Peláez, J. N. & Seco, D. (2018). Schatten classes of generalized Hilbert operators. *Collectanea Mathematica*, 69(1), pp. 83–105.

DOI: 10.1007/s13348-017-0195-5

SCHATTEN CLASSES OF GENERALIZED HILBERT OPERATORS

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Abstract. Let $\mathcal{D}_v$ denote the Dirichlet type space in the unit disc induced by a radial weight v for which $\widehat{v}(r) = \int_r^1 v(s)\,ds$ satisfies the doubling property $\int_r^1 v(s)\,ds \le C \int_{\frac{1+r}{2}}^1 v(s)\,ds$. In this paper, we characterize the Schatten classes $S_p(\mathcal{D}_v)$ of the generalized Hilbert operators

$$\mathcal{H}_g(f)(z) = \int_0^1 f(t)g'(tz)\,dt$$

acting on $\mathcal{D}_v$, where v satisfies the Muckenhoupt-type conditions

$$\sup_{0<r<1} \left(\int_r^1 \frac{\widehat{v}(s)}{(1-s)^2}\,ds\right)^{1/2} \left(\int_0^r \frac{1}{\widehat{v}(s)}\,ds\right)^{1/2} < \infty$$

and

$$\sup_{0<r<1} \left(\int_0^r \frac{\widehat{v}(s)}{(1-s)^4}\,ds\right)^{\frac{1}{2}} \left(\int_r^1 \frac{(1-s)^2}{\widehat{v}(s)}\,ds\right)^{\frac{1}{2}} < \infty.$$

For $p \ge 1$, it is proved that $\mathcal{H}_g \in S_p(\mathcal{D}_v)$ if and only if

$$\int_0^1 \left((1-r)\int_{-\pi}^{\pi} |g'(re^{i\theta})|^2\,d\theta\right)^{\frac{p}{2}} \frac{dr}{1-r} < \infty.$$

1. Introduction and main results

Let $\mathbb{D}$ denote the open unit disk of the complex plane, and let $H(\mathbb{D})$ be the class of all analytic functions on $\mathbb{D}$. A function $v : \mathbb{D} \to (0,\infty)$, integrable over $\mathbb{D}$, is called a weight. It is radial if $v(z) = v(|z|)$ for all $z \in \mathbb{D}$. The weighted Dirichlet space $\mathcal{D}_v$ consists of $f \in H(\mathbb{D})$ for which

$$\|f\|^2_{\mathcal{D}_v} = |f(0)|^2 + \int_{\mathbb{D}} |f'(z)|^2 v(z)\,dA(z) < \infty,$$

where $dA(z) = \frac{dx\,dy}{\pi}$ is the normalized Lebesgue area measure on $\mathbb{D}$. In this work, we will consider Dirichlet type spaces $\mathcal{D}_v$ induced by weights

Date: July 24, 2018.

2010 *Mathematics Subject Classification.* Primary: 47B10 Secondary: 31C25, 47G10.

Key words and phrases. Schatten class, Hilbert operator, Dirichlet spaces.

The research leading to these results has received partial funding from the Ramón y Cajal program of MICINN (Spain); from Ministerio de Economía y Competitividad, Spain, projects MTM2011-24606, MTM2011-25502 and MTM2014-52865-P; from La Junta de Andalucía, (FQM210) and (P09-FQM-4468); and from the European Research Council under the European Union's Seventh Framework Programme (FP/2007-2013) / ERC Grant Agreement n.291497.

in the class $\widehat{\mathcal{D}}$ of the radial weights v for which $\widehat{v}(r) = \int_r^1 v(s)\,ds$ satisfy $sup_{0<r<1}\frac{\widehat{v}(r)}{\widehat{v}(\frac{1+r}{2})} < \infty$. The standard radial weights $v(z) = (1-|z|)^\alpha, \alpha > -1$ meet this doubling property. We write $\mathcal{D}_\alpha$ for the Dirichlet type space induced by the standard weight $(1-|z|)^\alpha$. The Hardy space H^2 consists of $f \in H(\mathbb{D})$ for which $\|f\|_{H^2} = \lim_{r\to 1^-} M_2(r,f) < \infty$, where

$$M_2(r,f) = \left(\frac{1}{2\pi}\int_{-\pi}^{\pi} |f(re^{i\theta})|^2\,d\theta\right)^{\frac{1}{2}}.$$

The classical Littlewood-Paley formula says that $H^2 = \mathcal{D}_1$. We refer the reader to [6] for background information on this space. We denote by A^2_ω the Bergman space induced by a weight ω (see [12, Chapter 1]). Moreover, if ω is radial then $A^2_\omega = \mathcal{D}_{\omega^\star}$, where

$$\omega^\star(z) = \int_{|z|}^1 s\log\frac{s}{|z|}\omega(s)\,ds, \quad z \in \mathbb{D}\setminus\{0\}.$$

See [12, Theorem 4.2] for the details.

Every $g \in H(\mathbb{D})$ induces an operator, that we call *the generalized Hilbert operator* $\mathcal{H}_g$, defined by

$$\mathcal{H}_g(f)(z) = \int_0^1 f(t)g'(tz)\,dt, \quad f \in H(\mathbb{D}). \tag{1.1}$$

The sharp condition

$$\int_0^1 \frac{(1-s)^2}{\widehat{v}(s)}\,ds < \infty \tag{1.2}$$

ensures that the integral in (1.1) defines an analytic function for each $f \in \mathcal{D}_v$ (see Lemma 7 below).

The choice $g(z) = \log\frac{1}{1-z}$ in (1.1) gives an integral representation of the classical Hilbert operator $\mathcal{H}$. The Hilbert operator $\mathcal{H}$ is a model of Hankel operator, and has been the object of previous studies such as [1, 3, 5], where the authors dealt with questions related to the boundedness, the operator norm and the spectrum of $\mathcal{H}$. This has revealed a natural connection between $\mathcal{H}$ and other classical objects: the weighted composition operators, the Szegö projection and the Legendre functions of the first kind. The Hilbert operator is bounded on the classical Dirichlet type space $\mathcal{D}_\alpha$ if and only if $\alpha \in (0,2)$, as was shown in [4, 7]. In fact, if $\alpha \geq 2$ there is $f \in \mathcal{D}_\alpha$, $f \geq 0$ on $[0,1)$ such that $\int_0^1 f(t)\,dt = \infty$.

The generalized Hilbert operator $\mathcal{H}_g$ was introduced recently in [7], where it is provided, among other results, a description of the $g \in H(\mathbb{D})$ such that $\mathcal{H}_g$ is bounded, compact or Hilbert-Schmidt on $\mathcal{D}_\alpha, \alpha \in (0,2)$. In [11], the authors solve the question of when is $\mathcal{H}_g$ bounded or compact between weighted Bergman spaces A^p_ω and A^q_ω, $1 < p,q < \infty$, induced by a large class of radial weights.

The primary aim of this paper is to determine the membership in Schatten ideals $\mathcal{S}_p(\mathcal{D}_v)$ of generalized Hilbert operators $\mathcal{H}_g$ acting on Dirichlet type spaces $\mathcal{D}_v$, $v \in \widehat{\mathcal{D}}$. This leads us to consider the following spaces. For $0 < p < \infty$, the mixed norm space $\mathcal{B}(2,p)$ consists of $g \in H(\mathbb{D})$ such that

$$\|g\|^q_{\mathcal{B}(2,p)} = |g(0)|^p + \int_0^1 M_2^p(r,g')(1-r)^{\frac{p}{2}-1}\,dr < \infty.$$

Let us observe that $\mathcal{B}(2,2)$ is nothing else but the classical Dirichlet space $\mathcal{D}_0 = \mathcal{D}$. The space $\mathcal{B}(2,\infty)$ consists of $g \in H(\mathbb{D})$ such that

$$\|g\|_{\mathcal{B}(2,\infty)} = |g(0)| + \sup_{0<r<1} M_2(r,g')(1-r)^{\frac{1}{2}} < \infty.$$

A classical result of Hardy and Littlewood [6, Chapter 5] asserts that $\mathcal{B}(2,\infty)$ coincides with the mean Lipschitz space $\Lambda\left(2,\frac{1}{2}\right)$ of the $g \in H(\mathbb{D})$ having a non-tangential limit $g(e^{i\theta})$ almost everywhere and such that

$$\omega_2(g,t) = \mathrm{O}(t^{\frac{1}{2}}), \quad t \to 0,$$

where

$$\omega_2(g,t) = \sup_{0<h\le t}\left(\int_0^{2\pi} |g(e^{i(\theta+h)}) - g(e^{i\theta})|^2 \frac{d\theta}{2\pi}\right)^{1/2}$$

is the integral modulus of continuity of order 2.

