline{\pi}) - \max_{D \in \mathcal{M}_{\overline{\pi}}} -b'\lambda^D = \min_{D \in \mathcal{M}_{\overline{\pi}}} (b - \overline{b})'\lambda^D, \tag{15}$$

for all $b \in \mathrm{dom}\mathcal{F} \cap U_{\overline{b}}$, where $U_{\overline{b}}$ is as in Lemma 3. Consequently, if we denote

$$k^- := \min_{D \in \mathcal{M}_{\overline{\pi}}} \left\| \lambda^D \right\|_1 \text{ and } k^+ = \max_{D \in \mathcal{M}_{\overline{\pi}}} \left\| \lambda^D \right\|_1,$$

(where, as usual, $\left\| \lambda^D \right\|_1 = \sum_{t \in D} \lambda_t^D$) we deduce the following result saying that k^- and k^+ are, respectively, a calmness constant from below and above for our optimal value function $\vartheta_{\overline{c}}^R$.

Corollary 2 *Let $\overline{b} \in \mathrm{dom}\mathcal{F}$ and let $U_{\overline{b}}$ be as in Lemma 3. Then, for $b \in \mathrm{dom}\mathcal{F} \cap U_{\overline{b}}$, one has*
(i) $\vartheta(\overline{\pi}) - \vartheta(\overline{c}, b) \leq k^- \left\| b - \overline{b} \right\|_\infty$;
(ii) $\vartheta(\overline{c}, b) - \vartheta(\overline{\pi}) \leq k^+ \left\| b - \overline{b} \right\|_\infty$.

So, k^- and k^+ are, respectively, upper bounds on the calmness moduli from below and above of $\vartheta_{\overline{c}}^R$ at $\overline{b}$, given by

$$\underline{\mathrm{clm}}\vartheta_{\overline{c}}^R(\overline{b}) = \limsup_{b \underset{\mathrm{dom}\mathcal{F}}{\to} \overline{b}} \frac{\vartheta(\overline{\pi}) - \vartheta(\overline{c}, b)}{\left\| b - \overline{b} \right\|_\infty} \text{ and } \overline{\mathrm{clm}}\vartheta_{\overline{c}}^R(\overline{b}) = \limsup_{b \underset{\mathrm{dom}\mathcal{F}}{\to} \overline{b}} \frac{\vartheta(\overline{c}, b) - \vartheta(\overline{\pi})}{\left\| b - \overline{b} \right\|_\infty}.$$

The following theorem shows that k^- is always attained as the calmness modulus from below of $\vartheta_{\overline{c}}^R$. The counterpart for k^+ is no longer true, as Example 1 shows.

Theorem 4 *Let $\overline{b} \in \mathrm{dom}\mathcal{F}$. One has:*

$$\underline{\mathrm{clm}}\vartheta_{\overline{c}}^R(\overline{b}) = k^- \text{ and } \overline{\mathrm{clm}}\vartheta_{\overline{c}}^R(\overline{b}) \leq k^+.$$

Consequently,

$$k^- \leq \mathrm{clm}\vartheta_{\overline{c}}^R(\overline{b}) \leq k^+.$$

Proof. As commented above, it is clear that $\underline{\text{clm}}\vartheta_{\overline{c}}^{R}(\overline{b}) \leq k^{-}$ and $\overline{\text{clm}}\vartheta_{\overline{c}}^{R}(\overline{b}) \leq k^{+}$.

So, we only need to prove that $\underline{\text{clm}}\vartheta_{\overline{c}}^{R}(\overline{b}) \geq k^{-}$. Consider the sequence

$$b^r := \overline{b} + \frac{1}{r} 1_T, \text{ for all } r,$$

where $1_T \in \mathbb{R}^T$ represents the vector having all its coordinates equal to 1. Clearly, $\{b^r\} \subset \text{dom}\mathcal{F}$. Then, appealing to (15) we have

$$\begin{aligned} \underline{\text{clm}}\vartheta_{\overline{c}}^{R}(\overline{b}) &\geq \limsup_r \frac{\vartheta\left(\overline{c}, \overline{b}\right) - \vartheta\left(\overline{c}, b^r\right)}{\left\|b^r - \overline{b}\right\|_\infty} \\ &= \limsup_r \frac{\min_{D \in \mathcal{M}_{\overline{\pi}}} \left(b^r - \overline{b}\right)' \lambda^D}{\left\|b^r - \overline{b}\right\|_\infty} \\ &= \lim_r \frac{\frac{1}{r} \min_{D \in \mathcal{M}_{\overline{\pi}}} \left\|\lambda^D\right\|_1}{\frac{1}{r}} = k^{-}. \end{aligned}$$

■

The following proposition is intended to provide an alternative approach for determining $\left\|\lambda^D\right\|_1$, with $D \in \mathcal{M}_{\overline{\pi}}$. In it, u_D represents the projection of 0_n on $\text{aff}\{\overline{a}_t,\ t \in D\}$ in the Euclidean norm, provided that $D \in \mathcal{M}_{\overline{\pi}}$ (observe that $0_n \notin \text{aff}\{\overline{a}_t,\ t \in D\}$ as a consequence of the linear independence of $\{\overline{a}_t,\ t \in D\}$).

Proposition 1 *For each $D \in \mathcal{M}_{\overline{\pi}}$, one has*

$$\left\|\lambda^D\right\|_1 = \frac{-\overline{c}' u_D}{\left\|u_D\right\|_2^2}. \tag{16}$$

Proof. Take any $D \in \mathcal{M}_{\overline{\pi}}$. Since $-\left\|\lambda^D\right\|_1^{-1} \overline{c} \in \text{aff}\{\overline{a}_t,\ t \in D\}$, the definition of u_D yields

$$\left(-\left\|\lambda^D\right\|_1^{-1} \overline{c} - u_D\right)' u_D = 0,$$

which entails the aimed equality (16). ■

The following example shows that the calmness modulus from above of $\vartheta_{\overline{c}}^{R}$ can take any positive value less than or equal to k^{+}.

