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The role of inertia in the rupture of ultrathin liquid films

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ABSTRACT
Theory and numerical simulations of the Navier–Stokes equations are used to unravel the influence of inertia on the dewetting dynamics of an ultrathin film of Newtonian liquid deposited on a solid substrate. A classification of the self-similar film thinning regimes at finite Ohnesorge numbers is provided, unifying previous findings. We reveal that, for Ohnesorge numbers smaller than one, the structure of the rupture singularity close to the molecular scales is controlled by a balance between liquid inertia and van der Waals forces, leading to a self-similar asymptotic regime with \( h_{\text{min}} \propto \tau^{2/5} \) as \( \tau \to 0 \), where \( h_{\text{min}} \) is the minimum film thickness and \( \tau \) is the time remaining before rupture. The flow exhibits a three-region structure comprising an irrotational core delimited by a pair of boundary layers at the wall and at the free surface. A potential-flow description of the irrotational core is provided, which is matched with the vortical layers, allowing us to present a complete parameter-free asymptotic description of inertia-dominated film rupture.

I. INTRODUCTION

A growing number of emerging technologies involve the manipulation of liquid metals from millimeter to sub-micron scales.\(^7,8,15\) For instance, in additive manufacturing and electronics, liquid gallium has received special attention mainly due to its non-toxic character and its low melting point, just slightly above room temperature. In addition, plasmonic devices and many patterning and coating processes rely on metal or metal-like materials such as silver and gold. Many of these applications involve the presence of liquid jets, liquid droplets, and thin liquid films in critical intermediate stages. In particular, thin liquid films have been extensively studied due to their central role in many engineering devices, as well as in geological and physiological flows (see Refs.4–6 and Ref. 20 for thorough and excellent reviews). In most of the previous theoretical studies of liquid film dynamics, the effect of liquid inertia has been neglected, an approximation that is highly accurate when the characteristic length scale is small and the liquid viscosity is large. In contrast, inertia cannot be neglected when the working fluid is a liquefied metal, as clearly pointed out in a number of recent investigations.\(^1,2,3,5,6,14\)

In the present work, we report a theoretical and numerical study of the influence of liquid inertia on the instability, nonlinear dynamics, and rupture of ultra-thin liquid films placed on a solid impermeable substrate, which are known to become unstable in the non-wetting case due to the action of the long-range van der Waals (vdW) forces. In particular, we study and classify the self-similar regimes that are transiently achieved as gas–solid contact is approached, allowing us to provide a unified description of all previous theoretical findings. We also deduce a new non-slender inertial regime in which the minimum film thickness scales with the time to contact as \( h_{\text{min}} \propto \tau^{2/5} \), stemming from a balance between liquid inertia and vdW forces, the capillary force being asymptotically subdominant. The latter result complements the existing descriptions of the inertial limit, which predict \( h_{\text{min}} \propto \tau^{2/7} \) from a balance of liquid inertia with capillary and vdW forces,\(^10,25\) in that the latter slender flow regimes experience a crossover to the newly found solution due to the breakdown of slenderness when the molecular scale is approached.

This paper is organized as follows: In Sec. II, we present the mathematical model used to describe the inertial dewetting flow, and we present numerical integrations of the Navier–Stokes equations
aimed at establishing its main dynamical features. In Sec. III, we first provide an exhaustive classification of the asymptotic self-similar regimes during film thinning at finite Ohnesorge numbers. We then focus on the inertial near-rupture flow in Sec. IV, including the universal self-similar potential flow in the bulk of the liquid film, as well as the wall and free-surface boundary layers. Some concluding remarks are finally presented in Sec. V.

II. GOVERNING EQUATIONS AND NUMERICAL RESULTS

A. Navier–Stokes equations

We consider the incompressible flow resulting from the thinning of a dewetting two-dimensional thin liquid film of density \( \rho \), viscosity \( \mu \), and initial thickness \( h_0 \), initially resting on a solid substrate and separated from a passive immiscible ambient by an interface of constant surface tension coefficient \( \sigma \). Such films are known to become unstable to infinitesimal perturbations due to long-range intermolecular forces when the film thickness lies below a thickness threshold of about the 100 nm.\(^{1,2,3}\) The overall actions of such forces are modeled through a disjoining pressure term with the associated potential energy per unit volume \( A/(6h^5) \), where \( A \) is the Hamaker constant\(^{1} \) reflecting the strength of the intermolecular interactions. Note that this potential is derived for the particular case of the vdW force between two parallel surfaces. Although the rigorous way to compute these long-range forces would imply the need to solve an integro-differential problem for the coupled evolution of these forces and the flow, the latter task has never been attempted to study thin-film rupture, at least as far as we know. The simplification used in all hydrodynamic studies consists in substituting the exact integral of the vdW potential, either by the parallel-plane approximation used in the present manuscript, in most cases, or by introducing corrections to the disjoining pressure that take into account nonparallelism. It should be pointed out that the local flow in the immediate vicinity of the minimum interface radius could be very similar using the exact representation, since by definition, the local slope of the interface is zero at the minimum. However, the only way to check the latter hypothesis is performing an actual calculation of the full integro-differential description, a formidable task that we have not attempted. To describe the resulting unstable dynamics, we make use of the so-called augmented incompressible Navier–Stokes equations, which are non-dimensionalized upon taking

\[
\ell_c = a = \frac{A}{6\pi \sigma}, \quad v_c = \sqrt{\frac{A}{6\pi \rho a}}, \quad \ell_t = \frac{a}{v_c} = \sqrt{\frac{\rho a^3}{\sigma}}, \quad p_t = \phi_c = \frac{A}{6\pi a^3},
\]

as the characteristic length, velocity, time, pressure, and intermolecular potential scales, respectively, where \( a \) is the molecular length scale and \( v_c \) is a characteristic velocity straightforwardly derived from a balance between inertial and vdW forces (see Sec. III). Note also that \( t_c \) can be interpreted as the capillary time based on \( a \). We would like to point out that the reason for our selection of \( a \) as the relevant length scale, instead of the more common choice of \( h_0 \), is due to the fact that, here, we focus on the film thinning dynamics near its rupture. Thus, we expect that the local flow close to the singularity loses memory of the initial conditions, and in particular, the initial thickness \( h_0 \) should become irrelevant during the last stages. The use of the scales (1) yields

\[
\nabla \cdot \mathbf{v} = 0, \quad x \in \mathcal{V}, \quad (3a)
\]

\[
\frac{D\mathbf{v}}{Dt} = -\nabla h^{-3} + \nabla \cdot \mathbf{T}, \quad x \in \mathcal{V}, \quad (3b)
\]

