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GRADIENT BLOW-UP FOR A FOURTH-ORDER QUASILINEAR BOUSSINESQ-TYPE EQUATION

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ABSTRACT. The Cauchy problem for a fourth-order Boussinesq-type *quasilinear wave equation* (QWE–4) of the form

$$u_{tt} = -(|u|^n u)_{xxxx} \quad \text{in } \mathbb{R} \times \mathbb{R}_+, \quad \text{with a fixed exponent } n > 0,$$

and bounded smooth initial data, is considered. Self-similar single-point *gradient blow-up* solutions are studied. It is shown that such singular solutions exist and satisfy the case of the so-called self-similarity of the second type.

Together with an essential and, often, key use of numerical methods to describe possible types of gradient blow-up, a “homotopy” approach is applied that traces out the behaviour of such singularity patterns as $n \rightarrow 0^+$, when the classic *linear beam equation* occurs

$$u_{tt} = -u_{xxxx},$$

with simple, better-known and understandable evolution properties.

1. INTRODUCTION: QUASILINEAR BOUSSINESQ (WAVE) MODELS AND GRADIENT BLOW-UP

1.1. The fourth-order QWE and shock waves. The aim of this paper is to study the formation of basic shock singularities for higher-order quasilinear wave-type equations. In particular, we describe key features in formation of shock-type finite time singularities, in the form of *gradient blow-up behaviours* that occur at single points. Thus, in order to analyse these phenomena we consider the fourth-order *Boussinesq-type*, or *quasilinear wave equation* (QWE–4) of the form

$$(1.1) \quad u_{tt} = -(|u|^n u)_{xxxx} \quad \text{in } \mathbb{R} \times (0, T), \quad \text{with an exponent } n > 0.$$

As a particular case, we observe that $n = 2$ in (1.1) yields the cubic equation with an analytic nonlinearity:

$$(1.2) \quad u_{tt} = -(u^3)_{xxxx}.$$

In general, we can say that blow-up phenomena, as intermediate asymptotics and approximations of highly non-stationary processes, are common and well known in various areas of mechanics and physics. The origin of intensive systematic studies of such nonlinear effects was gas dynamics (since the end of the 1930s and 1940s) supported later in the 1960s by plasma physics (wave collapse) and nonlinear optics (self-focusing phenomena).

Finite-time blow-up singularities lie at the heart of several principal problems of PDE theory concerning existence, uniqueness, optimal regularity, and free-boundary propagation. The role of blow-up analysis in nonlinear PDE theory will increase when more complicated classes of higher-order degenerate parabolic, hyperbolic, nonlinear dispersion, and other equations of interest are involved in the framework of massive mathematical research and application.

One might find main applications of such equations in mathematical physics and mechanics in [11, 14], as well as key references on their mathematical studies in recent years. We must mention as well the

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work of Eggers & Fontelos [5] where a classification of singularities for several PDEs was carried out using similarity transformations. Moreover, in [11, 14], self-similar blow-up solutions, leading to discontinuous shock waves and, for other data, to smooth rarefaction waves, were studied. In particular, the author analysed a third order non-linear dispersion equation of the form $u_t = (uu_x)_{xx}$ studying the formation of shock singularities, such as blow-up formation, as $t \rightarrow T^- < \infty$, of shock waves when, for smooth initial data,

$$(1.3) \quad u(x, T^-) \text{ is a discontinuous function.}$$

If blow-up happens, the first key question is the behaviour of solutions as $t \rightarrow T^-$, that reflects both mathematical and physical-mechanical essence of these phenomena. Such singular limits create a class of one of the most difficult asymptotic problems in nonlinear PDE theory since the internal structure of these blow-up singularities can be rather complicated.

Moreover, we should mention that there are different types of blow-up behaviours such as *blow-up in a bounded localised domain*, also *global blow-up* i.e.

$$|u(x, t)| \rightarrow \infty \text{ as } t \rightarrow T^-, \text{ uniformly on any compact subset in } x,$$

and *single point blow-up*

$$|u(x, t)| \rightarrow \infty \text{ as } t \rightarrow T^-, \text{ at a single point and is uniformly bounded away from it.}$$

We will show that for equation (1.1) we find blow-up behaviour at a single point $x = 0$. Moreover, it is worth mentioning that a full classification of the blow-up patterns is still not available for these types of problems such as (1.1). Therefore, it is clear that many of current techniques associated with some remarkable and famous specific PDEs become non-aplicable, so that new ideas and approaches for non-linear PDEs are necessary in order to obtain key features for the phenomena appearing in problems such as (1.1). The results presented here are then new and shed some light in that direction.

1.2. Main results: gradient blow-up. In the present paper, we focus on the analysis of gradient blow-up behaviours that occur at a single point. To this end, we will use a self-similar expression of the equation (1.1) (see Section 2 for details), i.e.

$$(1.4) \quad u(x, t) = (T - t)^\alpha g(y), \quad y = x/(T - t)^\beta, \quad \beta = \frac{2 + \alpha n}{4},$$

such that (1.1) becomes

$$(1.5) \quad \begin{aligned} \beta^2 y^2 g'' + \mu y g' + \alpha(\alpha - 1)g &= -(|g|^n g)^{(4)} \text{ in } \mathbb{R}, \\ \mu = \beta(\beta + 1) - 2\alpha\beta &= \frac{\beta[6 + (n - 8)\alpha]}{4}. \end{aligned}$$

Moreover, we find its asymptotic behaviour at infinity (Section 2). First, the *minimal growth*,

$$g_{\min}(y) = C y^\gamma (1 + o(1)), \quad \gamma = \frac{\alpha}{\beta} > 0, \quad \text{with } C \neq 0 \text{ a constant,}$$

and also the *maximal growth*

$$g_{\max}(y) \sim B y^\delta (1 + o(1)), \quad \text{where } \delta = \frac{4}{n} > \frac{\alpha}{\beta},$$

with $B \equiv B(y)$ a bounded oscillatory function, as $y \rightarrow +\infty$. That envelope behaviour at infinity, bounded between the maximal and minimal growth, provides us with a gradient blow up at $x = 0$ as $t \rightarrow T^-$. Indeed, passing to the limit as $t \rightarrow T^-$ in (1.4) this asymptotic behaviour gives crucial information for our problem (1.1), in particular, allowing us to find such a gradient blow-up behaviour since, in the minimal case

$$y = \frac{x}{(T - t)^\beta} \rightarrow \infty,$$

and, hence,

$$u(x, t) = (T - t)^\alpha g(y) = (T - t)^\alpha C \left(\frac{x}{(T - t)^\beta} \right)^{\alpha/\beta} \rightarrow u(x, T^-) = Cx^{\frac{\alpha}{\beta}},$$

for any $x \geq 0$ or, similarly for $x < 0$, uniformly on compact subsets. Note that the above solutions $u(x, T^-) = Cx^{\frac{\alpha}{\beta}}$ correspond to gradient blow-up under the following condition:

$$\frac{\alpha}{\beta} < 1 \implies \alpha(4 - n) < 2.$$

Otherwise, the profile $u(x, T^-) = Cx^{\frac{\alpha}{\beta}}$ can be Lipschitz at $x = 0$ if $\frac{\alpha}{\beta} = 1$, or even $C^{k+\sigma}$ -smooth (or more) if $\frac{\alpha}{\beta} > 1$ and $k = \lfloor \frac{\alpha}{\beta} \rfloor$. However, in the latter case we also expect a discontinuous shock wave to occur for $t > 0$. Moreover, an interesting and “unusual” feature occurs for $t > T$, since solutions become smooth again and a shock wave does not appear. This is a new mechanism of formation for shock waves of such Boussinesq-type wave equations.

Consequently, we study a “counterpart” of (1.3), i.e. we construct blow-up self-similar solutions such that, a single point gradient blow-up occurs as $t \rightarrow T^-$, and

$$(1.6) \quad \boxed{u_x(x, T^-) \text{ has a singularity at } x = 0, \text{ but } u(x, t) \text{ is smooth for } t > T.}$$

In other words, we look for *gradient blow-up* via (1.6) at $t = T^-$. Then, the solutions get a singularity at $x = 0$ (and, possibly, u_x gets to infinite at this single point). Thus, solutions develop gradient singularities in finite time. We would like to mention that several other very important aspects for the third order equation $u_t = (uu_x)_{xx}$ were analysed in [14] for the formation of shock waves via gradient blow-up phenomena as we have studied here. Some of those qualities might be applied and extended here for equation (1.1). However, we will not focus on those aspects.

Therefore, after the asymptotic analysis performed in Section 2 for the self-similar solutions (1.4) corresponding to the self-similar equation (1.5) when $n > 0$ we study the linear problem, when $n = 0$. For $n = 0$, (1.1) becomes the 1D *linear beam equation* or the *fourth-order linear wave equation* (the LBE-4 or the LWE-4)

$$(1.7) \quad u_{tt} = -u_{xxxx} \quad \text{in } \mathbb{R} \times \mathbb{R}_+;$$

see the mathematical state-of-the-art concerning the operator and semigroup theory for (1.7) on a bounded x -interval in [8]. In \mathbb{R} , many crucial evolution, oscillatory, etc. properties can also be revealed by constructing the *fundamental solution* $b_0(x, t)$ and applying convolution techniques. Something we have performed in Sections 3 and 4.

We then claim that, by a homotopy deformation as $n \rightarrow 0^+$, these properties will persist for the quasilinear case (1.1) of small enough $n > 0$. Thus, we apply this basic idea to treat fundamental solutions for (1.1). Also, we expect that this “homotopy” argument can be a basis for a proper functional setting of the Cauchy problem for (1.1), using the clear well-posedness for the LWE (1.7).

To do so, we carry out an analysis of the homotopy deformation from the non-linear self-similar equation (1.5) to the corresponding one for the case when $n = 0$ that allows us to ascertain such gradient blow-up behaviour satisfying (1.6). Actually, we applied two main approaches to justify the existence of various types of gradient blow-up as $t \rightarrow T^-$ and extensions beyond, as $t \rightarrow T^+$ (and, eventually, (1.6)):

- (i) Section 5: a “homotopy-like” approach by passing to the limit as $n \rightarrow 0^+$. Thus, we are trying to perform a first step in describing singularity formation phenomena for the QWE-4 (1.1), where one of the basic tools for better understanding of such difficult nonlinear phenomena is:

$$(1.8) \quad \boxed{\text{a continuous “homotopic” connection as } n \rightarrow 0^+ .}$$

- (ii) Section 6: a careful and convincing numerical study of possible similarity solutions in both limits $t \rightarrow T^\pm$ is performed. We must point out that the numerics for the nonlinear eigenvalue problem (1.5) under the minimal growth at infinity presents an enormous challenge. The problem appears

due to the expansion of the minimal bundle since the oscillatory terms are buried within the expansion that depends, obviously, on the parameter α (coming from the self-similar solution). When, α is bigger the minimal bundle dominates the maximal one, as y gets larger. Then, finding the maximal behaviour becomes very difficult; details below.

Eventually, we show that, after a single-point gradient-like blow-up, there exists a unique similarity extension beyond.

The QWE-4: applications and on general theoretical demands. A sufficient theoretical demand and challenge for nonlinear PDEs such as (1.1) and (1.2) are already available from some areas of applications. For instance, concerning physical motivation of the nonlinear *modified dispersive Klein–Gordon equation* ($mKG(1, n, k)$),

$$(1.9) \quad u_{tt} + a(u^n)_{xx} + b(u^k)_{xxxx} = 0, \quad n, k > 1 \quad (u \geq 0),$$

see [17] for further applications and some exact travelling wave (TW) solutions constructed. For $b > 0$, (1.9) is of wave Boussinesq-type with a class of nonnegative solutions.

