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ON THE WANDERING PROPERTY IN DIRICHLET SPACES

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Abstract. We show that in a scale of weighted Dirichlet spaces D_α, including the Bergman space, given any finite Blaschke product B there exists an equivalent norm in D_α such that B satisfies the *wandering subspace property* with respect to such norm. This extends, in some sense, previous results by Carswell, Duren and Stessin [3]. As a particular instance, when $B(z) = z^k$ and $|\alpha| \leq \frac{\log(2)}{\log(k+1)}$, the chosen norm is the usual one in D_α.

1. Introduction

For isometries T acting on complex, separable, infinite dimensional Hilbert spaces $\mathcal{H}$, the classical Wold Decomposition Theorem asserts that whenever T is pure ($\bigcap_{n=0}^{\infty} T^n\mathcal{H} = \{0\}$), the closed subspace $\mathcal{K} = \mathcal{H} \ominus T\mathcal{H}$ has the *wandering subspace property* in $\mathcal{H}$: $\mathcal{H}$ coincides with the smallest closed invariant subspace under T generated by $\mathcal{K}$, denoted by $[\mathcal{K}]_T$. This is a consequence of the fact that $\mathcal{H}$ decomposes as the orthogonal direct sum of closed subspaces

$$\mathcal{H} = \mathcal{K} \oplus T\mathcal{K} \oplus T^2\mathcal{K} \oplus \dots$$

More generally, a subspace of a Hilbert space is called a *wandering subspace* of a given operator if it is orthogonal to its images under positive powers of the operator. In this regards, the Wold Decomposition Theorem says that every invariant subspace of a pure isometry is indeed, generated by a wandering subspace.

Well known examples arise when considering multiplication operators induced by inner functions in the classical Hardy space H^2. Recall that an inner function θ is an analytic function in the unit disc $\mathbb{D}$ with contractive values ($|\theta(z)| \leq 1$ for $z \in \mathbb{D}$) such that the boundary values

$$\theta(e^{it}) := \lim_{r \to 1^-} \theta(re^{it})$$

have modulus 1 for almost all t (they exist for almost every t with respect to Lebesgue measure on the unit circle). In such cases, every closed subspace $\mathcal{M}$ in H^2 invariant under multiplication by θ is wandering and

$$[\mathcal{M} \ominus \theta\mathcal{M}]_\theta = \mathcal{M}.$$

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Accordingly, θ is said to have the *wandering subspace property* (WSP).

Nevertheless, it is not completely understood yet which functions φ in H^∞ (the space of bounded analytic functions on $\mathbb{D}$) enjoy the WSP in H^2, that is, for which functions do the corresponding multiplication operators M_φ on H^2 satisfy

$$[\mathcal{M} \ominus \varphi\mathcal{M}]_{M_\varphi} = \mathcal{M}$$

for every closed invariant subspace $\mathcal{M}$. In [11], it was shown that a necessary condition is that φ be writable as the composition $G \circ h$, where h is an inner function and G is univalent in $\mathbb{D}$. Moreover, they also proved a sufficient condition, namely, $\varphi = G \circ h$ with G a weak-star generator of H^∞. Whether this last condition is in fact a necessary one is left open.

The question turns out to be drastically difficult to handle whenever the underlying Hilbert space is the Bergman space A^2. In a remarkable paper, Aleman, Richter and Sundberg [1] proved that $\varphi(z) = z$ possesses the WSP in A^2. However, for univalent functions, Carswell [4] showed the existence of bounded univalent functions φ in $\mathbb{D}$, vanishing at the origin and failing to have the WSP both in H^2 and A^2. Indeed, previously in [3], the authors had provided necessary conditions for H^∞ functions to have the WSP in A^2. They showed that in particular, not every inner function has this property in A^2 and even that infinite Blaschke products without it can be found. For finite Blaschke products, the question in the Bergman space remains open (see [5]).

The main goal of this work is showing that not only in the Bergman space but also in a scale of weighted Dirichlet spaces D_α including A^2, for every finite Blaschke product B, it is possible to renorm the space (with an equivalent norm) such that B enjoys the wandering subspace property. This seems to follow an opposite direction to a recent work by our third author [15], in which renormings were found of the same spaces allowing one to *disprove* the corresponding WSP for multiplication by some monomials. In sum, one conclusion of the present work is that the geometry of the space plays a significant role in order to deal with this question, since its answer depends strongly on the norm expression.