as the relevant equations of motion, where \( \mathbf{T} = -p l + \rho g (\nabla \mathbf{v} + \nabla ^\intercal \mathbf{v}) \) is the stress tensor of the fluid, \( \mathbf{D}/\mathbf{D}t = \partial_t + \mathbf{v} \cdot \nabla \) is the substantial derivative. \( Oh = \mu / \sqrt{\rho \sigma} = \mu (6\pi / (\rho^2 A))^{1/4} \) is the Ohnesorge number based on \( a, \mathcal{V} \) is the liquid film domain, \( \mathbf{x} = (x, y) \) are Cartesian coordinates, and \( \mathbf{v} = (u, v) \) is the velocity field, assumed to be two-dimensional. We note here that the first authors to include the disjoining pressure as an augmented pressure in the full Navier–Stokes equations were Ruckenstein and Jain,\(^{2,3}\) followed later on by many investigators such as Orón et al.,\(^{2,3}\) Craster and Matar,\(^{2,3}\) and Blossey.\(^{2,3}\) Note also that the exploring the equivalent axisymmetric configuration, leading to film breakup modes of hole or ring type,\(^{2,3}\) is certainly interesting but is out of the scope of the present study. At the free surface \( \partial \mathcal{V} \), we impose the kinematic and stress balance boundary conditions, which read

\[
\mathbf{n} \cdot (\partial_t \mathbf{x}_c - \mathbf{v}) = 0, \quad x \in \partial \mathcal{V}, \quad (4a)
\]

\[
\mathbf{T} \cdot \mathbf{n} = -n (\nabla \cdot \mathbf{n}), \quad x \in \partial \mathcal{V}, \quad (4b)
\]

respectively, where \( \mathbf{x}_c \) is the parameterization of the interface, located at \( y = h(x, t) \), and \( \mathbf{n} \) is the unit normal vector to the interface. At the solid substrate, \( y = 0 \), the no-slip and no-penetration boundary condition is enforced, \( \mathbf{v} = 0 \). As for the initial conditions, in the numerical simulations, we consider half wavelength of a spatially periodic liquid film, and thus, we impose the symmetry condition, that is, \( u = 0 \) and no shear stress \( \mathbf{T} \cdot \mathbf{e}_x = 0 \) at \( x = 0 \) and \( x = \pi / k \), where \( k < k_c = \sqrt{3/\pi h_0^4} \) is the dimensionless wavenumber of the initially perturbed interface \( \mathbf{x}_c = [x, h(x_c, 1 - \epsilon \cos kx)] \) imposed at \( t = 0 \). Here, \( k_c \) is the dimensionless cut-off wavenumber predicted by linear instability theory\(^{2} \) (also Appendix A), \( h_0 = h_0^* / a \) is the initial film thickness normalized with the molecular length scale, and \( \epsilon \) is a small positive constant that triggers the instability and induces the rupture of the liquid film at \( x = 0 \) and \( t = t_0 \).

Two non-dimensional parameters govern the flow at hand, namely, the dimensionless initial film thickness, \( h_0 \), and the molecular Ohnesorge number, \( Oh \). These two parameters can be written in terms of the Weber and Ohnesorge numbers defined when taking the global scales, thus the appropriate ones to describe the initial linear regime of the thinning. These are the same scales as in (1) but substitute \( a / h_0^* \) everywhere. In particular, the corresponding characteristic velocity \( v_0^* \) is given by the balance \( v_0^*/h_0^* \sim A/(6h_0^5) \Rightarrow v_0^* = A/(6\pi h_0^5) \) (see Appendix A). Using these global scales, the corresponding Weber and Ohnesorge numbers are \( We = v_0^* h_0^* / \sigma = h_0^{12} \) and \( Oh = \sqrt{Oh} / \tau_0 = Oh We^{1/4} \). Note that none of these Ohnesorge numbers are the most commonly used, \( Oh \sim \mu / \sqrt{\rho h_0^3} \sim Oh / \sqrt{\tau_0} = Oh We^{1/4} \), and thus \( Oh = Oh Oh_0 \), which, henceforth, will be called global Ohnesorge number.

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B. Numerical simulations of the flow evolution toward the singularity

The finite-element method recently employed by Ref. 19 was slightly modified to integrate (3) and (4) for a wide range of values of Oh and \( h_\text{in} \). More details about the numerical method can be found in Appendix C. As the film evolved toward rupture, we tracked the minimum film thickness \( h_{\text{min}} = h(0, t) \), the maximum streamwise velocity, \( u_{\text{max}} \), and the maximum wall shear stress, \( \tau_{xy} = \partial_y u \big|_{y=0} \). Note that the existence of a power law \( h_{\text{min}} \propto \tau^n \) for some value of \( n > 0 \), where \( \tau = \tau_R - \tau \) is the time remaining to rupture, is a local self-similarity test that can be easily extracted from the numerical simulations. Figure 1 shows several representative film evolutions revealing that \((h_{\text{min}}, u_{\text{max}}, \tau_{xy}) \propto (\tau^{2/5}, \tau^{-3/5}, \tau^{-11/10})\) as \( \tau \to 0 \) for all the values of \( h_\text{in} \) and sufficiently small values of \( Oh \). However, it is important to emphasize that the latter universal behavior is achieved for \((\tau, h_{\text{min}}) \leq (0.01, 0.16)\), for which the continuum approximation is compromised. We thus conclude that the 2/5 power law would never be fully established under realistic conditions. For \( \tau \gtrsim 0.01 \) and \( h_\text{in} \lesssim 30 \), the results of Fig. 1 show no sign of sustained power-law behavior for \( Oh \lesssim 1 \), as revealed by the instantaneous exponents \( n(\tau) = d \log_{10} h_{\text{min}} / d \log_{10} \tau \) plotted in Figs. 1(b) and 1(c).

In particular, Fig. 1(b) reveals that, for all the cases where \( Oh \ll 1 \), the value of \( n \) increases monotonically with \( n(\tau) \to 2/5 \) as \( \tau \to 0 \) independently of \( h_\text{in} \). It is also deduced that the function \( n(\tau) \) has no inflection points for the smallest values of \( h_\text{in} \), namely, \( h_\text{in} = 1 \) and \( h_\text{in} = 3.16 \), indicating the absence of intermediate self-similarity other than the 2/5 power law, which, as shown in Sec. III, is due to a balance between liquid inertia and vdW forces with negligible surface tension and viscous forces (regime IV of Table I). However, for

![Figure 1](https://example.com/figure1.png)

**FIG. 1.** (a)–(b) The function \( \log_{10} h_{\text{min}}(\log_{10} \tau) \) for different values of \( h_\text{in} \) and \( Oh \) indicated in the legend. The gray dashed lines show the self-similar thinning laws associated with regimes I–IV in Table I. The top inset displays the maximum streamwise velocity, \( u_{\text{max}} \), and the maximum shear stress at the wall \( \tau_{xy} \) for \( h_\text{in} = 1 \) and \( Oh = 10^{-2} \). The bottom inset shows the curvature at the minimum normalized with the minimum thickness, \( h_{\text{min}} \kappa \), for the different cases indicated in the legend (solid lines), together with the scaling law \( h_{\text{in}} \kappa \propto \tau^{-4/7} \) (dashed line), deduced from the scales of regime III.15 The instantaneous exponent \( n(\tau) \): (c)–(d) influence of \( h_\text{in} \) for \( Oh \ll 1 \) and influence of \( Oh \) for \( h_\text{in} = (10, 30) \).
TABLE I. The four thinning regimes during the rupture of ultrathin liquid films. In the balances, $\rho$, $\mu$, and $\sigma$ stand for the liquid inertia, the viscous forces, and the capillary forces, respectively. The references stand for Zhang and Lister\textsuperscript{25} (ZL99), Moreno-Boza et al.\textsuperscript{19} (MB20), and Garg et al.\textsuperscript{19} (GA17).

