

Classical picture of postexponential decay

E. Torrontegui,¹ J. G. Muga,¹ J. Martorell,² and D. W. L. Sprung³

¹*Departamento de Química Física, Universidad del País Vasco–Euskal Herriko Unibertsitatea, Apdo. 644, E-48080 Bilbao, Spain*

²*Departament d'Estructura i Constituents de la Matèria, Universitat de Barcelona, E-08028 Barcelona, Spain*

³*Department of Physics and Astronomy, McMaster University, Hamilton, Ontario L8S 4M1, Canada*

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Postexponential decay of the probability density of a quantum particle leaving a trap can be reproduced accurately, except for interference oscillations at the transition to the postexponential regime, by means of an ensemble of classical particles emitted with constant probability per unit time and the same half-life as the quantum system. The energy distribution of the ensemble is chosen to be identical to the quantum distribution, and the classical point source is located at the scattering length of the corresponding quantum system. A one-dimensional example is provided to illustrate the general argument.

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I. INTRODUCTION

Exponential decay is ubiquitous in quantum physics and constitutes the typical dynamical pattern for unstable systems. Theory predicts deviations at short times, related to the Zeno effect, and also at long times, with significant implications for cosmology [1], hidden variables [2], non-Hermitian formulations [3,4], and radioactive-dating methods [5]. On the experimental side, the postexponential regime, generally algebraic, has been elusive. There are few claims of having observed it [6], which has engendered much effort to explain why or to improve its observability [7,8].

Classically, exponential decay arises from a constant decay probability per unit time. What has been lacking so far has been a correspondingly simple, physically appealing classical picture of postexponential decay. The purpose of this article is to provide such a framework by following up on an old suggestion of Newton's [9,10].

The standard quantum-mechanical derivations of postexponential decay are not very helpful in suggesting such a picture. The early reliance upon the Paley-Wiener theorem [11] is not very illuminating from a physical perspective and it provides bounds, not precise predictions. Similarly postexponential decay can be attributed mathematically to the fact that the pole contribution to exponential decay is eventually comparable to or smaller than a line integral, whose value arises predominantly from a saddle point at threshold, associated with slow particles [7,12,13]. This is surely more intuitive, yet not fully satisfying for those seeking a pictorial, rather than a complex variable, understanding of the phenomenon [14]. In this vein, Hellund proposed an electrostatic analog [15] which relates quantum emission of radiation to the damped oscillation of a charge describable in purely classical (stochastic) terms and interprets the deviation from exponential decay as a “straggling phenomenon” characteristic of a diffusion process. However, quantum dynamics cannot generally be reduced to a classical diffusion process. Jacob and Sachs [16], in their field-theoretical analysis of a scalar particle coupled to two pions, found nonexponential terms decaying like $t^{-3/2}$ in the amplitude. Their explanation was geometrical: a particle produced at position \vec{r} , having velocity between v and dv , will appear after time t within a spherical shell of radius vt centered on \vec{r} and thickness $t dv$. The probability that it will be found in a

small volume element within the shell is inversely proportional to its volume, $4\pi t^3 v^2 dv$. Hence the probability amplitude is proportional to $t^{-3/2}$. This cannot be a universal explanation though, since the decay amplitude in one-dimensional (1D) models behaves generically like $t^{-3/2}$, whereas the previous argument translated to 1D would imply only $t^{-1/2}$.

Postexponential power-law behavior is sometimes interpreted as expressing the dominance of free motion [17,18]. However explicit calculations of the long-time propagator for specific potential scattering models show that in general both free and scattering terms are needed in the propagator to reproduce the correct results [19].

A general argument that justifies nonexponential decay is the so-called “initial state reconstruction” [20,21]. Deviations from exponential decay would be the consequence of Feynman paths returning to the initial state from the orthogonal, decay-product subspace. This argument alone, however, makes no quantitative predictions of the observed behavior and is applied to the survival probability, which is not always easy to observe.

In this article we take up and develop a frequently overlooked observation by Newton making use of classical mechanics [9,10]. Newton noted that, if a point source emits classical particles with an exponential decay law and with a suitable velocity distribution, the current density away from the source will eventually depend on time according to an inverse power law. Indeed, we shall show that by adjusting the parameters according to the quantum system, the classical model accurately reproduces the onset, power law, and intensity of postexponential decay of the quantum probability density of a particle escaping from a trap. For simplicity we assume that the particle is restricted to the half-line, $r \geq 0$, as in quantum s -wave scattering. We shall also assume that the initial quantum state is orthogonal to any bound states so that it must eventually decay (escape) fully from the trap.