Example 1 Consider the problem in $\mathbb{R}$ given by

$$\begin{array}{lll} \overline{\pi}: & \text{minimize} & x_1 \\ & \text{subject to} & -x_1 \leq 0,\ t = 1, \\ & & -2x_1 \leq 0,\ t = 2, \\ & & \theta x_1 \leq 0,\ t = 3, \end{array}$$

where $\theta > 0$. Trivially, $\mathcal{M}_{\overline{\pi}} := \{\{1\},\{2\}\}$, $\lambda^{\{1\}} = 1 = k^+$, $\lambda^{\{2\}} = \frac{1}{2} = k^-$. Let us check that

$$\overline{\mathrm{clm}}\vartheta_{\overline{c}}^{R}(\overline{b}) = \min\left\{1, \frac{1}{\theta}\right\}.$$

Observe that $\overline{b} = 0_3$. According to Corollary 1, in some neighborhood $U_{\overline{b}}$ of $\overline{b}$ we have

$$\vartheta\left(\overline{c}, b\right) = \max\{-b_1, -\tfrac{1}{2}b_2\},\ \text{for } b = (b_1, b_2, b_3)' \in \mathrm{dom}\mathcal{F} \cap U_{\overline{b}}.$$

Moreover, it is immediate that

$$b \in \mathrm{dom}\mathcal{F} \Leftrightarrow \max\{-b_1, -\tfrac{1}{2}b_2\} \leq \tfrac{1}{\theta}b_3.$$

So,

$$\frac{\vartheta\left(\overline{c}, b\right) - \vartheta\left(\overline{\pi}\right)}{\left\|b - \overline{b}\right\|_\infty} = \frac{\max\{-b_1, -\frac{1}{2}b_2\}}{\|b\|_\infty} \leq \frac{\frac{1}{\theta}b_3}{\|b\|_\infty} \leq \tfrac{1}{\theta},\ \text{for all } b \in \mathrm{dom}\mathcal{F} \cap U_{\overline{b}}.$$

Consequently, one always have

$$\overline{\mathrm{clm}}\vartheta_{\overline{c}}^{R}(\overline{b}) \leq \frac{1}{\theta}.$$

Now, we distinguish two cases:

Case 1: $\theta > 1$. Just consider the sequence

$$b^r = \left(-\frac{1}{\theta r}, -\frac{1}{\theta r}, \frac{1}{r}\right)',\ r = 1, 2...$$

It is clear that $b^r \in \mathrm{dom}\mathcal{F}$ for all r. Moreover, for r large enough (to ensure $b^r \in U_{\overline{b}}$) one has

$$\frac{\vartheta\left(\overline{c}, b^r\right) - \vartheta\left(\overline{\pi}\right)}{\left\|b^r - \overline{b}\right\|_\infty} = \frac{\frac{1}{\theta r}}{\frac{1}{r}} = \frac{1}{\theta},$$

So, $\overline{\mathrm{clm}}\vartheta_{\overline{c}}^{R}(\overline{b}) \geq \frac{1}{\theta}$.

Case 2: $0 < \theta \leq 1$, yielding $\frac{1}{\theta} \geq 1 = k^+ \geq \overline{\mathrm{clm}}\vartheta_{\overline{c}}^{R}(\overline{b})$. Consider the sequence

$$b^r = \left(-\frac{1}{r}, -\frac{1}{r}, \frac{1}{r}\right)',\ r = 1, 2...$$

One has $b^r \in \mathrm{dom}\mathcal{F}$ for all r and, for r large enough,

$$\frac{\vartheta\left(\overline{c}, b^r\right) - \vartheta\left(\overline{\pi}\right)}{\left\|b^r - \overline{b}\right\|_\infty} = \frac{\frac{1}{r}}{\frac{1}{r}} = 1.$$

Inspired by the previous example, the following proposition provides a sufficient condition for having the equality $\overline{\mathrm{clm}}\vartheta_{\overline{c}}^{R}(\overline{b}) = k^+$.

Proposition 2 *Let $\overline{b} \in \mathrm{dom}\mathcal{F}$. Assume that there exists some $\overline{D} \in \mathcal{M}_{\overline{\pi}}$, with $\left\|\lambda^{\overline{D}}\right\|_1 = k^+$, and some $\varepsilon > 0$ such that $b^\varepsilon \in \mathrm{dom}\mathcal{F}$, with*

$$b_t^\varepsilon := \begin{cases} \overline{b}_t - \varepsilon & \text{if } t \in \overline{D}, \\ \overline{b}_t + \varepsilon & \text{if } t \in T \backslash \overline{D}. \end{cases} \tag{17}$$

Then,

$$\overline{\mathrm{clm}}\vartheta_{\overline{c}}^R(\overline{b}) = \mathrm{clm}\vartheta_{\overline{c}}^R(\overline{b}) = k^+.$$

Proof. Take $\overline{D} \in \mathcal{M}_{\overline{\pi}}$ and $\varepsilon > 0$ verifying the assumptions of the current proposition. First, observe that for any $0 < \widetilde{\varepsilon} < \varepsilon$, the associated $b^{\widetilde{\varepsilon}}$ defined as in (17) (replacing ε with $\widetilde{\varepsilon}$) also verifies that $b^{\widetilde{\varepsilon}} \in \mathrm{dom}\mathcal{F}$, as far as $\mathrm{dom}\mathcal{F}$ is a convex set. Just observe that $b^{\widetilde{\varepsilon}} = \left(1 - \frac{\widetilde{\varepsilon}}{\varepsilon}\right)\overline{b} + \frac{\widetilde{\varepsilon}}{\varepsilon} b^\varepsilon$.

Let $U_{\overline{b}}$ be as in Lemma 3, and consider the sequence $\{b^{\frac{1}{r}}\}$, where $b^{\frac{1}{r}}$ comes again from replacing ε with $\frac{1}{r}$ in (17). Let r_0 be large enough to guarantee $\frac{1}{r_0} < \varepsilon$ (so, $b^{\frac{1}{r}} \in \mathrm{dom}\mathcal{F}$, $r \geq r_0$) and $b^{\frac{1}{r}} \in U_{\overline{b}}$, for all $r \geq r_0$. Then

$$\begin{aligned} \mathrm{clm}\vartheta_{\overline{c}}^R(\overline{b}) &\geq \limsup_r \frac{\vartheta\left(\overline{c}, b^{\frac{1}{r}}\right) - \vartheta\left(\overline{\pi}\right)}{\left\|b^{\frac{1}{r}} - \overline{b}\right\|_\infty} \\ &= \limsup_r \frac{\max_{D \in \mathcal{M}_{\overline{\pi}}}\left(-\left(b^{\frac{1}{r}} - \overline{b}\right)' \lambda^D\right)}{\frac{1}{r}} = \left\|\lambda^{\overline{D}}\right\|_1, \end{aligned}$$

where the last equality comes from the fact that

$$\left|-\left(b^{\frac{1}{r}} - \overline{b}\right)' \lambda^D\right| \leq \left\|b^{\frac{1}{r}} - \overline{b}\right\|_\infty \left\|\lambda^D\right\|_1 = \frac{1}{r}\left\|\lambda^D\right\|_1,$$

for all $D \in \mathcal{M}_{\overline{\pi}}$, and $-\left(b^{\frac{1}{r}} - \overline{b}\right)' \lambda^{\overline{D}} = \frac{1}{r}\left\|\lambda^{\overline{D}}\right\|_1$. ■

As a consequence of the previous proposition we have the following corollary under SCQ. It is well-known that SCQ at $\overline{b} \in \mathrm{dom}\mathcal{F}$ is equivalent to $\overline{b} \in \mathrm{int}\,\mathrm{dom}\mathcal{F}$.