The corresponding "little oh" mean Lipschitz space $b(2,\infty)$, usually denoted by $\lambda\left(2,\frac{1}{2}\right)$, consists of $g \in H(\mathbb{D})$ such that

$$\lim_{r\to 1^-} M_2(r,g')(1-r)^{1/2} = 0.$$

The next theorem is the main result of this paper.

Theorem 1. *Let $g \in H(\mathbb{D})$, $1 \le p \le \infty$ and $v \in \widehat{\mathcal{D}}$ which satisfies the conditions*

$$M_1(v) = \sup_{0<r<1}\left(\int_r^1 \frac{\widehat{v}(s)}{(1-s)^2}\,ds\right)^{1/2}\left(\int_0^r \frac{1}{\widehat{v}(s)}\,ds\right)^{1/2} < \infty, \tag{1.3}$$

and

$$M_2(v) = \sup_{0<r<1}\left(\int_0^r \frac{\widehat{v}(s)}{(1-s)^4}\,ds\right)^{\frac{1}{2}}\left(\int_r^1 \frac{(1-s)^2}{\widehat{v}(s)}\,ds\right)^{\frac{1}{2}} < \infty. \tag{1.4}$$

Then, the following conditions are equivalent:

(i) $\mathcal{H}_g \in S_p(\mathcal{D}_v)$;
(ii) $g \in \mathcal{B}(2,p)$.

Moreover,

$$\|\mathcal{H}_g\|_{S_p(\mathcal{D}_v)} \asymp \|g - g(0)\|_{\mathcal{B}(2,p)}.$$

It will also be proved (see Proposition 10 below) that $\mathcal{H}_g$ is compact on $\mathcal{D}_v$ if and only if $g \in b(2,\infty)$. Notice that the conditions that characterize the Schatten classes do not depend on the weight defining the space.

If $v(z) = (1-|z|)^\alpha$ is a standard weight, (1.3) holds if and only if $\alpha > 0$, and (1.4) is satisfied if and only if $\alpha < 2$, so both conditions hold simultaneously if and only if $\alpha \in (0,2)$. Therefore, in particular, Theorem 1 provides a characterization of Schatten classes of generalized Hilbert operators $\mathcal{H}_g$ acting on the Hardy space H^2 ($\alpha = 1$) and standard Bergman spaces A^2_β, $\beta \in (-1,0)$. Going further the next result follows from Theorem 1.

Corollary 2. *Let $g \in H(\mathbb{D})$, $1 \le p \le \infty$ and $\omega \in \widehat{\mathcal{D}}$ which satisfies the condition*

$$\sup_{0<r<1}\left(\int_0^r \frac{\widehat{\omega}(t)}{(1-t)^2}\,dt\right)^{\frac{1}{2}}\left(\int_r^1 \frac{1}{\widehat{\omega}(t)}\,dt\right)^{\frac{1}{2}} < \infty. \tag{1.5}$$

Then, the following conditions are equivalent:

(i) $\mathcal{H}_g \in S_p(A^2_\omega)$;

(ii) $g \in \mathcal{B}(2,p)$.

The Muckenhoupt-type condition (1.5) arises in the study of generalized Hilbert operators $\mathcal{H}_g$ on weighted Bergman spaces in [11], where the authors describe the $g \in H(\mathbb{D})$ such that $\mathcal{H}_g$ is bounded, compact or Hilbert-Schmidt on A^2_ω for the the subclass of $\widehat{\mathcal{D}}$ consisting of regular weights.

From now on, for each radial weight v and $x \in \mathbb{R}$ let us denote $\widehat{V}_x(r) = \frac{\widehat{v}(r)}{(1-r)^x}$. Our approach to prove the case $p = \infty$ of Theorem 1 reveals the role of Muckenhoupt type conditions (1.3) and (1.4). On one hand, (1.3) allows to prove that $L^2_{\widehat{V}_2}$ is a natural restriction of $\mathcal{D}_v$ to functions defined on $[0,1)$. On the other hand, the sublinear Hilbert operator defined by

$$\widetilde{\mathcal{H}}(f)(z) = \int_0^1 \frac{|f(t)|}{1-tz}\,dt$$

behaves like a kind of maximal function for all generalized Hilbert operators $\mathcal{H}_g$ such that $g \in \mathcal{B}(2,\infty)$, and hence, it will be essential to study its boundedness on $L^2_{\widehat{V}_2}$.

Theorem 3. *Let $v \in \widehat{\mathcal{D}}$ which satisfies the conditions* (1.2) *and* (1.3). *Then the following assertions are equivalent:*

(i) $\mathcal{H} : L^2_{\widehat{V}_2} \to \mathcal{D}_v$ *is bounded;*

(ii) $\widetilde{\mathcal{H}} : L^2_{\widehat{V}_2} \to \mathcal{D}_v$ *is bounded;*

(iii) v *satisfies the Muckenhoupt type condition* (1.4).

Moreover, if $M_2(v) < \infty$, then

$$\frac{M_2(v)}{M_1(v)} \lesssim \|\mathcal{H}\|_{\left(L^2_{\widehat{V}_2},\mathcal{D}_v\right)} \asymp \|\widetilde{\mathcal{H}}\|_{\left(L^2_{\widehat{V}_2},\mathcal{D}_v\right)} \lesssim M_1(v)M_2(v).$$

We obtain as a byproduct the following result which extends several results in the literature [4, 11].

Corollary 4. *Let $v \in \widehat{\mathcal{D}}$ which satisfies the conditions* (1.3) *and* (1.4). *Then, both the Hilbert operator $\mathcal{H}$ and the sublinear Hilbert operator $\widetilde{\mathcal{H}}$ are bounded on $\mathcal{D}_v$.*

Throughout the paper the letter $C = C(\cdot)$ will denote an absolute constant whose value depends on the parameters indicated in the parenthesis, and may change from one occurrence to another. We will use the notation $a \lesssim b$ if there exists a constant $C = C(\cdot) > 0$ such that $a \le Cb$, and $a \gtrsim b$ is understood in an analogous manner. In particular, if $a \lesssim b$ and $a \gtrsim b$, then we will write $a \asymp b$.

2. The Hilbert operator on $\mathcal{D}_v$

2.1. **Some results on weights.**

The following lemma provides useful characterizations of weights in $\widehat{\mathcal{D}}$. For a proof, see [13]. Given a radial weight v, we write $v_x = \int_0^1 s^x v(s)\,ds$, $x > -1$.

Lemma 5. *Let ω be a radial weight. Then the following conditions are equivalent:*

(i) $\omega \in \widehat{\mathcal{D}}$;

(ii) *There exist $C = C(\omega) \ge 1$ and $\beta = \beta(\omega) > 0$ such that*
$$\widehat{\omega}(r) \le C \left(\frac{1-r}{1-t}\right)^{\beta} \widehat{\omega}(t), \quad 0 \le r \le t < 1;$$

(iii) *There exist $C = C(\omega) > 0$ and $\gamma = \gamma(\omega) > 0$ such that*
$$\int_0^t \left(\frac{1-t}{1-s}\right)^{\gamma} \omega(s)\,ds \le C\widehat{\omega}(t), \quad 0 \le t < 1;$$

(iv) *There exist $C = C(\omega) > 0$ and $\eta = \eta(\omega) > 0$ such that*
$$\omega_x \le C \left(\frac{y}{x}\right)^{\eta} \omega_y, \quad 0 < x \le y < \infty;$$

(v) *There exists $C = C(\omega) > 0$ such that $\omega_n \le C\omega_{2n}$ for all $n \in \mathbb{N}$;*

(vi)
$$\omega_x \asymp \widehat{\omega}\left(1 - \frac{1}{x}\right), \quad x \in [1, \infty);$$

(vii) *There exists $\lambda = \lambda(\omega) \ge 0$ such that*
$$\int_{\mathbb{D}} \frac{\omega(z)\,dA(z)}{|1-\overline{\zeta}z|^{\lambda+1}} \asymp \frac{\widehat{\omega}(\zeta)}{(1-|\zeta|)^{\lambda}}, \quad \zeta \in \mathbb{D};$$

(viii) $\omega^{\star}(z) \asymp \widehat{\omega}(z)(1-|z|)$ *for* $|z| \ge \frac{1}{2}$.

