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A z^k -INVARIANT SUBSPACE WITHOUT THE WANDERING PROPERTY

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ABSTRACT. We study operators of multiplication by z^k in Dirichlet-type spaces D_α . We establish the existence of k and α for which some z^k -invariant subspaces of D_α do not satisfy the wandering property. As a consequence of the proof, any Dirichlet-type space accepts an equivalent norm under which the wandering property fails for some space for the operator of multiplication by z^k , for any $k \geq 6$.

1. INTRODUCTION

In the present paper we will be concerned with (closed) subspaces of so-called Dirichlet-type spaces, that remain invariant under the action of the operator of multiplication by z^k , for some $k \in \mathbb{N}$.

Definition 1.1. Let $\alpha \in \mathbb{R}$. We denote by D_α the *Dirichlet-type space* with parameter α , defined as

$$D_\alpha = \left\{ f \in \text{Hol}(\mathbb{D}) : f(z) = \sum_{k=0}^{\infty} a_k z^k, \right. \\ \left. \|f\|_\alpha^2 := \sum_{k=0}^{\infty} |a_k|^2 (k+1)^\alpha < \infty \right\}.$$

The particular case when $\alpha = -1$ is denoted by A^2 , and is often referred to as *the Bergman space*. It consists of all holomorphic functions over the unit disc \mathbb{D} with square integrable modulus with respect to the normalized Lebesgue area measure (with density denoted $dA(z)$). For a function f with Maclaurin coefficients $\{a_k\}_{k \in \mathbb{N}}$, the norm satisfies the identity

$$\|f\|_{-1}^2 = \sum_{k=0}^{\infty} \frac{|a_k|^2}{k+1} = \int_{\mathbb{D}} |f(z)|^2 dA(z).$$

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We refer to [9, 15] for further information on this space. Other relevant values of α are 0 and 1, when the spaces are, respectively, the Hardy space H^2 and the Dirichlet space D . See [8, 13] for information on H^2 and [3, 10, 21] for information on D .

The shift S is the operator taking a function f in D_α to the function $Sf(z) = zf(z)$. The invariant subspaces for the shift operator have attracted much interest from specialists, in part because of the connections with the Hilbert invariant subspace problem (see [1, 19]). We denote by M_g the operator of multiplication by an analytic function g . If M_g is bounded we say that g is a multiplier (of D_α). From now on, if the value of α is fixed, we denote $\{U\}_g$ the smallest closed subspace of D_α containing the set of functions U which is invariant under the action of the operator M_g . For instance, both A^2 and H^2 have the space of bounded analytic functions with the supremum norm as their space of multipliers, while D_α spaces for $\alpha > 1$ are closed under multiplication. Neither of this is true for D_α whenever $0 < \alpha \leq 1$. We say that U generates M (under g) if $\{U\}_g = M$. The present article deals with invariant subspaces under the action of $M_{z^k} = S^k$ for some $k \in \mathbb{N}$, and in particular with the question of whether or not some special subsets generate or not the whole subspace.

In [2], the authors found the correct generalization for A^2 of a classical result of Beurling for the Hardy space H^2 (see [4]):

Theorem 1.2 (Aleman, Richter, Sundberg). *Let $M \subset A^2$ be a closed subspace invariant under S . Then*

$$M = \{M \ominus SM\}_z.$$

This is, however, a more intricate situation than that of H^2 where each invariant subspace is generated by a single function. Examples of invariant subspaces not generated by a single element can be found in [14]. Theorem 1.2 justifies the following definition:

Definition 1.3. Let g be a multiplier in D_α , and let $M \subset D_\alpha$ be a closed subspace that is invariant under the operator M_g . We say that M has the *wandering property* (relative to g in D_α) if

$$M = \{M \ominus gM\}_g.$$

Shimorin ([22]) generalized Aleman-Richter-Sundberg's result to the shift in Dirichlet-type spaces for $-1 \leq \alpha \leq 1$ and to other particular operators. See also [20]. In the paper [18], the failure of the wandering property is shown for the shift in the spaces D_α for $\alpha \leq -5$, although these spaces are equipped with a different equivalent norm, arising from an integral representation. The result for $\alpha < -5$ is an application of

other results by [16]. In the interesting articles [17, 6], the authors consider the question of generalizing Theorem 1.2 to other bounded multiplication operators in H^2 and A^2 (respectively). In the latter, the following problem is proposed:

Problem 1.4. *Let $g \in H^\infty$ and consider M_g acting on A^2 . Assume $M \subset A^2$ is a closed subspace invariant under M_g . Is it true then that M satisfies the wandering property?*

The question has been studied in several papers including [6] and [7], and even though some counterexamples have been found, the problem remains open even for the case when $g(z) = z^k$ for some $k \in \mathbb{N}$, $k \geq 2$. The latter is mentioned specifically as an open problem, already on [6].

The purpose of this article is to construct z^k -invariant subspaces of D_α spaces that do not satisfy the wandering property. By doing this, we solve an analogue of Problem 1.4 in those spaces. Our main result is the following:

Theorem 1.5. *There exists $\epsilon > 0$ such that for all $\alpha \in (-16 - \epsilon, -16 + \epsilon)$ there exists a closed subspace $M \subset D_\alpha$ invariant under multiplication by z^6 , but without the wandering property.*

Notice that the wandering property depends a priori on the choice of equivalent norm for the ambient space D_α , even though the property of invariance is independent of this choice. This is in fact, a key feature and as a corollary of the proof of our main theorem, we will obtain the following result:

Corollary 1.6. *Let $k \geq 6$. There exist two polynomials F_1 and F_2 such that for each value of α , the space D_α admits a choice of equivalent norm under which the invariant subspace generated by F_1 and F_2 under multiplication by z^k does not have the wandering property.*

We will complete this with a proof that the behavior for large enough Dirichlet-type spaces (with their usual norms) is consistent with this:

Theorem 1.7. *Let $k \geq 10$, $k \in \mathbb{N}$. For each $\alpha < -(5k + \frac{700}{(k-9)^2})$, D_α contains a z^k invariant subspace without the wandering property.*

Although this result may seem stronger than Theorem 1.5, we consider the techniques used to establish Theorem 1.5 to be more capable of dealing with more general problems. Our numerical evidence suggests that the optimal threshold for each $k \geq 10$ (and in fact, $k \geq 6$) is much higher than that of Theorem 1.7, probably increasing with k , and staying somewhere between -5 and 0 for all values of k .

Notice that in H^2 with the classical norm, the z^k wandering property holds for $k \geq 1$, since the powers of the shift all behave like the shift itself (for instance, Shimorin's proof for $k = 1$ works for all k).

Remark 1.8. In the forthcoming paper [11], we will show among other results that for each $k \geq 1$ and each negative $\alpha \geq \frac{\log(2)}{\log(k+1)}$, the D_α spaces with their usual norms do have the z^k wandering property. In fact, for any $\alpha < 0$ and $k \geq 1$, there exists a choice of optimal norm under which the wandering property holds for z^k . Therefore the norm dependence is absolutely relevant to the problem.

In Section 2, we provide sufficient conditions on a pair of polynomial functions F_1 and F_2 and on the ambient space D_α for the subspace M generated by F_1 and F_2 to satisfy the claim in Theorem 1.5. Then, in Section 3, we show how to arrive to a choice of F_1 and F_2 that satisfies such sufficient conditions, for the case $\alpha = -16$ and $k = 6$. One of the conditions will be open and depend continuously on the parameters, and as a consequence there will exist an interval of possible values of α for which the same counterexample will work, therefore proving Theorem 1.5. We also provide, in Section 3.5, the proof of Corollary 1.6 and, in Section 3.6, that of Theorem 1.7. The values of α and k above may seem arbitrary but the method will give a clear idea of how k needs to be at least 6. For other values of α we have performed numerical computations, finding evidence suggesting that for all $\alpha \leq \alpha_0 = -4.999$ there exist invariant subspaces without the wandering property for z^6 . Changing k to a larger value requires a small modification of our method, but it allows to extend the bound on α to $\alpha_1 = -4.2$. We present these numerical results in Section 4. We will conclude with some further remarks in Section 5.

2. SUFFICIENT CONDITIONS FOR THE FAILURE OF THE WANDERING PROPERTY

We denote by \langle, \rangle the inner product in D_α , by $\omega_k = (k+1)^\alpha$ and given $h \in \text{Hol}(\mathbb{D})$ and $s \in \mathbb{N}$, $\hat{h}(s)$ will be the Taylor coefficient (always centered at 0) of order s of h .

Much of what we will do in this Section is applicable to any $k \geq 6$ and any Dirichlet-type space excepting H^2 and D . Those exceptions are the only ones in which ω_k is an affine function of k , which will make certain linear system become incompatible. Thus we take $\alpha \in \mathbb{R} \setminus \{0, 1\}$. Later we will concentrate on the case $\alpha = -16$ and $k = 6$.