Corollary 3 *Let $\overline{b} \in \mathrm{dom}\mathcal{F}$ and assume that SCQ holds at $\overline{b}$. Then*

$$\overline{\mathrm{clm}}\vartheta_{\overline{c}}^R(\overline{b}) = \mathrm{clm}\vartheta_{\overline{c}}^R(\overline{b}) = k^+.$$

4 Calmness modulus under canonical perturbations

This section is devoted to compute/estimate the calmness moduli from below and above of the optimal value function restricted to $\text{dom}\mathcal{F}^{op}$, $\vartheta^R : \text{dom}\mathcal{F}^{op} \to]-\infty, +\infty[$, at $\overline{\pi} \in \text{dom}\mathcal{F}^{op}$. Recall that they are respectively given by

$$\underline{\text{clm}}\vartheta^R(\overline{\pi}) = \limsup_{\pi \underset{\text{dom}\mathcal{F}^{op}}{\to} \overline{\pi}} \frac{\vartheta(\overline{\pi}) - \vartheta(\pi)}{\|\pi - \overline{\pi}\|} \text{ and } \overline{\text{clm}}\vartheta^R(\overline{\pi}) = \limsup_{\pi \underset{\text{dom}\mathcal{F}^{op}}{\to} \overline{\pi}} \frac{\vartheta(\pi) - \vartheta(\overline{\pi})}{\|\pi - \overline{\pi}\|},$$

and, roughly speaking, provide a measure of the maximum rate of decrease and increase, respectively, under perturbations of the data (regarding solvable problems only). These calmness moduli are shown to be closely related to the corresponding calmness moduli of $\vartheta_{\overline{c}}^R$ (where only perturbations of b are allowed). To start with, we establish the following lemma.

Lemma 4 *There exists $\overline{\delta} > 0$ such that if $\pi, \overline{\pi} \in \text{dom}\mathcal{F}^{op}$ satisfy $\|\pi - \overline{\pi}\| < \overline{\delta}$, with $\pi \equiv (c, b)$ and $\overline{\pi} \equiv (\overline{c}, \overline{b})$, then*

$$\mathcal{F}^{op}(\pi) \subset \mathcal{F}^{op}(\overline{c}, b).$$

Proof. Reasoning by contradiction, assume the existence of a sequence of problems $\{\pi^r \equiv (c^r, b^r)\} \subset \text{dom}\mathcal{F}^{op}$ converging to $\overline{\pi}$ and a sequence of points $\{x^r\} \subset \mathbb{R}^n$ such that $x^r \in \mathcal{F}^{op}(\pi^r) \setminus \mathcal{F}^{op}(\overline{c}, b^r)$ for all r. We have that

$$-c^r \in \text{cone}\{\overline{a}_t,\ t \in T_{b^r}(x^r)\}, \text{ for all } r.$$

We may assume (by taking a subsequence if necessary) that $T_{b^r}(x^r) = D$ for all r (not depending on r). Then

$$-\overline{c} \in \text{cone}\{\overline{a}_t,\ t \in D\},$$

(by closedness) which yields the contradiction $x^r \in \mathcal{F}^{op}(\overline{c}, b^r)$ for each r. ■

Theorem 5 *Let $\overline{\pi} \in \text{dom}\mathcal{F}^{op}$. Then*

$$\overline{\text{clm}}\vartheta^R(\overline{\pi}) = \overline{\text{clm}}\vartheta_{\overline{c}}^R(\overline{b}) + d(0_n, \mathcal{F}^{op}(\overline{\pi})).$$

Proof. Take $\overline{x} \in \mathcal{F}^{op}(\overline{\pi})$ with $\|\overline{x}\| = d(0_n, \mathcal{F}^{op}(\overline{\pi}))$. Fix arbitrarily $\varepsilon > 0$ and let $\pi \equiv (c, b) \in \text{dom}\mathcal{F}^{op}$ be such that $\|\pi - \overline{\pi}\| < \delta$ for a certain $\delta > 0$

satisfying the following:

$$\begin{array}{rcl} \delta & \leq & \overline{\delta} \text{ (the one from Lemma 4),} \\ \left\| b-\overline{b}\right\|_\infty & < & \delta \Rightarrow \left\{ \begin{array}{l} d\left(\overline{x},\mathcal{F}^{op}\left(\overline{c},b\right)\right) < \varepsilon \text{ (by Theorem 3),} \\ b\in U_{\overline{b}} \text{ (that of Lemma 3),} \\ \vartheta\left(\overline{c},b\right)-\vartheta\left(\overline{\pi}\right) \leq \left(\overline{\text{clm}}\vartheta_{\overline{c}}^{R}(\overline{b})+\varepsilon\right)\left\| b-\overline{b}\right\|_\infty . \end{array}\right. \end{array}$$