The following technical lemma will be used in the proof of Theorem 3.

Lemma 6. *Let v be a radial weight. If* (1.4) *holds, then*

$$\sup_{0\le r<1}\left(\int_0^r \frac{v(s)}{(1-s)^3}\,ds\right)^{\frac12}\left(\int_r^1 \frac{1}{\widehat{V}_2(s)}\,ds\right)^{\frac12}<\infty.$$

Proof. For $0<r<1$, Fubini's theorem gives

$$\int_0^r \frac{\widehat{v}(s)}{(1-s)^4}\,ds=\int_0^r \frac{v(s)[1-(1-s)^3]}{3(1-s)^3}\,ds+\widehat{v}(r)\frac{1-(1-r)^3}{3(1-r)^3}.$$

We may assume that $r\in[\frac12,1)$, and then we have

$$\begin{aligned}\int_0^r \frac{v(s)}{(1-s)^3}\,ds &\le C\int_0^{\frac12} v(s)\,ds+C\int_{\frac12}^r \frac{v(s)[1-(1-s)^3]}{(1-s)^3}\,ds\\ &\le C\int_0^{\frac12} v(s)\,ds+C\int_0^r \frac{\widehat{v}(s)}{(1-s)^4}\,ds.\end{aligned}$$

By (1.4), there is $C=C(v)$ such that $\int_0^1 \frac{1}{\widehat{V}_2(s)}\,ds\le CM_2(v)<\infty$. Therefore, the above inequality yields

$$\begin{aligned}&\sup_{\frac12\le r<1}\left(\int_0^r \frac{v(s)}{(1-s)^3}\,ds\right)^{\frac12}\left(\int_r^1 \frac{1}{\widehat{V}_2(s)}\,ds\right)^{\frac12}\\ &\le C\sup_{\frac12\le r<1}\left(\int_0^{\frac12} v(s)\,ds+\int_0^r \frac{\widehat{v}(s)}{(1-s)^4}\,ds\right)^{\frac12}\left(\int_r^1 \frac{1}{\widehat{V}_2(s)}\,ds\right)^{\frac12}\\ &\le C\left(\int_0^{\frac12} v(s)\,ds\int_0^1 \frac{1}{\widehat{V}_2(t)}\,dt<\infty\right)^{\frac12}+CM_2(v)\le CM_2(v)<\infty.\end{aligned}$$

This finishes the proof. □

2.2. **Hardy-Littlewood type inequalities.** The first result in this subsection gives a sharp condition that ensures that $\mathcal{H}_g$ is well defined on $\mathcal{D}_v$.

Lemma 7. *Let v be a radial weight which satisfies* (1.2)*. Then there is a positive constant $C(v)$ such that*

$$\int_0^1 |f(t)|\,dt\le C(v)\|f\|_{\mathcal{D}_v},\quad f\in H(\mathbb{D}). \tag{2.1}$$

Proof. For any $f(z)=\sum_{k=0}^\infty \widehat{f}(k)z^k\in H(\mathbb{D})$, it is clear that

(2.2)

$$\int_0^1 |f(t)|\,dt\le|\widehat{f}(0)|+\left(\sum_{k=1}^\infty k^2|\widehat{f}(k)|^2 v_{2k-1}\right)^{1/2}\left(\sum_{k=1}^\infty \frac{1}{k^2(k+1)^2 v_{2k-1}}\right)^{1/2}.$$

By Lemma 5(v)(vi)

$$\sum_{k=1}^{\infty}\frac{1}{k^2(k+1)^2 v_{2k-1}} \le C\sum_{k=1}^{\infty}\frac{1}{\widehat{v}\left(1-\frac{1}{k+1}\right)}\int_{1-\frac{1}{k+1}}^{1-\frac{1}{k+2}}(1-s)^2\,ds$$
$$\le C\int_{\frac{1}{2}}^{1}\frac{(1-s)^2}{\widehat{v}(s)}\,ds \le C\int_{0}^{1}\frac{(1-s)^2}{\widehat{v}(s)}\,ds \le C.$$

This together with (2.2) and the identity $\|f\|^2_{\mathcal{D}_v}=|\widehat{f}(0)|^2+2\sum_{k=1}^{\infty}|k\widehat{f}(k)|^2 v_{2k-1}$, implies (2.1). □

A classical result of Hardy-Littlewood ([6, Theorem 5.11]) says that

$$\int_0^1 M^2_\infty(r,f)\,dr \le C\|f\|^2_{H^2}.$$

See also the classical Féjer-Riesz inequality [6, Theorem 3.13]. Applying this inequality to dilated functions $f_r(z)=f(rz)$, $0<r<1$, and integrating respect to a radial weight ω, it can be easily obtained that

$$\int_0^1 M^2_\infty(r,f)\,\widehat{\omega}(r)\,dr \le C\|f\|^2_{A^2_\omega}. \tag{2.3}$$

The next result shows a Hardy-Littlewood type inequality in a setting of weighted Dirichlet spaces.

Lemma 8. *Let v be a radial weight which satisfies* (1.3)*. Then, there exists $C=C(v)>0$ such that*

$$\int_0^1 M^2_\infty(s,f)\frac{\widehat{v}(s)}{(1-s)^2}\,ds \le CM_1^2(v)\|f\|^2_{\mathcal{D}_v},\quad f\in H(\mathbb{D}). \tag{2.4}$$

Proof. By condition (1.3) there is a constant $C=C(v)>0$ such that $\int_0^1\frac{\widehat{v}(s)}{(1-s)^2}\,ds \le CM_1^2(v)<\infty$. Using [9, Theorem 1] with

$$U^2(s)=\begin{cases}\frac{\widehat{v}(s)}{(1-s)^2}, & 0\le s<1\\ 0, & s\ge 1\end{cases}$$

and

$$V^2(s)=\begin{cases}\widehat{v}(s), & 0\le s<1\\ 0, & s\ge 1\end{cases}.$$

we obtain

$$\begin{aligned}&\int_0^1 M^2_\infty(s,f)\frac{\widehat{v}(s)}{(1-s)^2}\,ds\\ &\le CM_1^2(v)|f(0)|^2+\int_0^1\left(\int_0^s M_\infty(r,f')\,dr\right)^2\frac{\widehat{v}(s)}{(1-s)^2}\,ds\\ &\lesssim CM_1^2(v)\left(|f(0)|^2+\int_0^1 M^2_\infty(s,f')\widehat{v}(s)\,ds\right).\end{aligned} \tag{2.5}$$

Joining (2.3) and (2.5), we get (2.4). □

It is worth mentioning that the inequality

$$M_\infty(r,f')\le C\frac{M_\infty\left(\frac{1+r}{2},f\right)}{1-r},\quad 0<r<1,$$

implies the reverse inequality of (2.5) for any $f\in H(\mathbb{D})$ and $v\in\widehat{\mathcal{D}}$.

2.3. **Proof of Theorem 3.** It is clear that (ii)⇒(i).

(i)⇒(iii). This part of the proof uses ideas from [9]. For $r\in[0,1)$, set $\phi_r(t)=\frac{1}{\widehat{V_2}(t)}\chi_{[r,1)}(t)$, so that $\phi_r\in L^2_{\widehat{V_2}}$ by (1.2). Here, as usual, χ_E stands for the characteristic function of a set E. Bearing in mind Lemma 8, we deduce

$$\|\mathcal{H}(\phi_r)\|_{L^2_{\widehat{V_2}}}\lesssim M_1(v)\|\mathcal{H}(\phi_r)\|_{\mathcal{D}_v}\le M_1(v)\|\mathcal{H}\|_{\left(L^2_{\widehat{V_2}},\mathcal{D}_v\right)}\|\phi_r\|_{L^2_{\widehat{V_2}}},$$

and hence

$$\int_0^1\widehat{V_2}(s)\left(\int_r^1\frac{1}{\widehat{V_2}(t)(1-ts)}\,dt\right)^2ds\lesssim\int_r^1\frac{1}{\widehat{V_2}(t)}\,dt. \tag{2.6}$$

On the other hand,

$$\int_0^r\widehat{V_2}(s)\left(\int_r^1\frac{1}{\widehat{V_2}(t)(1-ts)}\,dt\right)^2ds\ge\frac{1}{4}\left(\int_0^r\widehat{V_4}(s)\,ds\right)\left(\int_r^1\frac{1}{\widehat{V_2}(t)}\,dt\right)^2,$$

and this, together with (2.6), implies

$$M_2(v)\lesssim M_1(v)\|\mathcal{H}\|_{\left(L^2_{\widehat{V_2}},\mathcal{D}_v\right)}<\infty.$$