The rest of the manuscript is organized as follows. In Section 2 we recall some preliminaries, introducing the family of weighted Dirichlet spaces D_α, where our work takes place. We will recall Shimorin's Theorem [17], which provides a unified proof of the theorems of Beurling [2] and Aleman, Richter and Sundberg [1] and shows, in particular, that $\varphi(z) = z$ possesses the WSP in the scale of D_α spaces considered. In addition, we introduce some basic results justifying the direction of the proof of our main results, which will be proved in Section 3. Moreover, some consequences are derived, including the observation that for a range of α the WSP holds for $\varphi(z) = z^k$ $(k \geq 1)$ even with the original norm. Finally, and even though we were not able to answer this question for A^2 with its usual norm, we were able to establish the WSP for z^k acting on its finite codimensional subspace $z^k A^2$.

2. The setting

2.1. **Dirichlet-type spaces.** Let α be a real number. The Dirichlet-type space D_α consists of analytic functions $f(z) = \sum_{k=0}^{\infty} a_k z^k$ in $\mathbb{D}$ such that its norm

$$\|f\|_\alpha := \left(\sum_{k=0}^{\infty} |a_k|^2 (k+1)^\alpha \right)^{1/2}$$

is finite. Observe that particular instances of $\alpha's$ yield well-known Hilbert spaces of analytic functions in $\mathbb{D}$. More precisely, when $\alpha = -1$ we have the classical Bergman space A^2, $\alpha = 0$ corresponds to the Hardy space H^2, and $\alpha = 1$ to the Dirichlet space $\mathcal{D}$. Note that the continuous inclusion $D_\beta \subsetneq D_\gamma$ holds for all $\gamma < \beta$, i.e., $\|f\|_\gamma \leq \|f\|_\beta$ for all $f \in D_\beta$ and $\gamma < \beta$. Moreover, when $\beta > 1$ the spaces D_β are continuously embedded in the disc algebra $\mathcal{A}$.

Dirichlet-type spaces are particular instances of general *weighted Hardy spaces*, introduced by Shields [16] to study weighted shifts in ℓ^2. There is an extensive literature on these spaces, and we refer the reader to [7, Chapter 2], for instance.

Recall that an analytic function u in $\mathbb{D}$ is a *multiplier* of D_α, if the analytic Toeplitz operator $T_u : f \mapsto uf$ is defined everywhere on D_α (and hence bounded, by the Closed Graph Theorem). A well known fact about the Dirichlet space is that the algebra $\mathcal{M}(\mathcal{D})$ of all the multipliers of $\mathcal{D}$ is not easy to describe. In particular, the strict inclusion $\mathcal{M}(\mathcal{D}) \subset \mathcal{D} \cap H^\infty$ holds. Indeed, the elements of $\mathcal{M}(\mathcal{D})$ were characterized by Stegenga [18] in a notable paper, in terms of a condition involving the logarithmic capacity of their boundary values. We refer to [19] for multipliers and Carleson measures in Dirichlet spaces and to [8] for more on the subject of multipliers of D_α.

In any case, it is not difficult to prove that every finite Blaschke product is a multiplier of D_α for all $\alpha \in \mathbb{R}$. Recall that a finite Blaschke product is given by

$$B(z) = e^{i\theta} \prod_{i=1}^{N} \frac{z - \alpha_i}{1 - \overline{\alpha_i} z}, \qquad (z \in \mathbb{D})$$

where $\alpha_i \in \mathbb{D}$, counted according to its (prescribed) multiplicity. Finite Blaschke products play an important role in mathematics and connected areas such as complex geometry, linear algebra, operator theory and systems. We refer to the recent monograph [9] for a detailed account of these results.

In order to analyze whether any finite Blaschke product B satisfies the WSP in D_α, we begin by considering the concrete example $B(z) = z^2$ acting on the Bergman space A^2 (an open problem specifically posed in [5]).