Now pick an arbitrary $\widehat{x}\in\mathcal{F}^{op}\left(\pi\right)\subset\mathcal{F}^{op}\left(\overline{c},b\right)$ (because $\delta\leq\overline{\delta}$) and $\widetilde{x}\in\mathcal{F}^{op}\left(\overline{c},b\right)$ with $\left\|\widetilde{x}-\overline{x}\right\|<\varepsilon$. Clearly $\vartheta\left(\pi\right)=c'\widehat{x}\leq c'\widetilde{x}$ and $\vartheta\left(\overline{c},b\right)=\overline{c}'\widehat{x}=\overline{c}'\widetilde{x}$. Then we have

$$\begin{array}{rcl} \vartheta\left(\pi\right)-\vartheta\left(\overline{\pi}\right) & = & c'\widehat{x}-\overline{c}'\widehat{x}+\vartheta\left(\overline{c},b\right)-\vartheta\left(\overline{\pi}\right) \\ & \leq & c'\widetilde{x}-\overline{c}'\widetilde{x}+\left(\overline{\text{clm}}\vartheta_{\overline{c}}^{R}(\overline{b})+\varepsilon\right)\left\| b-\overline{b}\right\|_\infty \\ & \leq & \left\| c-\overline{c}\right\|_*\left(\left\|\overline{x}\right\|+\varepsilon\right)+\left(\overline{\text{clm}}\vartheta_{\overline{c}}^{R}(\overline{b})+\varepsilon\right)\left\| b-\overline{b}\right\|_\infty \\ & \leq & \left(\overline{\text{clm}}\vartheta_{\overline{c}}^{R}(\overline{b})+\left\|\overline{x}\right\|+2\varepsilon\right)\left\|\pi-\overline{\pi}\right\|. \end{array}$$

Since $\varepsilon>0$ has been arbitrarily chosen, we get $\overline{\text{clm}}\vartheta^{R}(\overline{\pi})\leq\overline{\text{clm}}\vartheta_{\overline{c}}^{R}(\overline{b})+d\left(0_n,\mathcal{F}^{op}(\overline{\pi})\right)$.

Next we show that the previous inequality holds as an equality. The case $0_n\in\mathcal{F}^{op}(\overline{\pi})$ is trivial, since in such a case we have $\overline{\text{clm}}\vartheta_{\overline{c}}^{R}(\overline{b})\leq\overline{\text{clm}}\vartheta^{R}(\overline{\pi})\leq\overline{\text{clm}}\vartheta_{\overline{c}}^{R}(\overline{b})+0$; i.e., $\overline{\text{clm}}\vartheta^{R}(\overline{\pi})=\overline{\text{clm}}\vartheta_{\overline{c}}^{R}(\overline{b})$. Hence, let us assume $0_n\notin\mathcal{F}^{op}(\overline{\pi})$. Let us consider a sequence $\{b^r\}_{r\in\mathbb{N}}\subset\text{dom}\mathcal{F}$ such that

$$\overline{\text{clm}}\vartheta_{\overline{c}}^{R}(\overline{b})=\lim_r\frac{\vartheta(\overline{c},b^r)-\vartheta(\overline{\pi})}{\left\| b^r-\overline{b}\right\|_\infty}.$$

Because of Theorem 3 we may assume $0_n\notin\mathcal{F}^{op}\left(\overline{c},b^r\right)$ for all r. The same theorem ensures the existence of $x^r\in\mathcal{F}^{op}\left(\overline{c},b^r\right)$ with $\left\| x^r\right\|=d\left(0_n,\mathcal{F}^{op}\left(\overline{c},b^r\right)\right)$, for all r, and $\left\| x^r\right\|\to\left\|\overline{x}\right\|$ (we do not need to guarantee $x^r\to\overline{x}$). More in detail, replacing $\{b^r\}$ with an appropriate subsequence (we do not relabel for simplicity) we could choose $x^r\in B\left(0_n,\left\|\overline{x}\right\|+\frac{1}{r}\right)$, i.e., the open ball centered at 0_n with radius $\left\|\overline{x}\right\|+\frac{1}{r}$, which is an open set containing $\overline{x}$; again considering an appropriate subsequence we may assume that $\{x^r\}$ converges to certain $z\in\text{cl}B\left(0_n,\left\|\overline{x}\right\|\right)$, and if $\left\| z\right\|<\left\|\overline{x}\right\|$ we attain a contradiction with Theorem 3. Now, for each r, we appeal to [7, Lemma 9] to ensure the existence of $u^r\in\mathbb{R}^n$ with $\left\| u^r\right\|_*=1$ such that

$$\left(u^r\right)'x\geq\left(u^r\right)'x^r=\left\| x^r\right\| \text{ for all } x\in\mathcal{F}^{op}\left(\overline{c},b^r\right). \tag{18}$$

Let us define $c^r:=\overline{c}+\left\| b^r-\overline{b}\right\|_\infty u^r$, which entails $\left\| c^r-\overline{c}\right\|_*=\left\| b^r-\overline{b}\right\|_\infty$. First we note that, for all $x\in\mathcal{F}^{op}\left(\overline{c},b^r\right)$, we have

$$\left(c^r\right)'x=\overline{c}'x+\left\| b^r-\overline{b}\right\|_\infty\left(u^r\right)'x\geq\vartheta(\overline{c},b^r)+\left\| c^r-\overline{c}\right\|_*\left\| x^r\right\|. \tag{19}$$

Our next step consists of establishing the existence of $r_0 \in \mathbb{N}$ such that

$$\pi^r \equiv (c^r, b^r) \in \mathrm{dom}\mathcal{F}^{op}, \text{ for } r \geq r_0. \tag{20}$$

Assuming for the moment that (20) holds, it yields $\mathcal{F}^{op}(\pi^r) \subset \mathcal{F}^{op}(\overline{c}, b^r)$ for $r \geq r_0$ large enough (to apply Lemma 4). Then, by repeating inequality (19) with any $x \in \mathcal{F}^{op}(\pi^r)$, we deduce $\vartheta(\pi^r) \geq \vartheta(\overline{c}, b^r) + \|c^r - \overline{c}\|_* \|x^r\|$ and therefore, recalling $\|c^r - \overline{c}\|_* = \left\|b^r - \overline{b}\right\|_\infty$, we have

$$\begin{aligned}
\overline{\mathrm{clm}}\vartheta^R(\overline{\pi}) &\geq \limsup_r \frac{\vartheta(\pi^r) - \vartheta(\overline{\pi})}{\|\pi^r - \overline{\pi}\|} \\
&= \limsup_r \frac{\vartheta(\pi^r) - \vartheta(\overline{c}, b^r)}{\|c^r - \overline{c}\|_*} + \lim_r \frac{\vartheta(\overline{c}, b^r) - \vartheta(\overline{\pi})}{\left\|b^r - \overline{b}\right\|_\infty} \\
&\geq \lim_r \|x^r\| + \overline{\mathrm{clm}}\vartheta^R_{\overline{c}}(\overline{b}) = \|\overline{x}\| + \overline{\mathrm{clm}}\vartheta^R_{\overline{c}}(\overline{b}),
\end{aligned}$$

which establishes the aimed equality $\overline{\mathrm{clm}}\vartheta^R(\overline{\pi}) = \overline{\mathrm{clm}}\vartheta^R_{\overline{c}}(\overline{b}) + d(0_n, \mathcal{F}^{op}(\overline{\pi}))$.