(iii)⇒(ii). For any $\phi\in L^2_{\widehat{V_2}}$,

$$\left(\widetilde{\mathcal{H}}(\phi)\right)'(z)=\int_0^1\frac{t|\phi(t)|}{(1-tz)^2}\,dt,$$

and so Minkowski's inequality in continuous form yields

$$\begin{aligned}M_2(r,\left(\widetilde{\mathcal{H}}(\phi)\right)')&=\left(\frac{1}{2\pi}\int_0^{2\pi}\left|\int_0^1\frac{|\phi(t)|t}{(1-tre^{i\theta})^2}\,dt\right|^2d\theta\right)^{\frac12}\\&\le\int_0^1|\phi(t)|\left(\int_0^{2\pi}\frac{d\theta}{|1-tre^{i\theta}|^4}\right)^{\frac12}dt\asymp\int_0^1\frac{|\phi(t)|}{(1-tr)^{3/2}}\,dt.\end{aligned}$$

Hence, decomposing the range of variation of t, we obtain

$$\|\widetilde{\mathcal{H}}(\phi)\|^2_{\mathcal{D}_v}\lesssim I_1(r)+I_2(r)+|\widetilde{\mathcal{H}}(\phi)(0)|^2 \tag{2.7}$$

where

$$I_1(r)=\int_0^1\left(\int_0^r\frac{|\phi(t)|}{(1-t)^{3/2}}\,dt\right)^2v(r)\,dr$$

and

$$I_2(r)=\int_0^1\left(\int_r^1\frac{|\phi(t)|}{(1-tr)^{3/2}}\,dt\right)^2v(r)\,dr.$$

By (1.4)

$$|\widetilde{\mathcal{H}}(\phi)(0)|^2 \le \|\phi\|^2_{L^2_{\widehat{V}_2}} \int_0^1 \frac{1}{\widehat{V}_2(s)}\,ds \le C(v)M_2^2(v)\|\phi\|^2_{L^2_{\widehat{V}_2}} < \infty. \tag{2.8}$$

The inequality

$$I_1(r) \lesssim \|\phi\|^2_{L^2_{\widehat{V}_2}} \tag{2.9}$$

can be written as

$$\int_0^1 \left(\int_0^r \Phi(t)\,dt\right)^2 U^2(r)\,dr \lesssim \int_0^1 \Phi^2(r)V^2(r)\,dr,$$

where

$$U^2(x) = \begin{cases} v(x), & 0 \le x < 1 \\ 0, & x \ge 1 \end{cases},$$

$$V^2(x) = \begin{cases} (1-x)\widehat{v}(x), & 0 \le x < 1 \\ 0, & x \ge 1 \end{cases},$$

and $\Phi(t) = \frac{|\phi(t)|}{(1-t)^{\frac{3}{2}}}$. From this, by [9, Theorem 1], (2.9) holds if and only if

$$M_3(v) = \sup_{0<r<1} \widehat{v}(r)^{\frac{1}{2}} \left(\int_0^r \frac{1}{(1-s)\widehat{v}(s)}\,ds\right)^{\frac{1}{2}} < \infty.$$

Using (1.3), we get

$$\begin{aligned}\int_0^r \frac{1}{(1-s)\widehat{v}(s)}\,ds &\le \frac{1}{1-r}\int_0^r \frac{1}{\widehat{v}(s)}\,ds \le \frac{M_1^2(v)}{(1-r)\int_r^1 \widehat{V}_2(s)\,ds} \\ &\le \frac{M_1^2(v)}{(1-r)\int_r^{\frac{1+r}{2}} \widehat{V}_2(s)\,ds} \le \frac{CM_1^2(v)}{\widehat{v}\left(\frac{1+r}{2}\right)} \le \frac{CM_1^2(v)}{\widehat{v}\,(r)},\end{aligned}$$

which implies that $M_3(v) \le CM_1(v)$, and so by [9, Theorem 1]

$$I_1(r) \le C(v)M_1^2(v)\|\phi\|^2_{L^2_{\widehat{V}_2}}. \tag{2.10}$$

Moreover, by applying [9, Theorem 2] with

$$U^2(x) = \begin{cases} \frac{v(x)}{(1-x)^3}, & 0 \le x < 1 \\ 0, & x \ge 1 \end{cases},$$

and

$$V^2(x) = \begin{cases} \widehat{V}_2(x), & 0 \le x < 1 \\ 0, & x \ge 1 \end{cases},$$

we deduce that

$$I_2(r) \lesssim \int_0^1 \left(\int_r^1 \phi(t)\,dt\right)^2 \frac{v(r)}{(1-r)^3}\,dr \le M_4^2(v)\|\phi\|^2_{L^2_{\widehat{V}_2}},$$

holds whenever

$$M_4(v) = \sup_{0\le r<1} \left(\int_0^r \frac{v(s)}{(1-s)^3}\,ds\right)^{\frac{1}{2}} \left(\int_r^1 \frac{1}{\widehat{V}_2(s)}\,ds\right)^{\frac{1}{2}} < \infty.$$

By Lemma 6, $M_4(v) \leq CM_2(v) < \infty$. This together with (2.7), (2.8) and (2.10) gives (iii)⇒(ii). Going further, since there is an absolute constant $K > 0$ such that $\min\{M_1(v), M_2(v)\} > K$, we get

$$\|\widetilde{\mathcal{H}}\|_{\left(L^2_{\widehat{V}_2}, \mathcal{D}_v\right)} \lesssim M_1(v) M_2(v).$$

□

Corollary 4 follows from Theorem 3 and Lemma 8.

3. Proof of Theorem 1

The right choice of the norm used is in many cases a key to a good understanding of how a concrete operator acts in a given space. Here the spaces $\mathcal{B}(2,p)$ will be equipped with an l^p- norm of the H^2 norms of dyadic blocks of the Maclaurin series. In fact, a calculation shows that

$$\|g\|^2_{\mathcal{B}(2,\infty)} \asymp |\widehat{g}(0)|^2 + \sup_{n\in\mathbb{N}} \left(\frac{1}{2^n} \sum_{k\in I(n)} k^2 |\widehat{g}(k)|^2 \right),$$

where $g(z) = \sum_{k=0}^{\infty} \widehat{g}(k) z^k$ and $I(n) = \{k \in \mathbb{N} : 2^n \leq k < 2^{n+1} - 1\},\ n \in \mathbb{N}$. The same techniques allow us to prove that

$$g \in b(2,\infty) \Leftrightarrow \lim_{n\to\infty} \frac{1}{2^n} \sum_{k\in I(n)} k^2 |\widehat{g}(k)|^2 = 0.$$

Throughout this section, the expression $\widetilde{g_k}$ will denote $k^2|\hat{g}(k)|^2$. Using [8, Theorem 1] it can also be proved that

$$\|g\|^p_{\mathcal{B}(2,p)} \asymp |\widehat{g}(0)|^p + \sum_{n=0}^{\infty} 2^{-\frac{np}{2}} \left(\sum_{k\in I(n)} \widetilde{g_k} \right)^{\frac{p}{2}}, \quad 0 < p < \infty.$$

3.1. Boundedness and compactness.

Proposition 9. *Let $g \in H(\mathbb{D})$ and $v \in \widehat{\mathcal{D}}$ which satisfies the conditions (1.3) and (1.4). Then $\mathcal{H}_g$ is bounded on $\mathcal{D}_v$ if and only $g \in \mathcal{B}(2,\infty)$. Moreover,*

$$\|\mathcal{H}_g\| \asymp \|g - g(0)\|_{\mathcal{B}(2,\infty)}.$$

Proof. We use that the norm of $\mathcal{H}_g(f)$ can be computed from the Taylor coefficients of g and the moments of $f\chi_{[0,1)}$

$$\|\mathcal{H}_g(f)\|^2_{\mathcal{D}_v} = \left|\widehat{g}(1) \int_0^1 f(t)\,dt\right|^2 + \sum_{j=1}^{\infty} j^2 \widetilde{g_{j+1}} \left|\int_0^1 t^j f(t)\,dt\right|^2 v_{2j-1}. \tag{3.1}$$