2.2. **Vector valued shifts.** The following approach is based on some ideas described in [12, 14]. We may consider the space A^2 as a direct (*orthogonal*) sum of two copies of itself, A^2_1 and A^2_2, where a function $f \in A^2$ is decomposed as

$$f(z) = f_1(z^2) + z f_2(z^2).$$

It is clear that $f \in A^2$ if and only if $f_1, f_2 \in A^2$, but we are imposing different equivalent norms on each copy of A^2. We may think either A^2_1 and A^2_2 equipped with their usual norms and their sum A^2 equipped with the norm arising from such sum, or on the contrary, A^2 with usual norm decomposed as sum of two subspaces which inherit some comparable norm. We consider here the first of those choices. By doing so, we may view the operator M_{z^2} as a diagonal matrix shift sending $(f_1, f_2) \in A^2 \oplus A^2$ to (Sf_1, Sf_2). In this sense, M_{z^2} may be expressed as

$$\begin{pmatrix} S & 0 \\ 0 & S \end{pmatrix}. \tag{1}$$

The techniques developed by Nordgren when trying to solve Problem 151 in [10] suggest a particular direction to study the problem we have in mind. If $B(z) = z^k$

does not satisfy the WSP, some closed invariant subspace $\mathcal{M}$ such that

$$\mathcal{M} \neq [\mathcal{M} \ominus z^k \mathcal{M}]_{z^k} \qquad (2)$$

could, perhaps, be described through a finite number of linear conditions. For instance, a finite number of generators $h_1, ..., h_r \in H^\infty$ multiplied by functions $f_1, ..., f_r \in A^2$, which, in addition, satisfy some finite number of restrictions on their Taylor coefficients. It seems difficult to come up with restrictions on the Taylor coefficients involving coefficients of degree higher than k, and still generate a non trivial closed invariant subspace of A^2. However, it appears plausible that a counterexample may be found for M_{z^2} looking at how the matrix operator (1) acts on the product space.

This is the idea behind the proofs of the following preliminary results, in which it is possible to guarantee that a closed invariant subspace $\mathcal{M}$ for M_{z^2}, that is, $\mathcal{M} \in Lat(M_{z^2})$, is generated by $\mathcal{M} \ominus z^2\mathcal{M}$ whenever either $\mathcal{M}$ is also invariant for the shift, or decomposable as direct sum of closed subspaces in each of the two copies of A^2, say $\mathcal{M}_1 \subset A^2_1$ and $\mathcal{M}_2 \subset A^2_2$.

Proposition 2.1. *Let $\mathcal{M} \in Lat(M_z)$, then $\mathcal{M} = [\mathcal{M} \ominus z^2\mathcal{M}]_{z^2}$.*

Proof. Since $\mathcal{M} \supset z\mathcal{M} \supset z^2\mathcal{M}$, we have the decomposition

$$\mathcal{M} \ominus z^2\mathcal{M} = (\mathcal{M} \ominus z\mathcal{M}) \oplus (z\mathcal{M} \ominus z^2\mathcal{M}).$$

Since $\mathcal{M} \in Lat(M_z)$, Aleman, Richter and Sundberg's Theorem [1] yields $\mathcal{M} = [\mathcal{M} \ominus z\mathcal{M}]_z$, or equivalently

$$\mathcal{M} = \overline{\{pf : p \in \mathcal{P}, f \in \mathcal{M} \ominus z\mathcal{M}\}},$$

where $\mathcal{P}$ denotes the space of all polynomials.

We decompose $\mathcal{P}$ as the span of $\widetilde{\mathcal{P}}_0 := \{p : p(z) = q(z^2), q \in \mathcal{P}\}$ and $\widetilde{\mathcal{P}}_1 := \{p : p(z) = zq(z^2), q \in \mathcal{P}\}$. Choosing the induced norm in each copy of A^2, we have that

$$\mathcal{M} \subset [\mathcal{M} \ominus z\mathcal{M}]_{z^2} \oplus [z\mathcal{M} \ominus z^2\mathcal{M}]_{z^2} \subset [\mathcal{M} \ominus z^2\mathcal{M}]_{z^2}.$$

The other inclusion ($[\mathcal{M} \ominus z^2\mathcal{M}]_{z^2} \subset \mathcal{M}$) is always satisfied. □

Proposition 2.2. *Let $\mathcal{M} = \mathcal{M}_1 \oplus \mathcal{M}_2$ with $\mathcal{M}_1 \subset A^2_1$, $\mathcal{M}_2 \subset A^2_2$. Then $\mathcal{M} = [\mathcal{M} \ominus z^2\mathcal{M}]_{z^2}$.*

Proof. Since $\mathcal{M}_1$, $\mathcal{M}_2$ are shift invariant, it will be a direct consequence of the main theorem in [1]:

$$\mathcal{M} = \mathcal{M}_1 \oplus \mathcal{M}_2 = [\mathcal{M}_1 \ominus z\mathcal{M}_1]_z \oplus [\mathcal{M}_2 \ominus z\mathcal{M}_2]_z.$$