So, it remains to prove (20). From [30, Lemma 4.1], we can write

$$\mathcal{F}(b^r) = \mathrm{conv}\,\mathcal{E}(b^r) + \left\{d \in \mathbb{R}^n : \overline{a}_t' d \leq 0,\ t \in T\right\},$$

(recall that $\mathcal{E}(b^r) := \mathrm{extr}\left(\mathcal{F}(b^r) \cap \mathrm{span}\{\overline{a}_t,\ t \in T\}\right.$ for all r) where the last term is the recession cone of $\mathcal{F}(b^r)$, which does not depend on r (only on the fact that $b^r \in \mathrm{dom}\mathcal{F}$). Let us write this (polyhedral) recession cone as $\mathrm{cone}\{d_1, ..., d_p\}$. On the other hand, Lemma 2(i) ensures that $\{\mathcal{E}(b^r)\}_{r\in\mathbb{N}}$ is a sequence of uniformly bounded nonempty compact sets. Assume, reasoning by contradiction, that $\vartheta(\pi^r) = -\infty$ for all r (replacing, if necessary, the sequence with an appropriate subsequence). Because of the compactness of $\mathrm{conv}\mathcal{E}(b^r)$, $\vartheta(\pi^r) = -\infty$ implies (again considering an appropriate subsequence, if necessary) that $(c^r)' d_k < 0$ for all r and some fixed $k \in \{1, ..., p\}$. Letting $r \to \infty$ we obtain $\overline{c}' d_k \leq 0$, which entails that d_k is not only a recession direction of $\mathcal{F}(b^r)$, but also of $\mathcal{F}^{op}(\overline{c}, b^r)$, for all r. This, together with $(c^r)' d_k < 0$ ensures that, for each $r \in \mathbb{N}$, $x \mapsto (c^r)' x$ is not bounded from below on $\mathcal{F}^{op}(\overline{c}, b^r)$, contradicting (19). ■

Remark 6 With respect to the proof of the previous theorem, when $\mathcal{F}^{op}(\overline{\pi})$ is bounded, one immediately has (20) as a consequence of [19, Lemma 10.2].

Theorem 6 *Let* $\overline{\pi} \in \mathrm{dom}\mathcal{F}^{op}$. *Then*

$$\underline{\mathrm{clm}}\vartheta^R(\overline{\pi}) \leq \underline{\mathrm{clm}}\vartheta^R_{\overline{c}}\left(\overline{b}\right) + e\left(\mathcal{E}^{op}(\overline{\pi}), 0_n\right), \tag{21}$$

where the last term represents the Hausdorff excess of $\mathcal{E}^{op}(\overline{\pi})$ *over* $\{0_n\}$, *which may alternatively be written as* $\max\{\|x\| \mid x \in \mathcal{E}^{op}(\overline{\pi})\}$; *i.e., the maximum norm in a finite set.*

Proof. For simplicity, write $\alpha := \underline{\mathrm{clm}}\vartheta_{\overline{c}}^{R}(\overline{b}) + e\left(\mathcal{E}^{op}\left(\overline{\pi}\right), 0_n\right)$. Reasoning by contradiction, assume the existence of a sequence $\{\pi^r \equiv (c^r, b^r)\} \subset \mathrm{dom}\mathcal{F}^{op}$ converging to $\overline{\pi}$ such that

$$\vartheta\left(\overline{\pi}\right) - \vartheta\left(\pi^r\right) > (\alpha + \varepsilon) \left\|\pi^r - \overline{\pi}\right\|$$

for all $r \in \mathbb{N}$ and some $\varepsilon > 0$. According to Lemma 2(ii), pick any $\overline{x} \in \mathrm{Lim\,sup}_r\, \mathcal{E}^{op}\left(\pi^r\right)$, and take $r_1 < r_2 < ... < r_k < ...$ and associated $x^k \in \mathcal{F}^{op}\left(\pi^{r_k}\right)$ such that $x^k \to \overline{x}$. According to Lemma 4 we may assume that $x^k \in \mathcal{F}^{op}\left(\overline{c}, b^{r_k}\right)$ for all $k \in \mathbb{N}$. Then, for k large enough guaranteeing $\left\|x^k - \overline{x}\right\| \leq \varepsilon$ and $b^{r_k} \in U_{\overline{b}}$ (see again Lemma 3 and Corollary 2), and taking Theorem 4 into account, we attain the following contradiction:

$$\begin{aligned}
\vartheta\left(\overline{\pi}\right) - \vartheta\left(\pi^{r_k}\right) &= \vartheta\left(\overline{\pi}\right) - \vartheta\left(\overline{c}, b^{r_k}\right) + \overline{c}' x^k - \left(c^{r_k}\right)' x^k \\
&\leq \underline{\mathrm{clm}}\vartheta_{\overline{c}}^{R}(\overline{b}) \left\|\overline{b} - b^{r_k}\right\|_\infty + \left\|\overline{c} - c^{r_k}\right\|_* \left\|x^k\right\| \\
&\leq \left(\underline{\mathrm{clm}}\vartheta_{\overline{c}}^{R}(\overline{b}) + \left\|\overline{x}\right\| + \varepsilon\right) \left\|\pi^{r_k} - \overline{\pi}\right\| \\
&\leq (\alpha + \varepsilon) \left\|\pi^{r_k} - \overline{\pi}\right\|.
\end{aligned}$$

■

The following example shows that inequality (21) may be strict.