<table>
<thead>
<tr>
<th>Regime</th>
<th>Validity</th>
<th>Balance</th>
<th>Realisability</th>
<th>$h_{\text{min}}(\tau)$</th>
<th>$x(\tau)h_{\text{min}}(\tau)$</th>
<th>References</th>
</tr>
</thead>
<tbody>
<tr>
<td>I</td>
<td>$e_1R_1 \ll 1, e_1 \ll 1$</td>
<td>vdW-$\mu$-$\sigma$</td>
<td>$Oh \geq h_o^{-3/2}$</td>
<td>$(\tau/Oh)^{1/5}$</td>
<td>$(\tau/Oh)^{-2/5}$</td>
<td>ZL99</td>
</tr>
<tr>
<td>II</td>
<td>$e_1R_1 \ll 1, e_1 \sim 1$</td>
<td>vdW-$\mu$</td>
<td>$Oh \geq h_o^{-3/2}$</td>
<td>$(\tau/Oh)^{1/3}$</td>
<td>$\sim 1$</td>
<td>MB20</td>
</tr>
<tr>
<td>III</td>
<td>$e_1R_1 \gg 1, e_1 \ll 1$</td>
<td>vdW-$p$-$\sigma$</td>
<td>$Oh \leq 1$</td>
<td>$\sim 2^{2/7}$</td>
<td>$\tau^{-4/7}$</td>
<td>GA17</td>
</tr>
<tr>
<td>IV</td>
<td>$e_1R_1 \gg 1, e_1 \sim 1$</td>
<td>vdW-$p$</td>
<td>$Oh \leq 1$</td>
<td>$\sim 2^{2/5}$</td>
<td>$\sim 1$</td>
<td>Present work</td>
</tr>
</tbody>
</table>

$h_o = 10$ and $h_o = 31.6$, an inflection point appears in the evolution of $n(\tau)$ near the value $n = 2/7$, which corresponds to the regime described by Garg et al.\textsuperscript{19} as a local balance between liquid inertia, vdW forces, and surface tension forces (regime III of Table I). Although for $h_o \leq 31.6$, the 2/7-regime is only established for less than a decade, its range of validity increases with increasing $h_o$. Indeed, Garg et al.\textsuperscript{19} reported Navier–Stokes simulations for $h_o = 1165$, the case in which the 2/7 regime is established for several decades for a shear-thinning liquid.

The effect of liquid viscosity on the rupture regimes is deduced from Fig. 1(c), where $n(\tau)$ is plotted for several values of $Oh$ and $h_o = (10, 30)$. As the value of $Oh$ increases, the film is seen to pass through a number of intermediate self-similar regimes described in previous studies, although none of them are clearly established except in the limit $\tau \rightarrow 0$. As $\tau$ decreases, the 1/3 power law described by Zhang and Lister\textsuperscript{25} (regime I of Table I) is first observed during a brief transient, followed by the 2/7-law for small enough values of $Oh$. For smaller values of $\tau$, the 1/3-law described by Moreno-Boza et al.\textsuperscript{19} (regime II in Table I) is clearly observed, holding until the singularity for $Oh = 31.62$. Finally, regime IV is reached as $\tau \rightarrow 0$ for $Oh \leq 1$.

As shown in Fig. 2(a) for $Oh = 10^{-2}$ and $h_o = 1$, the local rupture flow presents a distinctive multi-scale structure. The vorticity concentrates in boundary layers at the wall and at the free surface, surrounding a central irrotational core. The self-similar potential flow and the relevant scalings of the boundary layers are described in Sec. IV.

III. THE FILM THINNING REGIMES AT FINITE OHNESORGE NUMBERS

The results of Fig. 1 reveal the existence of different regimes during the thinning of the liquid film for finite values of $Oh$, which will now be explained using order-of-magnitude estimations of the

![FIG. 2. (a) Contours of longitudinal velocity, $\Phi(x, \eta)$, obtained from the self-similar potential flow. The free surface is wedge-shaped, with an angle $\theta_o = 45^\circ$ off the solid wall. (b) Snapshot of a numerical simulation performed for $Oh = 0.01$ and $h_o = 1$ at $\tau = 4.41 \times 10^{-4}$, with $\varepsilon = 0.01$ and $k = 0.3k_c$. Solid lines represent contours of longitudinal velocity, and colors represent contours of normalized vorticity, $|\omega|\tau^{10/9}$. The inset shows instantaneous re-scaled film shapes for $\tau = (0.027, 6.98 \times 10^{-3}, 2.98 \times 10^{-3}, 9.81 \times 10^{-4}, 8.13 \times 10^{-3}, 6.13 \times 10^{-3}, 4.13 \times 10^{-3}, 2.13 \times 10^{-3}, 4.41 \times 10^{-4})$ illustrating the self-similar collapse.](image-url)
governing equations (3) and (4). Hereinafter, the dimensional variables with a non-dimensional counterpart will be denoted with an asterisk. We begin by defining two key variables, namely, the local slenderness and the local reduced Reynolds number,

$$
\varepsilon_l(\tau_\ast) = \frac{\h_{\text{min}}^*}{x_{\ast}^*},
$$

(5a)

$$
\varepsilon_l/R_e(\tau_\ast) = \frac{\rho u_{\ast}^e \h_{\text{min}}^*}{\mu x_{\ast}^*} \sim \frac{\rho v_{\ast}^e h_{\text{min}}^*}{\mu} \sim \frac{\rho h_{\text{min}}^* \h_{\text{min}}^*}{\mu R_e},
$$

(5b)

respectively, where $x_{\ast}^*(\tau_\ast)$ and $u_{\ast}^e(\tau_\ast)$ are the characteristic longitudinal length scale and velocity, respectively. Note that, in the last two estimations of Eq. (5b), the continuity Eq. (3a) has been used, providing the estimate $v_{\ast}^e \sim \h_{\text{min}}^*/\varepsilon_l$, where $v_{\ast}^e/\h_{\text{min}}^*$ is the characteristic transverse velocity. Note also that the local curvature at the minimum film thickness, $\kappa^* = \partial^2 h^*/\partial x^2_{\ast}|_{x^* = 0}$, can be estimated as

$$
\kappa^*(\tau_\ast) \sim \frac{h_{\text{min}}^*}{x_{\ast}^* \tau_\ast} \Rightarrow \kappa^* \h_{\text{min}}^* \sim \varepsilon_l^2,
$$

(6)

and thus, the local curvature, measured with the instantaneous film thickness, is just the square of the slenderness parameter.