Since $\mathcal{M} \ominus z^2\mathcal{M}$ contains the direct sums $\mathcal{M}_1 \ominus z\mathcal{M}_1$ and $\mathcal{M}_2 \ominus z\mathcal{M}_2$, then $\mathcal{M}$ is generated by $\mathcal{M} \ominus z^2\mathcal{M}$. Again, the other inclusion ($[\mathcal{M} \ominus z^2\mathcal{M}]_{z^2} \subset \mathcal{M}$) is always satisfied. □

If we call $\mathcal{M}_1 = \mathcal{M} \cap A^2_1$ and $\mathcal{M}_2 = \mathcal{M} \cap A^2_2$, it is necessarily true that $\mathcal{M}_1 \perp \mathcal{M}_2$, $\mathcal{M}_1$ and $\mathcal{M}_2$ are shift invariant, and $\mathcal{M}_1 \oplus \mathcal{M}_2 \supset \mathcal{M}$ but $\mathcal{M}$ may be defined, for instance, through restrictions between the $\mathcal{M}_1$ and $\mathcal{M}_2$ components.

On the other hand, it is possible to provide invariant spaces $\mathcal{M} \in Lat(M_{z^2})$ not satisfying the hypotheses of Propositions 2.1 and 2.2, but such that $\mathcal{M} = [\mathcal{M} \ominus z^2\mathcal{M}]_{z^2}$:

Example 2.3. Let $a \in \mathbb{C}\backslash\{0\}$, $h(z) = 1 + az$, and $\mathcal{M} = [h]_{z^2}$. Then $\mathcal{M} = [\mathcal{M} \ominus z^2\mathcal{M}]_{z^2}$ since h is orthogonal to $z^2\mathcal{M}$ and generates $\mathcal{M}$.

Notice that in this case, if we denote $\mathcal{M}_i = \mathcal{M} \cap A_i^2$, we have $\mathcal{M}_1 = \mathcal{M}_2 = A^2$ but $1 \in A^2 \backslash \mathcal{M}$, and $\mathcal{M} \in Lat(M_{z^2}) \backslash Lat(M_z)$. It can be shown that any space generated by a finite collection of elements without any relations also provides similar examples (being the norm imposed on A^2 the product of k usual norms for collections of k elements).

2.3. **Shimorin's Theorem.** The main contribution regarding the wandering subspace property in a variety of spaces was carried out by Shimorin in [17]. In particular, he showed that $\varphi(z) = z$ satisfies the WSP in D_α for $\alpha \in [0, 1]$ since the operators of multiplication by z are *concave*, i.e., for every $x \in D_\alpha$, $\|T^2x\|^2 - 2\|Tx\|^2 + \|x\|^2 \leq 0$. For $\alpha \in [-1, 0)$, the WSP follows as a consequence of the following result:

Theorem 1 (Shimorin)**.** *Let T be a bounded operator in a Hilbert space H such that the following hold:*

(i) $\bigcap_{n\in\mathbb{N}} T^n H = \{0\}$

(ii) *For x, $y \in H$, we have*

$$\|x + Ty\|^2 \leq 2(\|Tx\|^2 + \|y\|^2).$$

Then T has the wandering subspace property in H.

Observe that Shimorin's approach only applies to the *usual norms* in D_α (those described above). In the recent paper [15], Seco has shown for each $\alpha \in \mathbb{R}$ and each positive integer $k \geq 6$, the existence of an equivalent norm $\| \cdot \|$ in D_α and $\mathcal{M} \in Lat(M_{z^k})$ that fails to have the wandering property with respect to the norm $\| \cdot \|$, that is,

$$\mathcal{M} \neq [\mathcal{M} \ominus z^k \mathcal{M}]_{z^k} \text{ respect to } \| \cdot \|.$$

In particular, this is shown in some cases to be the usual norm for D_α: for instance when $k \geq 10$ and $\alpha < -(5k + \frac{700}{(k-9)^2})$, or when $\alpha \in (-16-\varepsilon, -16+\varepsilon)$ and $k = 6$, but numerical results hint that for $\alpha < -4.2$ there might be k large enough providing counterexamples (see also [13] for related results in this direction). The results in the next Section will establish, nevertheless, that by means of renormings it is possible to have the WSP for any finite Blaschke product.