Example 2 Consider the nominal problem, in $\mathbb{R}^2$ with the Euclidean norm,

$$\overline{\pi} : \text{minimize } x_2 \text{ s.t. } x_1 \leq -1, \quad -x_2 \leq 1.$$

Clearly $\mathcal{E}^{op}\left(\overline{\pi}\right) = \left\{(-1,-1)'\right\}$. Let us see that the specification of (21) to this case reads as $2 \leq 1 + \sqrt{2}$. For $\left\|\pi - \overline{\pi}\right\| < 1$ one has $\pi \equiv (c, b) \in \mathrm{dom}\mathcal{F}^{op}$ if and only if $c_1 \leq 0$, where $c = (c_1, c_2)'$. For convenience, let us write $c = (\varepsilon_1, 1 + \varepsilon_2)'$ and $b = (-1 + \varepsilon_3, 1 + \varepsilon_4)'$, with

$$\left\|\pi - \overline{\pi}\right\| = \max\left\{\sqrt{\varepsilon_1^2 + \varepsilon_2^2}, \left|\varepsilon_3\right|, \left|\varepsilon_4\right|\right\} =: \varepsilon < 1.$$

Then, provided that $\varepsilon_1 \leq 0$, we have

$$\vartheta\left(\overline{\pi}\right) - \vartheta\left(\pi\right) = -1 - \varepsilon_1\left(-1 + \varepsilon_3\right) - \left(1 + \varepsilon_2\right)\left(-1 - \varepsilon_4\right) \leq 2\varepsilon + \varepsilon^2,$$

and, accordingly, by letting $\varepsilon \searrow 0$, we have $\underline{\mathrm{clm}}\vartheta^R(\overline{\pi}) \leq 2$. Indeed, by taking $\varepsilon_1 = \varepsilon_3 = 0$ and $\varepsilon_2 = \varepsilon_4 = \varepsilon$, we see that $\underline{\mathrm{clm}}\vartheta^R(\overline{\pi}) = 2$. A simpler calculation with $c = \overline{c}$ shows that $\underline{\mathrm{clm}}\vartheta_{\overline{c}}^{R}(\overline{b}) = 1$.

Theorem 7 *Let* $\overline{\pi} \in \mathrm{dom}\mathcal{F}^{op}$ *with* $\mathcal{F}^{op}(\overline{\pi})$ *bounded. Then*

$$\begin{aligned}\underline{\mathrm{clm}}\vartheta^R(\overline{\pi}) &= \underline{\mathrm{clm}}\vartheta^R_{\overline{c}}\left(\overline{b}\right) + e\left(\mathrm{extr}\mathcal{F}^{op}(\overline{\pi}), 0_n\right) \\ &= \underline{\mathrm{clm}}\vartheta^R_{\overline{c}}(\overline{b}) + e\left(\mathcal{F}^{op}(\overline{\pi}), 0_n\right).\end{aligned}$$

Proof. The last equality follows a standard argument by using the convexity of the norm. Let us observe that the boundedness of $\mathcal{F}^{op}(\overline{\pi})$ entails $\mathrm{span}\{\overline{a}_t,\ t \in T\} = \mathbb{R}^n$, so that we have to prove that (21) holds as an equality in this case. Let us consider, similarly to the proof of Theorem 5, a sequence $\{b^r\}_{r\in\mathbb{N}} \subset \mathrm{dom}\mathcal{F}$ such that

$$\underline{\mathrm{clm}}\vartheta^R_{\overline{c}}\left(\overline{b}\right) = \lim_r \frac{\vartheta(\overline{\pi}) - \vartheta(\overline{c}, b^r)}{\left\|b^r - \overline{b}\right\|_\infty}.$$

From Theorem 3 we easily deduce

$$e\left(\mathcal{F}^{op}(\overline{c}, b^r), 0_n\right) \to e\left(\mathcal{F}^{op}(\overline{\pi}), 0_n\right),$$

as $r \to \infty$. Let us write, for each r, $e\left(\mathcal{F}^{op}(\overline{c}, b^r), 0_n\right) = \|x^r\|$ with $x^r \in \mathcal{F}^{op}(\overline{c}, b^r)$. For each r, let $u^r \in \mathbb{R}^n$ with $\|u^r\|_* = 1$ be such that $(u^r)' x^r = \|x^r\|$ and let $c^r := \overline{c} - \left\|b^r - \overline{b}\right\|_\infty u^r$. Similarly to Remark 6, we have $\pi^r \equiv (c^r, b^r) \in \mathrm{dom}\mathcal{F}^{op}$ for r large enough (say for all r). Clearly $\|\pi^r - \overline{\pi}\| = \left\|b^r - \overline{b}\right\|_\infty$. Choose for each r any $\widehat{x}^r \in \mathcal{F}^{op}(\pi^r)$, which, for r large enough (say again for each r) satisfies $\widehat{x}^r \in \mathcal{F}^{op}(\overline{c}, b^r)$ by virtue of Lemma 4. Observe that

$$(u^r)' \widehat{x}^r \leq \|u^r\|_* \|\widehat{x}^r\| = \|\widehat{x}^r\| \leq \|x^r\| = (u^r)' x^r,$$

due to the choice of x^r and u^r. Consequently,

$$\begin{aligned}(c^r)' x^r &= \overline{c}' x^r - \left\|b^r - \overline{b}\right\|_\infty (u^r)' x^r \\ &\leq \overline{c}' \widehat{x}^r - \left\|b^r - \overline{b}\right\|_\infty (u^r)' \widehat{x}^r = (c^r)' \widehat{x}^r = \vartheta(\pi^r).\end{aligned}$$

In other words, $x^r \in \mathcal{F}^{op}(\pi^r)$. Thus,

$$\begin{aligned}\underline{\mathrm{clm}}\vartheta^R(\overline{\pi}) &\geq \limsup_r \frac{\vartheta(\overline{\pi}) - \vartheta(\pi^r)}{\|\pi^r - \overline{\pi}\|} \\ &= \lim_r \frac{\vartheta(\overline{\pi}) - \vartheta(\overline{c}, b^r)}{\left\|b^r - \overline{b}\right\|_\infty} + \lim_r \frac{\overline{c}' x^r - (c^r)' x^r}{\left\|b^r - \overline{b}\right\|_\infty} \\ &= \underline{\mathrm{clm}}\vartheta^R_{\overline{c}}(\overline{b}) + \lim_r \|x^r\| \\ &= \underline{\mathrm{clm}}\vartheta^R_{\overline{c}}(\overline{b}) + e\left(\mathcal{F}^{op}(\overline{\pi}), 0_n\right).\end{aligned}$$

■

5 Conclusions

First, we summarize in Table 1 the main contributions of the present work in relation to the calmness moduli from below and above of the optimal value functions ϑ^R and $\vartheta^R_{\overline{c}}$, where $\vartheta^R : \mathrm{dom}\mathcal{F}^{op} \to]-\infty, +\infty[$ is the restriction of ϑ to the set of solvable (equivalently, bounded) problems and $\vartheta^R_{\overline{c}} : \mathrm{dom}\mathcal{F} \to]-\infty, +\infty[$ is the optimal value function depending only on parameter b. Recall that

$$k^- := \min_{D \in \mathcal{M}_{\overline{\pi}}} \left\| \lambda^D \right\|_1 \text{ and } k^+ := \max_{D \in \mathcal{M}_{\overline{\pi}}} \left\| \lambda^D \right\|_1,$$

where $\mathcal{M}_{\overline{\pi}}$ is the set of minimal KKT subsets of indices at $\overline{\pi}$.