The space M we will construct is the smallest closed subspace of D_α invariant under multiplication by z^k and containing two functions F_1

and F_2 , which are taken to be polynomials. That is

$$M = \{F_1, F_2\}_{z^k}. \quad (1)$$

Our objective is to show the existence of spaces M as above for which the wandering property fails, that is, spaces M that satisfy

$$\{M \ominus z^k M\}_{z^k} \neq M. \quad (2)$$

By the definition of M in (1), M is invariant under z^k , and hence M satisfying (2) may only exist provided that M contains at least one function not in $\{M \ominus z^k M\}_{z^k}$. If such a function exists, one of the generators must be an example. In fact, we will construct F_1 and F_2 such that F_1 will belong to $(M \setminus \{M \ominus z^k M\}_{z^k})$.

We could take a very general polynomial function and find what is exactly needed. This is not our approach, which focus on taking only a few degrees of freedom so that equations become solvable in finite time, explicitly.

Let us define F_1 by

$$F_1(z) = \sum_{i=0}^4 a_i z^i + \sum_{i=0}^3 a_{k+i} z^{k+i}, \quad (3)$$

and F_2 , by

$$F_2(z) = \sum_{i=0}^3 b_i z^i + b_5 z^5, \quad (4)$$

where a_i, b_i are complex constant coefficients to be determined later in terms of k and α , and where a_4 and b_5 are non-null.

Remark 2.1. Notice that a_4 and b_5 will play a different role than other parameters and that they are the only coefficients whose degree is not congruent with 0, 1, 2 or 3 mod k .

This special role of a_4 and b_5 is justified by the following result which will allow us to make use of a form of Fourier analysis on the elements of M .

Lemma 2.2. *Let M be as in (1), where F_1 and F_2 are defined as in (3) and (4), and let $a_4 \neq 0 \neq b_5$. Then*

$$M = \{f(z) = f_1(z^k)F_1(z) + f_2(z^k)F_2(z) : f_1, f_2 \in D_\alpha\}.$$

Proof. Denote by M_0 the right-hand side in the statement of the Lemma. To see $M_0 \subset M$, notice that F_1, F_2 are polynomials and hence, multipliers in A^2 . Baring this in mind, let $f \in M_0$ be given by the functions f_1 and f_2 in D_α in the same way as in the description of M in the statement, and denote the sequence of Taylor polynomials of f_1 as $\{p_n\}_{n \in \mathbb{N}}$

and those of f_2 by $\{q_n\}_{n \in \mathbb{N}}$. Then $g_n(z) = p_n(z^k)F_1(z) + q_n(z^k)F_2(z)$ is in M for all $n \in \mathbb{N}$ and g_n converges in D_α norm to f . Since M is a closed subspace, $f \in M$.

On the other hand, to see the other inclusion, let $f(z) = f_1(z^k)F_1(z) + f_2(z^k)F_2(z)$ and suppose, firstly, that $f_1 \notin D_\alpha$. Then the Taylor coefficients of f of order $kt + 4$, $t \in \mathbb{N}$, must be

$$\hat{f}(kt + 4) = \hat{f}_1(t)a_4.$$

Recall that we took $a_4 \neq 0$, and by our hypothesis

$$\|f_1\|_\alpha^2 = \sum_{t \in \mathbb{N}} |\hat{f}_1(t)|^2 (t+1)^\alpha = +\infty.$$

The necessary conclusion is that

$$\|f\|_\alpha^2 \geq \sum_{s \in k\mathbb{N}+4} |\hat{f}(s)|^2 (s+1)^\alpha = +\infty,$$

which is a contradiction with the assumption that $f \in M \subset D_\alpha$. The same argument works for f_2 instead of f_1 replacing a_4 by b_5 and $kt + 4$ by $kt + 5$ where necessary, showing that if $f_2 \notin D_\alpha$ then $f \notin M$, which finishes the proof of the Lemma. \square

The proof of Lemma 2.2 tells us that we can recover, for any $f \in M$, the sequence $\{\hat{f}_1(t)\}_{t \in \mathbb{N}}$ and $\{\hat{f}_2(t)\}_{t \in \mathbb{N}}$ that serve as coordinates of f as an element of M . We may think of such coefficients as the *Fourier coefficients* of f , and of f_1 and f_2 as a form of *Fourier transform* of f . This will be the main role of a_4 and b_5 in our construction. Going forward, our next task will be to understand which functions belong to $M \ominus z^k M$ depending on the values of the coefficients a_i and b_i . Suppose that $f \in M \ominus z^k M$. Then, for any $s \in \mathbb{N}$, $s \geq 1$, and for $j = 1, 2$ we must have

$$\langle f, z^{ks} F_j \rangle = 0.$$

This is, in fact, easily seen to be a characterization of the elements in $M \ominus z^k M$. It turns out to be useful to express these relations in terms of the Fourier coefficients $\{\hat{f}_i(t)\}_{n \in \mathbb{N}, i=1,2}$. We introduce the following notation:

Definition 2.3. We will denote

(A1)

$$A_{s,1} = \langle z^{k(s-1)} F_1, z^{ks} F_1 \rangle = \sum_{h=0}^3 \overline{a_h} a_{k+h} \omega_{ks+h}.$$

(A2)

$$A_{s,2} = \langle z^{ks} F_1, z^{ks} F_2 \rangle.$$

$$(A3) \quad A_{s,3} = \|z^{ks} F_1\|_\alpha^2.$$

$$(A4) \quad A_{s,4} = \|z^{ks} F_2\|_\alpha^2.$$

$$(A5) \quad A_{s,5} = \langle z^{k(s-1)} F_1, z^{ks} F_2 \rangle = \sum_{h=0}^3 \overline{b_h} a_{k+h} \omega_{ks+h}.$$

The basic property that they will satisfy is a recurrence relation:

Lemma 2.4. *Let $f(z) = f_1(z^k)F_1(z) + f_2(z^k)F_2(z)$ for some functions $f_1, f_2 \in D_\alpha$. Then $f \in M \ominus z^k M$ if and only if for all $s \geq 1$ we have both:*

$$(a) \quad 0 = \hat{f}_1(s+1)\overline{A_{s+1,1}} + \hat{f}_2(s+1)\overline{A_{s+1,5}} + \hat{f}_1(s)A_{s,3} + \hat{f}_2(s)\overline{A_{s,2}} + \hat{f}_1(s-1)A_{s,1}.$$

$$(b) \quad 0 = \hat{f}_1(s)A_{s,2} + \hat{f}_2(s)A_{s,4} + \hat{f}_1(s-1)A_{s,5}.$$

Proof. By Lemma 2.2, $f \in M$, so the only condition that needs checking is that f is orthogonal to the functions $z^{ks} F_j$, for $s \geq 1$ and $j = 1, 2$. (a) is equivalent to $f \perp z^{ks} F_1$ whereas (b) is equivalent to $f \perp z^{ks} F_2$, since any other combinations of Taylor coefficients that may appear on the expressions for the inner products are equal to zero. \square

From now on, until the end of this Section we will concentrate on giving conditions, in terms of the relevant quantities $A_{s,r}$ (which only depend on F_1 and F_2), under which the space M will be as in the statement of the Theorem 1.5.

The previous Lemma may be interpreted as a recurrence scheme relating the coefficients of functions in $M \ominus z^k M$. If we make certain choices of parameters a_i and b_i , we will be able to use these recurrences to our end. The first step is apparently innocent but later on, it will allow us to forget about the whole tail of Fourier coefficients of degrees greater or equal to 2:

Lemma 2.5. *Suppose that F_1 and F_2 are such that*

$$A_{2,1} = A_{3,1} = A_{2,5} = A_{3,5} = 0. \quad (5)$$

Then, for any $f \in M \ominus z^k M$, the Fourier functions f_1 and f_2 must satisfy

$$\hat{f}_1(2) = \hat{f}_2(2) = 0. \quad (6)$$

Proof. Consider the case $s = 2$ in Lemma 2.4, and suppose that (5) holds. The equations in Lemma 2.4 will be satisfied if and only if the following linear system is satisfied:

$$\begin{pmatrix} \|z^{2k} F_1\|_\alpha^2 & \langle z^{2k} F_2, z^{2k} F_1 \rangle \\ \langle z^{2k} F_1, z^{2k} F_2 \rangle & \|z^{2k} F_2\|_\alpha^2 \end{pmatrix} \cdot \begin{pmatrix} \hat{f}_1(2) \\ \hat{f}_2(2) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

By the Cauchy-Schwarz inequality, the determinant of this system is strictly positive (F_1 and F_2 can't be multiples of each other, because of the presence of a_4 and b_5). Therefore, the only solution is the trivial one (6). \square

We are finally ready to give insight into the role of the recurrence relations in describing $M \ominus z^k M$:

Lemma 2.6. *Suppose F_1 and F_2 satisfy (5) in Lemma 2.5, and moreover suppose that*

$$A_{1,1} = 0. \tag{7}$$

Then $M \ominus z^k M$ is spanned by F_2 and F_3 where

$$F_3(z) = F_1(z) + \frac{z^k A_{1,5}}{|A_{1,2}|^2 - A_{1,3} A_{1,4}} (A_{1,3} F_2(z) - \overline{A_{1,2}} F_1(z)).$$

Before proceeding with the proof, notice that the denominator of the second term of the right-hand side is different from 0 by Cauchy-Schwarz inequality, and so F_3 is well defined. Also, notice that we could also say that $M \ominus z^k M$ is generated by F_2 and F_4 where

$$F_4(z) = F_1(z) \left(1 - \frac{z^k A_{1,5} \overline{A_{1,2}}}{|A_{1,2}|^2 - A_{1,3} A_{1,4}} \right).$$

Proof. Let $f \in M \ominus z^k M$. Since the hypothesis of Lemma 2.5 is met, we can assume the Fourier coefficients of order 2 of the function f satisfy (6).