Another actively developing PDE and application area deals with some related nonlinear 2D *dispersive Boussinesq equations* denoted by the operator $B(m, n, k, p)$ (see [22] and references therein), such that

$$(1.10) \quad B(m, n, k, p)u \equiv (u^m)_{tt} + \alpha(u^n)_{xx} + \beta(u^k)_{xxxx} + \gamma(u^p)_{yyyy} = 0 \quad \text{in } \mathbb{R}^2 \times \mathbb{R}.$$

See also [16, Ch. 4-6] for more references and examples of exact solutions on invariant subspaces of QWEs of various types and orders.

It is clear that, for formation of crucial patterns and singularities for PDEs such as (1.9) and (1.10), the principal quasilinear higher-order operators are key. This leads to the canonical equations (1.1) or (1.2).

Unlike (1.9) posed for nonnegative solutions $u \geq 0$, the absolute value in the nonlinear dispersivity coefficient $|u|^n u$ in (1.1) is necessary while dealing with solutions of changing sign. Obviously, for (1.2) putting $|u|^2 \equiv u^2$ is not necessary.

It is worth mentioning now that fourth-order PDEs such as (1.1) “almost always” admit solutions of changing sign. So that their nonnegative smooth compacton-type solutions, that have been found in a number of papers, either are not solutions at all, or, at least, nonnegative compactons *are not robust*, i.e., they are not stable with respect to small perturbations of the parameters and/or nonlinear coefficients involved. Moreover, dynamical systems induced by such higher-order nonlinear dispersion PDEs, admitting nonnegative compactons, *are not structurally stable*; see more details in [13, 15]. In other words, dealing with complicated infinitely oscillatory solutions of various higher-order parabolic, hyperbolic and nonlinear dispersion PDEs is a common and unavoidable trend of modern PDE theory.

Concerning the QWE-4 (1.1), which is not a hyperbolic system, it seems that there are still not many results on proper functional settings for entropy solutions. Even though formation of shocks for similar equations (1.1) has been studied in [5, 11, 14], while types of gradient blow-up were essentially unknown. In other words, there is not a proper description of all types of singularities, that can occur in such a nonlinear dispersion evolution equation.

It seems that it is still not widely recognised that formal compactons, peakons, compact breathers and other localised dissipative structures, constructed for a number of nonlinear PDEs of higher orders, demand special mathematical tools to specify which free-boundary or the Cauchy problems these are solutions of; see [16, p. 199-200] as an invitation to such a discussion. In particular, it is difficult to find that such PDEs, having a mechanism of *nonlinear dispersion*, can produce *shocks or shock waves*, or other types of discontinuous/singular solutions in finite time, quite similarly to the QWE-2

$$u_{tt} = (|u|^n u)_{xx}$$

and the *nonlinear dispersion equation* (NDE-3) [9, 14, 15]

$$(1.11) \quad u_t = (uu_x)_{xx} \quad \text{in } \mathbb{R} \times \mathbb{R}_+ \quad (\text{or the NDE-5 } u_t = -(uu_x)_{xxxx}, [12, 13]).$$

2. “GRADIENT” BLOW-UP SIMILARITY SOLUTIONS FOR $n > 0$

2.1. Self-similar solutions. Our first goal is to show that formation of locally bounded gradient solutions of (1.1) can be performed via self-similar blow-up patterns which occur due to its natural scaling-invariant nature. Indeed, using the following scaling invariance in (1.1):

$$x := \lambda^\beta \bar{x}, \quad t := \lambda \bar{t}, \quad u := \lambda^{\frac{4\beta-2}{n}} \bar{u}, \quad \text{with}$$

$$\frac{\partial u}{\partial t} = \frac{\nu}{\lambda} \frac{\partial \bar{u}}{\partial \bar{t}}, \quad \frac{\partial^2 u}{\partial t^2} = \frac{\nu}{\lambda^2} \frac{\partial^2 \bar{u}}{\partial \bar{t}^2}, \quad \frac{\partial u}{\partial x_i} = \frac{\nu}{\mu} \frac{\partial \bar{u}}{\partial \bar{x}_i}, \quad \frac{\partial^2 u}{\partial x_i^2} = \frac{\nu}{\mu^2} \frac{\partial^2 \bar{u}}{\partial \bar{x}_i^2},$$

and substituting those expressions in (1.1) yields for invariance $\nu := \lambda^{\frac{4\beta-2}{n}}$ so that

$$\frac{\nu}{\lambda^2} \frac{\partial^2 \bar{u}}{\partial \bar{t}^2} = -\frac{\nu^{n+1}}{\mu^4} (|\bar{u}|^n \bar{u})_{\bar{x}\bar{x}\bar{x}\bar{x}}.$$

To keep this equation invariant, the following equalities must be fulfilled:

$$\frac{\nu}{\lambda^2} = \frac{\nu^{n+1}}{\mu^4} \implies \text{if } \mu := \lambda^\beta \implies \nu := \lambda^{\frac{4\beta-2}{n}}, \quad \text{so that}$$

$$u(x, t) := \lambda^{\frac{4\beta-2}{n}} \bar{u}(\bar{x}, \bar{t}) = (-t)^{\frac{4\beta-2}{n}} \bar{u}\left(\frac{x}{\lambda^\beta}\right), \quad \text{where } \lambda = -t.$$

Therefore, we arrive at the self-similar blow-up patterns of the form

$$(2.1) \quad u(x, t) = (-t)^\alpha g(y), \quad y = x/(-t)^\beta, \quad \beta = \frac{2+\alpha n}{4},$$

where α is a real parameter such that $\alpha = \frac{4\beta-2}{n} > 0$ if $\beta > \frac{1}{2}$ and g solves the ODE

$$(2.2) \quad \beta^2 y^2 g'' + \mu y g' + \alpha(\alpha-1)g = -(|g|^n g)^{(4)} \quad \text{in } \mathbb{R},$$

$$\mu = \beta(\beta+1) - 2\alpha\beta = \frac{\beta[6 + (n-8)\alpha]}{4}.$$

Here the blow-up time is taken at $T = 0$ for simplicity, so that solutions (2.1) generate gradient blow-up as $t \rightarrow 0^-$. In the case of blow-up time T , we will have the following self-similar patterns

$$u(x, t) = (T-t)^\alpha g(y), \quad y = x/(T-t)^\beta, \quad \beta = \frac{2+\alpha n}{4}.$$

2.2. Minimal and Maximal growth at infinity. We next need special “boundary conditions” at infinity, i.e. as $y \rightarrow \infty$. This allows us to formulate a condition at infinity or, denoted by “radiation-type conditions” (we use a standard term from dispersion theory) for the nonlinear eigenvalue problem (2.2). Thus we will look for solutions of the nonlinear eigenvalue problem (2.2) so that those conditions at infinity are satisfied. Hence, our nonlinear ODE (2.2) admits two kinds of asymptotic behaviour at infinity, i.e. when $y \rightarrow +\infty$ and we state the following.

Proposition 2.1. *For any $\alpha > 0$, the solutions $g(y)$ of the ODE (2.2) admit:*

(i) *A minimal growth*

$$(2.3) \quad g_{\min}(y) = C y^\gamma (1 + o(1)), \quad \gamma = \frac{\alpha}{\beta} > 0, \quad \text{as } y \rightarrow +\infty,$$

where $C \neq 0$ is an arbitrary constant.

(ii) *Moreover, there exist solutions of (2.2) with a maximal growth*

$$(2.4) \quad g_{\max}(y) \sim B y^\delta (1 + o(1)), \quad \text{where } \delta = \frac{4}{n} > \frac{\alpha}{\beta},$$

as $y \rightarrow +\infty$ and with $B \equiv B(y)$ a bounded oscillatory function.

Proof. This result follows from balancing of linear and nonlinear operators in this ODE (2.2), though a rigorous justification is rather involved and technical (although we support the results numerically). A formal derivation is straightforward:

- (i) The *minimal* behaviour at infinity corresponds to the linear operator on the left-hand side of (2.2). Since, the nonlinear one is negligible as $y \rightarrow \infty$. Then, calculating the orders of the algebraic growth for both terms we see that

$$(|g|^n g)^{(4)} \sim y^{\frac{\alpha}{\beta}(n+1)-4} \ll g \sim y^{\frac{\alpha}{\beta}}, \text{ since } \frac{\alpha n}{\beta} = \frac{4\alpha n}{2+\alpha n} < 4.$$

To obtain (2.3), we look for algebraically growing solutions of (2.2) with the form $g(y) = y^\nu$, arriving at the following characteristic equation:

$$\beta^2 \nu(\nu - 1) + (\beta(\beta + 1) - 2\alpha\beta)\nu + \alpha(\alpha - 1) = 0,$$

which, obviously, possesses the necessary root $\nu = \frac{\alpha}{\beta}$ (the second one is irrelevant).

We are only interested in the minimal behaviour (2.3). One can see that since, passing to the limit as $t \rightarrow 0^-$ in (2.1), we obtain a locally bounded finite-time profile of the form

$$(2.5) \quad u(x, 0^-) = Cx^{\frac{\alpha}{\beta}} \quad \text{for any } x \geq 0 \quad (\text{or, similarly, for } x < 0).$$

- (ii) On the other hand, (2.2) admits the *maximal* behaviour (2.4), determined by the nonlinear term on the right-hand side, as $y \rightarrow +\infty$.

$$g_{\max}(y) \sim B y^\delta (1 + o(1)), \quad \text{where } \delta = \frac{4}{n} > \frac{\alpha}{\beta},$$

Here, (2.4) indicates a monotone algebraic “envelope” of an oscillatory asymptotic bundle of maximal solutions. Namely, B is actually a bounded oscillatory function. Indeed, substituting (2.4) into (2.2) and assuming that $B \neq 0$ is a constant yields

$$(2.6) \quad \begin{aligned} & \beta^2 \delta(\delta - 1) + \mu\delta + \alpha(\alpha - 1) \\ & = -|B|^n \delta(n+1)[\delta(n+1) - 1][\delta(n+1) - 2][\delta(n+1) - 3]. \end{aligned}$$

Thus, the right hand side is negative, since $\delta(n+1) - 3 = 1 + \frac{4}{n} > 0$, whilst the left hand side in (2.6) is equal to $\frac{4}{n^2} + \frac{2}{n} > 0$. Therefore, a constant solution B is not possible and we conclude that it is a function. □

Remarks.

- Similar minimal and maximal conditions are posed as $y \rightarrow -\infty$ (possibly with different constants $C \neq 0$ and B , if necessary).
- The notions of minimal/maximal are justified as follows: as $y \rightarrow +\infty$,

$$y^\gamma \ll y^\delta, \quad \text{since } \delta = \frac{4}{n} > \gamma = \frac{\alpha}{\beta}.$$

Therefore, solutions with a maximal growth do not produce any finite (non-singular) trace, as in (2.5), at the blow-up time $t = 0$.

- As usual in self-similar approaches and ODE theory for these problems, such conditions are known to be defined and associated with self-similar solutions of the *second kind*, a term, which was introduced by Ya.B. Zel’dovich in 1956 [23].

We note that there are two main types of self-similar solutions. For solutions of the first kind the similarity variable y can be determined *a priori* from dimensional considerations and conservation laws, such as the conservation of mass or momentum.

For solutions of the second kind the exponent β (and by relations the exponent α) in the similarity variable must be obtained along with the solution by solving a nonlinear eigenvalue problem of the form (2.2).

Many of such ODE problems (but indeed, easier) have been rigorously solved since then. For instance, we might cite [1] where the self-similar nonlinear eigenvalue problem associated with the following *thin film equation* (TFE) $u_t = -\nabla \cdot (|u|^n \nabla \Delta u)$ was analysed.

Consequently, to obtain proper gradient blow-up profiles, we pose the “nonlinear eigenvalue problem” (2.2) with “nonlinear eigenvalues” $\alpha > 0$ so that:

$$(2.7) \quad \boxed{\alpha : \text{solve (2.2) with the minimal condition (2.3).}}$$

Then, gradient blow-up profiles should appear from the (nonlinear eigenvalue) problem (2.7). Moreover, one can see that a well-posedness of this problem crucially depends on the dimension of the maximal (expected to be stable) and minimal asymptotic bundles as $y \rightarrow \pm\infty$.