The momentum Eq. (3b) admits two limiting balances depending on the value of $\varepsilon_l R_e$; namely, the vDW–viscous balance when $\varepsilon_l R_e \ll 1$ and the vDW–inertial balance when $\varepsilon_l R_e \gg 1$. These two limiting cases will be explained in Subsections III A–III D.

A. Regimes under dominant viscous forces

The viscous regime prevails when $\varepsilon_l R_e \ll 1$, leading to the vDW–viscous balance

$$
\frac{A}{\h_{\text{min}}^* x_{\ast}^*} \sim \frac{\mu u_{\ast}^e}{\h_{\text{min}}^*} \sim \frac{\mu x_{\ast}^*}{\h_{\text{min}}^*} \Rightarrow x_{\ast}^* \sim \left(\frac{A}{\mu} \frac{\tau_\ast}{\h_{\text{min}}^*}\right)^{1/2},
$$

(7)

with an associated local slenderness

$$
\varepsilon_l \sim \left(\frac{\mu}{A}\right)^{1/2} \left(\frac{\tau_\ast}{\h_{\text{min}}^*}\right)^{3/2} \h_{\text{min}}^*.
$$

(8)

Assuming the similarity ansatz $\h_{\text{min}}^* \propto \tau_{\ast}^a$, it is deduced that

$$
\varepsilon_l \sim \tau_{\ast}^{(3a-1)/2}
$$

(9)

so that asymptotic slenderness as $\tau \to 0$ requires that $a > 1/3$. Two different values of a leading to self-similar regimes are possible depending on the value of $\varepsilon_l$, as described in Secs. III A 1 and III A 2.

1. The slender viscous regime

When $\varepsilon_l \ll 1$, the local flow is slender, $x_{\ast}^* \gg \h_{\text{min}}^*$, and thus, the capillary pressure gradient is written as

$$
\sigma \frac{\partial x_{\ast}^*}{\partial x} \sim \sigma \h_{\text{min}}^* / x_{\ast}^*,
$$

(10)

and its relative importance over the driving vDW force is $O[(\sigma/A)\h_{\text{min}}^*/x_{\ast}^2] \sim (\mu A/\sigma)\h_{\text{min}}^*/\tau_\ast \propto \tau_{\ast}^{1-1/5}$, implying that $a \geq 1/5$. Note that vDW, viscous, and surface tension forces are in balance when $a = 1/5$, leading to the rupture law

$$
\h_{\text{min}}^* \sim \left(\frac{A^2}{\mu \sigma}\right)^{1/5} \tau_{\ast}^{1/5},
$$

(11)

discovered by Zhang and Lister. In the latter regime, which is referred to as Regime I in Table I, the local slenderness and the local Reynolds number scale correspondingly as

$$
\varepsilon_l \sim \left(\frac{\mu^2 A}{\sigma^2}\right)^{1/5} \tau_{\ast}^{-2/5},
$$

(12a)

$$
\varepsilon_l R_e \sim \frac{\rho A^{4/3}}{\mu^{2/3} \sigma^{2/5}} \tau_{\ast}^{-3/5}.
$$

(12b)

2. The non-slender viscous regime

The self-similar solution (11) fails when $\varepsilon_l \sim 1$ due to the breakdown of slenderness. The latter condition holds at the crossover time,

$$
\tau_\ast = \tau_{\ast c} \sim \frac{\mu A^{1/2}}{\sigma^{1/2}},
$$

(13)

with the associated minimum thickness

$$
\h_{\text{min}}^* (\tau_\ast c) \sim \left(\frac{A}{\sigma}\right)^{1/2} \Rightarrow a \Rightarrow \h_{\text{min}}(\tau_\ast) \sim 1.
$$

(14)

Equation (14) implies a breakdown of the 1/5 power law close to the molecular scale, as studied in detail by Moreno-Boza et al. Indeed, note that the breakdown of slenderness that occurs when $\varepsilon_l \sim 1$ gives rise to a regime given by the balance $A/(\h_{\text{min}}^* h_{\text{min}}^* \sim \mu c_{\ast}/\h_{\text{min}}^* \sim \mu/(\h_{\text{min}} \tau_{\ast})$, yielding the asymptotic thinning law

$$
\h_{\text{min}}^* \sim \left(\frac{A}{\mu}\right)^{1/3} \tau_{\ast}^{-1/3},
$$

(15)

discovered by Moreno-Boza et al. where a detailed description is given in their work.

B. The viscous-inertial crossover

Apart from the breakdown of slenderness described in Sec. III A 2, the self-similar solution (11) may also fail when $\varepsilon_l R_e \sim 1$ due to the onset of inertial effects at a crossover time given by

$$
\tau_\ast = \tau_{\ast o} \sim \frac{3^{5/3} A^{4/3}}{\mu^{2/3} \sigma^{3/3}},
$$

(16)

with the associated minimum thickness

$$
\h_{\text{min}}^*(\tau_{\ast o}) \sim \frac{A^{1/3}}{\mu^{2/3} \sigma^{1/3}} \Rightarrow \h_{\text{min}}(\tau_{\ast}) \sim \text{Oh}^{-2/3}.
$$

(17)

Equation (17) determines an inertial crossover at a length scale that depends on Oh with two relevant limiting cases. When $\h_{\text{min}}(\tau_{\ast}) \gtrsim h_0 \Rightarrow \text{Oh} \lesssim h_0^{-3/2}$, liquid inertia is important from the onset film thinning, the case in which the 1/5 power law (11) is never established. On the other hand, when $\h_{\text{min}}(\tau_{\ast}) \lesssim 1 \Rightarrow \text{Oh} \gtrsim 1$, liquid inertia

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becomes important at molecular scales, and the 1/5-law holds until $h_{\text{min}} \sim 1$, when slenderness breaks down and Eq. (15) holds. In the intermediate case $h_{\text{min}}^{3/2} \lesssim Oh \lesssim 1$, the 1/5 law experiences the inertial crossover at $h_{\text{min}} \sim Oh^{-2/3}$.

C. Regimes under dominant inertia

As happened in the viscous case, the possible self-similar inertial regimes that are established when $\varepsilon_{\text{v}}Re_{\text{v}} \gg 1$ depend on the local slenderness. Let us now describe the two corresponding limiting cases separately.