Now, we turn to the case $s = 1$ in Lemma 2.5, and we make use of (6), eliminating the terms on $\hat{f}_1(2)$ or $\hat{f}_2(2)$. We then obtain a linear system which can be written as

$$\begin{pmatrix} A_{1,3} & \overline{A_{1,2}} \\ A_{1,2} & A_{1,4} \end{pmatrix} \cdot \begin{pmatrix} \hat{f}_1(1) \\ \hat{f}_2(1) \end{pmatrix} = \begin{pmatrix} 0 \\ -f_1(0) A_{1,5} \end{pmatrix}.$$

By Cauchy-Schwartz inequality, the system is invertible. This means that for each value of $\hat{f}_1(0)$ there is a unique value of $\hat{f}_1(1)$ and $\hat{f}_2(1)$ while $\hat{f}_2(0)$ is free. We can then see that any element in $M \ominus z^k M$ has a set of coefficients of order 0, 1 and 2 that is a linear combination of those for F_2 and those for F_3 , since these ones arise as the solution when taking $\hat{f}_2(0) \neq 0 = \hat{f}_1(0)$ (for F_2) and, respectively for F_3 , $\hat{f}_2(0) =$

$0 \neq \hat{f}_1(0)$. Moreover, F_2 and F_3 satisfy all the cases $s \geq 3$ of Lemma 2.4 trivially, which implies that F_2 and F_3 are elements of $M \ominus z^k M$. Therefore, any other element G of $M \ominus z^k M$ must be of the form $G = \lambda_0 F_2 + \lambda_1 F_3 + F_5$, for some constants $\lambda_0, \lambda_1 \in \mathbb{C}$, where the Fourier coefficients of F_5 are non-zero only for order $s \geq 3$. Bearing in mind that F_2 and F_3 are in $M \ominus z^k M$, we can see that if $G \in M \ominus z^k M$, then, also F_5 must belong to $M \ominus z^k M$, but at the same time, F_5 must be in $z^k M$. Hence, F_5 is orthogonal to itself, i.e., it is identically 0. We have seen that indeed, F_2 and F_3 span all the elements of $M \ominus z^k M$. \square

It could very well be that $A_{1,5}A_{1,2} = 0$ and then $F_3 = F_1$ (from the previous Lemma). In that case, M would definitely have the wandering property. A crucial step will be to show that under certain assumptions (that obviously will have to imply $A_{1,5}A_{1,2} \neq 0$) F_1 is not generated as a combination of F_2 and F_3 . This may be regarded as the main point in the proof:

Lemma 2.7. *Suppose F_1 and F_2 satisfy (5) in Lemma 2.5, and (7) in Lemma 2.6. Additionally, suppose that*

$$A_{1,5}A_{1,2} \neq 0 \tag{8}$$

and

$$A_{1,3}A_{1,4} - |A_{1,2}|^2 < |A_{1,5}A_{1,2}|. \tag{9}$$

Then $F_1 \notin \{M \ominus z^k M\}_{z^k}$. In particular, M does not have the wandering property.

Before we proceed with the proof, some remarks are in order:

Remark 2.8. (1) Since the left-hand side of (9) is strictly positive (once more due to Cauchy-Schwartz inequality), the property (9) clearly implies (8). However, this assumption made explicit justifies studying the quotient of both quantities

$$\frac{A_{1,3}A_{1,4} - |A_{1,2}|^2}{|A_{1,5}A_{1,2}|}.$$

- (2) It seems relevant to interpret condition (9) in terms of Cauchy-Schwartz inequality. This condition tells us then that F_1 and F_2 are *almost parallel*. However, the remaining conditions (5) and (7) play against this as an obstacle.
- (3) Conditions (5), (7) and (8) do not depend at all on the choice of a_4 and b_5 , although in order for the existence of the Fourier expressions we needed them to be non-null. This, together with the fact that (9) is an open condition (a strict inequality between 2 quantities that are continuous as functions of a_4 and b_5 , seen

as variables), simplifies the problem: if we can find a solution to (5), (7), (8) and (9) with $a_4 = b_5 = 0$, then a small enough perturbation changing only the values of these 2 coefficients, will still meet all the conditions.

Now we are going to show the proof of Lemma 2.7.

Proof. Denote

$$c = \frac{A_{1,3}A_{1,4} - |A_{1,2}|^2}{|A_{1,5}A_{1,2}|}.$$

c is well defined because of (8). From Lemma 2.6, F_2 and F_3 span $M \ominus z^k M$, and thus we can infer that if $F_1 \in \{M \ominus z^k M\}_{z^k}$, then F_1 can be arbitrarily well approximated in the D_α norm by functions of the form

$$\left(\sum_{i=0}^t \lambda_{i,t}(z^k + c)z^{ki} \right) F_1(z) + f_2(z^k)F_2(z),$$

where $f_2 \in D_\alpha$ and $\lambda_{i,s}$ are constants for each i and s . Since the terms of order $ks + 4$ ($s \geq 1$) of F_1 are all null, this means that $(\sum_{i=0}^t z^{ki} \lambda_{i,t})(z^k + c)F_1(z)$ must approximate F_1 in D_α -norm arbitrarily well. Now we will use rather basic properties of (classical) shift-invariant subspaces of the spaces D_α . The preliminaries for these may be found in [5] and [10]. Firstly, $(z^k + c)F_1(z)$ generates the same z -invariant subspace of D_α as F_1 . This implies that $z^k + c$ is a cyclic function for the shift operator acting on D_α , which in turn, implies that $z^k + c$ must have no zeros in \mathbb{D} . This is only possible if $|c| \geq 1$, which contradicts the assumption (9), completing the proof. \square

Remark 2.9. Although for $\alpha > 1$ the cyclicity of $z^k + c$ would imply the stronger condition that $|c| > 1$, when including a_4 and b_5 in the computations we need to allow for additional space, so the condition that we are using seems sharp from this point of view in all spaces.

Remark 2.10. What remains, in order to complete the proof of Theorem 1.5, is to find a suitable choice of $a_0, \dots, a_3, a_k, \dots, a_{k+3}, b_0, \dots, b_3$ satisfying all the conditions (5), (7), (8) and (9).

3. A PARTICULAR CHOICE OF PARAMETERS MEETING THE SUFFICIENT CONDITIONS

In the previous Section we have identified the key optimization problem of determining whether

$$\inf \frac{A_{1,3}A_{1,4} - |A_{1,2}|^2}{|A_{1,5}A_{1,2}|} < 1, \quad (10)$$

where the infimum runs through all nonzero vectors

$$(a_0, \dots, a_3, a_k, \dots, a_{k+3}, b_0, \dots, b_3) \in \mathbb{C}^{12}$$

for which the equations (5), (7), (8) hold. If this is the case, then the space M will fail to have the wandering property (M is described in (1), where F_1 and F_2 are as in (3) and (4), the remaining notation is in Definition 2.3).

Since $A_{1,2}, A_{1,3}, A_{1,4}, A_{1,5}$, are all 2-homogeneous, the objective function in (10) is 0-homogeneous. Hence, we could in principle choose, for example,

$$A_{1,5}A_{1,2} = 1 \tag{11}$$

and minimize the numerator. Even though the road we will follow will be different, we think that this expression in terms of an optimization problem is of interest in itself. In this Section the main objective will therefore be to show the following result:

Theorem 3.1. *For $\alpha = -16$ and $k = 6$,*

$$\inf A_{1,3}A_{1,4} - |A_{1,2}|^2 < 1$$

where the infimum runs through all the non-null possible vectors of coefficients

$$(a_0, \dots, a_3, a_k, \dots, a_{k+3}, b_0, \dots, b_3) \in \mathbb{C}^{12}$$

such that (5), (7), (8) and (11) hold.

Denote by B_0 the function $\mathbb{C}^{12} \rightarrow \mathbb{R}$ given by

$$B_0 = \frac{A_{1,3}A_{1,4} - |A_{1,2}|^2}{|A_{1,2}A_{1,5}|}. \tag{12}$$

We will show the validity of (10), which is equivalent to Theorem 3.1. We start with several further reductions of the problem that are still applicable in big generality.

We will later change twice the objective function that we will optimize, but bounds for B_0 will follow from bounds for those other functions B_1 and B_2 . We will find an explicit bound for B_2 , while B_1 will reappear on Section 4 on numerical results.