2.3. Numerical periodic oscillatory profiles. We now show some of the profiles for those minimal and maximal bundles numerically.

Maximal bundle.

We seek here to demonstrate that $B = B(s)$, with $s = \ln y$, possesses periodic oscillations as $y \rightarrow \infty$ and that the bundle of maximal solutions is 2D. It is convenient to introduce

$$h(y) = |g(y)|^n g(y), \quad \text{so that} \quad g(y) = |h(y)|^{-\frac{n}{n+1}} h(y),$$

and then (2.2) becomes

$$|h|^{\frac{n}{n+1}} h^{(4)} + \frac{\beta^2}{(n+1)} y^2 \left(h'' - \frac{n}{(n+1)} \frac{(h')^2}{h} \right) + \frac{\mu}{(n+1)} y h' + \alpha(\alpha-1)h = 0.$$

We consider solutions for the maximal behaviour in the form

$$g(y) = B(y)y^{\frac{4}{n}}, \quad h(y) = H(y)y^{4+\frac{4}{n}},$$

so that $H(y) = |B(y)|^n B(y)$ or $B(y) = |H(y)|^{-\frac{n}{n+1}} H(y)$. The resulting ODE for $H(y)$ (or equivalently $B(y)$) is equidimensional in y , so that on using $s = \ln y$, we obtain for $H(s)$ the autonomous ODE

$$(2.8) \quad \ddot{H} + a_1 \dot{H} + a_2 \ddot{H} + a_3 \dot{H} + a_4 H + |H|^{-\frac{n}{n+1}} \left(\frac{\beta^2}{(n+1)} \left(\ddot{H} - \frac{n\dot{H}^2}{(n+1)H} \right) + \frac{\beta(n+4)}{n(n+1)} \dot{H} + \frac{2(n+2)}{n^2} H \right) = 0,$$

where $\dot{\cdot}$ denotes d/ds and

$$a_1 = \frac{2(5n+8)}{n}, \quad a_2 = \frac{(35n^2+120n+96)}{n^2}, \\ a_3 = \frac{2}{n^3}(5n+8)(5n^2+20n+16), \quad a_4 = \frac{8}{n^4}(n+1)(n+2)(3n+4)(n+4).$$

We may recover $B(s)$ from $B(s) = |H(s)|^{-\frac{n}{n+1}} H(s)$. Figure 1 illustrates numerical solutions of (2.8) as an IVP with $H(0) = 1, \dot{H}(0) = \ddot{H}(0) = \ddot{H}(0) = 0$, using Matlab solver ode15s with AbsTol=RelTol= 10^{-6} . Shown is the case $n = 1$ with selected β . The profiles are heavily damped, indicating that the maximal bundle (2.4) has behaviour subdominant to $O(y^{\frac{4}{n}})$.

Minimal bundle.

Next, we analyse the asymptotic bundle of the minimal solutions. For practical reasons, we pose two anti-symmetry conditions at the origin,

$$(2.9) \quad (|g|^n g)(0) = (|g|^n g)''(0) = 0 \implies g(-y) = -g(y) \quad \text{for } y < 0,$$

which allows us to consider the ODE (2.2) for $y > 0$ only. Since the problem is scaling invariant, we also pose a normalisation condition, say (any other ones will also do it) $(|g|^n g)'(0) = 1$. Thus, we have two possibilities:

(i) The bundle of minimal solutions as $y \rightarrow +\infty$ is 1D. Then, in order to shoot/match TWO boundary conditions at $y = 0$ in (2.9), one needs to add to C the only left parameter $\alpha > 0$, so we arrive at a well-posed shooting $2 \mapsto 2$. This is a classical situation of the self-similarity of the second type, where

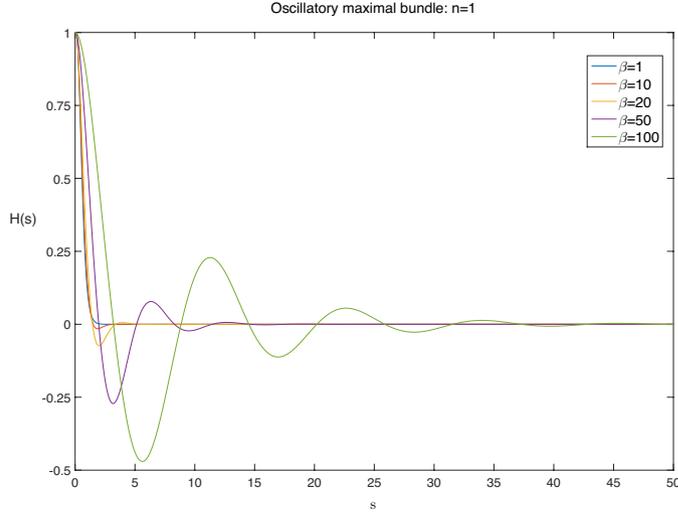


FIGURE 1. Illustrative numerical solutions of the oscillatory function $H(s)$ from (2.8) in the case $n = 1$ and selected β .

the nonlinear spectrum is discrete. As it happens here and we will prove it via a homotopy deformation from the linear self-similar equation when $n = 0$

(ii) The bundle of minimal solutions is 2D. Then a shooting $2 \mapsto 2$ is well-posed without using α , so we can try to construct proper similarity profiles for any α (the case of a continuous spectrum).

However, for $n = 0$, as we will see in the next section, the case (i) holds, and it persists for small $n > 0$. In this case, we arrive at the situation of a proper $2 - 2$ shooting with two parameters (cf. the case $n = 0$ below)

$$\nu = (|g|^n g)'''(0) \quad \text{and} \quad \alpha > 0,$$

to delete 2D bundle of maximal asymptotics from the behaviour as $y \rightarrow +\infty$. This corresponds as well to a classical case of self-similarity of the second kind, which is well-known from the literature since the 1940-50s.

Subsequently, we ascertain that the bundle of minimal solutions is 1D, with the correction term obtained by linearisation about the leading term in g_{\min} to obtain the inhomogeneous equation. Indeed, $g = g_{\min} + Z$ implies that

$$(n+1)(|g_{\min}|^n Z)^{(4)} + \beta y^2 Z'' + \mu y Z' + \alpha(\alpha-1)Z \sim -(|g_{\min}|^n g_{\min})^{(4)},$$

as $y \rightarrow +\infty$. Taking $g_{\min} = C y^{\frac{\alpha}{\beta}}$, we obtain a unique (non-oscillatory) solution,

$$(2.10) \quad Z(y) = D y^{(n+1)\frac{\alpha}{\beta} - 4} + \dots,$$

where $D \in \mathbb{R}$ is determined by C in (2.3) and is given explicitly by

$$(2.11) \quad \left(\frac{1}{2}\alpha n + 1\right)^3 D = -2|C|^n C(n+1) \left(\frac{1}{2}\alpha n + \alpha - 1\right) \left(\frac{3}{2}\alpha n + 2\alpha - 1\right) \left(\frac{1}{2}\alpha n + 2\alpha - 3\right).$$

Further terms in this expansion may be obtained by proceeding in a similar manner, and thus a converging series for $g_{\min}(y)$ for $y \gg 1$ can be constructed, containing one free parameter C .

3. SIMILARITY SOLUTIONS IN THE LINEAR CASE $n = 0$

Curiously, the linear case $n = 0$ allows us to detect some crucial “geometric” properties that might be extended to the nonlinear ODE (2.2) for, at least, small $n > 0$; see Section 5.

Thus, for $n = 0$, the solutions (2.1) take a simpler form

$$u(x, t) = (-t)^\alpha g(y), \quad y = \frac{x}{\sqrt{-t}}, \quad \beta = \frac{1}{2},$$

and the ODE for g is now:

$$(3.1) \quad \frac{1}{4} y^2 g'' + \mu y g' + \alpha(\alpha - 1)g = -g^{(4)} \quad \text{in } \mathbb{R}, \quad \mu = \frac{3-4\alpha}{4}.$$

Indeed, following the analysis performed above for (2.2) we note that in this case, with $n = 0$, the parameter α remains undetermined while $\beta = 1/2$ is the lower bound for the β 's obtained in order to have positive values of α .

Minimal behaviour at infinity when $n = 0$. Again, ‘‘boundary conditions’’ at infinity, i.e. as $y \rightarrow \infty$, are very simple now. Indeed, a *minimal* behaviour at infinity, as $y \rightarrow +\infty$, is

$$(3.2) \quad g_{\min}(y) = C y^\gamma (1 + o(1)), \quad \text{with } \gamma = 2\alpha > 0,$$

where $C \neq 0$ is a constant. This bundle of minimal solutions is 1D. To this end, we perform a linearisation about g_{\min} to get an inhomogeneous equation :

$$(3.3) \quad g = g_{\min} + Z \implies Z^{(4)} + \frac{1}{4} y^2 Z'' + \mu y Z' + \alpha(\alpha - 1)Z = -g_{\min}^{(4)} \sim y^{2\alpha-4}$$

as $y \rightarrow +\infty$. One can see from (3.3) that there exists a unique (non-oscillatory) solution,

$$Z(y) = D y^{2\alpha-4} + \dots,$$

where $D \in \mathbb{R}$ is uniquely determined by C in (3.2) and given explicitly by

$$D = -2\alpha(\alpha - 1)(2\alpha - 1)(2\alpha - 3)C.$$

These expressions agree with (2.10) and (2.11) for $n = 0$. As in the $n > 0$ case, we may continue the expansion to construct a converging series for $g_{\min}(y)$, for $y \gg 1$. Specifically,

$$(3.4) \quad g_{\min} = C y^{2\alpha} \sum_{k=1}^{\infty} D_k y^{-4(k-1)},$$

where $D_1 = 1$ and

$$D_{k+1} = -D_k \frac{(2\alpha + 4 - 4k)(2\alpha + 3 - 4k)(2\alpha + 2 - 4k)(2\alpha + 1 - 4k)}{k(4k - 2)},$$

for $k \geq 1$. It is noteworthy that this series truncates when 2α is a (non-negative) integer.

Maximal behaviour at infinity when $n = 0$. Maximal behaviour is now exponential:

$$(3.5) \quad g(y) \sim y^{(-1-2\alpha)} e^{ay^2} \implies a^2 = -\frac{1}{16} \implies a_{\pm} = \pm i \frac{1}{4}.$$

Therefore, the general bundle as $y \rightarrow +\infty$ is 3D:

$$(3.6) \quad g(y) \sim C y^{2\alpha} (1 + D_2 y^{-4} + D_3 y^{-8} + \dots) + y^{-1-2\alpha} \left(C_1 \cos\left(\frac{y^2}{4}\right) + C_2 \sin\left(\frac{y^2}{4}\right) \right) + \dots,$$

and there exists a 2D bundle of maximal solutions (since C_1, C_2 are arbitrary), which are oscillatory as $y \rightarrow \infty$.

Let us finally discuss the possibility of finding a proper solution of the linear ODE (3.1) for some values of eigenvalues $\alpha > 0$. To this end, we perform a shooting procedure, from $y = 0$ to $y = +\infty$, by posing the normalization and the anti-symmetry conditions:

$$(3.7) \quad g'(0) = 1, \quad g(0) = g''(0) = 0.$$

Now let us count the total number of conditions and parameters. We are left with two parameters only, these are

$$(3.8) \quad \nu = g'''(0) \quad \text{and} \quad \alpha,$$

and we need to satisfy two conditions at infinity in (3.6):

$$(3.9) \quad C_1 = C_1(\nu, \alpha) = 0 \quad \text{and} \quad C_2 = C_2(\nu, \alpha) = 0.$$

Thus we observe a well-posed 2 – 2 shooting problem. Moreover, in view of the analyticity of all the coefficients involved, we arrive at:

Proposition 3.1. *The linear eigenvalue problem (3.1) admits no more than a countable family of solution pairs $\{g_k(y), \alpha_k\}_{k \geq 1}$, with the only concentration point at infinity, when $\alpha_k \rightarrow +\infty$ as $k \rightarrow \infty$.*

We will show that these basic, but fundamental, properties can be extended for small $n > 0$.