Assume that $g \in \mathcal{B}(2,\infty)$, and let us see first that

$$\sum_{j=2}^{\infty} j^2 \widetilde{g_{j+1}} \left|\int_0^1 t^j f(t)\,dt\right|^2 v_{2j-1} \leq C\|g - g(0)\|^2_{\mathcal{B}(2,\infty)} \|f\|^2_{\mathcal{D}_v}. \tag{3.2}$$

Indeed, the left-hand side of the above can be decomposed in dyadic pieces in terms of the parameter j, and is therefore dominated by

$$
\begin{aligned}
&C\sum_{n=1}^{\infty} v_{2^{n+1}-1}2^{2n}\left(\int_0^1 t^{2^n}|f(t)|\,dt\right)^2 \sum_{j\in I(n)} \widetilde{g_{j+1}}\\
&\leq C\|g-g(0)\|^2_{\mathcal{B}(2,\infty)}\sum_{n=1}^{\infty} v_{2^{n+1}-1}2^{3n}\left(\int_0^1 t^{2^n}|f(t)|\,dt\right)^2\\
&\leq C\|g-g(0)\|^2_{\mathcal{B}(2,\infty)}\sum_{m=0}^{\infty}\sum_{j\in I(m)} j^2\left(\int_0^1 t^j|f(t)|\,dt\right)^2 v_{2j-1}\\
&\leq C\|g-g(0)\|^2_{\mathcal{B}(2,\infty)}\|\widetilde{\mathcal{H}}(f)\|^2_{\mathcal{D}_v}.
\end{aligned}
$$

By Corollary 4, the last quantity is less or equal than $C\|g-g(0)\|^2_{\mathcal{B}(2,\infty)}\|f\|^2_{\mathcal{D}_v}$, showing the validity of (3.2).

Moreover, using Corollary 4 again, the remaining terms in (3.1) involving $\hat{g}(1)$ and $\hat{g}(2)$ can easily be controlled by $\|g-g(0)\|^2_{\mathcal{B}(2,\infty)}\|f\|^2_{\mathcal{D}_v}$. This together with (3.1) and (3.2), implies that $\mathcal{H}_g$ is bounded on $\mathcal{D}_v$ with

$$\|\mathcal{H}_g\| \lesssim \|g-g(0)\|_{\mathcal{B}(2,\infty)}.$$

Reciprocally, assume that $\mathcal{H}_g$ is bounded on $\mathcal{D}_v$. For each $N\in\mathbb{N}$, denote $a_N = 1-2^{-N}$ and consider the function f_N defined, for $z\in\mathbb{D}$ as follows:

$$f_N(z) = (1-a_N)^{\frac{\lambda}{2}}\widehat{v}(a_N)^{-1/2}(1-a_N z)^{\frac{1-\lambda}{2}},$$

By Lemma 5(ii)(vii), $\lambda>0$ can be choosen big enough so that

$$\sup_N \|f_N\|_{\mathcal{D}_v} < \infty. \tag{3.3}$$

We are going to see now that

$$\|\mathcal{H}_g(f_N)\|^2_{\mathcal{D}_v} \geq C2^{-N}\sum_{j\in I(N)}\widetilde{g_{j+1}}, \quad N\in\mathbb{N}.$$

By (3.1), the left hand side above is larger or equal than

$$
\begin{aligned}
&\sum_{n=0}^{\infty} 2^{2n}\left(\int_0^1 t^{2^{n+1}} f_N(t)\,dt\right)^2 v_{2^{n+2}-1}\sum_{j\in I(n)}\widetilde{g_{j+1}}\\
&\geq 2^{2N}\left(\int_0^1 t^{2^{N+1}} f_N(t)\,dt\right)^2 v_{2^{N+2}-1}\sum_{j\in I(N)}\widetilde{g_{j+1}}\\
&\geq C2^{2N}\left(\int_{a_N}^1 t^{2^{N+1}} f_N(t)\,dt\right)^2 \widehat{v}(a_N)\sum_{j\in I(N)}\widetilde{g_{j+1}}\\
&\geq C2^{-N}\sum_{j\in I(N)}\widetilde{g_{j+1}}
\end{aligned}
$$

which together with (3.3) and the inequality $\int_0^1 f_0(t)\,dt \geq C > 0$ gives that

$$\|g - g(0)\|_{\mathcal{B}(2,\infty)} \lesssim \|\mathcal{H}_g\|.$$

□

Now, we deal with the compactness of generalized Hilbert operators $\mathcal{H}_g$.

Proposition 10. *Let $g \in H(\mathbb{D})$ and $v \in \widehat{\mathcal{D}}$ which satisfies the conditions (1.3) and (1.4). Then $\mathcal{H}_g$ is compact on $\mathcal{D}_v$ if and only $g \in b(2,\infty)$.*

We will need the following lemma, which can be easily proved by using (1.2), Hölder's inequality and (2.4).

Lemma 11. *Let v be a radial weight such that (1.2) and (1.3) are satisfied. Let $\{f_j\}_{j=1}^{\infty}$ be a sequence in $\mathcal{D}_v$ such that $\sup_j \|f_j\|_{\mathcal{D}_v} = K < \infty$ and $f_j \to 0$, as $j \to \infty$, uniformly on compact subsets of $\mathbb{D}$. Then the following assertions hold:*

(i) $\lim_{j\to\infty} \int_0^1 |f_j(t)|\,dt = 0$;
(ii) *$\mathcal{H}_g(f_j) \to 0$, as $j \to \infty$, uniformly on compact subsets of $\mathbb{D}$ for each $g \in H(\mathbb{D})$.*

We also remind the reader that the norm convergence in $\mathcal{D}_v$, $v \in \widehat{\mathcal{D}}$, implies the uniform convergence on compact subsets of $\mathbb{D}$ by [13, Lemma 3.2]. With these tools and from the proof of Theorem 9, Proposition 10 can be shown using standard techniques. Therefore, its proof will be omitted. See [7, Section 7] or [11, Section 7] for further details.

3.2. **Hilbert-Schmidt operators.** First, we observe that

$$\mathcal{B}(2,2) \subset \mathcal{D}_v \quad \text{if } v \in \widehat{\mathcal{D}} \text{ satisfies that } \int_0^1 \widehat{V_2}(s)\,ds < \infty\,. \tag{3.4}$$

In fact, by Lemma 5(vi)

$$\begin{aligned}\sup_{j\in\mathbb{N}}(j+1)v_{2j+1} &\asymp \sup_{j\in\mathbb{N}}(j+1)\widehat{v}\left(1-\frac{1}{2j+1}\right)\\ &\asymp \sup_{j\in\mathbb{N}}\int_{1-\frac{1}{j+1}}^{1-\frac{1}{j+2}} \widehat{V_2}(s)\,ds \leq \int_0^1 \widehat{V_2}(s)\,ds < \infty,\end{aligned}$$

which implies (3.4).

Proposition 12. *Let $g \in H(\mathbb{D})$ and $v \in \widehat{\mathcal{D}}$ which satisfies the conditions (1.3) and (1.4). Then $\mathcal{H}_g \in S_2(\mathcal{D}_v)$ if and only if $g \in \mathcal{B}(2,2)$. Moreover,*

$$\|\mathcal{H}_g\|_{S_2(\mathcal{D}_v)} \asymp \|g - g(0)\|_{\mathcal{B}(2,2)}.$$

Proof. Denote $e_0(z) = 1$,

$$e_n(z) = \frac{z^n}{\|z^n\|_{\mathcal{D}_v}} = \frac{z^n}{\sqrt{2n^2 v_{2n-1}}}, \quad n \in \mathbb{N}\backslash\{0\},$$

and consider the basis $\{e_n\}_{n\in\mathbb{N}}$ of $\mathcal{D}_v$. If $g(z)=\sum_0^\infty \widehat{g}(k)z^k \in H(\mathbb{D})$, since $\mathcal{H}_g(e_0)(z)=\frac{g(z)-g(0)}{z}$, by (3.4)

$$\|\mathcal{H}_g(e_0)\|^2_{\mathcal{D}_v} \asymp \|g-g(0)\|^2_{\mathcal{D}_v} \lesssim \|g-g(0)\|^2_{\mathcal{B}(2,2)}.$$

On the other hand,

$$\begin{aligned}\|\mathcal{H}_g(e_n)\|^2_{\mathcal{D}_v} &= \frac{|\widehat{g}(1)|^2}{2(n+1)^2n^2v_{2n-1}} \\ &\quad + \frac{1}{2n^2v_{2n-1}}\sum_{k=1}^\infty \frac{k^2\widetilde{g_{k+1}}}{(n+k+1)^2}v_{2k-1}, \quad n\in\mathbb{N}.\end{aligned} \tag{3.5}$$