3. The wandering subspace property under renormings

In this section, we show that in any D_α with $\alpha \in [-1, 1]$ (where $\varphi(z) = z$ meets the WSP), given any finite Blaschke B product, it is possible to renorm the space (with an equivalent norm) such that B also has the WSP.

Before that, observe that Example 2.3 may be generalized to the case where instead of z^2 we make use of any finite Blaschke product. For a function $f \in H^2$, given any finite Blaschke product, B, it is clear from the Wold decomposition that we can express f as

$$f(z) = \sum_{k=0}^{\infty} B^k(z) h_k(z),$$

where h_k are functions in the model space $H^2 \ominus BH^2$, and the norm of f may be found from those of h_k. Indeed,

$$\|f\|_0^2 = \sum_{k=0}^{\infty} \|h_k\|_0^2,$$

where, recall that $\|\cdot\|_0$ corresponds to the H^2-norm.

In [6], the authors find an analogous expansion for the D_α spaces:

Theorem 3.1 (Chalendar, Gallardo-Gutiérrez, Partington)**.** *Let $\alpha \in [-1,1]$ and B any finite Blaschke product. Then $f \in D_\alpha$ if and only if $f = \sum_{k=0}^{\infty} h_k B^k$ (convergence in D_α norm) with $h_k \in K_B$ and*

$$\sum_{k=0}^{\infty}(k+1)^\alpha \|h_k\|_0^2 < \infty.$$

Remark 3.2. The previous theorem was stated in [6] for $\alpha \in \{-1,0,1\}$ and $B(0) = 0$, but the same scheme of proof works bearing in mind two key facts:

(i) Multiplication by any function in the model space $H^2 \ominus BH^2$ is a bounded operator.

(ii) Composition with the Blaschke product is a bounded operator in D_α.

These are both easy to check and the only parts of the proof that generalize in a non-obvious way. The assumption $B(0) = 0$ is not really necessary since the spaces $B^n K_B$ are still mutually orthogonal in H^2, and hence, linearly independent finite-dimensional spaces.

We are now in a position to state the following:

Theorem 3.3. *Let $\alpha \in [-1,1]$ and B a finite Blaschke product. Then there exists a norm $\|\cdot\|_B$ under which B has the wandering subspace property in D_α, that is, for any $\mathcal{M} \in Lat(M_B)$ we have*

$$[\mathcal{M} \ominus B\mathcal{M}]_B = \mathcal{M} \quad \text{with respect to } \|\cdot\|_B.$$

Moreover, for $B(z) = z^k$ and $|\alpha| \in [0, \log(2)/\log(k+1)]$, the norm $\|\cdot\|_B$ coincides with the usual D_α norm $\|\cdot\|_\alpha$.

Proof. Given B a finite Blaschke product, let $\|\cdot\|_B$ denote the norm defined by the corresponding expression arising from Theorem 3.1. Then, the multiplication operator induced by B acts exactly as the shift operator M_z acts on D_α with respect to $\|\cdot\|_B$; therefore it satisfies property (ii) in Shimorin's Theorem. Consequently, M_B has the WSP.

The property (ii) for the shift in such spaces can be checked by showing the following two properties:

$$\omega_1 \geq 1/2,$$

$$\omega_n(\omega_{n-1} + \omega_{n+1}) \leq 2\omega_{n-1}\omega_{n+1}, \quad n \geq 1,$$

where $\omega_n = (n+1)^\alpha$.

Finally, assume $B(z) = z^k$ and consider the usual norm $\|\cdot\|_\alpha$. Let $\alpha \in [-\frac{\log(2)}{\log(k+1)}, 0)$ and notice that in this case, the proof of the second inequality above works in the same way as for M_z with ω_{n-1} substituted by ω_{n-k} and ω_{n+1} substituted by ω_{n+k}. The first inequality is satisfied substituting ω_1 by ω_k precisely because $|\alpha| \leq \frac{\log(2)}{\log(k+1)}$. If $\alpha \geq 0$ apply the same reasoning to $1/\omega_k$ to see that the operator is concave. □

It seems worth mentioning that if we take $B(z) = z^2$ in Theorem 3.3, the range of values of α for which the result holds without renorming can actually be improved by moving the lower bound from $\alpha \geq -\log(2)/\log(3) \approx -0.6309$ to $\alpha \geq -\log(2/3)/\log(5/3) \approx -0.7937$:

Proposition 3.4. *Let $\alpha \in [-\log(2/3)/\log(5/3), 0]$. Then the wandering subspace property holds for the operator of multiplication by z^2 in D_α equipped with its usual norm $\|\cdot\|_\alpha$.*

Proof. First note that we can define a norm in D_α, given by a weight ω that makes multiplication by z^2 on D_α space satisfy the Shimorin condition just by changing the weights on the first coordinate ($\omega_0 = \|1\|^2$): Indeed, define the weight ω by $\omega_k = (k+1)^\alpha$ for $k \geq 1$ and ω_0 will be determined later. Condition (i) in Shimorin's Theorem is trivially satisfied and condition (ii) is equivalent to meeting all of the following:

(a) $\omega_0 \leq 2\omega_2$.
(b) $\omega_1 \leq 2\omega_3$.
(c) $(1/\omega_{n-2} + 1/\omega_{n+2} - 2/\omega_n) \leq 0$ for all $n \geq 2$.

Property (b) is equivalent to $2^\alpha \leq 2^{2\alpha+1}$, which is immediately checked since $\alpha \geq -1$. Standard calculus techniques show the validity of (c), for $n \geq 3$ and we are left with finding ω_0 such that

$$\frac{1}{2 \cdot 3^{-\alpha} - 5^{-\alpha}} \leq \omega_0 \leq 2 \cdot 3^\alpha.$$

Therefore, if we assume

$$1 \leq (2 \cdot 3^{-\alpha} - 5^{-\alpha})(2 \cdot 3^\alpha),$$

there is a valid choice of ω_0 such that ω defines a norm in D_α for which the WSP holds. The latter equation is equivalent to

$$\alpha \geq \frac{\log(2/3)}{\log(5/3)}.$$

Now we know that for any z^2-invariant $\mathcal{M}$, the space $\mathcal{M} \ominus z^2\mathcal{M}$ is exactly the same under the original norm and the new norm, and so even if the norm is different, whether or not the WSP holds does not change. So we get the desired result under the original norm. □

Remark 3.5. One could be inclined to think that the WSP for z^k on A^2 follows from that on z^kA^2, shown in the previous proposition, based on its finite codimension as a subspace of A^2. However, it follows from [15] that $\mathbb{C}^{22} \oplus z^6A^2$ fails to have the z^6 WSP if we equip $\mathbb{C}^{22}$ with the weight $\omega_t = (t+1)^{-16}$ for $t = 0, ..., 21$. This space still contains z^6A^2 as a finite codimension subspace.

Proposition 3.6. *Let $k \in \mathbb{N}$, $\alpha \in [-1, 0]$ and $\mathcal{M} = z^kD_\alpha$. Then z^k has the wandering subspace property in $\mathcal{M}$.*

Proof. First, we can assume $\alpha < 0$. For $s \in \mathbb{N}$ denote by $\omega_s = (s+1)^\alpha$. The condition (ii) of Shimorin's Theorem becomes equivalent to

(a) $\omega_s \leq 2\omega_{k+s}$, for all $s = k, ..., 2k-1$, and
(b) $(1/\omega_s + 1/\omega_{s+2k} - 2/\omega_{s+k}) \leq 0$ for all $s \geq k$.

To see (a), notice that the minimum of $(s+1)/(k+s+1)$ for $s = k, ..., 2k-1$ is achieved at $s = k$, that such minimum is therefore bigger than $1/2$ and that $\alpha \geq -1$. In order to check (b), it suffices to see that the quantity

$$g(s) = (s+1)^{-\alpha} - (s+k+1)^{-\alpha}$$

is negative and increasing on s. Negativity is clear since the exponent $-\alpha$ is positive and $(s+k+1) \geq (s+1)$. Moreover $g'(s) = |\alpha|((s+1)^{-\alpha-1} - (s+k+1)^{-\alpha-1})$, which is positive since $\alpha \geq -1$. □

Remark 3.7. Proposition 3.6 may be interpreted as a property of the *subspace* $z^k D_\alpha$ or as a property of the *equivalent norm on* D_α given by $\|f\| := \|S^k f\|_\alpha$, that is, as a property of D_α with this particular choice of equivalent norm. In this sense, it yields a different proof of Theorem 3.3 for the case when B is a monomial.

We conclude with the following result, which reduces the question further.