4. SPECTRAL PROPERTIES OF A QUADRATIC PENCIL

In this section, we now present different aspects to develop the spectral theory that is appropriate for the linear PDE (1.7). Surprisingly, this theory turned out to be difficult and comprised of several delicate aspects of spectral analysis of new non self-adjoint operators and corresponding quadratic pencils. We note that this linear operator/pencil analysis is unavoidable for a proper understanding of first features of the quasilinear PDE (1.1). Indeed, we will connect them via a homotopy deformation as $n \rightarrow 0$ (1.8) extending several qualitative properties of the linear case when $n = 0$ to the quasilinear range $n > 0$.

Thus, we present here some spectral concepts associated with the linear PDE (1.7).

4.1. Fundamental solution when $n = 0$. Considering the LW E–4 (1.7) due to the Fourier Transform, one can obtain its *fundamental solution* of the self-similar form

$$(4.1) \quad b_0(x, t) = \sqrt{t} F_0(y), \quad y = x/\sqrt{t}, \quad \text{satisfying} \quad b_0(x, 0) = 0, \quad b_{0t}(x, 0) = \delta(x)$$

in the sense of distributions. The symmetric rescaled kernel $F_0 = F_0(|y|)$ solves the ODE

$$(4.2) \quad \mathbf{B}F_0 \equiv -F_0^{(4)} - \frac{1}{4} F_0'' y^2 - \frac{1}{4} F_0' y + \frac{1}{4} F_0 = 0 \quad \text{in} \quad \mathbb{R}, \quad \int F_0 = 1.$$

Integrating (4.2) once yields

$$(4.3) \quad -F_0''' - \frac{1}{4} F_0' y^2 + \frac{1}{4} F_0 y = 0 \quad \text{in} \quad \mathbb{R} \quad (F_0'(0) = F_0'''(0) = 0).$$

Finally, this gives the symmetric kernel

$$F_0(y) = \frac{1}{2\pi} \int_0^\infty \frac{\sin z \cos(\sqrt{z}y)}{z^{3/2}} dz.$$

Thus, for initial data from $\mathcal{D}'(\mathbb{R})$, $u(x, 0) = u_0(x)$, $u_t(x, 0) = u_1(x)$ in \mathbb{R} , the LW E–4 has a unique weak solution given by the convolutions

$$(4.4) \quad u(x, t) = (b(x - \cdot, t) * u_0(\cdot))_t + b(x - \cdot, t) * u_1(\cdot).$$

This is similar to the classic *d'Alembert–Kirchhoff–Poisson formula* for the linear second-order wave equation (the LW E–2) admitting a simpler fundamental solution in 1D, in the sense that

$$(4.5) \quad u_{tt} = u_{xx} \quad \implies \quad b(x, t) = \frac{1}{2} \theta\left(1 - \frac{|x|}{t}\right) \equiv F_0(y), \quad y = \frac{x}{t},$$

where θ is the Heaviside function, so that the kernel is now $F_0(y) = \frac{1}{2} \theta(1 - |y|)$.

The formula (4.4) gives a more complicated behaviour for (1.7) than that for (4.5). Indeed, the kernel $F(y)$ in (4.1) is highly oscillatory. Namely, a WKB J-type asymptotic analysis of the ODE (4.2) yields the following asymptotics:

$$(4.6) \quad F_0(y) \sim y^{-2} \left(C_1 \cos \frac{y^2}{4} + C_2 \sin \frac{y^2}{4} \right) \quad \text{as} \quad y \rightarrow +\infty,$$

where $C_{1,2}$ are some constants. It then follows that $F_0(y)$ is integrable over \mathbb{R} in the improper sense, so the normalisation $\int F_0 = 1$ in (4.2) makes sense. Moreover, $F_0 \notin L^1(\mathbb{R})$, but $F_0 \in L^2(\mathbb{R})$. Actually, the full 4D bundle of solutions of (4.2) is as follows:

$$F(y) \sim C_1 y^{-2} \cos\left(\frac{y^2}{4} + C_2\right) + C_3 \frac{1}{y} + C_4 y \quad \text{as } y \rightarrow +\infty,$$

where, due to the integrability of the fundamental rescaled profile $F_0(y)$ one has to have $C_3 = C_4 = 0$. Therefore, two remaining parameters $C_{1,2}$ are sufficient to shoot two symmetry conditions at the origin given in (4.3).

4.2. Rescaled variables and equation. Namely, we introduce the rescaled variables associated with the similarity structure in (4.1) satisfying:

$$(4.7) \quad u(x, t) = \sqrt{t} w(y, \tau), \quad y = x/\sqrt{t}, \quad \tau = \ln t \Rightarrow w_{\tau\tau} - w_{\tau y} y = \mathbf{B}w \quad \text{in } \mathbb{R} \times \mathbb{R},$$

where \mathbf{B} is the operator in (4.2) defining the kernel F_0 of the fundamental solution (4.1). For the 2-vector $W = (w, w_\tau)^T$, (4.7) is a dynamical system (DS)

$$(4.8) \quad W_\tau = \mathbf{C}W, \quad \text{with the matrix } \mathbf{C} = \begin{bmatrix} 0 & 1 \\ \mathbf{B} & yD_y \end{bmatrix}.$$

4.3. Semigroup and eigenfunction expansion. To expand (4.4) with a kind of ‘‘mean convergence’’ rather than a formal representation of a weak solution, we need to assume that data u_0, u_1 belong to the adjoint space $L_{\rho^*}^2$ (cf. (4.18) below)

$$(4.9) \quad u_0, u_1 \in L_{\rho^*}^2(\mathbb{R}), \quad \rho^*(y) = e^{ay^2} > 0, \quad a > 0, \quad \text{so } (L_{\rho^*}^2)^* = L_\rho^2,$$

where $\rho = \frac{1}{\rho^*}$ and L_ρ^2 is the space of linear functionals defined in $L_{\rho^*}^2$. This duality will help us to identify eigenfunctions of operators and pencils that will be introduced.

Next we show the explicit representation of the semigroup induced by the dynamical system (4.8).

Lemma 4.1. *Assume the dynamical system $W_\tau = \mathbf{C}W$ denoted by (4.8). Then, the semigroup induced by the dynamical system (4.8) possesses the following explicit representation*

$$(4.10) \quad w(y, \tau) = \frac{1}{2} e^{-\tau} \int F_0(\cdot) u_0(z) dz - \frac{1}{2} e^{-\tau} \int F_0'(\cdot)(\cdot) u_0(z) dz + \int F_0(\cdot) u_1(z) dz,$$

where (\cdot) denotes the variable $y - e^{-\frac{\tau}{2}} z$.

Proof. It follows from (4.4) that

$$(4.11) \quad \sqrt{t} w(y, \tau) = \frac{1}{2\sqrt{t}} \int F_0(\cdot) u_0(z) dz - \frac{1}{2\sqrt{t}} \int F_0'(\cdot)(\cdot) u_0(z) dz + \sqrt{t} \int F_0(\cdot) u_1(z) dz,$$

where (\cdot) denotes the variable $(x - z)/\sqrt{t}$. In terms of the independent rescaled variables (4.7) with $t = e^\tau$, (4.11) reads (4.10) where now $(\cdot) = y - e^{-\frac{\tau}{2}} z$. This is the actual explicit representation of the semigroup induced by the dynamical system (4.8), proving the lemma. \square

Furthermore, in view of the time variable $\tau = \ln t$, stating the Cauchy problem for (4.7) with initial data at $\tau = 0$, it follows that

$$(4.12) \quad \begin{aligned} w(y, 0) &= w_0(y) \quad \text{and} \quad w_\tau(y, 0) = w_1(y), \quad \text{i.e.,} \\ w_0(y) &= u(y, 1) \quad \text{and} \quad w_1(y) = u_\tau(y, 1) - \frac{1}{2} w_0(y) + \frac{1}{2} w_0'(y). \end{aligned}$$

Now, we ascertain an expression for the solutions $w(y, \tau)$ depending on a set of eigenfunctions which, as we show below, they are the eigenfunctions of a *quadratic pencil operator* corresponding to the PDE in (4.7); see Markus [19] for necessary concepts and general pencil operator theory.

Lemma 4.2. *Solutions of the problem (4.7) might be represented by the eigenfunction expansion*

$$(4.13) \quad \begin{aligned} w(y, \tau) &= \psi_0(y)M_0(u_1) + e^{-\frac{\tau}{2}}\psi_1(y)M_1(u_1) + e^{-\tau}[\phi_2(y)M_0(u_0) + \psi_2(y)M_2(u_1)] \\ &+ e^{-\frac{3\tau}{2}}[\phi_3(y)M_1(u_0) + \psi_3(y)M_3(u_1)] \\ &+ \sum_{(k \geq 4)} e^{-\frac{k\tau}{2}}[\phi_k(y)M_{k-2}(u_0) + \psi_k(y)M_k(u_1)], \end{aligned}$$

where we introduce two sets of eigenfunctions, ϕ_k and ψ_k ,

$$(4.14) \quad \left\{ \begin{aligned} \phi_2(y) &= \frac{1}{2}[F_0(y) - yF_0^1(y)], \\ \phi_3(y) &= -\frac{1}{2}yF_0^2(y), \\ \phi_k(y) &= \frac{(-1)^k}{2} \frac{1}{\sqrt{(k-2)!}} [(1-k)F_0^{(k-2)}(y) - yF_0^{(k-1)}(y)] \quad \text{for } k \geq 4, \\ \psi_k(y) &= \frac{(-1)^k}{\sqrt{k!}} F_0^{(k)}(y) \quad \text{for } k \geq 0. \end{aligned} \right.$$

For convenience, we put $\phi_0 = \phi_1 = 0$, that gives us the whole eigenfunction set $\Phi = \{\psi_k, \phi_k, k \geq 0\}$. The coefficients $\{M_k(\cdot)\}$ in (4.13) are the moments of initial data,

$$(4.15) \quad M_j(v) = \frac{1}{\sqrt{j!}} \int z^j v(z) dz, \quad j \geq 0.$$

Proof. First, we use the Taylor expansion of the analytic kernel $F_0(\cdot)$ that yields

$$(4.16) \quad F_0(y - e^{-\frac{\tau}{2}}z) = \sum_{k \geq 0} \frac{(-1)^k}{k!} F_0^{(k)}(y) e^{-\frac{k\tau}{2}} z^k.$$