1. The slender inertial regime

We consider first the case with $\varepsilon_1 \ll 1$ for which the balance of liquid inertia and vdW forces yields

$$
pu_c x_c^2 \sim \frac{A}{h_{\text{min}}^3},
$$

so that the long-wave approximation breaks down when

$$
\tau_* = \frac{\rho l^2 A^{1/4}}{\sigma^{5/4}} \Rightarrow h_{\text{min}}^* \sim \left( \frac{A}{\sigma} \right)^{1/2} \sim a_*.
$$

D. Classification of the self-similar film thinning regimes

The previous development demonstrates the existence of four possible self-similar regimes during the vdW-induced rupture of the film at finite values of Oh, which are summarized in Table I. In dimensionless terms, these regimes are as follows: (I) When $\varepsilon_1Re_{\text{v}} \ll 1$ and $\varepsilon_1 \ll 1$, the slender viscous regime discovered by Zhang and Lister holds, with a minimum thickness $h_{\text{min}} \sim (\alpha Oh)^{1/5}$ and a local curvature $\kappa \sim (\alpha Oh)^{-3/5}$. (II) When $\varepsilon_1Re_{\text{v}} \ll 1$ and $\varepsilon_1 \sim 1$, the non-slender Stokes flow described by Moreno-Boza et al. takes place, with $h_{\text{min}} \sim (\alpha Oh)^{1/3}$ and $\kappa \sim h_{\text{min}}^{-4/3}$. (III) When $\varepsilon_1Re_{\text{v}} \gg 1$ and $\varepsilon_1 \ll 1$, the slender inertial regime reported by Garg et al. prevails, with $h_{\text{min}} \sim \tau^{1/2}$ and $\kappa \sim \tau^{-6/7}$. (IV) Finally, when $\varepsilon_1Re_{\text{v}} \gg 1$ and $\varepsilon_1 \sim 1$, an ultimate non-slender inertial regime takes over, with $h_{\text{min}} \sim \tau^{2/7}$ and $\kappa \sim h_{\text{min}}^{-1}$. Since the latter regime is reported for the first time in the present paper, Sec. IV will be devoted to a detailed analysis of its main features.

Finally, let us provide a simple argument for the asymptotic sub-dominance of surface tension forces in the non-slender regimes II and IV. The intuition behind the latter fact is simply that, in these regimes, there is only one characteristic length scale, namely, $x_c \sim y_c \sim h_{\text{min}}$. Thus, the local curvature $\kappa^* \sim (h_{\text{min}})^{-4/3} \rightarrow \infty$ as $\tau_* \rightarrow 0$, which diverges as the singularity is approached. What remains finite, however, is the local curvature scaled with the instantaneous minimum thickness, i.e., $\kappa^* h_{\text{min}} \sim 1$ as $\tau_* \rightarrow 0$, explaining the fact that, in regimes II and IV, the shape of the interface is self-similar when appropriate variables are used to describe the local flow. Although surface tension forces diverge as $(h_{\text{min}})^{-4/3}$ toward the singularity, their sub-dominance is simply explained by the fact that vdW forces diverge as $(h_{\text{min}})^{-1}$ as $\tau_* \rightarrow 0$ and thus become much larger than surface tension forces in the last stages of the collapse close to the continuum limit.

IV. THE UNIVERSAL SELF-SIMILAR INERTIAL REGIME

Suggested by the flow evolution shown in Fig. 1 and by the order-of-magnitude analysis presented in Sec. III, the universal behavior $h_{\text{min}} = 1.1 \tau^{2/7}$ associated with regime IV in Table I will now be unveiled using similarity theory.

A. The self-similar potential flow

Letting $x = \tau^{\beta} \xi$, $y = \tau^{\alpha} \eta$, $u = \tau^\beta U$, $v = \tau^{\alpha + \beta} V$, $p = \tau^{-3a} P$, and $h = \tau^f(\xi)$, where $\alpha$, $\beta$, and $\gamma$ are real numbers, a consistent leading-order dominant balance between inertia and vdW forces can indeed be found by substituting the similarity test into (3) and (4) and performing the limit $\tau \rightarrow 0$, yielding the exponents $\alpha = \beta = 2/5$ and $\gamma = \beta - 1 = -3/5$, in agreement with the results of Sec. III. To unravel the structure of the leading-order potential flow as $\tau \rightarrow 0$, we make use of the velocity potential $\Phi$ such that $U = \Phi_\xi$ and $V = \Phi_\eta$, reducing the description to the integration of the Laplace equation,

$$
\Phi_{\xi\xi} + \Phi_{\eta\eta} = 0,
$$

in $0 < \xi < \infty$ and $0 < \eta < f(\xi)$, where $f(\xi)$ is the $a$ priori unknown shape of the free surface. Note that our strategy to unravel the non-slender inertial regime resembles that employed by Moreno-Boza
et al.\textsuperscript{12} in the viscous case, where the biharmonic equation substitutes the Laplace equation. Hereinafter, subscripts denote partial derivatives unless stated otherwise. The leading-order contribution of the stress balance at the interface reduces to an Euler–Bernoulli-like boundary condition to be imposed together with the kinematic condition,
\begin{align}
\frac{1}{f^3} + \frac{1}{5} \Phi + \frac{2}{5} (\xi \Phi_\xi + \eta \Phi_\eta) + \frac{1}{2} (\Phi_\xi^2 + \Phi_\eta^2) &= 0,
\end{align}
and
\begin{align}
\frac{2}{5} f + \Phi_\eta - f \xi (\frac{2}{5} \xi + \Phi_\xi) &= 0,
\end{align}
along \( \eta = f \), together with the no-penetration condition \( \mathbf{n} \cdot \nabla \Phi = 0 \) at \( \eta = 0 \) and \( \xi = 0 \), where the gradient operator is applied with respect to \( \xi \) and \( \eta \). The far-field boundary condition at \( \xi \gg 1 \) and \( 0 < \eta < f \) would require uniform matching with the inner flow solution for \( x/r^{2/5} \ll 1 \) and therefore is replaced by (24a) with the Hamaker term, i.e., \( 1/f^3 \), switched off, which may alternatively be interpreted as a zero-pressure boundary condition. This enables us to give a fairly reasonable approximation to the real flow that is to be expected in the vicinity of the singularity. Problems of similar mathematical nature were derived, for instance, by Ref. \textsuperscript{24} in an opening angle \( \theta_0 \approx 45^\circ \) off the substrate, where \( r^2 = (\xi - \xi_0)^2 + \eta^2 \) and \( \theta = \arctan \eta/\xi \) are polar coordinates. Examination of the far-field revealed that \( \xi_0 = 0 \) and \( f(0) \approx 1.08 \). Also, a consistent radial decay of the potential of the form \( \Phi \sim r^{-\lambda} \) along the ray \( \theta = \theta_0 \) was observed. The value of \( \lambda \) was seen to adjust quite well to \( 1/2 \) from the numerical computation (see the inset of Fig. 3(a)). This is indeed the only value of \( \lambda \) that ensures compatibility in terms of powers of \( r \) in (24a). A noteworthy aspect of the result shown in Fig. 2 is the displacement effect of the wall boundary layer on the outer irrotational core. Indeed, the Navier–Stokes solution of Fig. 2(b) is seen to be slightly displaced in the positive \( y \)-direction with respect to the potential flow of Fig. 2(a).

The non-slender universal regime described in the present section is the inertial counterpart of the self-similar Stokes flow reported by Moreno-Boza et al.\textsuperscript{19} classified as regimes IV and II in Table I. It is noteworthy that the local interface shape in the Stokes case is also a wedge, but with an opening angle of \( 37^\circ \) instead of the larger angle of \( 45^\circ \) associated with inertial breakup.