3.1. General method. The conditions (5) and (7), which are required for vectors $(a_0, \dots, a_3, a_k, \dots, a_{k+3}, b_0, \dots, b_3)$ to be eligible to the minimization problem in (10), may be expressed altogether in a pair of

matrix systems as follows:

$$N \begin{pmatrix} a_0 \overline{a_k} \\ a_1 \overline{a_{k+1}} \\ a_2 \overline{a_{k+2}} \\ a_3 \overline{a_{k+3}} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \quad N \begin{pmatrix} b_0 \overline{a_k} \\ b_1 \overline{a_{k+1}} \\ b_2 \overline{a_{k+2}} \\ b_3 \overline{a_{k+3}} \end{pmatrix} = \begin{pmatrix} \overline{A_{1,5}} \\ 0 \\ 0 \end{pmatrix}, \quad (13)$$

where N is the 3×4 matrix given by

$$N = \begin{pmatrix} \omega_k & \omega_{k+1} & \omega_{k+2} & \omega_{k+3} \\ \omega_{2k} & \omega_{2k+1} & \omega_{2k+2} & \omega_{2k+3} \\ \omega_{3k} & \omega_{3k+1} & \omega_{3k+2} & \omega_{3k+3} \end{pmatrix}. \quad (14)$$

Remark 3.2. Whenever the ambient space is either the Hardy space or the Dirichlet space, the above system is incompatible with the equation (8), since the last row is a linear combination of the previous ones. This problem only appears with the *usual norm* in those spaces, given by $\omega_k = 1$ or $\omega_k = k + 1$, respectively, and linear combinations of these.

We also introduce the following notation which will simplify the linear expressions:

$$N_0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ \omega_k & \omega_{k+1} & \omega_{k+2} & \omega_{k+3} \\ \omega_{2k} & \omega_{2k+1} & \omega_{2k+2} & \omega_{2k+3} \\ \omega_{3k} & \omega_{3k+1} & \omega_{3k+2} & \omega_{3k+3} \end{pmatrix}, \quad (15)$$

and

$$N_1 = \begin{pmatrix} \omega_{k+1} & \omega_{k+2} & \omega_{k+3} \\ \omega_{2k+1} & \omega_{2k+2} & \omega_{2k+3} \\ \omega_{3k+1} & \omega_{3k+2} & \omega_{3k+3} \end{pmatrix}. \quad (16)$$

By introducing two variables Z_1 and Z_2 , that will play a mute role, we may convert the system (13) into a square linear system with the matrix N_0 instead of N :

$$N_0 \begin{pmatrix} a_0 \overline{a_k} \\ a_1 \overline{a_{k+1}} \\ a_2 \overline{a_{k+2}} \\ a_3 \overline{a_{k+3}} \end{pmatrix} = \begin{pmatrix} \overline{Z_1} \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad N_0 \begin{pmatrix} b_0 \overline{a_k} \\ b_1 \overline{a_{k+1}} \\ b_2 \overline{a_{k+2}} \\ b_3 \overline{a_{k+3}} \end{pmatrix} = \begin{pmatrix} \overline{Z_2} \\ \overline{A_{1,5}} \\ 0 \\ 0 \end{pmatrix}. \quad (17)$$

Notice N_0 is invertible whenever N_1 is. This should be checked in each case: for us, the values corresponding to $\alpha = -16$ and $k = 6$ give a determinant of N_1 that is not 0 and so N_0 is invertible (for spaces other than Hardy or Dirichlet, invertibility will typically hold for all values of k except perhaps a finite number of cases). If Z_1 and Z_2 are non-null, then so are b_0, b_1, b_2 , and b_3 . Hence, we can express the equations in

terms of the elements of N_0^{-1} . Denote the elements of the first two rows of N_0^{-1} by

$$\begin{pmatrix} 1 & 0 \\ E_1 & G_1 \\ E_2 & G_2 \\ E_3 & G_3 \end{pmatrix}, \quad (18)$$

and denote

$$Z_3 := x + iy := \frac{Z_2}{A_{1,5}},$$

which is well-defined if we assume (8) ($Z_3 \in \mathbb{C}, x, y \in \mathbb{R}$). Since $\omega = \{\omega_k\}_{k \in \mathbb{N}} \subset \mathbb{R}$, then all of E_1, E_2, E_3, G_1, G_2 and G_3 are also real numbers (and for each fixed space and k they are a specific and computable set of numbers). Suppose that $E_i \neq 0$ for $i = 1, 2, 3$. Then the system (17) is equivalent to the following one, which gives the values of all the parameters b_0, \dots, b_3 and a_k, \dots, a_{k+3} in terms of those of a_0, \dots, a_3 :

$$\begin{pmatrix} a_k \\ a_{k+1} \\ a_{k+2} \\ a_{k+3} \end{pmatrix} = \begin{pmatrix} \frac{Z_1}{a_0} \\ \frac{E_1 Z_1}{a_1} \\ \frac{E_2 Z_1}{a_2} \\ \frac{E_3 Z_1}{a_3} \end{pmatrix}, \quad \begin{pmatrix} b_0 \\ b_1 \\ b_2 \\ b_3 \end{pmatrix} = \begin{pmatrix} a_0 \frac{\overline{A_{1,5} Z_3}}{Z_1} \\ a_1 \frac{\overline{A_{1,5}}}{Z_1} \left(\overline{Z_3} + \frac{G_1}{E_1} \right) \\ a_2 \frac{\overline{A_{1,5}}}{Z_1} \left(\overline{Z_3} + \frac{G_2}{E_2} \right) \\ a_3 \frac{\overline{A_{1,5}}}{Z_1} \left(\overline{Z_3} + \frac{G_3}{E_3} \right) \end{pmatrix}.$$

In this case, any choice of $Z_3 \in \mathbb{C}$ and $a_0, \dots, a_3, Z_1, A_{1,5} \in \mathbb{C} \setminus \{0\}$ will yield the value of the remaining variables so that (5) and (7) are met. We can express the *objective function* B_0 (as in (12)) in terms of these 7 variables only, and then find a good choice of those 7 variables to provide a low value of B_0 .

To do so, we translate each of the significant quantities for B_0 , that is $A_{1,3}, A_{1,4}$ and $A_{1,2}$, in terms of our variables. In this way, $A_{1,3}$ is given by:

$$\begin{aligned} A_{1,3} = & |a_0|^2 \omega_k + |a_1|^2 \omega_{k+1} + |a_2|^2 \omega_{k+2} + |a_3|^2 \omega_{k+3} + \\ & + |Z_1|^2 (|a_0|^{-2} \omega_{2k} + |a_1|^{-2} E_1^2 \omega_{2k+1} + |a_2|^{-2} E_2^2 \omega_{2k+2} + |a_3|^{-2} E_3^2 \omega_{2k+3}). \end{aligned}$$

We introduce the following notations:

$$\begin{aligned} d_i &= |a_i|^2, \quad i = 0, 1, 2, 3. \\ H_0 &:= 1, \quad H_i := E_i^2, \quad i = 1, 2, 3. \\ C_1 &= \sum_{i=0}^3 d_i \omega_{k+i}, \quad C_2 = \sum_{i=0}^3 \frac{H_i \omega_{2k+i}}{d_i}. \end{aligned}$$

This notation will be convenient because C_1 and C_2 only depend, for a fixed space and k , on the choice of d_i (which can be taken to be any positive real number). By making this choice, the expression for $A_{1,3}$ becomes much simpler:

$$A_{1,3} = C_1 + |Z_1|^2 C_2.$$

On the other hand, if we make use of the same notation for $A_{1,4}$, and defining

$$D_i = -\frac{G_i}{E_i}, \quad i = 1, 2, 3,$$

we may see that

$$A_{1,4} = \frac{|A_{1,5}|^2}{|Z_1|^2} \left(d_0 |Z_3|^2 \omega_k + \sum_{i=1}^3 d_i |Z_3 - D_i|^2 \omega_{k+i} \right).$$

To follow with previous notation, we give names of the form C_t to quantities that depend only on d_0, \dots, d_3 and on the parameters arising from the choice of space and the value of k :

$$C_3 = 2 \sum_{i=1}^3 D_i d_i \omega_{k+i}, \quad C_4 = \sum_{i=1}^3 D_i^2 d_i \omega_{k+i}.$$

These quantities can be used to simplify the formula for $A_{1,4}$:

$$A_{1,4} = \frac{|A_{1,5}|^2}{|Z_1|^2} (C_1 |Z_3|^2 - C_3 x + C_4).$$

Finally, $|A_{1,2}|^2$ is given by

$$|A_{1,2}|^2 = \frac{|A_{1,5}|^2}{|Z_1|^2} \left| C_1 Z_3 - \frac{C_3}{2} \right|^2.$$

Observe that the value of $|A_{1,5} A_{1,2}|$ can also be obtained now as a function of $A_{1,5}$ and the other free values d_0, \dots, d_3 , Z_3 and Z_1 . Now we are ready to give the desired form of B_0 :

$$B_0 = \frac{|Z_1|^2 C_2 (C_1 |Z_3|^2 - C_3 x + C_4) + C_1 C_4 - \frac{C_3^2}{4}}{|Z_1| \left| C_1 Z_3 - \frac{C_3}{2} \right|}.$$

The first conclusion from here is that the arguments of a_i and Z_1 do not play any role so we can work directly with the positive real variables d_0, d_1, d_2, d_3 and assume $Z_1 \in \mathbb{R}^+$. Since H_i is always positive, so are C_1, C_2 and C_4 .