Clearly,

$$\begin{aligned}\sum_{n=1}^\infty \frac{1}{n^2(n+k+1)^2v_{2n-1}} &\gtrsim \frac{1}{v_{2k+1}}\sum_{m=1}^\infty \sum_{n=m(k+1)}^{(m+1)(k+1)} \frac{1}{n^2(n+k+1)^2} \\ &\asymp \frac{1}{(k+1)^3v_{2k+1}}, \quad k\in\mathbb{N}.\end{aligned}$$

The opposite inequality also holds, since $v\in\widehat{\mathcal{D}}$, from Lemma 5(vi) and (1.3), we get

$$\begin{aligned}\sum_{n=1}^k \frac{1}{n^2(n+k+1)^2v_{2n-1}} &\asymp \frac{1}{(k+1)^2}\sum_{n=1}^k \frac{1}{\widehat{v}\left(1-\frac{1}{2n-1}\right)}\int_{1-\frac{1}{n}}^{1-\frac{1}{(n+1)}} ds \\ &\asymp \frac{1}{(k+1)^2}\int_0^{1-\frac{1}{(k+1)}} \frac{1}{\widehat{v}\,(s)}\,ds \\ &\lesssim \frac{1}{(k+1)^2\int_{1-\frac{1}{(k+1)}}^{1-\frac{1}{(k+2)}} \widehat{V}_2(s)\,ds} \asymp \frac{1}{(k+1)^3v_{2k}}, \quad k\in\mathbb{N}.\end{aligned}$$

For the rest of the values of n, using again Lemma 5, and (1.4)

$$\begin{aligned}\sum_{n=k+1}^\infty \frac{1}{n^2(n+k+1)^2v_{2n-1}} &\asymp \sum_{n=k+1}^\infty \frac{1}{n^2v_{2n-1}}\int_{1-\frac{1}{n}}^{1-\frac{1}{(n+1)}} ds \asymp \int_{1-\frac{1}{k+1}}^1 \frac{1}{\widehat{V}_2(s)}\,ds \\ &\lesssim \frac{1}{\int_0^{1-\frac{1}{k+1}} \widehat{V}_4(s)\,ds} \lesssim \frac{1}{(k+1)^3v_{2k}}, \quad k\in\mathbb{N}.\end{aligned}$$

From here, using Lemma 5(iv) and (3.5), we deduce

$$\begin{aligned}\sum_{n=1}^\infty \|\mathcal{H}_g(e_n)\|^2_{\mathcal{D}_v} &= |\widehat{g}(1)|^2\sum_{n=1}^\infty \frac{1}{2(n+1)^2n^2v_{2n-1}} \\ &\quad + \sum_{k=1}^\infty k^2\widetilde{g_{k+1}}v_{2k-1}\sum_{n=1}^\infty \frac{1}{2n^2(n+k+1)^2v_{2n-1}} \\ &\asymp |\widehat{g}(1)|^2 + \sum_{k=1}^\infty (k+1)|\hat{g}(k+1)|^2 \asymp \|g-g(0)\|_{\mathcal{B}(2,2)},\end{aligned}$$

proving our assertion. □

3.3. **Proof of Theorem 1. (ii) ⇒ (i).** In order to obtain this implication in Theorem 1 we use results from [10] on complex interpolation for the mixed norm spaces $\mathcal{B}(2,p)$ and from [14, Theorem 2.6] for Schatten classes. Given (X_0, X_1) a compatible pair of Banach spaces, we denote by $(X_0, X_1)_{[\theta]}$ the complex interpolating space of exponent $\theta \in [0,1]$. With the notation from [10] for D and $A^{p,q}_{\delta,k}$, if one chooses the particular case $D = \mathbb{D}$ in [10], then $\mathcal{B}(2,q) = A^{2,q}_{1,1}$. In this way, the following result on complex interpolation on the mixed norm space $\mathcal{B}(2,q)$ is a consequence of [10, Theorem 3.1] (see also [10, Theorem B]).

Theorem A. *Let $0 < q_0 < q_1 \leq \infty$ and $\theta \in (0,1)$. If $\frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}$, then*

$$(\mathcal{B}(2,q_0), \mathcal{B}(2,q_1))_{[\theta]} = \mathcal{B}(2,q).$$

Proposition 13. *Let $2 < p < \infty$ and $v \in \widehat{\mathcal{D}}$ satisfying (1.3) and (1.4). If $g \in \mathcal{B}(2,p)$, then $\mathcal{H}_g \in S_p(\mathcal{D}_v)$ and $\|\mathcal{H}_g\|_{S_p(\mathcal{D}_v)} \lesssim \|g - g(0)\|_{\mathcal{B}(2,p)}$.*

Proof. Let us consider the linear operator $T(g) = \mathcal{H}_g$. By Proposition 12 the operator T is bounded from $\mathcal{B}(2,2)$ to $S_2(\mathcal{D}_v)$ with

$$\|T(g)\|_{S_2(\mathcal{D}_v)} = \|\mathcal{H}_g\|_{S_2(\mathcal{D}_v)} \asymp \|g - g(0)\|_{\mathcal{B}(2,2)}.$$

Analogously, by Proposition 9, T is bounded from $\mathcal{B}(2,\infty)$ to $S_\infty(\mathcal{D}_v)$ with

$$\|T(g)\|_{S_\infty(\mathcal{D}_v)} = \|\mathcal{H}_g\|_{S_\infty(\mathcal{D}_v)} \asymp \|g - g(0)\|_{\mathcal{B}(2,\infty)}.$$

So, the previous inequalities together with [2, Theorem 4.1.2, p. 88], Theorem A and [14, Theorem 2.6] imply that $T : \mathcal{B}(2,p) \to S_p(\mathcal{D}_v)$ is bounded. This finishes the proof. □

In order to deal with the case $0 < p < 2$, we will need two technical lemmas.

Lemma 14. *Let $v \in \widehat{\mathcal{D}}$ satisfying (1.4). Then, for any $q > 0$ there is a constant $C(q,v)$ such that*

$$\sum_{n=k}^{\infty} 2^{-3qn} v_{2^{n+1}}^{-q} \leq C(q,v) 2^{-3qk} v_{2^{k+1}}^{-q}$$

for all $k \in \mathbb{N} \cup \{0\}$.

Proof. First, we prove that there exists $\gamma = \gamma(v) \in (0,1)$ such that

$$\int_{\frac{1+r}{2}}^{1} \frac{(1-s)^2}{\widehat{v}(s)}\, ds \leq \gamma \int_r^1 \frac{(1-s)^2}{\widehat{v}(s)}\, ds, \quad 0 < r < 1. \tag{3.6}$$

Taking into account that $v \in \widehat{\mathcal{D}}$ and (1.4)

(3.7)
$$\int_{\frac{1+r}{2}}^{1} \frac{(1-s)^2}{\widehat{v}(s)}\,ds \le C(v)\left(\int_0^{\frac{1+r}{2}} \widehat{V}_4(s)\,ds\right)^{-1}$$
$$\le C(v)\frac{(1-r)^3}{\widehat{v}\left(\frac{1+r}{2}\right)} \le C(v)\int_r^{\frac{1+r}{2}} \frac{(1-s)^2}{\widehat{v}(s)}\,ds, \quad 0<r<1,$$

which is equivalent to (3.6). Then, Lemma 5 and (3.7) yield

$$\sum_{n=k}^{\infty} 2^{-3qn} v_{2^{n+1}}^{-q} \le C(q,v)\sum_{n=k}^{\infty}\left(\frac{1}{\widehat{v}\left(1-\frac{1}{2^{n+1}}\right)}\int_{1-\frac{1}{2^{n+1}}}^{1}(1-s)^2\,ds\right)^q$$
$$\le C(q,v)\left(\int_{1-\frac{1}{2^{k+1}}}^{1}\frac{(1-s)^2}{\widehat{v}(s)}\,ds\right)^q \sum_{n=k}^{\infty}\gamma^{n-k}.$$

Since the last sum is convergent, all of the above is controlled by

$$C(q,v)\left(\frac{1}{2^{3k}\widehat{v}\left(1-\frac{1}{2^{k+1}}\right)}\right)^q \le C(q,v)\left(\frac{1}{2^{3k}v_{2^{k+1}}}\right)^q, \quad k\in\mathbb{N}\cup\{0\}.$$

□

Lemma 15. *Let $v \in \widehat{\mathcal{D}}$ satisfying (1.3). Then, for any $q>0$ there is a constant $C(q,v)$ such that*

$$\sum_{n=1}^{k} 2^{-qn} v_{2^{n+1}}^{-q} \le C(q,v) 2^{-qk} v_{2^{k+1}}^{-q}$$

for all $k \in \mathbb{N}$.