II. CLASSICAL AND QUANTUM SOURCES AND DECAY

Consider first a source at $r = 0$ which emits classical particles with a definite velocity v from $t_0 = 0$ so that the fraction of particles emitted between t_0 and $t_0 + dt_0$ is

$$P(t_0)dt_0 = \frac{dt_0}{\tau} e^{-t_0/\tau}, \quad (1)$$

where τ is the emission lifetime. The spatial probability density observed at point r , at time t , is

$$P_{c,v}(r,t) = \frac{1}{\tau v} e^{-(t-r/v)/\tau} \theta\left(t - \frac{r}{v}\right), \quad (2)$$

where θ is the step function. For the more general case in which the emitted particles have a velocity distribution $\rho(v)$,

$$P_c(r,t) = \int_{r/t}^{\infty} dv \rho(v) \frac{1}{v\tau} e^{-(t-r/v)/\tau}, \quad (3)$$

or, using $r = v(t - t_0)$,

$$P_c(r,t) = \int_0^t dt_0 \frac{1}{(t-t_0)\tau} \rho\left(\frac{r}{t-t_0}\right) e^{-t_0/\tau}. \quad (4)$$

The behavior for $t \gg \tau$ of this integral can be expressed as an asymptotic series,

$$P_c(r,t) \sim \sum_{n=0}^m \tau^n [g^{(n)}(0) - g^{(n)}(t) e^{-t/\tau}], \quad (5)$$

where

$$g(t_0) \equiv \frac{1}{t-t_0} \rho\left(\frac{r}{t-t_0}\right), \quad (6)$$

and $g^{(n)}$ is its n th derivative with respect to t_0 . The leading term asymptotically is

$$P_c(r,t) \sim g(0) = \frac{1}{t} \rho\left(\frac{r}{t}\right). \quad (7)$$

This is equivalent to Newton's result [9,10] (we use the probability density rather than the current density). To advance from here, consider now the decay from a quantum trap of a system prepared in a normalized nonstationary state $|\Psi_0\rangle$. The wave function of this state at a point r and time t is

$$\Psi(r,t) = \langle r | e^{-iHt/\hbar} | \Psi_0 \rangle, \quad (8)$$

with corresponding probability density $P_q(r,t) = |\Psi(r,t)|^2$. Using stationary states normalized in energy $u_E(r)$ [such that $\langle u_E | u_E \rangle = \delta(E - E')$] and inserting the completeness relation,

$$\Psi(r,t) = \int_0^{\infty} dE \langle r | u_E \rangle \langle u_E | \Psi_0 \rangle e^{-iEt/\hbar}. \quad (9)$$

The u_E are solutions of the s -wave, radial Schrödinger equation,

$$\left[\frac{d^2}{dr^2} - v(r) + k^2 \right] u_E(r) = 0, \quad (10)$$

where $v(r) = (2m/\hbar^2)V(r)$ and $k^2 = (2m/\hbar^2)E$. As in [22], it is convenient to define new solutions $w_k(r) = \hbar \sqrt{\frac{k}{m}} u_E(r)$ normalized as $\langle w_k | w_k \rangle = \delta(k - k')$, which obey the boundary condition

$$\lim_{r \rightarrow \infty} w_k(r) = \sqrt{\frac{2}{\pi}} \sin[kr + \delta(k)], \quad (11)$$

where $\delta(k)$ is the phase shift of the s partial wave. These solutions are related to the regular solutions $\hat{\phi}_k(r)$ [which behave like the Riccati-Bessel function $\hat{j}_0(kr)$ as $r \rightarrow 0$],

$$w_k(r) = \sqrt{\frac{2}{\pi}} \frac{\hat{\phi}_k(r)}{|f(k)|}, \quad (12)$$

where $f(k) = |f(k)| \exp(-i\delta)$ is the Jost function, as defined, for example, in Taylor [23]. It gives the relative normalization between solutions having unit incoming flux at infinity and solutions that have slope k at the origin. The partial-wave S -matrix element is $S(k) = f(-k)/f(k)$. Zeroes of $f(k)$ in the upper half complex momentum plane correspond to bound states, while those in the lower half plane are associated with scattering resonances.

For an initially localized nonstationary state

$$\langle r | \Psi_0 \rangle = 0 \quad \text{for } r > r_a, \quad (13)$$

the wave function can be written as

$$\Psi(r,t) = \frac{2m}{\pi\hbar^2} \int_0^{\infty} dE \frac{1}{k} \hat{\phi}_k(r) \frac{\langle \hat{\phi}_k | \Psi_0 \rangle}{|f(k)|^2} e^{-iEt/\hbar}, \quad (14)$$

which has a form similar to the survival amplitude obtained in [22]. They have generically the same asymptotic behavior at long times, which corresponds to an energy distribution $\rho(E) = |\langle u_E | \Psi_0 \rangle|^2 \sim E^{1/2}$, as $E \rightarrow 0$. This long-time asymptotic behavior is governed by the properties when $k \rightarrow 0$ of the integrand of Eq. (14). For "well-behaved" potentials, (those falling off faster than r^{-3} when $r \rightarrow \infty$ and less singular than $r^{-3/2}$ at the origin), the $\ell = 0$ Jost function tends to a constant when $k \rightarrow 0$. [In the exceptional case that a zero energy resonance occurs, $f(k=0) = 0$.] The $\hat{\phi}_k$ behave near the origin as Riccati-Bessel functions, $\hat{j}_0(kr)$, and are therefore linear in k . The behavior of the integrand near threshold is thus $\sim E^{1/2}$. Following the same steps as in the derivation of the asymptotic behavior of the survival amplitude in [22], one finds that the position probability density behaves like $P_q(r,t) \sim t^{-3}$ at long times.