Table 1: Summary of results

	Calmness from below	Calmness from above
Perturbing b	$\underline{\mathrm{clm}}\vartheta^R_{\overline{c}}(\overline{b}) = k^-$	$\overline{\mathrm{clm}}\vartheta^R_{\overline{c}}(\overline{b}) \underset{(1)}{\leq} k^+$
Perturbing b and c	$\underline{\mathrm{clm}}\vartheta^R(\overline{\pi}) \underset{(2)}{\leq} k^- + e(\mathcal{E}^{op}(\overline{\pi}), 0_n)$	$\overline{\mathrm{clm}}\vartheta^R(\overline{\pi}) = \overline{\mathrm{clm}}\vartheta^R_{\overline{c}}(\overline{b}) + d(0_n, \mathcal{F}^{op}(\overline{\pi})) \leq k^+ + d(0_n, \mathcal{F}^{op}(\overline{\pi}))$

(with $\mathcal{E}^{op}(\overline{\pi})$ being defined in (10)).

So, to start with, we observe that $\vartheta^R_{\overline{c}}$ and ϑ^R are always calm from below and above, and hence calm. Moreover, by combining the previous results in the table, we have

$$k^- \leq \mathrm{clm}\vartheta^R_{\overline{c}}(\overline{b}) \leq k^+$$

and

$$d(0_n, \mathcal{F}^{op}(\overline{\pi})) \leq \mathrm{clm}\vartheta^R(\overline{\pi}) \leq \max\{k^- + e(\mathcal{E}^{op}(\overline{\pi}), 0_n),\ k^+ + d(0_n, \mathcal{F}^{op}(\overline{\pi}))\}.$$

Secondly, we comment that inequalities (1) and (2) in Table 1 might be strict as examples 1 and 2 show. Going further, the paper shows that (1) is held as an equality under SCQ, while (2) becomes an equality when $\mathcal{F}^{op}(\overline{\pi})$ is bounded.

Consequently, under these two conditions (SCQ at $\bar{b}$ together with the boundedness of $\mathcal{F}^{op}(\overline{\pi})$), equivalently, when $\overline{\pi} \in \operatorname{int}\operatorname{dom}\mathcal{F}^{op}$, we have the exact formulae for all moduli, which are gathered in the following theorem. In it, we have also taken into account the fact that $\overline{\pi} \in \operatorname{int}\operatorname{dom}\mathcal{F}^{op}$ turns out to be equivalent to the simultaneous nonemptiness and boundedness of both nominal optimal sets $\mathcal{F}^{op}(\overline{\pi})$ and $\Lambda^{op}(\overline{\pi})$; indeed, for $\overline{\pi} \in \operatorname{dom}\mathcal{F}^{op}$, the boundedness of $\Lambda^{op}(\overline{\pi})$ is equivalent to SCQ (see [19, Th. 6.1(v)]). In this case we can write $k^{+} = \max_{\lambda \in \operatorname{extr}\Lambda^{op}(\overline{\pi})} \|\lambda\|_1 = \max_{\lambda \in \Lambda^{op}(\overline{\pi})} \|\lambda\|_1$. Moreover, one always has $k^{-} = \min_{\lambda \in \operatorname{extr}\Lambda^{op}(\overline{\pi})} \|\lambda\|_1 = \min_{\lambda \in \Lambda^{op}(\overline{\pi})} \|\lambda\|_1$, because of the linearity of $\|\cdot\|_1$ on Λ. Recall also that the boundedness of $\mathcal{F}^{op}(\overline{\pi})$ entails $\mathcal{E}^{op}(\overline{\pi}) = \operatorname{extr}\mathcal{F}^{op}(\overline{\pi})$ and $e(\operatorname{extr}\mathcal{F}^{op}(\overline{\pi}), 0_n) = e(\mathcal{F}^{op}(\overline{\pi}), 0_n) = \max_{x \in \mathcal{F}^{op}(\overline{\pi})} \|x\|$. So, according to these comments, the results in Table 1 give rise to the following theorem.

Theorem 8 *Let $\overline{\pi} \in \operatorname{int}\operatorname{dom}\mathcal{F}^{op}$. Then, we have*

$$\begin{aligned}\operatorname{clm}\vartheta(\overline{\pi}) &= \operatorname{clm}\vartheta^{R}(\overline{\pi}) \\ &= \max\{k^{-} + e(\mathcal{F}^{op}(\overline{\pi}), 0_n),\ k^{+} + d(0_n, \mathcal{F}^{op}(\overline{\pi}))\} \\ &= \max\left\{\min_{\lambda \in \Lambda^{op}(\overline{\pi})} \|\lambda\|_1 + \max_{x \in \mathcal{F}^{op}(\overline{\pi})} \|x\|,\ \max_{\lambda \in \Lambda^{op}(\overline{\pi})} \|\lambda\|_1 + \min_{x \in \mathcal{F}^{op}(\overline{\pi})} \|x\|\right\}.\end{aligned}$$

Looking again at the previous summary of results in Table 1, we immediately derive the following corollary under the uniqueness of optimal solution. It is stated without the SCQ assumption. In fact, if we have both, $\mathcal{F}^{op}(\overline{\pi}) = \{\overline{x}\}$ and SCQ at $\overline{\pi}$, then we additionally have an exact formulae for $\overline{\operatorname{clm}}\vartheta^{R}_{\overline{c}}(\bar{b})$ $(= k^{+})$.