Proof. Now, we prove that there exists $\gamma = \gamma(v) \in (0,1)$ such that

(3.8)
$$\int_{\frac{1+r}{2}}^{1} \widehat{V}_2(s)\,ds \le \gamma \int_r^1 \widehat{V}_2(s)\,ds, \quad 0<r<1.$$

By (1.3)

(3.9)
$$\int_{\frac{1+r}{2}}^{1} \widehat{V}_2(s)\,ds \le \frac{C(v)}{\int_0^{\frac{1+r}{2}}\frac{1}{\widehat{v}(s)}\,ds} \le \frac{C(v)}{\int_r^{\frac{1+r}{2}}\frac{1}{\widehat{v}(s)}\,ds}$$
$$\le C(v)\frac{\widehat{v}(r)}{1-r} \le C(v)\int_r^{\frac{1+r}{2}} \widehat{V}_2(s)\,ds, \quad 0<r<1,$$

which is equivalent to (3.8). So, by (3.9) and Lemma 5

$$\begin{aligned}\sum_{n=1}^{k} 2^{-qn} v_{2^{n+1}}^{-q} &\le C(q,v) \sum_{n=1}^{k} 2^{-qn} \widehat{v}\left(1-\frac{1}{2^{n+1}}\right)^{-q} \\ &\le C(q,v) \sum_{n=1}^{k} \left(\int_{1-\frac{1}{2^{n+1}}}^{1} \widehat{V_2}(s)\,ds\right)^{-q} \\ &\le C(q,v) \left(\int_{1-\frac{1}{2^{k+1}}}^{1} \widehat{V_2}(s)\,ds\right)^{-q} \sum_{n=1}^{k} \gamma^{k-n} \\ &\le C(q,v) \left(\int_{1-\frac{1}{2^{k+1}}}^{1-\frac{1}{2^{k+2}}} \widehat{V_2}(s)\,ds\right)^{-q} \\ &\le C(q,v) 2^{-qk} \widehat{v}\left(1-\frac{1}{2^{k+1}}\right)^{-q} \le C(q,v) 2^{-qn} v_{2^{k+1}}^{-q}, \quad k \in \mathbb{N}.\end{aligned}$$

□

Now we are ready to prove the remaining case of (ii)⇒(i) in Theorem 1.

Proposition 16. *Let* $v \in \widehat{\mathcal{D}}$ *satisfying* (1.3) *and* (1.4)*. If* $0 < p < 2$ *and* $g \in \mathcal{B}(2,p)$*, then* $\mathcal{H}_g \in \mathcal{S}_p(\mathcal{D}_v)$ *and* $\|\mathcal{H}_g\|_{S_p(\mathcal{D}_v)} \lesssim \|g-g(0)\|_{\mathcal{B}(2,p)}$.

Proof. Let $g(z) = \sum_{k=0}^{\infty} \widehat{g}(k) z^k \in \mathcal{B}(2,p)$. We use the orthonormal basis $\{e_n\}_{n=0}^{\infty}$, where

$$e_0(z) = 1 \quad \text{and} \quad e_n(z) = \frac{\sum_{k+1\in I(n)} z^k}{\left(\sum_{k+1\in I(n)} k^2 v_{2k-1}\right)^{1/2}}, \quad n \in \mathbb{N}. \tag{3.10}$$

Since $0 < p \le 2$, $\mathcal{B}(2,p) \subset \mathcal{B}(2,2)$. Thus, by Proposition 12, $\mathcal{H}_g \in S_2(\mathcal{D}_v)$, and in particular $\mathcal{H}_g$ is a compact operator. Therefore, by [14, Theorem 1.26] and [14, Corollary 1.32] (applied to $\mathcal{H}_g^\star \mathcal{H}_g$)

$$\|\mathcal{H}_g\|_{S^p(\mathcal{D}_v)}^p = \|\mathcal{H}_g^\star \mathcal{H}_g\|_{S^{\frac{p}{2}}(\mathcal{D}_v)}^{\frac{p}{2}} \le \sum_{n=0}^{\infty} \langle \mathcal{H}_g e_n, \mathcal{H}_g e_n\rangle_{\mathcal{D}_v}^{\frac{p}{2}}.$$

Since $\mathcal{H}_g(e_0)(z) = \frac{g(z)-g(0)}{z}$, by (3.4)

$$\|\mathcal{H}_g(e_0)\|_{\mathcal{D}_v}^2 \asymp \|g-g(0)\|_{\mathcal{D}_v}^2 \lesssim \|g-g(0)\|_{\mathcal{B}(2,p)}^2.$$

Moreover, for $n \in \mathbb{N}$

$$\begin{aligned}\mathcal{H}_g(e_n)(z) &= \sum_{j=0}^{\infty} (j+1)\hat{g}(j+1) \left(\int_0^1 t^j e_n(t)\,dt\right) z^j \\ &= \left(\sum_{k+1\in I(n)} (k+1)^2 v_{2k+1}\right)^{-1/2} \sum_{j=0}^{\infty} (j+1)\hat{g}(j+1) \left(\sum_{m+1\in I(n)} \frac{1}{m+j+1}\right) z^j.\end{aligned}$$

So, it is enough to prove

$$\sum_{n=0}^{\infty} \langle \mathcal{H}_g e_n, \mathcal{H}_g e_n \rangle_{\mathcal{D}_v}^{\frac{p}{2}} \lesssim \|g - g(0)\|_{\mathcal{B}(2,p)}^p. \tag{3.11}$$

By Lemma 5(v), $\sum_{k+1\in I(n)} (k+1)^2 v_{2k+1} \asymp 2^{3n} v_{2^{n+1}}$, yielding

$$\begin{aligned} \sum_{n=1}^{\infty} \langle \mathcal{H}_g e_n, \mathcal{H}_g e_n \rangle_{\mathcal{D}_v}^{\frac{p}{2}} &\lesssim \sum_{n=1}^{\infty} 2^{-\frac{3pn}{2}} v_{2^{n+1}}^{-\frac{p}{2}} \left| \widehat{g}(1) \sum_{m+1\in I(n)} \frac{1}{m+1} \right|^p \\ &+ \sum_{n=1}^{\infty} 2^{-\frac{3pn}{2}} v_{2^{n+1}}^{-\frac{p}{2}} \left(\sum_{j=1}^{\infty} j^2 \widetilde{g_{j+1}} \left(\sum_{m+1\in I(n)} \frac{1}{m+j+1} \right)^2 v_{2j-1} \right)^{\frac{p}{2}} \\ &\leq J_1 + J_2 + J_3, \end{aligned} \tag{3.12}$$

where J_1 is the first sum on the right hand side, and the second sum is decomposed in $j \leq 2^{n+1} - 1$ (J_2) and $j \geq 2^{n+1}$ (J_3). By Lemma 14,

$$J_1 \lesssim |\widehat{g}(1)|^p \sum_{n=1}^{\infty} 2^{-\frac{3pn}{2}} v_{2^{n+1}}^{-\frac{p}{2}} \lesssim \|g - g(0)\|_{\mathcal{B}(2,p)}^p. \tag{3.13}$$

We now estimate for J_2, which satisfies

$$J_2 \asymp \sum_{n=1}^{\infty} 2^{-\frac{3pn}{2}} v_{2^{n+1}}^{-\frac{p}{2}} \left(\sum_{k=0}^{n} \sum_{j\in I(k)} j^2 \widetilde{g_{j+1}} v_{2j-1} \right)^{\frac{p}{2}}. \tag{3.14}$$