Differentiating and substituting (4.16) into the ODEs (4.2) or (4.3), it is not difficult to see that the derivatives of the kernel $F_0(y)$ satisfy the estimates for all $k \geq 0$

$$|F^{(k)}(y)| \leq c^k (1 + y^2)^{\frac{k}{2}} \quad \text{in } \mathbb{R} \quad (c > 0),$$

so the power series (4.16) converges uniformly on any $[a, b]$ to the analytic kernel $F_0(\cdot)$, which is not surprising of course. Substituting (4.16) into (4.10) yields the expansion

$$\begin{aligned} w(y, \tau) &= \frac{1}{2} e^{-\tau} \int \sum_{k \geq 0} \frac{(-1)^k}{k!} F_0^{(k)}(y) e^{-\frac{k\tau}{2}} z^k u_0(z) dz \\ &- \frac{1}{2} e^{-\tau} \int \sum_{k \geq 0} \frac{(-1)^k}{k!} F_0^{(k+1)}(y) e^{-\frac{k\tau}{2}} z^k (y - e^{-\frac{\tau}{2}}z) u_0(z) dz \\ &+ \int \sum_{k \geq 0} \frac{(-1)^k}{k!} F_0^{(k)}(y) e^{-\frac{k\tau}{2}} z^k u_1(z) dz, \end{aligned}$$

and, hence, thanks to (4.15) it follows that

$$\begin{aligned}
w(y, \tau) &= \frac{1}{2} \sum_{k \geq 0} \frac{(-1)^k}{\sqrt{k!}} F_0^{(k)}(y) e^{-\frac{(k+2)\tau}{2}} M_k(u_0) \\
&\quad - \frac{1}{2} \sum_{k \geq 0} \frac{(-1)^k}{\sqrt{k!}} y F_0^{(k+1)}(y) e^{-\frac{(k+2)\tau}{2}} M_k(u_0) \\
&\quad + \frac{1}{2} \sum_{k \geq 0} \frac{(-1)^k (k+1)}{\sqrt{(k+1)!}} F_0^{(k+1)}(y) e^{-\frac{(k+3)\tau}{2}} M_{k+1}(u_0) \\
&\quad + \sum_{k \geq 0} \frac{(-1)^k}{\sqrt{k!}} F_0^{(k)}(y) e^{-\frac{k\tau}{2}} M_k(u_1).
\end{aligned}$$

Therefore, we arrive at

$$w(y, \tau) = \sum_{k \geq 0} e^{-\frac{k\tau}{2}} \phi_k(y) M_{k-2}(u_0) + \sum_{k \geq 0} e^{-\frac{k\tau}{2}} \psi_k(y) M_k(u_1),$$

which leads us to (4.13) having the whole set of eigenfunctions $\Phi = \{\psi_k, \phi_k, k \geq 0\}$, given by (4.14) with $\phi_0 = \phi_1 = 0$. \square

Remark 4.1. *Note that the moments of initial data (4.15) are well defined for data satisfying (4.9). Otherwise, as usual, these moments should be treated as values of linear functionals ψ_j^* being j th-order polynomials as the elements for the adjoint space $L_{\rho^*}^2$, and then (4.13) is treated as a formal expansion of a weak (generalised) solution. We are not going to develop here such a dual theory of expansions of weak solutions, and will be stucked to classical concepts for further use.*

4.4. Quadratic pencil. We observe that (4.13) is the eigenfunction expansion of a *quadratic pencil* corresponding to the PDE in (4.7). Indeed, to find the eigenfunctions, we set

$$(4.17) \quad w(y, \tau) = e^{\lambda_k \tau} \Phi_k(y) \implies \mathbf{C}(\lambda_k) \Phi_k \equiv \mathbf{B} \Phi_k - \lambda_k^2 \Phi_k + \lambda_k \Phi_k' y = 0.$$

Hence, (4.14) implies that, for initial data (4.9), there exists a family of eigenfunctions Φ_k given by (4.14) for the quadratic pencil operator in (4.17), and, thanks to the expression (4.13) we can say that all the eigenvalues are real:

$$\lambda_k = -\frac{k}{2}, \quad k = 0, 1, 2, \dots$$

Functional setting for the Quadratic Pencil. We now need to clarify a suitable functional setting for the linear operator \mathbf{B} given in (4.2). Note that \mathbf{B} is not self-adjoint, and, in particular,

$$\mathbf{B}^* = \mathbf{B} - \mathbf{E}, \quad \mathbf{E} = \frac{1}{2} y D_y + \frac{1}{4} I \quad \text{in the metric of } L^2(\mathbb{R}).$$

We define \mathbf{B} in the weighted space $L_\rho^2 = L_\rho^2(\mathbb{R})$, where

$$(4.18) \quad \rho(y) = \frac{1}{\rho^*(y)} = e^{-ay^2} > 0 \quad \text{and } a > 0 \text{ is a constant.}$$

We will use the following notations (for both weights ρ and the adjoint one ρ^*):

$$\begin{aligned}
\langle \cdot, \cdot \rangle &\text{ is the (dual, see below) scalar product in } L^2; \\
\langle \cdot, \cdot \rangle_\rho \text{ and } \|\cdot\|_\rho &\text{ are scalar product and induced norm in } L_\rho^2(\mathbb{R}).
\end{aligned}$$

Thus, $L^2(\mathbb{R})$ becomes the natural dual space for the weighted ones L_ρ^2 and the adjoint $L_{\rho^*}^2$ of linear functionals. Indeed, for any $v \in L_\rho^2$ and $w \in L_{\rho^*}^2$, the Cauchy-Buniakovskii-Schwarz inequality yields (cf. (4.9))

$$|\langle v, w \rangle| = \left| \int_{\mathbb{R}} v(y) w(y) dy \right| = \left| \int_{\mathbb{R}} \sqrt{\rho} v(y) \frac{1}{\sqrt{\rho}} w(y) dy \right| \leq \|v\|_\rho \|w\|_{\rho^*}.$$

We next introduce a weighted Sobolev space, which is a Hilbert space H_ρ^4 of functions, with the inner product and the norm

$$\langle v, w \rangle_{4,\rho} = \int_{\mathbb{R}} \rho \sum_{k=0}^4 v^{(k)} w^{(k)} dy \quad \text{and} \quad \|v\|_{4,\rho}^2 = \int_{\mathbb{R}} \rho \sum_{k=0}^4 |v^{(k)}|^2 dy.$$

Then $H_\rho^{2m} \subset L_\rho^2 \supset L^2$, and $\mathbf{B} : H_\rho^4 \rightarrow L_\rho^2$ is bounded, since (see a similar proof in [6])

$$\int_{\mathbb{R}} \rho (y^2 v'')^2 dy \leq C \|v\|_{4,\rho}^2 \quad \text{for any } v \in H_\rho^4, \quad \text{with a constant } C > 0.$$

4.5. First discrete spectrum of the pencil. Note that a generalised “radiation”-like condition at infinity shown in Section 3 is necessary to guarantee discreteness of the spectrum. This means that the algebraically decaying solutions of

$$\mathbf{B}\psi = \lambda\psi, \quad \text{with} \quad \psi(y) \sim y^{\pm\sqrt{4-\lambda}},$$

(hence with the behaviour containing the factor $\sim \cos \ln |y|$), as $y \rightarrow \infty$, are not allowed. Hence, more oscillatory asymptotics such as in (4.6) or (4.14) are the only possible ones. Consequently, the above given properties of \mathbf{B} imply that the quadratic pencil operator in (4.17) satisfies the following:

Lemma 4.3. *There exists the first discrete spectrum of the pencil $\mathbf{C}(\lambda)$ in L_ρ^2 consisting of real eigenvalues given by $\sigma_1(\mathbf{C}) = \{\lambda_k = -\frac{k}{2}, k = 0, 1, 2, \dots\}$, where λ_k have multiplicity not more than two with eigenfunctions given by (4.14).*

Observe that the results for Lemma 4.3 follow from the expansion (4.13) that gives all eigenvalues and eigenfunctions of the pencil operator (4.17) in L_ρ^2 for data from the adjoint space L_ρ^{2*} .

Remark 4.2. a) *The completeness and closure of the eigenfunction set $\Phi = \{\psi_k, \phi_k\}$ (to be used in eigenfunction expansion approaches) are discussed in [10]. However, to be concrete, the expansion (4.13) of the action of a continuous semigroup suggests that the eigenfunction set Φ in (4.14) is complete in L^2 and L_ρ^2 . Also, we next define the linear subspace \tilde{L}_ρ^2 of eigenfunction expansions, i.e.,*

$$v \in \tilde{L}_\rho^2 \quad \text{iff} \quad v = \sum (c_k \psi_k + d_k \phi_k) \quad \text{with convergence in } L_\rho^2,$$

as the closure of the subset of finite sums $\{\sum_{|\beta| \leq K} (c_k \psi_k + d_k \phi_k), K \in \mathbb{N}\}$ in the L_ρ^2 -norm. Then the eigenfunctions set of (4.14) is closed in \tilde{L}_ρ^2 in the usual sense.

b) *The resolvent $(\mathbf{C} - \lambda \text{Id})^{-1}$ is compact in L_ρ^2 (for $\lambda \neq \lambda_k$) due to the compact embedding $H_\rho^4 \subset L_\rho^2$.*

4.6. On adjoint polynomials. The eigenfunction expansion (4.13) with moments (4.15) shows that there exists a “bi-orthonormal” basis $\Phi^* = \{\psi_k^*(y), \phi_k^*(y), k \geq 0\}$ of adjoint eigenfunctions that are finite polynomials. The actual structure of those polynomials can be derived from the moments in (4.13) by using the relation (4.12) to initial data for $w(y, \tau)$. In a natural sense, these polynomials can be treated as eigenfunctions of the adjoint pencil operator \mathbf{C}^* ; see analogies in [6] (the corresponding radiation condition as $y \rightarrow \infty$ now prohibits oscillatory asymptotics as in (4.6); then the spectrum becomes discrete).

Remark 4.3. *For the fixed (dual) metric of L^2 , Φ is a Riesz basis, and the corresponding adjoint basis Φ^* is also a Riesz basis; see Naimark [20, § 5] for the case of non self-adjoint ordinary differential operators. For the present singular case of the operator \mathbf{B} and the pencil operator \mathbf{C} , the functional meaning of \mathbf{B}^* , \mathbf{C}^* , their spectral properties, and completeness/closure of the eigenfunction set Φ^* need further study.*

5. GRADIENT BLOW-UP PROFILES AS $n \rightarrow 0^+$

This is a common idea developed by the authors in a number of papers for various nonlinear PDEs; see [2, 1, 3]. In particular, we perform a homotopic approach as $n \rightarrow 0^+$ via branching theory based on the Lyapunov–Schmidt methods in order to obtain relevant results and properties for the solutions of the self-similar equation (2.2). This homotopic-like approach is based upon the spectral properties

of the linear associated problem. However, on the contrary to what happens for the *thin film equations* analysed [2, 1, 3], here we do not possess the complete spectral theory for the linear problem (3.1) so that, we can only perform such a branching-analysis for the first eigenvalue.

Thus, for the QWE-4 (1.1), we construct an analogy of the fundamental solution (4.1) of the linear PDE (1.7). Indeed, the QWE-4 (1.1) is shown to admit a smooth (for $t > 0$) global “fundamental solution” with measures as initial data, which has the form

$$b_n(x, t) = t^{\frac{2}{n+4}} F_n(y), \quad y = \frac{x}{t^{(n+2)/(n+4)}},$$

where F_n satisfies an ODE that is similar to the shock one (2.2).

As $n \rightarrow 0^+$, F_n is shown in the subsequent sections, both analytically and numerically, to be continuously transformed into the rescaled kernel F_0 for the linear operator in (1.7). Performing a homotopy deformation in (1.1) as $n \rightarrow 0^-$ leads to the necessity of studying spectral and other properties of a related to (1.7) non self-adjoint linear operator and of a quadratic pencil of linear operators, which was analysed in the previous section. This allows us to show a formal branching at $n = 0$ for such highly oscillatory and changing sign functions. Note that, each zero is a singular point for the equivalent integral operator. However, a rigorous proof of this fact for the zeroes is not planned and expected to be very difficult.