B. The boundary layers

The potential solution described in Sec. IV A constitutes the outer flow for two vortical boundary layers sitting at the wall and at the free surface, briefly described in Secs. IV B 1 and IV B 2.
1. The wall boundary layer

The slipping velocity at the wall $\Phi_i|_{\eta=\delta}$ and the streamwise pressure gradient $P_\xi|_{\eta=\delta}$ evaluated numerically and presented in Fig. 3(b), serve as boundary conditions for the viscous wall boundary layer (WBL). The thickness of the WBL, $\delta_w^*$, is given by the balance $\mu u^*_w/\kappa^*$ $\sim \mu u^*_w/\delta_w^*$, whence

$$\delta_w^* \sim \frac{\mu U^*}{\rho} \Rightarrow \delta_w = \mathcal{O}1/2 \tau^{1/2}. \quad (25)$$

The wall shear stress (WSS) $\tau_{xy}^* \sim \mu u^*_w/\delta_w^*$, yielding

$$\tau_{xy}^* \sim \frac{\mu U^*/\rho}{\mathcal{O}^{1/4}\delta_w^*} \Rightarrow \tau_{xy}^* \sim \mathcal{O}^{-1/2} \tau^{-11/10}, \quad (26)$$

as confirmed by the evolution of the shear stress in Fig. 1.

The substitution into (3) of the standard boundary-layer change in variables,\textsuperscript{21} namely,

$$x = \zeta^2/\zeta, \quad y = (\zeta^2/\zeta)^{1/3}, \quad u = \zeta^{1/3} \tilde{U}, \quad v = \zeta^{1/3} \tilde{V}, \quad p = \zeta^{1/3} \tilde{P}, \quad h = \zeta^{-1/2} \tilde{h}(\xi), \quad (27)$$

yields the leading-order problem

$$\tilde{U}_\xi + \tilde{V}_\zeta = 0, \quad (28a)$$

$$\frac{3}{5} \tilde{U} + \left( \frac{2\zeta}{5} + \tilde{U} \right) \tilde{U}_\xi + \left( \frac{\zeta}{2} + \tilde{V} \right) \tilde{V}_\xi = -\tilde{P}_\xi + \tilde{U}_\zeta, \quad (28b)$$

$$\tilde{P}_\zeta = 0, \quad (28c)$$

describing the WBL so that the pressure gradient is given by the external potential flow as $-\tilde{P}_\xi = \left[ (\tilde{U}_\xi + 2\xi^2/5)\tilde{U}_\xi + 3\xi^2/5 \right]_{\xi=0}$. The boundary conditions accompanying the system (28) are $\tilde{U} = \tilde{V} = 0$ at $\zeta = 0$, $\tilde{U} \rightarrow 0$ as $\zeta \rightarrow \infty$, and $\tilde{U} = 0$ at $\zeta = 0$. The parabolic system (28) is integrated using the method of lines, whose details are described in Appendix C. To obtain the initial condition, we take advantage of the self-similar nature of the boundary layer for $\xi \ll 1$. Indeed, the substitution of a stream function $\tilde{G}(\xi)$ into (28a) and (28b) yields

$$G'''' + \left( G - \zeta/2 \right) G''' - G' = 0 \quad (29)$$

with the boundary conditions $G = G' = 0$ at $\zeta = 0$ and $G' \rightarrow \xi^{-1} \Phi_2$ as $\xi \rightarrow \infty$. Equation (29) admits a self-similar solution for $\xi \ll 1$ in which $\xi^{-1} \Phi_2 \rightarrow -0.3802$ and $\xi^{-1} \Phi_1 \rightarrow 0.2965$, which was computed using a pseudospectral collocation technique.\textsuperscript{7}

2. The free-surface boundary layer

Finally, for conciseness, let us just provide the characteristic scales of the free-surface boundary layer (FSBL). Its thickness, $\delta_f^*$, can be estimated by taking into account that the free-surface vorticity is $\omega^*_f = -2\partial_t v^*_f + 2u^*_f \kappa^*$ in a two-dimensional unsteady flow,\textsuperscript{17} where $\partial_t$ is the derivative along the interface, $u^*_f$ and $v^*_f$ are the tangential and normal velocity components at the interface, and $\kappa^*$ is the twice the mean curvature. We now take into account that, in regime IV, $u^*_f \sim v^*_f \sim u^*_f \sim h_{\text{min}}/\tau_s$, and $\partial_t \sim \kappa^* \sim 1/\alpha$, providing $\omega^*_f \sim h_{\text{min}}^*/(\alpha \tau_s)$. Since, by definition, the vorticity is $\omega^*_f \sim \Delta u^*/\delta^*$, we deduce that $\delta^*_f \sim \alpha \Delta u^*/h_{\text{min}}$, where $\Delta u^*$ is the characteristic velocity increment across the FSBL. In turn, the value of $\Delta u^*$ may be estimated by balancing the convective acceleration with the capillary pressure gradient, $\mu u^*/\Delta x^* \sim \sigma \kappa^*/\Delta x^* \Rightarrow \Delta u^* \sim \sigma/(\rho u^*_f \kappa^*)$. Thus, we deduce that $\delta^*_f \sim \sigma/(\rho u^*_f \kappa^*)$, finally yielding

$$\delta^*_f \sim \frac{\sigma}{\rho^{3/2} A^{1/3}} \tau_s^{1/3} \Rightarrow \delta^*_f \sim \tau_s^{1/3}, \quad (30)$$

for the FSBL thickness that, as deduced from Eq. (30), does not depend on $\mathcal{O}h$ in contrast with the WBL thickness (25). We finally note that $\delta^*_f \rightarrow 0$ as $\tau \rightarrow 0$ much faster than the WBL, in agreement with the results shown in Fig. 2.

V. CONCLUDING REMARKS

New insights into the inertia-driven dewetting of unstable ultrathin films of Newtonian liquids have been gained through theoretical analysis and numerical integration of the Navier–Stokes equations. We have shown that when the Ohnesorge number $Oh \lesssim 1$, the final approach of the flow toward the rupture singularity close to the molecular scales is self-similar, with a non-dimensional minimum film thickness $h_{\text{min}} = 1.1 \tau^{-2/5}$ due to a dominant balance between liquid inertia and van der Waals forces, with asymptotically subdominant surface tension forces. The spatial structure of the flow in this new regime presents a distinguished three-region structure characterized by a potential core separated by two boundary layers, namely, a vortical curvature-induced layer sitting at the free surface and a viscous layer adhered to the solid substrate. It is interesting to note that when $Oh \ll 1$, the latter three-region structure is already present from the onset of the vdW-induced instability, as demonstrated in Appendix A by making use of linearized theory. Upon introducing appropriate self-similar variables, the irrotational core has been described as a universal solution to the Euler equations, featuring a wedge-shaped interface with an opening angle of 45° off the solid. The latter non-slender solution applies when the minimum film thickness approaches the molecular scale, as obtained similarly by Moreno-Boza et al.\textsuperscript{19} in the context of viscous dewetting. A parameter-free description of the viscous wall layer of non-dimensional thickness $\delta_w \sim \mathcal{O}1/2 \tau^{1/2}$ has been provided by integrating Prandtl’s equations using self-similar boundary-layer variables. We have also explained why the free-surface vortical layer is much thinner than the viscous wall layer, since its thickness scales as $\delta^*_f \sim \tau_s^{1/3} \ll \delta_w$ as $\tau \rightarrow 0$.