Moreover, B_0 is positive: This is clear from its definition on equation (12), if we bear in mind Definition 2.3 and Cauchy-Schwarz inequality.

As a function of Z_1 (considering other variables as constants), B_0 takes the form:

$$B_0 = \frac{e_0}{B_1} + e_1 B_1,$$

where e_0 and e_1 do not depend on Z_1 . Now, notice that this implies that both e_0 and e_1 are non-negative independently of the choices for $d_0, \dots, d_3, Z_3, A_{1,5}$. Otherwise, there would be choices of Z_1 for which B_0 is negative, which is a contradiction. It can be shown that in fact, both e_0 and e_1 are strictly positive but in fact, if one of them or both is null, then we can take a value of Z_1 such that B_0 is arbitrarily close to 0, and the Theorem 3.1 would hold. Therefore, $B_0 = B_0(Z_1)$ is a differentiable function of $Z_1 \in (0, +\infty)$ with

$$\lim_{Z_1 \rightarrow 0} B_0(Z_1) = \lim_{Z_1 \rightarrow +\infty} B_0(Z_1) = +\infty,$$

and so it must have a global minimum on \mathbb{R}^+ . This must be at the only point at which $0 = \frac{\partial B_0}{\partial Z_1}$ which is at

$$Z_1^* = \sqrt{\frac{e_0}{e_1}}.$$

There, we obtain the corresponding value for B_0 :

$$B_0(Z_1^*) = 2\sqrt{e_0 e_1}.$$

We have reduced the problem of minimizing B_0 on 2 variables since we know which value to choose for Z_1 and $A_{1,5}$ does not play any role anymore on the result (provided it is not 0). Now it is the moment to study e_0 and e_1 and find a good choice for Z_3 and for d_0, \dots, d_3 . We introduce one more bit of notation:

$$C_5 = C_1 C_4 - \frac{C_3^2}{4},$$

so that e_0 is given by

$$e_0 = \frac{C_5}{|C_1 Z_3 - \frac{C_3}{2}|},$$

whereas e_1 , by

$$e_1 = \frac{C_2 (C_1 |Z_3|^2 - C_3 x + C_4)}{|C_1 Z_3 - \frac{C_3}{2}|}.$$

Our objective is to show that B_0 can be made smaller than 1, which will hold if and only if $B_0^2 < 1$ for some values of Z_3, d_0, \dots, d_3 . For this reason, we intend to show that

$$4e_0 e_1 < 1.$$

The left-hand side above can now be described only in terms of C_j ($j = 1, \dots, 5$) and Z_3 :

$$4e_0e_1 = \frac{4C_2C_5}{C_1} \left(1 + \frac{C_5}{|C_1Z_3 - \frac{C_3}{2}|^2} \right).$$

C_5 is positive as may be seen from the expressions for C_1 , C_3 and C_4 . Thus, the existence of a set of values Z_3, d_0, \dots, d_3 such that $4e_0e_1 < 1$ is equivalent with the existence of (perhaps different values) d_0, \dots, d_3 so that $4C_2C_5/C_1 < 1$. To see this, observe that this last condition is necessary because C_5 is positive, and so $4e_0e_1 > \frac{4C_2C_5}{C_1}$. Also, suppose that for some choice of d_0, \dots, d_3 and for some $\varepsilon > 0$, we have

$$\frac{4C_2C_5}{C_1} < 1 - \varepsilon.$$

Since C_1, C_2, C_3, C_5 do not depend on Z_3 , we can choose Z_3 large enough in modulus so that

$$\frac{C_5}{|C_1Z_3 - \frac{C_3}{4}|^2} < \varepsilon.$$

This yields that

$$4e_0e_1 < 1 - \varepsilon^2 < 1.$$

We can define the *second objective function*

$$B_1 := \frac{4C_2C_5}{C_1} \tag{19}$$

(and it is a function of d_0, \dots, d_3 only). We have reduced our original problem to the much simpler one of proving

$$\inf_{d_0, d_1, d_2, d_3 \in \mathbb{R}^+} B_1 < 1. \tag{20}$$

Let us remark that C_1 , C_2 and C_5 are respectively 1–, (–1)– and 2– homogeneous, and so B_1 is 0-homogeneous. Therefore, from now on we fix $d_0 = 1$ and we do not lose anything (even if the best choice is for $d_0 = 0$, the condition we want to show is open and depends continuously on the parameters).

From the definition of C_5 we can see that $B_1 < 4C_2C_4$. In the next subsection, we will find values for which

$$B_2(d_1, d_2, d_3) := 4C_2C_4(1, d_1, d_2, d_3) \tag{21}$$

is bounded above by 1, therefore showing (20). Hence we give a name to B_2 as well: we call it the *third objective function*. Our final objective

becomes the following:

$$\inf_{d_1, d_2, d_3 \in \mathbb{R}^+} B_2 < 1.$$

As mentioned earlier, we will use again B_1 in the numerical computations section, since it is easier to obtain a value below 1 for a particular set of parameters for B_1 , but B_2 is easier to deal with in abstract because of its simpler expression. We are left with a 3 variable optimization problem concerning d_1 , d_2 and d_3 in \mathbb{R}^+ . To proceed from here, one needs a good understanding of the quantities $E_1, E_2, E_3, G_1, G_2, G_3$ defining C_1, C_2 and C_5 . Recall the definitions of N_0 and N_1 in (15) and (16). Through (18), we defined E_1, E_2, E_3 as the solution to

$$N_0 \begin{pmatrix} 1 \\ E_1 \\ E_2 \\ E_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

The value 1 can be integrated into the independent term, giving the equivalent system

$$\begin{pmatrix} E_1 \\ E_2 \\ E_3 \end{pmatrix} = -N_1^{-1} \begin{pmatrix} \omega_k \\ \omega_{2k} \\ \omega_{3k} \end{pmatrix}. \quad (22)$$

In an analogous way, we can obtain G_1, G_2, G_3 as the solution to

$$\begin{pmatrix} G_1 \\ G_2 \\ G_3 \end{pmatrix} = N_1^{-1} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}. \quad (23)$$

Cramer's rule tells us how to obtain the values of E_1, E_2, E_3, G_1, G_2 and G_3 provided that

$$\det N_1 \neq 0.$$

Equations (23) and (22) allow to obtain the values of

$$G_1 = \frac{\begin{vmatrix} \omega_{2k+2} & \omega_{2k+3} \\ \omega_{3k+2} & \omega_{3k+3} \end{vmatrix}}{\det N_1}; \quad E_1 = -\frac{\begin{vmatrix} \omega_k & \omega_{k+2} & \omega_{k+3} \\ \omega_{2k} & \omega_{2k+2} & \omega_{2k+3} \\ \omega_{3k} & \omega_{3k+2} & \omega_{3k+3} \end{vmatrix}}{\det N_1};$$

$$G_2 = \frac{\begin{vmatrix} \omega_{2k+3} & \omega_{2k+1} \\ \omega_{3k+3} & \omega_{3k+1} \end{vmatrix}}{\det N_1}; \quad E_2 = -\frac{\begin{vmatrix} \omega_{k+1} & \omega_k & \omega_{k+3} \\ \omega_{2k+1} & \omega_{2k} & \omega_{2k+3} \\ \omega_{3k+1} & \omega_{3k} & \omega_{3k+3} \end{vmatrix}}{\det N_1};$$

$$G_3 = \frac{\begin{vmatrix} \omega_{2k+1} & \omega_{2k+2} \\ \omega_{3k+1} & \omega_{3k+2} \end{vmatrix}}{\det N_1}; \quad E_3 = -\frac{\begin{vmatrix} \omega_{k+1} & \omega_{k+2} & \omega_k \\ \omega_{2k+1} & \omega_{2k+2} & \omega_{2k} \\ \omega_{3k+1} & \omega_{3k+2} & \omega_{3k} \end{vmatrix}}{\det N_1}.$$

We need to compute these 7 determinants to go further and for this we need to particularize on the values of k and α of our choice. Notice, however, that the 12 numbers determining the matrix N in (14) also determine the answer to the optimization problems presented.

The numerical observations of Section 4 will suggest that for each $k \geq 6$ there exists a small enough α_0 such that for all $\alpha \leq \alpha_0$, some z^k -invariant subspace $M \subset D_\alpha$ fails to have the wandering property. The numerical construction will be a minor modification of the one presented up to this point. We prove that this is indeed the case for $k = 6$ and $\alpha = -16$ in the next subsection.

3.2. Values for $k = 6$ and $\alpha = -16$. At this point we choose k equal to 6 and $\omega = \{\omega_t\}_{t \in \mathbb{N}}$ to be defined by

$$\omega_t = (t + 1)^{-16}.$$

Since the scale of all the numbers appearing in the matrix N is dominated by 7^{-16} , it is numerically useful to express the elements of N as 7^{-16} times other numbers. Going forward, for each figure we write the decimal numbers that are the correct rounding number to the nearest decimal. When the precision to which a number is determined is below this level, this is expressed. In this way, we obtain the values of ω_t indicated in Table 3.2, which are given also compared to $\omega_k = 7^{-16} = 3.0090635 \cdot 10^{-14}$.