5.1. Construction of the first nonlinear eigenfunction for $n > 0$. We construct the first non-linear eigenfunction when $n > 0$ and provide some numerical evidence of the profiles for those non-linear eigenfunctions depending on n . This construction, together with the numerical evidence, justify the continuous homotopy approach as $n \rightarrow 0$. The solution denoted by $b_n(x, t)$, is of similar to (4.1) for $n = 0$, i.e.

$$(5.1) \quad b_n(x, t) = t^\alpha F_n(y), \quad y = \frac{x}{t^\beta}, \quad \text{where } \beta = \frac{2 + \alpha n}{4},$$

and comes from the dimension analysis of the linear eigenvalue problem. In order to obtain

$$(5.2) \quad b_n(x, t) = t^{\frac{2}{n+4}} F_n(y), \quad y = \frac{x}{t^{(n+2)/(n+4)}},$$

i.e. the necessary value of the parameter α , which plays a role of a *nonlinear eigenvalue*, we need to declare the “conservation law” that is valid for both linear (1.7) and nonlinear (1.1) wave equations. Namely, integrating the PDEs over \mathbb{R} yields

$$\frac{d^2}{dt^2} \int u(x, t) dx = 0 \quad \implies \quad \int u(x, t) dx = C_1 t + C_0,$$

so the finite “mass” of solutions changes linearly with time. For (5.1) this yields

$$(5.3) \quad \int b_n(x, t) dx \sim t^{\alpha+\beta} = t \quad \implies \quad \alpha + \beta = 1.$$

Hence, combining (5.3) with the relation between the parameters α and β obtained after the construction of the self-similar solution of non-linear eigenfunctions, i.e. $\beta = \frac{2+\alpha n}{4}$, it follows that the parameters (the first nonlinear eigenvalue $\alpha_0 = \alpha_0(n)$)

$$(5.4) \quad \alpha_0(n) = \frac{2}{n+4} \quad \text{and hence} \quad \beta_0(n) = \frac{n+2}{n+4},$$

providing (5.2). For $n = 0$, these are exponents $\alpha_0(0) = \beta_0(0) = \frac{1}{2}$ of the fundamental solution (4.1).

On the other hand, exactly as in the linear case (4.1), our computations guarantee that the fundamental solution $b_n(x, t)$, for $n > 0$, can be defined as the one having the following measures as initial data:

$$b_n(x, 0) = 0 \quad \text{and} \quad b_{nt}(x, 0) = \delta(x) \quad \text{in the sense of bounded measures in } \mathbb{R}.$$

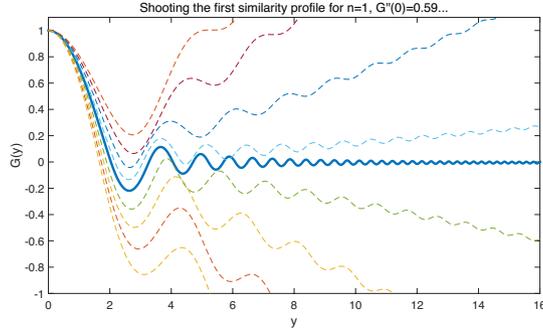


FIGURE 2. Shooting the first similarity profile satisfying (5.5), (5.7) for $n = 1$.

In the nonlinear case, we should check that an appropriate “fundamental” profile $F_n(y)$ actually exists. We substitute (5.1), with parameters (5.4), into (2.2) to get the ODE

$$(5.5) \quad F_n : \quad (|F|^n F)^{(4)} + \beta_0^2 F'' y^2 + \beta_0 \frac{2(n+1)}{n+4} F' y - \frac{2n+4}{(n+4)^2} F = 0 \quad \text{in } \mathbb{R}, \quad \int F = 1.$$

Again, one can see that, for $n = 0$, this yields the ODE (4.2). Similarly, (5.5) admits one integration leading to the simpler third-order ODE (cf. (4.3)),

$$(5.6) \quad F_n : \quad (|F|^n F)''' + \beta_0^2 F' y^2 - \beta_0 \frac{2}{n+4} F y = 0 \quad \text{in } \mathbb{R}.$$

Thus, the shooting procedure starts at the origin $y = 0$ with the symmetry condition

$$(5.7) \quad F'(0) = 0.$$

We then fix $F(0) = 1$, so that the only shooting parameter is $F''(0) \in \mathbb{R}$. Hence, as above, we arrive at a typical 1D shooting problem that is solved in a standard manner. In Figure 2, we show the actual numerical shooting for $n = 1$, where we take,

$$G(y) = |F_n(y)|^n F_n(y), \quad \text{and the normalization is } G(0) = 1.$$

Thus $F_n(y) = |G(y)|^{-\frac{n}{n+1}} G(y)$ and (5.6) becomes

$$G : \quad G''' + |G|^{-\frac{n}{n+1}} \left(\beta_0^2 G' y^2 - \beta_0 \frac{2}{n+4} G y \right) = 0 \quad \text{in } \mathbb{R}.$$

We use the regularisation $|G(y)| = (G(y)^2 + \delta^2)^{\frac{1}{2}}$, with $\delta = 10^{-6}$ and implement in MATLAB using the solver ODE15s with tolerances AbsTol=RelTol= 10^{-10} .

In the Figure 3, we demonstrate the first similarity profiles for various n including the linear fundamental profile when $n = 0$, as well as negative $n = -\frac{1}{2}$.

It is clearly seen that this negative n profile exhibits the largest oscillations amongst those presented. As a key feature, these Figures show that the dependence of the similarity profiles $F_n(y)$ on the parameter of nonlinearity $n > -1$ is clearly *continuous*, most plausibly, in the uniform metric. In other words, for such good solutions, it is possible to pass to the limit $n \rightarrow 0$ in the QWE-4 (1.1) to get the LWE-4 (1.7), so the equations are *homotopic* in the sense that they can be continuously deformed; see below.

5.2. Branching of nonlinear eigenfunction at $n = 0$ (a formal approach). Following classic bifurcation-branching theory [18, Ch. 6], [21, 4], we discuss the possibility of branching for the solutions $F_n(y)$ of the ODE (5.5) at $n = 0$ from the rescaled kernel $F_0(y)$ of the fundamental solution (4.1), which satisfies the much simpler linear equation (4.2). To this end, we perform the formal linearization in (5.5) by setting

$$(5.8) \quad \beta_0(n) = \frac{1}{2} + \frac{1}{8} n + O(n^2), \quad |F|^n = 1 + n \ln |F| + O(n^2).$$

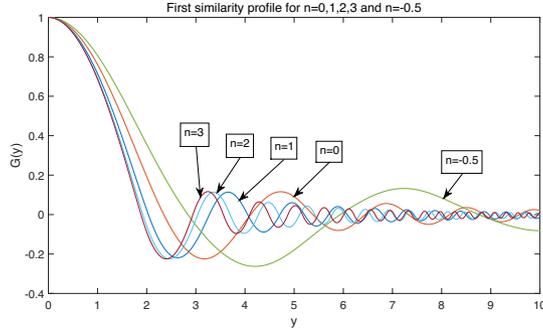


FIGURE 3. The first similarity profiles satisfying (5.5), (5.7) for $n = 3, 2, 1, 0$ and $n = -0.5$.

The last expansion needs extra justification. This expansion is not true uniformly on any bounded interval containing zeros of $F_n(y)$. As clearly seen from Figure 3 (and it is proven rigorously by asymptotic expansion) $F_n(y)$ always has a countable number of isolated zeros with the only concentration point at $y = \infty$. Bearing in mind that we actually deal with equivalent integral equations for F_n with continuous and, moreover, compact operators, we claim that such a violation of the expansion in (5.8) on a subset of an arbitrarily small measure around those transversal zeros does not spoil final conclusions on the analysis.

Hence, the second expansion in (5.8) cannot be interpreted pointwise for oscillatory changing sign solutions $F(y)$, though now these functions are assumed to have a *finite* number of zero surfaces.

However, since the possible zeros are isolated, they can be localised in arbitrarily small neighbourhoods. Indeed, it is clear that when $|F| > \delta > 0$, for any $\delta > 0$, there is no problem in approximating of $|F|^n$ as in (5.8), i.e., $|f|^n = O(n)$ as $n \rightarrow 0^+$. However, when $|F| \leq \delta$ for $\delta \geq 0$ sufficiently small, the proof of such an approximation is far from clear unless the zeros of the f 's are all transversal in a natural sense. In view of the expected finite oscillatory nature of solutions $F(y)$, this should allow one to obtain a weak convergence. Indeed, the second expansion in (5.8) remains true in a weak sense provided that the zeros belonging to zero set are sufficiently transversal in a natural sense, i.e., $\frac{|F|^n - 1}{n} \rightarrow \ln |F|$, as $n \rightarrow 0^+$, in L_{loc}^∞ since then, the singularity $\ln |F(y)|$ is not more than *logarithmic*. Equivalently we are dealing with the limit $n \ln^2 |F| \rightarrow 0$, as $n \downarrow 0^+$, at least in a very weak sense, since by the expansion (5.8) we have that $\frac{|F|^n - 1}{n} - \ln |F| = \frac{1}{2} n \ln^2 |F| + \dots$

Furthermore, in the present “blow-up” case, we do not need such subtle oscillatory properties of solutions close to interfaces, which are not known in complicated geometries. The point is that, due to the minimal growth condition (2.3) at infinity, we are looking for solutions $F(y)$ exhibiting finite oscillatory and sign changing properties, which are similar to those for linear combinations of eigenfunctions (4.14). Hence, we can suppose that their zeros (zero surfaces) are transversal a.e., so we find that, for $n > 0$ and any $\delta = \delta(n) > 0$ sufficiently small, $n |\ln |F|| \gg 1$, if $|F| \leq \delta(n)$ and, hence, on such subsets, $F(y)$ must be exponentially small:

$$|\ln |F|| \gg \frac{1}{n} \implies \ln |F| \ll -\frac{1}{n} \implies |F| \ll e^{-\frac{1}{n}}.$$

Thus, we can control the singular coefficients in (5.8), and, in particular, see that $\ln |F| \in L_{loc}^1(\mathbb{R}^N)$. Recall that this happens also in exponentially small neighbourhoods of the transversal zeros.

Thus, substituting (5.8) into (5.5) yields, on any subset uniformly bounded away from zeros of $F_n(y)$,

$$(5.9) \quad -F^{(4)} + \frac{1}{4} y^2 F'' + \frac{n+1}{n+4} y F' - \frac{2n+4}{(n+4)^2} F + n \left[(\ln |F| F)^{(4)} + \frac{1}{8} y^2 F'' + \frac{n+1}{4(n+4)} y F' \right] + O(n^2) = 0,$$

such that, when passing to the limit as $n \rightarrow 0^+$ yields $\mathbf{B}F = 0$ where \mathbf{B} is the operator (4.2) of the fundamental solution. Hence, we next set for small $n > 0$

$$(5.10) \quad F_n(y) = F_0(y) + nG(y) + o(n),$$

with the unknown function $G(y)$, which, substituting it into (5.9) yields the following equation for G

$$(5.11) \quad \mathbf{B}G = h \equiv [(\ln|F_0|F_0)^{(4)} + \frac{1}{8}y^2F_0'' + \frac{1}{16}yF_0'].$$

Just using the fact that $\mathbf{B}F_0(y) = 0$ and passing to the limit as $n \rightarrow 0^+$.

In the previous section, the non self-adjoint operator \mathbf{B} was discussed together with the related linear pencil operator, as well as the properties of the operator \mathbf{B} . Not going into details in this formal analysis, we refer to [6] for key results for similar (but not entirely the same) operators. In particular, we have shown that \mathbf{B} is a bounded linear operator defined in the weighted space L_ρ^2 with the domain being the Sobolev space H_ρ^4 , and has good spectral properties. At the present moment, we use a crucial fact that follows from existence and uniqueness of the fundamental solution: there exists its one-dimensional kernel

$$\ker \mathbf{B} = \text{Span} \{\Phi_0 = F_0\}.$$

Then, the perturbation in (5.10) must satisfy

$$(5.12) \quad G \perp F_0,$$

where we mean bi-orthogonality relative to the adjoint basis of \mathbf{B}^* . Recall that the adjoint basis $\{\Phi_k^*\}$ consists of finite polynomials and, in particular $\Phi_0^*(y) \equiv 1$ ($\Phi_0(y) = F_0(y)$).