For finite values of the Ohnesorge number, an order-of-magnitude analysis allowed us to classify all the possible regimes of dewetting of Newtonian liquids, thereby unifying previous studies. The slender viscous regime discovered by Zhang and Lister,\textsuperscript{16} with $h_{\text{min}} = 0.913 (\tau/Oh)^{1/3}$, prevails for $Oh \gtrsim 1$ and $h_{\text{min}}(\tau) \gg 1$, provided that the initial thickness $h_0 \gtrsim 30$. The non-sleender viscous regime reported by Moreno-Boza et al.,\textsuperscript{19} where $h_{\text{min}} = 0.665 (\tau/Oh)^{1/3}$, applies for $Oh \gtrsim 1$ and $h_{\text{min}}(\tau) \lesssim 4$, independently of the value of $h_0$. The slender inertial regime described by Garg et al.\textsuperscript{14} for shear-thinning fluids has been shown to apply also for Newtonian fluids, but only for $Oh \lesssim 1$, $h_0 \gtrsim 30$, and $h_{\text{min}}(\tau) \gg 1$. The latter regime, where $h_{\text{min}} = 1.0 \tau^{2/7}$, experiences a crossover to the inertial regime discovered in the present work, which prevails for $Oh \lesssim 1$ and $h_{\text{min}}(\tau) \lesssim 4$ for any value of $h_0$.\textsuperscript{15}
TABLE II. Physical properties, molecular Ohnesorge number, Oh, and global Ohnesorge number Ohg = μ/√(ρσh∞) for different liquid metals at their melting temperature, Tm, and at a higher working temperature, Tw. A value of h∞ = 40 nm has been used to compute Ohg.

<table>
<thead>
<tr>
<th>Metal</th>
<th>Tw–Tm (K)</th>
<th>a (nm)</th>
<th>μ (mPa)</th>
<th>ρ (kg m⁻³)</th>
<th>σ (N m⁻¹)</th>
<th>Oh</th>
<th>Ohg</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cu</td>
<td>2000–1356</td>
<td>0.338–0.277</td>
<td>1.88–4.00</td>
<td>7400–8020</td>
<td>1.20–1.78</td>
<td>1.09–2.01</td>
<td>0.1–0.167</td>
</tr>
<tr>
<td>Ag</td>
<td>2000–1234</td>
<td>0.571–0.522</td>
<td>1.72–5.00</td>
<td>8671–9346</td>
<td>0.78–0.93</td>
<td>0.88–2.35</td>
<td>0.105–0.268</td>
</tr>
<tr>
<td>Ni</td>
<td>2000–1728</td>
<td>0.350–0.343</td>
<td>3.40–4.90</td>
<td>6700–7810</td>
<td>1.65–1.71</td>
<td>1.73–2.29</td>
<td>0.162–0.212</td>
</tr>
<tr>
<td>Rb</td>
<td>950–312</td>
<td>0.913–0.775</td>
<td>0.18–0.67</td>
<td>1460</td>
<td>0.064–0.088</td>
<td>0.62–2.12</td>
<td>0.094–0.295</td>
</tr>
<tr>
<td>Cs</td>
<td>930–302</td>
<td>1.138–0.965</td>
<td>0.19–0.68</td>
<td>1443–1843</td>
<td>0.041–0.057</td>
<td>0.73–2.13</td>
<td>0.123–0.33</td>
</tr>
<tr>
<td>Si</td>
<td>2000–1685</td>
<td>0.275–0.258</td>
<td>0.3–0.8</td>
<td>2500–2570</td>
<td>0.7–0.8</td>
<td>0.43–1.09</td>
<td>0.036–0.088</td>
</tr>
</tbody>
</table>
leading to an estimated non-dimensional rupture time

\[ t_R = \frac{(3 + \sqrt{19})^{5/2}}{24(5 + \sqrt{19})} h_0^{1/2} \ln(e^{-1}) + O(h_0^{1/2}), \quad (A3) \]

defined by \( t_R = \ln(e^{-1})/\omega_m \), where \( \omega_m \) is the maximum growth rate, obtained by applying the condition \( \partial \omega/\partial k = 0 \). Equation (A3) reveals a strong dependence of the rupture time on the initial film thickness, \( t_R \propto h_0^{1/2} \). Conversely, when \( Oh \gg 1 \), the result for the rupture time obtained in Ref. 19 reads in inertial scales, \( t_R = 4/3Oh h_0^{1/2} \).

The results of Garg et al.²⁶ are obtained by taking the value of the Weber number as \( We = 7.37 \times 10^{-7} \) and thus \( h_0 = 1164.84 \). The experimental values of the physical parameters of the film considered therein are \( A = 10^{-8} \), \( \sigma = 0.05 \) Nm⁻¹, and \( \mu = 35.6 \) Pa s in the Newtonian case. The latter values yield \( a = 0.326 \) nm, and thus, the initial height of the film considered in Ref. 10 is \( h^*_0 = 379.43 \) mm. It is important to note that the latter values of \( h_0 \) and \( h^*_0 \) are unrealistically large, and the vdw forces may not even trigger the instability. The values of the molecular and global Ohnesorge numbers are \( Oh = 8.88 \times 10^4 \) and \( Oh_0 = 7.62 \times 10^5 \). Hence, the rupture time can be estimated by the Stokes limit reported in Refs. 19 and 23, which in dimensional form reads \( t_R^* = \frac{4}{3}(\mu a / \sigma)(h^*_0 / a)^3 \). Taking the latter values, one deduces the unrealistically large value of \( t_R^* = 6.63 \times 10^8 \) s \( \approx 21 \) years.

APPENDIX B: DIMENSIONAL ANALYSIS

The velocity components, \((u^*, v^*)\), and the pressure, \( p^*\), depend on the set of variables \((x^*, y^*, \tau^*, \rho, \mu, \sigma, h^*_0, \ldots)\), while the film thickness \( h^* \) depends on the same variables except for \( y^* \). Taking \((\tau^*, \rho, A)\) as the dimensional basis, the Buckingham II theorem yields the reduced functional forms

\[ \left[ u^*, v^* \right] = \tau_{e}^{-3/5}(A/\rho)^{1/5} \left[ \Pi_o, \Pi_o \right], \quad (B1a) \]
\[ p^* = \tau_{e}^{-6/5}(A^5/\rho) \left[ \Pi_o \right], \quad (B1b) \]
\[ h^* = \tau_{e}^{2/5}(A/\rho)^{3/5} \left[ \Pi_o \right], \quad (B1c) \]

where \( \left[ \Pi_o = \tau_e^2/5(\rho/A)^{1/5}[x^*, y^*, h^*_0] \right], \Pi_o = \tau_e^{4/5} \sigma / (\rho^{2/5} A^{3/5}), \) and \( \Pi_o = \tau_e^{6/5} \mu / (\rho^{3/5} A^{2/5}) \). Note that \( \Pi_o^{-1} \) and \( \Pi_o^{-2} \) can be interpreted as the local Reynolds and Weber numbers, respectively (see Sec. III). In the limit \( \tau_e \rightarrow 0, \Pi_o \rightarrow 0, \Pi_o \rightarrow 0, \) and \( \Pi_o \rightarrow \infty \), indicating that as rupture is approached, viscous and surface tension forces become negligible and that the flow becomes independent of \( h_0 \), thereby suggesting the existence of a similarity solution of the first kind for \( \tau_e \rightarrow 0 \), with liquid inertia balancing vDW forces. In addition, the fact that \( x^* \) and \( y^* \) have the same asymptotic scaling indicates that the near-rupture local flow is non-slender and that lubrication theory, which assumes that the characteristic length in the \( x \) direction is much larger than the film thickness, necessarily fails to describe the singularity.