For each of the determinants to be computed precisely, it is convenient to use the multiplicative properties of determinants to extract the factors of the form 7^{-16} , as well as powers of 10 in order to make the numbers in the determinants, mesoscopic. This will yield

$$\det N_1 = 7^{-48} \cdot 10^{-16} \cdot \begin{vmatrix} 11.8\dots & 1.793\dots & 0.3323 \\ 15.25\dots & 5.0595\dots & 1.8015\dots \\ 5.07\dots & 2.323\dots & 1.1035\dots \end{vmatrix}.$$

From here on, all the bounds are obtained from classical interval arithmetic. The above implies

$$\det N_1 = (1.6207616 \pm 5 \cdot 10^{-8}) \cdot 10^{-56}.$$

TABLE 1. Values of ω_t in the matrix N

t	ω_t	$\omega_t \cdot 7^{16}$
k	7^{-16}	1
$k + 1$	8^{-16}	$1.1806708702 \cdot 10^{-1}$
$k + 2$	9^{-16}	$1.793446761 \cdot 10^{-2}$
$k + 3$	10^{-16}	$3.32329305 \cdot 10^{-3}$
$2k$	13^{-16}	$4.99430433671 \cdot 10^{-5}$
$2k + 1$	14^{-16}	$1.52587890625 \cdot 10^{-5}$
$2k + 2$	15^{-16}	$5.05951042777 \cdot 10^{-6}$
$2k + 3$	16^{-16}	$1.80156077608 \cdot 10^{-6}$
$3k$	19^{-16}	$1.15215530802 \cdot 10^{-7}$
$3k + 1$	20^{-16}	$5.0709427749 \cdot 10^{-8}$
$3k + 2$	21^{-16}	$2.32305731254 \cdot 10^{-8}$
$3k + 3$	22^{-16}	$1.10358489374 \cdot 10^{-8}$

With this same philosophy we may obtain that

$$\begin{aligned} G_1 &= (7.8126227 \pm 6 \cdot 10^{-7}) \cdot 10^{14}, & E_1 &= -16.37478 \pm 2 \cdot 10^{-6}, \\ G_2 &= (-4.3037417 \pm 3 \cdot 10^{-7}) \cdot 10^{15}, & E_2 &= 65.63437 \pm 7 \cdot 10^{-6}, \\ G_3 &= (5.4695405 \pm 4 \cdot 10^{-7}) \cdot 10^{15}, & E_3 &= -73.35945 \pm 2 \cdot 10^{-5}. \end{aligned}$$

These results give upper bounds for the values appearing in the definitions of C_2 and C_4 : H_i and D_i^2 . In particular, we may see that

$$\begin{aligned} H_1 &= E_1^2 \leq 268.13349, & D_1^2 &= \frac{G_1^2}{E_1^2} \leq 2.276371 \cdot 10^{27}, \\ H_2 &= E_2^2 \leq 4307.8715, & D_2^2 &= \frac{G_2^2}{E_2^2} \leq 4.29962 \cdot 10^{27}, \\ H_3 &= E_3^2 \leq 5381.61, & D_3^2 &= \frac{G_3^2}{E_3^2} \leq 5.55892 \cdot 10^{27}. \end{aligned}$$

Now we have all elements of the third objective function defined on (21). We can obtain the bounds

$$C_2 \leq 13^{-16} + \frac{14^{-16} \cdot 268.13349}{d_1} + \frac{15^{-16} \cdot 4307.8715}{d_2} + \frac{16^{-16} \cdot 5381.61}{d_3},$$

which can be translated into

$$C_2 \leq 3.01 \cdot 10^{-20} \cdot \left(49.944 + \frac{4091.393}{d_1} + \frac{21795.721}{d_2} + \frac{9695.298}{d_3} \right). \quad (24)$$

For C_4 , the bound we obtain is

$$C_4 \leq 3.01 \cdot 10^{11} \cdot (26.877d_1 + 7.712d_2 + 1.848d_3). \quad (25)$$

Putting together (24) and (25) provides estimates for the third objective function in (21). At this point, we could find the best possible values for d_i but if we make an educated guess and choose $d_1 = 1$, $d_2 = 4$ and $d_3 = 6$ we will already obtain a good enough estimate:

$$B_4(1, 4, 6) \leq 0.02795 \quad (26)$$

This concludes the proof of the Theorem 3.1.

3.3. Extrapolation to an interval. What remains to complete the proof of Theorem 1.5 is to extend the solution to a small interval around $\alpha = -16$. Notice that in the proof of Theorem 3.1, every bound that we need depends only on quotients of minors of the matrix N . All these determinants vary continuously with the values of the sequence ω , and since they do not become 0 when $\alpha = -16$, all the bounds depend continuously on the value of α around $\alpha = -16$. On the other hand, following the same path described in the proof for a different choice of ω will also lead to a solution to the linear equations in the system (13), that determine which values of parameters we can try in the minimization problems. It is only the values of E_i and G_i (and hence those of their derived quantities, H_i , D_i and C_i) that will vary, affecting the minimization problem.

Because of this, the optimal value for the minimization problems will depend continuously on each value ω_t , for $t \in \mathbb{N}$ (as well as on the choice of d_1 , d_2 and d_3). This means that there must exist an interval around $\alpha = -16$ for which B_2 stays below 1. Theorem 1.5 is now completely proved.

3.4. Explicit recovery of the parameters. It is our intention now to recover explicitly the numerical values of the parameters intervening in the previous subsections, leading to the proof of Theorem 1.5, although the latter has already been established. We do this for completion, and also, in order to pave the path for a blind but very simple and explicit proof of the Theorem 3.1. Such proof consists on simply plugging the obtained values for a_i and b_i into the equations (13) and then check that the corresponding value of B_0 is less than 1.

At the end of the previous subsection we chose the values $d_1 = 1$, $d_2 = 4$ and $d_3 = 6$ which implies that we can take $a_1 = 1$, $a_2 = 2$ and $a_3 = \sqrt{6}$ to solve the optimization problem in (10) (recall a_i were taken positive real and d_i were their squares). We had also previously chosen $d_0 = 1$, and hence, $a_0 = 1$. The objective function was bounded in (26) by $4C_2C_5/C_1 < 1 - \varepsilon = 0.02795$, which yields a value of $\varepsilon = 0.97205$. From these values we can bootstrap and recover each number that was needed in the proof, explicitly. We give the results we obtain in

TABLE 2. Values of the parameters in the proof

Notation	Approximate value	Upper bound
C_4	$2.07 \cdot 10^{13}$	
C_2	$3.372 \cdot 10^{-16}$	
C_1	$3.379494 \cdot 10^{-14}$	
C_3	0.355785	
Z_3	$-2 \cdot 10^{13}$	
$ C_1 Z_3 - C_3/2 $	1.03168	
C_5	0.66791	
$\frac{C_5}{ C_1 Z_3 - \frac{C_3}{2} ^2}$		0.628
B_0^2		0.0463
B_0		0.216
e_0	0.6474	
e_1	0.01351	
Z_1	6.92	
$A_{1,5}$	2.59	
a_k	6.92	
a_{k+1}	-113.3	
a_{k+2}	227.1	
a_{k+3}	-207.2	
b_0	$-7.5 \cdot 10^{12}$	
b_1	$-2.53 \cdot 10^{13}$	
b_2	$-1.28 \cdot 10^{14}$	
b_3	$-8.67 \cdot 10^{13}$	

Table 3.4. Each number on that table can be found before those other numbers below it. The choice for Z_3 is arbitrary within some region, and recall that $D_i \in \mathbb{R}^+$. The value of $A_{1,5}$ can be obtained if we assume (11), from the fact that

$$|A_{1,2}|^2 = \frac{|A_{1,5}|^2}{|Z_1|^2} |C_1 Z_3 - C_3/2|^2 = \frac{1}{|A_{1,5}|^2}.$$

One can then, solve to obtain $A_{1,5}$ as a function of the already known parameters.

These are some values that will solve all the equations (5), (7), (8) and (9). One could also want to find an explicit choice of a_4 and b_5 that makes everything work as described in Remark 2.8(3). It is clear that only $A_{1,3}$ and $A_{1,4}$ will be affected by a permutation in those coefficients (as compared to $A_{1,1}, A_{1,2}$ and $A_{1,5}$). One can check that choosing $a_4 = b_5 = 1$ will not affect that much the value of $A_{1,3}$ and

$A_{1,4}$ and will in fact be enough to keep satisfying (9) (notice that in the expression of the norm $A_{1,3}$, very small weight 11^{-16} is assigned to the coefficient $|a_4|^2$, as well as to $A_{1,4}$ respectively with 12^{-16} and $|b_5|^2$).