Moreover, denoting $\ker \mathbf{B}^* = \text{span} \{\Phi_0^* = 1\}$ and by Y_0^* the complementary invariant subspace, orthogonal to Φ_0 , so that

$$V_0^* \in Y_0^* \quad \text{and} \quad V_0^*(y) := nG(y) + o(n),$$

we find that $F_n(y) = F_0(y) + V_0^*$. Then, it also follows that, by Fredholm's theorem, equation (5.11) has a unique solution satisfying (5.12) under the orthogonality condition

$$(5.13) \quad \langle h, \Phi_0^* \rangle \equiv \int h(y) \cdot 1 \, dy = 0,$$

which is obviously true and is checked via integration by parts and using the orthogonality of the eigenfunction Φ_0^* with respect to the operator \mathbf{B} .

Actually, the reduction to the third-order ODE (4.3) was performed by using this conservation law. The orthogonality (5.13) concludes that the branching at $n = 0$ given by (5.10) is very plausible.

5.3. Multiple eigenvalues for $k \geq 1$. The extension of the analysis for the first eigenfunction to the rest of the spectrum represents an open problem, which is particularly difficult for $k \geq 4$, where F_k satisfy truly fourth-order ODEs that cannot be reduce to the third-order one, as (5.5) is reduced to (5.6) (this is possible for a few first eigenvalues $\alpha_{0,1,2,3}$ only). On the other hand, a countable number of such n -branches of nonlinear eigenvalues can be detected by branching at $n = 0$ from eigenfunctions of the quadratic pencil operator (4.17).

The above branching analysis then corresponds to the first branch that is originated from the fundamental kernel $F_0(y)$ at $n = 0$ with $\lambda_0 = 0$.

Subsequently, we provide a formal analysis which will show that different branches emanating from $n = 0$ depend on the dimension of the eigenspace corresponding for the k -th eigenvalue of the *quadratic pencil operator* \mathbf{C} denoted by (4.17).

We expect that the appropriate similarity profiles in (3.1) denoted by $\{\Phi_k, k \geq 0\}$ exist for a *countable* sequence of nonlinear eigenvalues $\{(\alpha_k, \beta_k)\}$. Thus, we set

$$(5.14) \quad \Psi = \sum_{|\beta|=k} c_\beta \hat{\Phi}_\beta + V_k^* = \Phi_k + V_k^*,$$

for every $k \geq 1$. Moreover, we denote by $\{\hat{\Phi}_\beta\}_{|\beta|=k} = \{\hat{\Phi}_1, \dots, \hat{\Phi}_{M_k}\}$ the natural basis of the M_k^* -dimensional eigenspace

$$\ker(\mathbf{C})_{(\alpha_k(0), \beta_k(0))} \quad \text{and set} \quad \Phi_k = \sum_{|\beta|=k} c_\beta \hat{\Phi}_\beta,$$

where $\mathbf{C} \equiv \ker(\mathbf{C})_{(\alpha_k(0), \beta_k(0))}$ is the pencil operator denoted by (4.17). Thanks to the spectral analysis for the case $n = 0$ we find a family-pair for the parameters (α_k, β_k)

$$\alpha_k(0) := \frac{1}{2} + \lambda_k = \frac{1}{2} - \frac{k}{2},$$

where $\lambda_k = -k/2$, for $k = 0, 1, 2, 3, \dots$, given by Lemma 4.3. To obtain such a family for the parameter α we have used the *quadratic pencil operator* \mathbf{C} (4.17). Then, in terms of the parameter α and due to equation (3.1) we find the following relation

$$\frac{1}{4} - \lambda_k = \frac{3}{4} - \alpha \Rightarrow \alpha_k := \frac{1}{2} + \lambda_k.$$

Then, we introduce the next expression for the parameter α when $n = 0$ is away from zero, but sufficiently close,

$$(5.15) \quad \alpha_k(n) = \frac{2}{n+4} + \lambda_k, \quad \text{with} \quad k = 0, 1, 2, \dots$$

Moreover, since $\beta = \frac{2+\alpha n}{4}$ we have that

$$(5.16) \quad \beta_k(n) = \frac{1}{2} + \frac{n}{4} \alpha_k(n) = \frac{1}{2} + \frac{n}{2(n+4)} + n\lambda_k, \quad \text{with} \quad k = 0, 1, 2, \dots$$

Remark 5.1. *Note that even though $\alpha_k(0) < 0$ for any $k \geq 2$ we still have gradient blow-up behaviour at the single point $x = 0$ as shown in Proposition 2.1. Also, for $k = 1$ we find that $\alpha_1(0) = 0$ and as mentioned above in the introduction we expect to have gradient blow-up at $x = 0$ as well. For $k = 0$ we are reduced to the previously analysed case in Subsection 5.2. In this case we observe that $\mathbf{B} \equiv \mathbf{C}$.*

Subsequently, we define the expansions for the expressions (5.15) and (5.16) when n is very close to zero as

$$(5.17) \quad \alpha_k(n) = \alpha_k(0) + \mu_{1,k}n + o(n),$$

where $\mu_{j,k}$ are real coefficients for the k -th parameter and corresponding to the j -th term. Moreover, due to (5.16) and (5.17) it yields the next expansion for the k -th β -parameter

$$\begin{aligned} \beta_k(n) &= \frac{1}{2} + \frac{n}{4} \alpha_k(n) = \frac{1}{2} + \frac{n}{2(n+4)} + n\lambda_k + \frac{n^2}{4} \mu_{1,k} + o(n^2) \\ &= \beta_k(0) + n\lambda_k + \frac{n^2}{4} \mu_{1,k} + o(n^2), \end{aligned}$$

with $\beta_k(0) = \frac{n+2}{n+4}$. Consequently, it follows that

$$\beta_k(n) = \frac{1}{2} + \left(\frac{1}{8} + \lambda_k \right) n + o(n).$$

In addition, we set $V_k^* \in Y_k^*$ and $V_k^* = \sum_{|\beta|>k} c_\beta \hat{\Phi}_\beta^*$, where Y_k^* is the complementary invariant subspace of $\ker(\mathbf{C})_{(\alpha_k(0), \beta_k(0))}$. We also denote V_k^* by

$$(5.18) \quad V_k^*(y) := nG_k(y) + o(n).$$

Subsequently, substituting (5.14), with (5.18), into (2.2) yields

$$\begin{aligned}
& -\Phi_k^{(4)} - \frac{1}{4}y^2\Phi_k'' - \frac{n+2}{2(n+4)}y\Phi_k' + \frac{2n+4}{(n+4)^2}\Phi_k + \lambda_k y\Phi_k' - \lambda_k^2\Phi_k - \frac{4}{n+4}\lambda_k\Phi_k + \lambda_k\Phi_k \\
& + n \left[-G_k^{(4)} - \frac{1}{4}y^2G_k'' - \frac{n+2}{2(n+4)}yG_k' + \lambda_k yG_k' + \frac{2n+4}{(n+4)^2}G_k - \lambda_k^2G_k - \frac{4}{n+4}\lambda_kG_k + \lambda_kG_k \right] \\
& + n[-(\ln|\Phi_k|\Phi_k)^{(4)} - \left(\frac{1}{8} + \lambda_k\right)y^2\Phi_k'' - \left(\frac{1}{8} - \frac{n+2}{n+4}\right)\lambda_k y\Phi_k' + \mu_{1,k}y\Phi_k' \\
& - \frac{n+2}{8(n+4)}y\Phi_k' + \mu_{1,k}\Phi_k \left(1 - 2\lambda_k - \frac{4}{n+4}\right)] + O(n^2).
\end{aligned}$$

Then, passing to the limit as $n \downarrow 0^+$ we arrive at the *quadratic pencil operator* (4.17) corresponding to the k -th eigenfunction Φ_k , i.e. $\mathbf{C}\Phi_k = 0$. Now, evaluating the first order terms we obtain the following equation:

$$(5.19) \quad (\mathbf{C}_{(\alpha_k, \beta_k)})G_k = \mathcal{N}_k\Phi_k + O(n),$$

where

$$\begin{aligned}
\mathcal{N}_k\Phi_k &= (\ln|\Phi_k|\Phi_k)^{(4)} + \left(\frac{1}{8} + \lambda_k\right)y^2\Phi_k'' + \left(\frac{1}{8} - \frac{n+2}{n+4}\right)\lambda_k y\Phi_k' \\
&\quad - \mu_{1,k}y\Phi_k' + \frac{n+2}{8(n+4)}y\Phi_k' - \mu_{1,k}\Phi_k \left(1 - 2\lambda_k - \frac{4}{n+4}\right).
\end{aligned}$$

Indeed, passing to the limit as $n \rightarrow 0^+$ we have that

$$\mathbf{C}G_k = (\ln|\Phi_k|\Phi_k)^{(4)} + \left(\frac{1}{8} + \lambda_k\right)y^2\Phi_k'' - \frac{3}{8}\lambda_k y\Phi_k' - \mu_{1,k}y\Phi_k' + \frac{1}{16}y\Phi_k' - 2\lambda_k\mu_{1,k}\Phi_k.$$

Therefore, applying the Fredholm alternative [4], a unique $V_k^* \in Y_k^*$ exists if and only if the right-hand side of (5.19) is orthogonal to $\ker(\mathbf{C}_{(\alpha_k(0), \beta_k(0))})$. Multiplying the right-hand side of (5.19) by Φ_k^* , for every k , in the topology of the dual space L^2 , we obtain an algebraic system of $M_k^* + 1$ equations and unknowns, $\{c_\beta, |\beta| = k\}$ and $\mu_{1,k}$:

$$(5.20) \quad \langle \mathcal{N}_k \left(\sum_{|\beta|=k} c_\beta \hat{\Phi}_\beta \right), \hat{\Phi}_\beta^* \rangle = 0 \quad \text{for all } |\beta| = k,$$

and under the natural ‘‘normalising’’ constraint $\sum_{|\beta|=k} c_\beta = 1$.

Thus, the expression (5.20) is indeed the Lyapunov–Schmidt branching equation [21]. Through that algebraic system we shall ascertain the coefficients of the expansions (5.17) and, hence, eventually the directions of branching, as well as the number of branches. However, a full solution of the *non-variational* algebraic system (5.20) is a very difficult issue, though we claim that the number of branches is expected to be related to the dimension of the eigenspace $\ker(\mathbf{C}_{(\alpha_k(0), \beta_k(0))})$, as we did for a *fourth-order thin film equation* in [1].