Using Lemma 5 and Lemma 14, the above can be bounded by

$$\begin{aligned} &\sum_{n=1}^{\infty} 2^{-\frac{3pn}{2}} v_{2^{n+1}}^{-\frac{p}{2}} \left(\sum_{k=0}^{n} 2^{2k} v_{2^{k+1}} \sum_{j\in I(k)} \widetilde{g_{j+1}} \right)^{\frac{p}{2}} \\ &\lesssim \sum_{n=1}^{\infty} 2^{-\frac{3pn}{2}} v_{2^{n+1}}^{-\frac{p}{2}} \sum_{k=0}^{n} 2^{pk} v_{2^{k+1}}^{p/2} \left(\sum_{j\in I(k)} \widetilde{g_{j+1}} \right)^{\frac{p}{2}} \\ &\lesssim \sum_{k=0}^{\infty} 2^{pk} v_{2^{k+1}}^{p/2} \left(\sum_{j\in I(k)} \widetilde{g_{j+1}} \right)^{\frac{p}{2}} \sum_{n=k}^{\infty} 2^{-\frac{3pn}{2}} v_{2^{n+1}}^{-\frac{p}{2}} \\ &\lesssim \sum_{k=0}^{\infty} 2^{-\frac{pk}{2}} \left(\sum_{j\in I(k)} \widetilde{g_{j+1}} \right)^{\frac{p}{2}} \lesssim \|g - g(0)\|_{\mathcal{B}(2,p)}^p. \end{aligned} \tag{3.15}$$

To complete the estimation, we turn to J_3, which satisfies

$$J_3 \asymp \sum_{n=1}^{\infty} 2^{-\frac{pn}{2}} v_{2^{n+1}}^{-\frac{p}{2}} \left(\sum_{k=n+1}^{\infty} 2^{-2k} \sum_{j\in I(k)} j^2 \widetilde{g_{j+1}} v_{2j-1} \right)^{\frac{p}{2}}.$$

By Lemma 5 and Lemma 15, then J_3 is controlled by

$$\begin{aligned}
&\sum_{n=1}^{\infty} 2^{-\frac{pn}{2}} v_{2^{n+1}}^{-\frac{p}{2}} \left(\sum_{k=n+1}^{\infty} v_{2^{k+1}} \sum_{j\in I(k)} \widetilde{g_{j+1}} \right)^{\frac{p}{2}} \\
&\lesssim \sum_{n=1}^{\infty} 2^{-\frac{pn}{2}} v_{2^{n+1}}^{-\frac{p}{2}} \sum_{k=n+1}^{\infty} v_{2^{k+1}}^{p/2} \left(\sum_{j\in I(k)} \widetilde{g_{j+1}} \right)^{\frac{p}{2}} \\
&= \sum_{k=2}^{\infty} v_{2^{k+1}}^{p/2} \left(\sum_{j\in I(k)} \widetilde{g_{j+1}} \right)^{\frac{p}{2}} \sum_{n=1}^{k-1} 2^{-\frac{pn}{2}} v_{2^{n+1}}^{-\frac{p}{2}} \\
&\lesssim \sum_{k=2}^{\infty} 2^{-\frac{kp}{2}} \left(\sum_{j\in I(k)} \widetilde{g_{j+1}} \right)^{\frac{p}{2}} \lesssim \|g-g(0)\|^p_{\mathcal{B}(2,p)}.
\end{aligned}$$

This together with (3.12), (3.13), (3.14) and (3.15) gives (3.11). □

3.4. **Proof of Theorem 1.** (**i**) ⇒ (**ii**). In order to show the remaining part of the proof of Theorem 1 we will employ [14, Theorem 1.28].

Proposition 17. *Let* $g \in H(\mathbb{D})$ *and* $v \in \widehat{\mathcal{D}}$ *satisfying* (1.3) *and* (1.4)*. If* $1 \le p < \infty$ *and* $\mathcal{H}_g \in \mathcal{S}_p(\mathcal{D}_v)$*, then* $g \in \mathcal{B}(2,p)$ *and*

$$\|g-g(0)\|_{\mathcal{B}(2,p)} \lesssim \|\mathcal{H}_g\|_{S_p(\mathcal{D}_v)}.$$

Proof. Let $\{e_n\}_{n=0}^{\infty}$ be the orthonormal basis defined in (3.10). We write $M_0 = |\hat{g}(1)|$ and

$$M_n = M_n(g,v) = \left(\sum_{k+1\in I(n)} k^2 \widetilde{g_{k+1}} v_{2k-1} \right)^{1/2}, \quad n \in \mathbb{N}.$$

Let us consider $N_g = \{n \in \mathbb{N} : M_n \neq 0\}$ and

$$\sigma_n = M_n^{-1} \sum_{k+1\in I(n)} (k+1)\widehat{g}(k+1) z^k, \quad n \in N_g.$$

The set $\{\sigma_n\}_{n\in N_g}$ is orthonormal. Denote now $h_k := k^2 v_{2k-1}\widetilde{g_{k+1}}$. By Lemma 5,

$$
\begin{aligned}
&|\langle \mathcal{H}_g e_n, \sigma_n\rangle_{\mathcal{D}_v}| \\
&= \left(\sum_{k+1\in I(n)} k^2 v_{2k-1}\right)^{-\frac{1}{2}} M_n^{-1} \sum_{k+1\in I(n)} h_k \left(\sum_{m+1\in I(n)} \frac{1}{m+k+1}\right) \\
&\asymp \left(\sum_{k+1\in I(n)} k^2 v_{2k-1}\right)^{-\frac{1}{2}} \left(\sum_{k+1\in I(n)} h_k\right)^{\frac{1}{2}} \\
&\asymp \left(2^{3n} v_{2^{n+1}}\right)^{-\frac{1}{2}} \left(\sum_{k+1\in I(n)} h_k\right)^{\frac{1}{2}} \asymp \left(2^{-n} \sum_{k+1\in I(n)} \widetilde{g_{k+1}}\right)^{\frac{1}{2}}, \quad n\in N_g,\ n\geq 1.
\end{aligned}
$$

Hence, by [14, Theorem 1.28]

$$
\begin{aligned}
\infty > \|\mathcal{H}_g\|^p_{S_p(\mathcal{D}_v)} &\geq \sum_{n\in N_g} |\langle \mathcal{H}_g e_n, \sigma_n\rangle_{\mathcal{D}_v}|^p \\
&\geq \sum_{n\in N_g} \left(2^{-n} \sum_{k+1\in I(n)} \widetilde{g_{k+1}}\right)^{\frac{p}{2}} \gtrsim \|g-g(0)\|^p_{\mathcal{B}(2,p)}.
\end{aligned}
$$

□

Finally, Theorem 1 follows from Propositions 9, 12, 13, 16 and 17.

3.5. **Proof of Corollary 2.** Since $\omega \in \widehat{\mathcal{D}}$, by [12, Theorem 4.2] and Lemma 5,

$$
\begin{aligned}
\|f\|^2_{A^2_\omega} &= |f(0)|^2\omega(\mathbb{D}) + \|f'\|^2_{A^2_{\omega^*}} \\
&\asymp |f(0)|^2 + \int_{\mathbb{D}} |f'(z)|^2(1-|z|)\widehat{\omega}(|z|)\,dA(z) \qquad (3.16)\\
&= \|f\|^2_{\mathcal{D}_v}, \qquad f\in H(\mathbb{D}),
\end{aligned}
$$

where $v(|z|) = (1-|z|)\omega(|z|)$. Since $\omega \in \widehat{\mathcal{D}}$, we have that $v \in \widehat{\mathcal{D}}$ and $\widehat{v}(|z|) \asymp (1-|z|)^2\omega(|z|)$, $z\in\mathbb{D}$. So, using that $\widehat{\omega}$ is a non-decreasing function

$$
\begin{aligned}
&\sup_{0<r<1} \left(\int_r^1 \frac{\widehat{v}(s)}{(1-s)^2}\,ds\right)\left(\int_0^r \frac{1}{\widehat{v}(s)}\,ds\right) \\
&\asymp \sup_{0<r<1} \left(\int_r^1 \widehat{\omega}(s)\,ds\right)\left(\int_0^r \frac{1}{(1-s)^2\widehat{\omega}(s)}\,ds\right) < \infty.
\end{aligned}
$$

Moreover, by (1.5)

$$\begin{aligned}
&\sup_{0<r<1}\left(\int_0^r \frac{\widehat{v}(s)}{(1-s)^4}\,ds\right)^{\frac{1}{2}}\left(\int_r^1 \frac{(1-s)^2}{\widehat{v}(s)}\,ds\right)^{\frac{1}{2}}\\
&\asymp \sup_{0<r<1}\left(\int_0^r \frac{\widehat{\omega}(s)}{(1-s)^2}\,ds\right)^{\frac{1}{2}}\left(\int_r^1 \frac{1}{\widehat{\omega}(s)}\,ds\right)^{\frac{1}{2}}<\infty.
\end{aligned}$$

Therefore, v satisfies both conditions, (1.3) and (1.4). This together with (3.16) and Theorem 1, finishes the proof.

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