Corollary 4 *Let $\overline{\pi} \in \operatorname{dom}\mathcal{F}^{op}$ with $\mathcal{F}^{op}(\overline{\pi}) = \{\overline{x}\}$. Then*

$$\underline{\operatorname{clm}}\vartheta^{R}(\overline{\pi}) = \underline{\operatorname{clm}}\vartheta^{R}_{\overline{c}}(\bar{b}) + \|\overline{x}\| \quad \text{and} \quad \overline{\operatorname{clm}}\vartheta^{R}(\overline{\pi}) = \overline{\operatorname{clm}}\vartheta^{R}_{\overline{c}}(\bar{b}) + \|\overline{x}\|.$$

Consequently,

$$\operatorname{clm}\vartheta^{R}(\overline{\pi}) = \operatorname{clm}\vartheta^{R}_{\overline{c}}(\bar{b}) + \|\overline{x}\|.$$

5.1 Calmness modulus and distance to infeasibility

To finish this work, we analyze the relationship between $\operatorname{clm}\vartheta(\overline{\pi})$ and the well studied concept of distance to infeasibility; the reader is addressed to [33, 34] for details on this distance in the context of conic linear systems and to [7] (where it is called *distance to ill-posedness)* in the linear semi-infinite setting. Specifically, [33, Theorem 1.1] provides a certain Lipschitz type inequality for ϑ which immediately yields an upper bound on clm$\vartheta(\overline{\pi})$, provided that $\overline{\pi} \in \text{dom}\mathcal{F}^{op}$. This upper bound has the distance to infeasibiltiby in the denominator. Let us recall some details: paper [33] deals with conic linear problems in the form

$$\begin{array}{ll} \text{Inf} & c^*x \\ \text{s.t.} & b - Ax \in C_Y, \\ & x \in C_X, \end{array} \tag{22}$$

where $C_X \subset X$ and $C_Y \subset Y$ are closed convex cones in X and Y, respectively. X is a reflexive normed space while Y is an arbitrary normed space. The norms in both spaces are denoted by $\|\cdot\|$. Here $b \in Y$, $A : X \to Y$ is a (continuous) linear operator, with norm $\|A\| := \sup\{\|Ax\| \mid \|x\| = 1\}$, and $c^* : X \to \mathbb{R}$ is an element of the dual space of X, i.e., a continuous linear functional, with $\|c^*\| := \sup\{c^*x \mid \|x\| = 1\}$. The parameter space of all problems (22) is endowed with the product norm

$$\|(A, b, c^*)\| := \max\{\|A\|, \|b\|, \|c^*\|\}.$$

Our parametrized problem (1) may be translated into the conic format, just by taking $X = C_X := \mathbb{R}^n$, $Y = \mathbb{R}^T$, $C_Y := \mathbb{R}^T_+$, and considering a fixed matrix A (which remains unperturbed). In this way, the results of [33] apply into our LP context, where we are considering $\|\cdot\|_\infty$ for measuring the perturbations of b (indeed, the reader is addressed to [9] for details about the convenience of this norm when dealing with polyhedral cones).

Following the notation of [33], we consider

$$Pri\emptyset := \mathbb{R}^T \setminus \text{dom}\mathcal{F} \text{ and } Dual\emptyset := \mathbb{R}^n \setminus \text{dom}\Lambda,$$

corresponding, respectively, to the set of parameter b and c associated with primal and dual inconsistent (infeasible) problems. In this way,

$$d(b, Pri\emptyset) := \inf\{\|b - b^1\| \mid b^1 \in Pri\emptyset\}$$

represents the distance from $b \in \mathbb{R}^T$ to primal infeasibility, while $d(c, Dual\emptyset)$, analogously defined, denotes the corresponding distance to dual infeasibility. Observe that

$$\pi = (c, b) \in \text{int}\,\text{dom}\mathcal{F}^{op} \Leftrightarrow \min\{d(b, Pri\emptyset), d(c, Dual\emptyset)\} > 0.$$

Theorem 9 (See [33, Theorem 1.1(5)]) *Let* $\overline{\pi} \in \operatorname{int}\operatorname{dom}\mathcal{F}^{op}$. *Then, for any* $\pi = (c, b) \in \mathbb{R}^n \times \mathbb{R}^T$ *such that*

$$\left\|b - \overline{b}\right\| < d\left(\overline{b}, Pri\emptyset\right) \text{ and } \|c - \overline{c}\| < d\left(\overline{c}, Dual\emptyset\right),$$

we have

$$\begin{aligned} |\vartheta(\pi) - \vartheta(\overline{\pi})| \quad \leq \quad & \left\|b - \overline{b}\right\| \frac{\|\overline{c}\| + \|c - \overline{c}\|}{d\left(\overline{b}, Pri\emptyset\right) - \|\pi - \overline{\pi}\|} \frac{\|\overline{\pi}\|}{d\left(\overline{c}, Dual\emptyset\right)} \\ & + \|c - \overline{c}\| \frac{\left\|\overline{b}\right\| + \left\|b - \overline{b}\right\|}{d\left(\overline{c}, Dual\emptyset\right) - \|\pi - \overline{\pi}\|} \frac{\|\overline{\pi}\|}{d\left(\overline{b}, Pri\emptyset\right)}. \end{aligned} \tag{23}$$

Now if we divide both members of (23) by $\|\pi - \overline{\pi}\|$ and let $\|\pi - \overline{\pi}\| \to 0$, we inmediately derive the following Corollary.

Corollary 5 *Let* $\overline{\pi} \in \operatorname{int}\operatorname{dom}\mathcal{F}^{op}$. *One has*

$$\operatorname{clm}\vartheta(\overline{\pi}) \leq \frac{\|\overline{c}\|}{d\left(\overline{b}, Pri\emptyset\right)} \frac{\|\overline{\pi}\|}{d\left(\overline{c}, Dual\emptyset\right)} + \frac{\left\|\overline{b}\right\|}{d\left(\overline{c}, Dual\emptyset\right)} \frac{\|\overline{\pi}\|}{d\left(\overline{b}, Pri\emptyset\right)}. \tag{24}$$

Remark 7 Observe that Theorem 8 constitutes a refinement of Corollary 5, as far as inequality (24) can be strict. In fact, its right-hand side (upper bound on $\operatorname{clm}\vartheta(\overline{\pi})$) might be much greater than $\operatorname{clm}\vartheta(\overline{\pi})$ when $\overline{\pi}$ approaches the primal/dual infeasibility. Just consider the example, in $\mathbb{R}^2$ endowed with the Euclidean norm,

$$\pi^r : \text{minimize } x_1 + \frac{1}{r}x_2 \text{ s.t. } -x_1 \leq 0, \quad -x_2 \leq \frac{1}{r}, x_2 \leq \frac{1}{r}.$$

One easily sees that $b^r \to 0_3$, $d\left(b^r, Pri\emptyset\right) \to 0$, and so the right-hand side of (24) goes to $+\infty$, while (appealing to Theorem 8)

$$\operatorname{clm}\vartheta(\pi^r) = \left\|\left(1, \frac{1}{r}, 0\right)'\right\|_1 + \frac{1}{r} \to 1.$$

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