Corollary 3.8. *Let $k \geq 1$, $\alpha \in [-1, 0]$, and $\mathcal{M}$ be a z^k-invariant subspace of D_α. Then*

$$\mathcal{M} = [\mathcal{M} \ominus z^{2k}\mathcal{M}]_{z^k}.$$

Proof. Denote $T := M_{z^k}$ acting on D_α. Let $\mathcal{M}$ be a closed T-invariant subspace of D_α. Then $\mathcal{N} := T\mathcal{M}$ is a T-invariant subspace. Moreover, $\mathcal{N} \subset TD_\alpha$ and hence, by Proposition 3.6 we have

$$\mathcal{N} = [\mathcal{N} \ominus T\mathcal{N}]_T.$$

Now we can see that

$$\mathcal{M} \ominus T^2\mathcal{M} = (\mathcal{M} \ominus T\mathcal{M}) \oplus (T\mathcal{M} \ominus T^2\mathcal{M}).$$

So the smallest closed T-invariant subspace containing $\mathcal{M} \ominus T^2\mathcal{M}$ contains both $\mathcal{M} \ominus \mathcal{N}$ and $\mathcal{N}$, and so, it is $\mathcal{M}$. □

References

[1] Aleman, R., Richter, S., and Sundberg, C., Beurling's theorem for the Bergman space, *Acta Math.* **177** no. 2 (1996) 275–310.

[2] Beurling, A., On two problems concerning linear transformations in Hilbert space, *Acta Math.* **81** (1949) 239–255.

[3] Carswell, B. J., Duren, P. L., and Stessin, M. I., Multiplication invariant subspaces of the Bergman space, *Indiana Univ. Math. J.* **51** no. 4 (2002) 931–961.

[4] Carswell, B. J., Univalent mappings and invariant subspaces of the Bergman and Hardy spaces, *Proc. Amer. Math. Soc.* **131** no. 4 (2003) 1233–1241.

[5] Carswell, B. J. and Weir, R. J., Weighted reproducing kernels and the Bergman space, *J. Math. Anal. Appl.* **399** (2013) 617–624.

[6] Chalendar, I., Gallardo-Gutiérrez, E. A., and Partington, J. R., Weighted composition operators on the Dirichlet space: Boundedness and spectral properties, *Math. Ann.* **363** (2015) 1265–1279.

[7] Cowen, C. C. and MacCluer, B. D, *Composition Operators on Spaces of Analytic Functions*, Studies in Advanced Mathematics. CRC Press, Boca Raton, FL, 1995.

[8] El Fallah, O., Kellay, K., Mashreghi, J., and Ransford, T., *A primer on the Dirichlet space*, Cambridge University Press, 2014.

[9] Garcia, S. R., Mashreghi, J., and Ross, W., *Finite Blaschke Products and their connections*, Springer, 2018.

[10] Halmos, P. R., *A Hilbert space problem book*, Van Nostrand, Princeton, NJ, 1967.

[11] Khavinson, D., Lance, T. L. and Stessin, M. I., Wandering property in the Hardy space, *Michigan Math. J.* **44** no. 3 (1997) 597–606.

[12] Nordgren, E. A., Invariant subspaces of a direct sum of weighted shifts, *Pacific J. Math.* **27** no. 3 (1968) 587–598.

[13] Nowak, M. T., Rososzczuk, R. and Wołoszkiewicz-Cyll, M., Extremal functions in weighted Bergman spaces, *Complex Var. Elliptic Equ.* **62** no. 1 (2017) 98–109.

[14] Partington, J. R., *Linear operators and linear systems: an analytical approach to control theory*, Cambridge University Press, 2004.

[15] Seco, D., A z^k-invariant subspace without the wandering property, *J. Math. Anal. Appl.* **472** no. 2 (2019) 1377–1400.
[16] Shields, A. L. Weighted shift operators and analytic function theory *Topics in Operator Theory, Math. Surveys Monographs* Amer. Math. Soc., Providence, RI, **13** (1974) 49–128.
[17] Shimorin, S., Wold-type decompositions and wandering subspaces for operators close to isometries, *J. Reine Angew. Math.* **531** (2011) 147–189.
[18] Stegenga, D. A., Multipliers of the Dirichlet space, *Illinois J. Math.* **24** no.1 (1980) 113–139.
[19] Wu, Z., Carleson measures and multipliers for Dirichlet spaces, *J. Funct. Anal.* **169** (1999) 148–163.

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