COMPUTATIONS FOR BRANCHING OF SOLUTIONS. We ascertain some expressions for those coefficients in the particular case when $|\beta| = 1$, $N = 2$ and $M_1^* = 2$ (as an example) so that, in our notations,

$$\{\Phi_\beta\}_{|\beta|=1} = \{\hat{\Phi}_1, \hat{\Phi}_2\}, \quad \text{such that } \Phi_1 = c_1\hat{\Phi}_1 + c_2\hat{\Phi}_2.$$

and $\lambda_1 = -\frac{1}{2}$. Consequently, in this case, we obtain the following algebraic system:

$$(5.21) \quad \begin{aligned} & c_1\langle \hat{\Phi}_1^*, h_1 \rangle - c_1\frac{3}{8}\langle \hat{\Phi}_1^*, y^2\hat{\Phi}_1'' \rangle + c_1\mu_{1,1} + c_2\langle \hat{\Phi}_1^*, h_2 \rangle - c_2\frac{3}{8}\langle \hat{\Phi}_1^*, y^2\hat{\Phi}_2'' \rangle \\ & + \frac{c_1}{4}\langle \hat{\Phi}_1^*, y\hat{\Phi}_1' \rangle - c_1\mu_{1,1}\langle \hat{\Phi}_1^*, y\hat{\Phi}_1' \rangle + \frac{c_2}{4}\langle \hat{\Phi}_1^*, y\hat{\Phi}_2' \rangle - c_2\mu_{1,1}\langle \hat{\Phi}_1^*, y\hat{\Phi}_2' \rangle = 0, \end{aligned}$$

$$\begin{aligned}
& c_1 \langle \hat{\Phi}_2^*, h_1 \rangle - c_1 \frac{3}{8} \langle \hat{\Phi}_2^*, y^2 \hat{\Phi}_1'' \rangle + c_2 \mu_{1,1} + c_2 \langle \hat{\Phi}_2^*, h_2 \rangle - c_2 \frac{3}{8} \langle \hat{\Phi}_2^*, y^2 \hat{\Phi}_2'' \rangle \\
& + \frac{c_1}{4} \langle \hat{\Phi}_2^*, y \hat{\Phi}_1' \rangle - c_1 \mu_{1,1} \langle \hat{\Phi}_2^*, y \hat{\Phi}_1' \rangle + \frac{c_2}{4} \langle \hat{\Phi}_2^*, y \hat{\Phi}_2' \rangle - c_2 \mu_{1,1} \langle \hat{\Phi}_2^*, y \hat{\Phi}_2' \rangle = 0, \\
& c_1 + c_2 = 1,
\end{aligned}$$

where

$$h_1 := -[\ln(c_1 \hat{\Phi}_1 + c_2 \hat{\Phi}_2) \hat{\Phi}_1]^{(4)}, \quad h_2 := -[\ln(c_1 \hat{\Phi}_1 + c_2 \hat{\Phi}_2) \hat{\Phi}_2]^{(4)},$$

and, c_1 , c_2 and $\mu_{1,1}$ are the coefficients that we would like to calculate. Now, from the third equation we have $c_2 = 1 - c_1$, so that substituting it into the first two equations of (5.21) gives

$$(5.22) \quad \begin{cases} 0 = N_1(c_1, \mu_{1,1}) - c_1 \frac{3}{8} [\langle \hat{\Phi}_1^*, y^2 \hat{\Phi}_1'' \rangle - \langle \hat{\Phi}_1^*, y^2 \hat{\Phi}_2'' \rangle] + \frac{3}{8} \langle \hat{\Phi}_1^*, y^2 \hat{\Phi}_2'' \rangle \\ \quad - \mu_{1,1} \langle \hat{\Phi}_1^*, y \hat{\Phi}_2' \rangle + \frac{c_1}{4} [\langle \hat{\Phi}_1^*, y \hat{\Phi}_1' \rangle - \langle \hat{\Phi}_1^*, y \hat{\Phi}_2' \rangle] + \frac{1}{4} \langle \hat{\Phi}_1^*, y \hat{\Phi}_2' \rangle \\ 0 = N_2(c_1, \mu_{1,1}) - c_1 \frac{3}{8} [\langle \hat{\Phi}_2^*, y^2 \hat{\Phi}_1'' \rangle - \langle \hat{\Phi}_2^*, y^2 \hat{\Phi}_2'' \rangle] + \frac{3}{8} \langle \hat{\Phi}_2^*, y^2 \hat{\Phi}_2'' \rangle \\ \quad - \mu_{1,1} \langle \hat{\Phi}_2^*, y \hat{\Phi}_2' \rangle + \frac{c_1}{4} [\langle \hat{\Phi}_1^*, y \hat{\Phi}_1' \rangle - \langle \hat{\Phi}_1^*, y \hat{\Phi}_2' \rangle] + \frac{1}{4} \langle \hat{\Phi}_1^*, y \hat{\Phi}_2' \rangle + \mu_{1,1}, \end{cases}$$

where

$$\begin{aligned}
N_1(c_1, \mu_{1,1}) &:= c_1 \langle \hat{\Phi}_1^*, h_1 \rangle + \langle \hat{\Phi}_1^*, h_2 \rangle - c_1 \mu_{1,1} [\langle \hat{\Phi}_1^*, y \hat{\Phi}_1' \rangle - \langle \hat{\Phi}_1^*, y \hat{\Phi}_2' \rangle] \\
&\quad - c_1 \langle \hat{\Phi}_1^*, h_2 \rangle + c_1 \mu_{1,1}, \\
N_2(c_1, \mu_{1,1}) &:= c_1 \langle \hat{\Phi}_2^*, h_1 \rangle + \langle \hat{\Phi}_2^*, h_2 \rangle - c_1 \mu_{1,1} [\langle \hat{\Phi}_2^*, y \hat{\Phi}_1' \rangle - \langle \hat{\Phi}_2^*, y \hat{\Phi}_2' \rangle] \\
&\quad - c_1 \langle \hat{\Phi}_2^*, h_2 \rangle - c_1 \mu_{1,1}
\end{aligned}$$

represent the nonlinear parts of the algebraic system, with h_0 and h_1 just depending on c_1 in this case.

We observe that the nonlinear algebraic system (5.22) is rather difficult to solve. Note that multiplicity results are extremely difficult to obtain. Indeed, to ascertain the number of solutions for those nonlinear finite-dimensional algebraic problems is rather complicated. However, we expect and conjecture that this is somehow related to the dimension of the corresponding eigenspace $\ker(\mathbf{C}_{(\alpha_k, \beta_k)})$, $k \geq 1$ (the eigenspace of the *quadratic pencil operator* \mathbf{C}).

Lemma 5.1. *The nonlinear system (5.21) possesses at least one non-degenerate solution for the unknowns c_1 , c_2 and $\mu_{1,1}$ if the determinant of the linear of the system (5.22) is different from zero and the following is satisfied:*

$$(5.23) \quad \text{either } \frac{3}{8} \langle \hat{\Phi}_1^*, y^2 \hat{\Phi}_2'' \rangle + \frac{1}{4} \langle \hat{\Phi}_1^*, y \hat{\Phi}_2' \rangle \neq 0, \quad \text{or } \frac{3}{8} \langle \hat{\Phi}_2^*, y^2 \hat{\Phi}_2'' \rangle + \frac{1}{4} \langle \hat{\Phi}_1^*, y \hat{\Phi}_2' \rangle \neq 0.$$

Proof. To detect solutions for the system (5.21) we apply the Brouwer fixed point theorem to (5.22); see further details on this analysis in [1] for a *thin film equation*. Then, we suppose that the values c_1 and $\mu_{1,1}$ are the unknowns in a sufficiently big disc $D_R(\hat{c}_1, \hat{\mu}_{1,1})$, centered in a possible non-degenerate zero $(\hat{c}_1, \hat{\mu}_{1,1})$. Therefore, following this argument we can say that if the matrix associated with linear part of the system (5.22) possesses a determinant different from zero and (5.23) is satisfied the nonlinear algebraic system (5.22) has at least one non-degenerate solution. \square

6. FINITE TIME BLOW-UP FORMATION OF GRADIENT BLOW-UP: NUMERICAL EVIDENCE

The eigenvalue problem (2.7) presents a formidable challenge. This is most clearly seen even in the $n = 0$ case, where the two free parameters α and $g'''(0)$ in (3.8) are determined by requiring (3.9) to hold, in order to remove the maximal bundle. The difficulty arises due to the expansion (3.6), where the oscillatory terms are buried within the expansion of the minimal behaviour g_{\min} . As α increases, more terms in the series (3.4) of g_{\min} dominate the maximal behaviour (3.5) for large y . Extracting the maximal behaviour from numerical solutions is thus problematic.

Shown in Figure 4 is the numerical solution of (3.1) with (3.7) in the case $\nu = g'''(0) = -1$, $\alpha = 0.5$, using Matlab solver ode15s with AbsTol=RelTol= 10^{-10} . The oscillatory behaviour is eliminated when

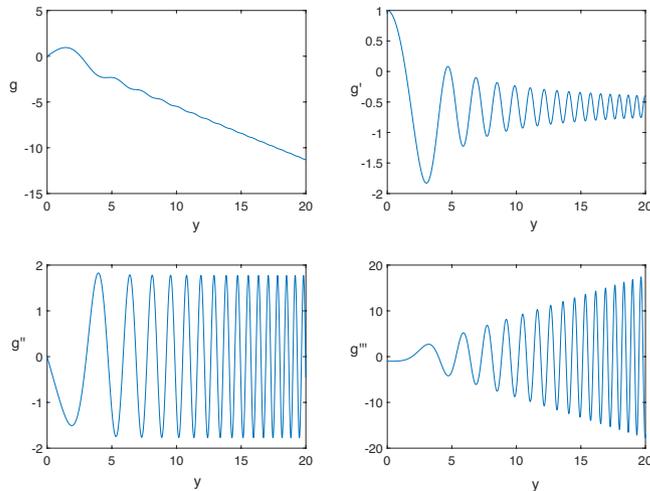


FIGURE 4. Numerical solution in the $n = 0$ case of (3.1) with (3.7) and $\nu = g'''(0) = -1, \alpha = 0.5$.

$\nu = g'''(0) = 0, \alpha = 0.5$, there being the exact solution $g(y) = y$. Determining eigenvalues for larger α is difficult.

7. EXTENSION OF BLOW-UP SOLUTIONS FOR $t > 0$: $t \mapsto -t$

This is obvious, since our hyperbolic equation is symmetric under the reflection

$$(7.1) \quad t \mapsto -t.$$

Therefore, we arrive at the same ODE for such profiles $g(y)$:

$$(7.2) \quad u_+(x, t) = t^\alpha g(y), \quad y = x/t^\beta, \quad \beta = \frac{2+\alpha n}{4}, \quad \text{where}$$

$$(7.3) \quad \beta^2 y^2 g'' + \mu y g' + \alpha(\alpha - 1)g = -(|g|^n g)^{(4)} \quad \text{in } \mathbb{R}, \quad \mu = \frac{\beta[6+(n-8)\alpha]}{4}.$$

Recall that, here, $\alpha \in \{\alpha_k\}$ has been obtained earlier by the above focusing blow-up analysis.

Thus, such a gradient blow-up does not lead to a discontinuous shock waves persisting for some $t > 0$, unlike the true shocks studied in [11].

8. FINAL CONCLUSIONS

The present research can be considered as a first step towards explaining specific mathematical difficulties that should be expected when dealing with higher-order quasilinear degenerate wave and Boussinesq models. Using a particular PDE (1.1), we discuss some ideas, concepts, and results around shock and rarefaction waves and nonlinear fundamental solutions. In particular, the obtained structure and formation of shocks for (1.1) may be useful for future development of a possible “entropy” theory for (1.1), which is expected to be very difficult and represents an extremely hard problem. We must admit that a formal branching homotopy approach (1.8) is rather technical and it demands a difficult spectral theory of a quadratic pencil of linear operators, which is not fully developed here. However, it seems that still no traces of any more rigorous results for such nonlinear PDEs are available in the vast literature on nonlinear equations.

In Section 2, we show that, as a key intrinsic feature of such QWEs, the nonlinear dispersion mechanism in equations such as (1.1) creates a gradient blow-up via similarity (“focusing”-like) solutions. Structural stability of such singularity phenomena demands further study based on extremely difficult ideas involving

centre and stable manifold techniques that are not properly developed still for such PDEs; cf. [15, § 6] for the NDE (1.11).

Also, we construct other fundamental solutions of (1.1) that appear at $t = 0$ from typical $\delta(x)$ -measures as initial data.

As a by-product of our analysis, we believe that the idea of *homotopic* continuous deformations (1.8) as $n \rightarrow 0^+$ in the QWE-4 (1.1) leading to the linear PDE (1.7) with simple properties, can be useful for a correct description and understanding crucial properties of its solutions. This remains reasonable even at the formal level, since proving rigorously this branching phenomenon for such nonlinear degenerate operators, with unknown functional setting and compact properties, is essentially out of reach for modern bifurcation-branching theory. Such ideas have been recently applied to the sixth-order thin film equation (TFE-6) [7] (as well as [2, 1, 3] for other similar TFEs)

$$(8.1) \quad u_t = \nabla \cdot (|u|^n \nabla \Delta^2 u),$$

which on passage to the limit $n \rightarrow 0$ gives the *tri-harmonic* (parabolic) equation $u_t = \Delta^3 u$. In particular, this homotopy approach helps to understand the nature of oscillatory sign-changing solutions of the TFE-6 (8.1) corresponding to the Cauchy problem.

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