Remark 3.3. A shorter but blind proof is possible. Once recovered the values of the parameters as obtained in the previous subsection, it would be enough, in order to prove Theorem 3.1, to check its validity for those parameters obtained. Although this may very well yield a valid proof, we find more illustrative to describe the method to obtain explicitly those parameters.

3.5. Proof of Corollary. As mentioned in Section 3.3, the proof of Theorem 3.1 (and thus, that of Theorem 1.5) depend only on properties of the sequence ω : indeed let $\omega \subset \mathbb{R}^+$ be a sequence with $\omega_0 = 1$ and such that

$$\lim_{k \rightarrow \infty} \frac{\omega_k}{\omega_{k+1}} = 1.$$

We refer to these as *Hardy-type spaces*. Such spaces include the D_α spaces, indeed, for the choice $\omega_k = (k+1)^\alpha$ and they have been studied with regards to invariant subspace properties such as cyclicity for the shift operator in [?]. The proof of the case $\alpha = -16$ in Theorem 1.5 extends directly without any relevant changes to any space for which the corresponding matrix N (that depends on the sequence ω) is the same as for D_{-16} . Notice that modifying the sequence ω in a finite number of points will not affect *which functions form the space*, only the choice of equivalent norm is affected. Hence, fixed $k \geq 6$, one can modify any space D_α to an equivalent norm by changing only 12 numbers of the sequence ω : those appearing in the definition of the matrix N , namely, $\omega_k, \dots, \omega_{k+3}, \omega_{2k}, \dots, \omega_{2k+3}, \omega_{3k}, \dots, \omega_{3k+3}$, by the corresponding ones for D_{-16} .

3.6. Proof for large spaces. We follow the notation from the previous subsections. By choosing $d_1 = d_2 = d_3 = 1$ and fixing $k \geq 9$ we are going to find that the method described also yields the existence of $\alpha_k^* = -(5k + \frac{700}{(k-9)^2})$ such that, for all $\alpha \leq \alpha_k^*$ there exists a z^k invariant subspace in D_α without the corresponding wandering property, as stated in Theorem 1.7. Denote $\beta = |\alpha|$ and we can suppose $\beta \geq 1$. If we compute the determinants giving the values of $G_1, G_2, G_3, E_1, E_2, E_3$ and denote by $1/d$ the determinant of N_1 , we obtain:

$$|G_1| = d((2k+3)(3k+3))^\alpha \left(\left(1 - \frac{1}{3k+4}\right)^\beta - \left(1 - \frac{1}{2k+4}\right)^\beta \right)$$

A standard estimate will give

$$|G_1| \leq d((2k+3)(3k+3))^\alpha \frac{\beta k}{(2k+4)(3k+4)}.$$

In an analogous manner we can estimate all the other elements G_i to obtain

$$|G_2| \leq ((2k+4)(3k+4))^\alpha \frac{2d\beta k}{(2k+2)(3k+2)},$$

$$|G_3| \leq ((2k+2)(3k+2))^\alpha \frac{d\beta k}{(2k+3)(3k+3)}.$$

Denote $E_0 = 1$. E_i can also be described in terms of d and ω in a similar way by developing the determinants.

For E_1 , for instance, we can see that

$$|E_1| = d(p_{1,1}(k)^\alpha - p_{2,1}(k)^\alpha + p_{3,1}(k)^\alpha - p_{4,1}(k)^\alpha + p_{5,1}(k)^\alpha - p_{6,1}(k)^\alpha),$$

where each $p_{i,1}$ is a polynomial of degree 3 of the form $p_{i,1}(k) = 6k^3 + ak^2 + bk + 12$. Since $\beta = -\alpha$ is very large, we can see that the polynomial making the biggest contribution to the determinant in the numerator of E_1 will be the smallest of them. In this case, $p_{1,1} = (k+1)(2k+3)(3k+4) = 6k^3 + 23k^2 + 29k + 12$. The second smallest is $p_{2,1} = (k+1)(2k+4)(3k+3) = 6k^3 + 24k^2 + 30k + 12$, and we are going to see that $p_{1,1}^\alpha$ is in fact much larger than $2p_{2,1}^\alpha > p_{2,1}^\alpha + p_{4,1}^\alpha$, where $p_{4,1}^\alpha > p_{6,1}^\alpha$ is the next biggest. Let $s \in (0, 1)$:

$$sp_{1,1}^\alpha \geq 2p_{2,1}^\alpha. \tag{27}$$

Then,

$$p_{1,1}^\alpha > (1-s)p_{1,1}^\alpha + p_{2,1}^\alpha + p_{4,1}^\alpha.$$

The other terms will make a positive contribution, since $p_{3,1}^\alpha$ is summed to the result and $p_{5,1}^\alpha > p_{6,1}^\alpha$. Altogether, this gives that

$$|E_1| \geq d(1-s)p_{1,1}^\alpha. \tag{28}$$

This is in fact, quite sharp, since we always have $|E_1| \leq 3dp_{1,1}^\alpha$. Let us determine what are the values of $s \in (0, 1)$ for which (27) holds: We want

$$\frac{p_{1,1}(k)}{p_{2,1}(k)} \leq \left(\frac{s}{2}\right)^{1/\beta}.$$

The left hand side is equal to $(1 - \frac{k}{(2k+4)(3k+3)})$. Taking logarithms on both sides and using the standard estimate $\log(x+1) \leq x$, will give the following sufficient condition for s to satisfy (27):

$$\beta \geq 6 \frac{(k+1)}{k} (k+2) \log(2/s). \tag{29}$$

The same principle may be applied to $|E_2|$ and $|E_3|$. However, what polynomials play the role of $p_{1,1}$ and $p_{2,1}$ will depend on each case. In any case, we will obtain sufficient conditions analogous to (29) for s to satisfy an analogous to (28), and it can be checked that these sufficient conditions are actually less restrictive than (29). This method may also be applied to estimate d . The best upper estimate for d will be

$$d \leq \frac{p_{1,*}^\beta}{1-s}, \quad (30)$$

where

$$p_{1,*}(k) = (k+2)(2k+3)(3k+4) = 6k^3 + 29k^2 + 46k + 24.$$

A first conclusion is that for all $i = 0, 1, 2, 3$ we have the bounds

$$(1-s)/3 \leq |E_i| \leq \frac{3}{1-s} \left(\frac{p_{1,3}(k)}{p_{1,*}(k)} \right)^\alpha. \quad (31)$$

Now we are ready to study the objective function $4C_2C_4$.

The bounds

$$C_2 \leq 4\omega_{2k} \sup_{i=0,1,2,3} H_i$$

$$C_4 \leq 3\omega_{k+1} \sup_{i=1,2,3} D_i^2$$

provide a bound for the objective function:

$$4C_2C_4 \leq 48\omega_{2k}\omega_{k+1} \sup_{i,j=0,1,2,3} \left(\frac{G_i E_i}{E_j} \right)^2.$$

The supremum among G_i^2 is clearly dominated by

$$G^2 := 4(\beta kd)^2((2k+2)(3k+2))^{2\alpha-2} \leq \frac{d^2\beta^2}{9k^2} \omega_{2k+1}^2 \omega_{3k+1}^2.$$

From this we can obtain another estimate for the objective function:

$$4C_2C_4 < \frac{48(\beta\omega_{2k+1}\omega_{3k+1}d)^2\omega_{2k}\omega_{k+1}}{9k^2} \sup_{i,j=0,1,2,3} \left(\frac{E_i}{E_j} \right)^2.$$

Now we may plug in the estimates from above and below for $|E_i|$ obtained in (31), which yields

$$4C_2C_4 < \frac{432\beta^2\omega_{2k}}{(1-s)^4 k^2 \omega_{k+1}} \left(\frac{q_2(k)}{q_1(k)} \right)^{2\beta},$$

where

$$q_1(k) = (k+1)^4(k+2/3), \quad q_2(k) = (k+2)(k+3/2)^2(k+4/3)^2.$$

TABLE 3. Solutions for $k \in [10, 17]$

k	β	s	Objective	β
10	530	0.05	0.994	293
11	165	0.3	0.612	162
12	120	0.6	0.387	110
13	104	0.8	0.490	89
14	98	0.9	0.556	83
15	90	0.93	0.562	84
16	87	0.94	0.864	87
17	88	0.97	0.502	88

This means that the objective function can be bounded by the factor $\frac{432\beta^2}{(1-s)^4 k^2} a^\beta$ where a is of the form

$$a = \left(\frac{1}{2} + \frac{3}{2(2k+1)}\right) \cdot \left(1 + \frac{1}{k+1}\right)^2 \cdot \left(1 + \frac{1}{2(k+1)}\right)^4 \cdot \left(1 + \frac{1}{3(k+1)}\right)^2 \cdot \left(1 + \frac{2}{3k+2}\right)^2.$$

Each of the factors on the right hand side is decreasing on k and the limit as k increases to ∞ is clearly $1/2$, so for some $k_0 \in \mathbb{N}$ and all $k \geq k_0$, a^β will decay exponentially fast with β . This is in fact true for $k \geq 10$, since $a(10) < 1$. We have now 2 conditions on s, k, β that if satisfied, guarantee the existence of a z^k invariant subspace in $D_{-\beta}$ without the wandering property. The conditions (29) and that $\frac{432\beta^2}{(1-s)^4 k^2} a^\beta < 1$.