APPENDIX C: NUMERICAL TECHNIQUES

We describe the numerical techniques employed to solve the Navier–Stokes equations (3)–(4b), the self-similar potential problem (23)–(24b), and the boundary-layer problems (28a), (28b), and (29). Most of the equations were written in a weak form upon convenient use of Green’s identities, rendering them amenable for the use of finite elements for the spatial discretization. The implementation was carried out in the finite-element solver COMSOL MULTIPHYSICS.

1. Navier–Stokes equations

The Navier–Stokes equations of motion and corresponding boundary conditions (3) and (4b) were first written in a weak form with the use of suitable test functions for velocity and pressure, i.e., \( \hat{\phi} \) and \( \hat{p} \), yielding the bilinear form

\[ 0 = \int_{\Omega} \left[ \frac{\partial u}{\partial t} - \phi \nabla \cdot \mathbf{u} + \frac{\sigma}{\rho} \nabla \nabla \cdot \mathbf{u} + \mathbf{D} \left( \nabla \mathbf{u} + \nabla \mathbf{u}^T \right) : \nabla \mathbf{u} + \hat{p} (\nabla \mathbf{u} + \nabla \mathbf{u}^T) \right] dV \]
\[ + \int_{\Gamma} \left( \nabla \cdot \mathbf{u} + h^{-1} \mathbf{n} \cdot \mathbf{u} \right) d\Sigma, \quad (C1) \]

where the surface gradient operator is defined as \( \nabla_s = (1 - \mathbf{n} \mathbf{n}) \cdot \nabla \), with \( \mathbf{n} \) being the identity tensor, and \( \mathbf{n} \) is the outward-pointing normal unit vector. Note that the disjoining pressure term has been conveniently rewritten as a surface stress after judicious application of Gauss’s theorem. A schematic of the numerical domain and boundary conditions is presented in Fig. 5. Equation (C1) was discretized using Taylor–Hood triangular elements for pressure and velocity (also the corresponding test functions) to ensure numerical stability. The use of an arbitrary Lagrangian–Eulerian (ALE) technique for the tracking of the interface allowed us to impose the kinematic boundary condition (4a) along \( \Sigma \) by prescribing the normal velocity of the mesh to that of the fluid. The displacement of the mesh...
elements was computed by solving a Laplace equation for the displacement field \( q = (q_x, q_y) \), namely \( \nabla^2 q = 0 \), with suitable boundary conditions. The entire system is advanced in time using a variable-order BDF technique with an adaptive time step until numerical rupture.

2. Potential flow

The elliptic system comprised of the Laplace Eq. (23) and the boundary conditions (24a) and (24b) were again written in a weak form, producing

\[
\int_{\mathcal{V}} \nabla \Phi \cdot \nabla \Phi \, d\mathcal{V} = 0,
\]

which is complemented with the boundary condition \( \Phi = \Phi_l \) at \( \Sigma_i \), where \( \Phi_l \) is the yet-unknown value of the potential at the free surface, leveraged as a Lagrange multiplier to solve the boundary PDE

\[
\int_{\Sigma_i} \left[ \frac{1}{f} \Phi + \frac{2}{5} (\xi \Phi_x + \eta \Phi_y) + \frac{1}{2} (\Phi_x^2 + \Phi_y^2) \right] \Phi_l \, d\Sigma_i = 0,
\]

where \( \Phi_l \) is the test function for the potential at the free surface, which effectively enforces the Euler–Bernoulli condition at the interface. The ALE method was once again used for the displacement of the mesh. This allowed us to achieve the kinematic condition (24b) by introducing an extra degree of freedom for the vertical displacement of the deformable mesh \( \delta \) at \( \Sigma_i \). The boundary condition for the displacement of the free surface then becomes \( q = \delta \), at \( \Sigma_i \), and the corresponding restriction for \( \delta \) may be viewed as a surface PDE of the form

\[
\int_{\Sigma_i} \left[ \frac{2}{5} \delta + \Phi_{\eta} - f(\frac{2}{5} \xi + \Phi_l) \right] \delta \, d\Sigma_i = 0,
\]

where \( \delta \) is the test function for the vertical displacement \( \delta \) discretized using first-order Lagrange elements and \( d\Sigma_i \) is the line element along \( \Sigma_i \). This additional equation enabled us to use a standard Newton–Raphson root-finding algorithm to iteratively solve all the unknowns upon a normalized tolerance, fixed to \( 10^{-6} \), provided a suitable initial guess. A summary of the procedure is sketched in Fig. 6.

3. Boundary layer equations

Equations (28a) and (28b) were treated using \( \xi \) as a temporal variable, thereby making the system parabolic. Finite elements were employed across the transversal direction \( \xi \) upon writing the corresponding weak form, namely,

\[
\int_0^\xi \left( \left( \frac{3}{5} \delta + \frac{2}{5} \xi + \delta \right) \delta \right) \delta \, d\xi = 0,
\]

where \( U \) and \( V \) are quadratic test functions for the variables \( U \) and \( V \), respectively, \( P_l \) is the known external pressure gradient whose expression is given in the main text, and \( \xi_{\infty} = 60 \) is the numerical infinity. The system was integrated with the boundary conditions \( U = V = 0 \) at \( \xi = 0 \) and marched in time using a BDF algorithm provided the initial condition for \( \xi = 0 \). Such a condition was obtained by solving the problem (29) in terms of the local stream function \( \xi G(\xi) \) using a pseudospectral collocation technique, as described in the main text.

FIG. 5. Sketch of the numerical domain \( \mathcal{V} \) bounded by \( \Sigma_b \) (symmetry axis at \( x = 0 \)), \( \Sigma_o \) (bottom wall), \( \Sigma_m \) (symmetry axis at \( x = n_k \)), and \( \Sigma_f \) (free surface). The corresponding boundary conditions for the velocity, stresses, and mesh displacement are also included in the schematic.

FIG. 6. Sketch of the initial guess (left-hand side panel) for the computation of the self-similar film shape and final converged solution (right-hand side panel). Note that in particular, \( f(\xi) \) and \( \theta_e \) are obtained as part of the solution.
DATA AVAILABILITY

The data that support the findings of this study are available from the corresponding author upon reasonable request.

REFERENCES