For any $k \leq 18$, the choices $s = 0,985$ and $\beta = 5k$ achieves both things (the bound on the objective function will be decaying on k and it holds for $k = 18$). For $k = 10, \dots, 17$ we provide a list of the corresponding choices for β and s that will satisfy both conditions. Any choice of β larger than the one provided will make the objective function decay as well. The lower bound given for β guarantees (29). See Table 3.6. All the corresponding choices of β are bounded by $5k + \frac{700}{(k-9)^2}$. This concludes the proof of the Theorem 1.7.

4. OTHER NUMERICAL RESULTS

In the proof of the Theorem 3.1, we have made an assumption that we have not mentioned anything about: the fact that the degrees for which F_1 and F_2 have non-zero coefficients are of the form $0, \dots, 5$ and

TABLE 4. Numerical results for $k = 6$

α	ϕ_2	ϕ_3	d_1	d_2	d_3	B_1
-16	0	0	1	4	6	0.02324
-16	0	3	1	10	2000	0.00667
-12	1	4	1	20	5000	0.02397
-8	1	8	1	10	5000	0.1525
-7	2	12	1	20	10000	0.31668
-6	3	17	0.2	13	16000	0.5635
-5	2	34	4	11	100000	0.99826
-4.999	2	34	4	11	100000	0.999006

$k, \dots, k + 3$. The same proof we followed could a priori work for any other functions F_1 and F_2 of the form:

$$F_1(z) = \sum_{i=0}^4 a_i z^{\gamma_i} + \sum_{i=0}^3 a_{k+i} z^{k+\gamma_i},$$

and

$$F_2(z) = \sum_{i=0}^3 b_i z^{\gamma_i} + b_5 z^{\gamma_5},$$

provided that a_i and b_i satisfy some equations similar to (13) but modified accordingly and that $\gamma_0, \dots, \gamma_5$ satisfy, whenever $i \neq j$, that

$$\gamma_i \not\equiv \gamma_j \pmod{k}.$$

We ran numerical experiments on the second objective function B_3 (as in (19)) for different values of α , k , d_1, d_2, d_3 and $\gamma_0, \dots, \gamma_3$. We indicate in Table 4 the most significant results. Recall that $B_1 < B_2$ and that a value for B_1 below 1 implies the existence of a z^k invariant subspace without the wandering property. The values of γ_i have been chosen of the form

$$\gamma_i = \phi_i k + i.$$

The values indicated there for several of the variables were selected in order to guarantee the objective function is clearly below 1. We keep $d_0 = 1, \gamma_0 = 0$, and $\gamma_1 = 1$ in all cases. Hence, we only show the values of $\alpha, \phi_2, \phi_3, d_1, d_2, d_3$ and B_1 , and a larger value of B_1 typically indicates that for that value of α it was more difficult to find a suitable example. Table 4 only contains values that achieved the objective for $k = 6$.

Without varying k from $k = 6$ to higher values it was very costly to determine adequate values for the other parameters for values of $\alpha > -5$. Allowing for higher values of k made it possible for the

TABLE 5. Numerical results for $k > 6$

α	ϕ_2	ϕ_3	d_1	d_2	d_3	k	B_1
-5	2	34	4	11	100000	7	0.875
-5	3	35	2.2	16	130000	10	0.71312
-4.5	3	50	1000	11	150000	12	0.96775
-4.25	3	97	70	0.54	70000	47	0.99436
-4.22	3	150	5000	0.2	150000	74	0.986
-4.2	3	166	10000	0.142	150000	88	0.999

barrier to be moved to $\alpha = -4.2$, as can be seen in the following Table 4.

Since we did not perform validation of the numerics for the experiments in this Section, the results should be taken not as a proof but rather as a hint and the values for which B_1 is close to 1 should be taken with special care. However, our results suggest that increasing α makes the problem harder and there is no particular reason to expect a much higher threshold than $\alpha = -4.2$. In any case, several choices have been made in our strategy of proof that limit the scope of the method. The chosen values of ϕ_2 , ϕ_3 , d_1 , d_2 , d_3 , and k for each α were chosen by experimentation with a variational spirit. We were unable to find any rationale on their behavior.

5. FURTHER REMARKS

- (1) Corollary 1.6 shows that the wandering property is not a property of the ambient space, as a set, but rather of the choice of equivalent norm. In this way, the Wold decomposition seems to be an obstruction only when we are dealing with the Hardy space H^2 (and perhaps spaces where the sequence ω is increasing).
- (2) It seems that a key feature of the values $k = 6$ and $\alpha = -16$ is that $|\alpha|$ is large enough for k . This is clearly the case in the values of $\alpha < -(5k + \frac{700}{(k-9)^2})$ when $k \geq 10$, from Theorem 1.7. One reasonable direction of research from here is to determine whether the wandering property in D_α spaces also fails for $k = 1, \dots, 9$. In any case, it seems plausible from all our observations that for each k there should be a value α_k^* , such that for all $\alpha < \alpha_k^*$, there exist z^k invariant subspaces in D_α (with their usual norms) without the wandering property, while for $\alpha > \alpha_k^*$ the wandering property holds. Perhaps a study of the derivatives of C_2 and C_4 in the proof of the Theorem 3.1 will

give information about this question. To deepen in this idea, let us comment that, of the possible spaces of the form H_ω^2 , we have seen that many contain a z^k invariant subspace without the wandering property. If we consider the set of all eligible sequences ω , this is true of at least a set of codimension 12. In fact, for each $t \in \mathbb{N}$ and for each ω for which there is a solution, variation of the minimization problem with respect to ω_t will also be continuous, so a small interval may be taken around ω_t that does not affect the existence of the non-wandering subspace. In Section 4, even more ways to generate such subspaces are described. This seems to point in the direction that if ω_t *decays fast enough, enough times* there should be z^k -invariant subspaces without the wandering property. The opposite direction (not fast enough decay or not enough times, implying wandering property) will be studied in more detail in [11].

- (3) F_1 and F_2 were chosen as polynomials with a very particular restriction on their coefficients (in Theorem 1.5, F_2 is of degree 5, and F_1 of degree $k + 3$ with several more constraints). It seems plausible to find other arrangements where, for instance, F_2 is of the form

$$F_2(z) = b_k z^k + \dots + b_{k+3} z^{k+3} + b_{k+5} z^{k+5}$$

instead, or where both functions include larger order coefficients, or where these depend more on the values of α and k . One may consider, as well, spaces generated by 3 or more functions, or non-polynomial functions. Each of these routes complicates the computations but provides hope of solving more cases than the ones presented here. We also deliberately ignored some degenerate solutions to some of the linear systems without exploring them completely. In any case, our proof does rely on the terms described and the proposed modifications would likely affect the procedure considerably.

- (4) The main role of the special coefficients a_4 and b_5 is to ensure a “register” of the functions f_1 and f_2 defining a function f in the subspace M . This register was encoded here in the Taylor coefficients of f using the fact that the closed span of $\{z^{ks+4}\}_{s \in \mathbb{N}}$ is an invariant subspace under z^k in D_α (respectively, $\{z^{ks+5}\}_{s \in \mathbb{N}}$ for b_5). This proved useful in Lemma 2.2 and several times after that. If one can find another way of doing this register without blocking so many coefficients only for that purpose, they may be able to increase the efficacy of our method to make it work for $k = 4$ or $k = 5$. This could be based on some division

lemmas in the style of Lemma 5.2 in [12]. Notice that some steps would need to be reproved like the fact that Cauchy-Schwarz inequality was strict when applied on Lemma 2.5 and Lemma 2.6. Our method does not seem to go any further than $k = 4$ since we need the matrix N to be at least of 4 columns wide for nontrivial solutions to exist to the linear system it induces, unless there are redundancies in the matrix N . Therefore, for the cases $k = 2$ and $k = 3$ an essential modification of our method seems necessary.

- (5) Up to an equivalent norm, D_{-16} is the same space as the weighted Bergman space

$$A_{15}^2 = \{f \in Hol(\mathbb{D}) : \|f\|^2 = \int_{\mathbb{D}} |f(z)|^2 (1 - |z|^2)^{15} dA(z) < \infty\},$$

which is another example of a H_{ω}^2 space. One seems inclined to wonder whether the counterexample presented here still fails to have the wandering property in this other norm, and if not, whether any other counterexample may be built in a similar spirit. The failure of the wandering property for the shift (the case $k = 1$) was shown for A_{β}^2 for all values of $\beta \geq 4$ in [18], so it seems plausible that the same threshold can be achieved for other values of k . In general, the same questions make sense for the standard Bergman-type spaces A_{β}^2 (given by weights $(1 - |z|^2)^{\beta}$, for $\beta > -1$). Only when $\beta = 0$ this coincides with the norm of the corresponding D_{α} space ($\alpha = -\beta - 1$). Our numerical results in the previous section (for D_{α} spaces and some $k \geq 6$) seem to agree with the results of [18] for $k = 1$ and the corresponding A_{β}^2 spaces.

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