The energy distribution corresponds asymptotically to a velocity distribution since all particles are eventually released. The two distributions are related by

$$\varrho(E)dE = \rho(v)dv. \quad (15)$$

Setting $E = \frac{1}{2}mv^2$, m being the mass of the emitted particles, makes $\varrho(E) \sim E^{1/2} \Rightarrow \rho(v) \sim v^2$. Going back to Eq. (7) and considering the long-time regime $t \gg \tau$, the classical particle velocity can be approximated by $v = r/t$, so $\rho(r/t) = \rho(v)$, which implies, as expected, that at large t the main contribution to the position probability density is from slow particles. If we consider the same dependence as in the quantum case, $\rho(v) \sim v^2$, Eq. (7) implies an asymptotic behavior $P_c(r,t) \sim t^{-3}$, that is, the classical model leads to the same power-law dependence as the quantum one. Moreover, in the following we shall see that it can be adjusted to provide the correct amplitude factor as well.

Let us return to Eq. (14) and write the bra-ket factor as

$$\langle \hat{\phi}_k | \Psi_0 \rangle = \sqrt{\frac{\pi}{2}} |f(k)| \langle w_k | \Psi_0 \rangle, \quad (16)$$

Using Eq. (12) and the asymptotic behavior given by Eq. (11), the wave function for $r \rightarrow \infty$ may be written as

$$\Psi(r,t) \sim \sqrt{\frac{2}{\pi}} \frac{m}{\hbar^2} \int_0^{\infty} dE \frac{1}{k} \langle w_k | \Psi_0 \rangle \sin[kr + \delta(k)] e^{-iEt/\hbar}. \quad (17)$$

At low energy, the phase shift $\delta(k)$ is well described by the effective range expansion

$$k \cot \delta(k) = -\frac{1}{a_0} + \frac{1}{2}r_0k^2 + \dots, \quad (18)$$

where a_0 is the scattering length while r_0 is called the effective range of the potential function. The asymptotic form of the resulting integral can be obtained from the Riemann-Lebesgue lemma [24]. Only the main term which depends on the $k \rightarrow 0$ behavior of the integrand is kept. This gives a probability density of the form

$$P_q(r,t) \sim \beta [r - a_0]^2 \frac{1}{t^3}, \quad (19)$$

where β is the strength factor for the asymptotic dependence of the velocity distribution, $\rho(v) \sim \beta v^2$, which will depend on the particular state and potential. It has units of $[\beta] \sim v^{-3}$. In the approximation $v = r/t$ we can write

$$\rho(v) \sim \beta \frac{r^2}{t^2}. \quad (20)$$

Introducing this in Eq. (7) we obtain for the classical probability density at long times

$$P_c(r,t) \sim \beta \frac{r^2}{t^3}. \quad (21)$$

Compare now Eqs. (19) and (21), and to avoid confusion, let us rewrite $r \rightarrow r_q$ for the quantum case and $r \rightarrow r_c$ for the classical one. We see that if the classical coordinate is shifted by a_0 , $r_c = r_q - a_0$, the classical model will reproduce the quantum probability density. Equivalently, $P_c(r) = P_q(r)$ at long times if the classical source is not at the origin but is displaced by the scattering length a_0 .

In the exceptional case of a potential with a zero energy resonance $a_0 \rightarrow \infty$, and therefore the first term on the r.h.s. of Eq. (18) is absent. This causes the Jost function to have a simple zero at $k = 0$ (see [23]), and therefore $|\langle w_{k=0} | \Psi_0 \rangle|^2$ is nonvanishing. We then find, instead of Eq. (19), that

$$P_q(r,t) \sim |\langle w_{k=0} | \Psi_0 \rangle|^2 m / (\hbar t), \quad (22)$$

which is again in agreement with the classical expression (7) taking $\rho(v = 0) = |\langle w_{k=0} | \Psi_0 \rangle|^2 m / \hbar$.

III. MODEL CALCULATION

We now check the general argument developed in Secs. I and II, in the specific case of Winter's δ -barrier model [13] which is described in Fig. 1. The initial state is an eigenstate of the infinite square well potential,

$$\langle r | \Psi_0 \rangle = \begin{cases} \sqrt{\frac{2}{L}} \sin\left(\frac{n\pi r}{L}\right) & r \leq L \\ 0 & r \geq L \end{cases}, \quad (23)$$

and the k -normalized basis functions are

$$\langle r | w_k \rangle = e^{-i\delta(k)} \sqrt{\frac{2}{\pi}} \times \begin{cases} \sin(kr)/f(k) & r \leq L \\ (i/2)[e^{-ikr} - S(k)e^{ikr}] & r \geq L \end{cases}, \quad (24)$$

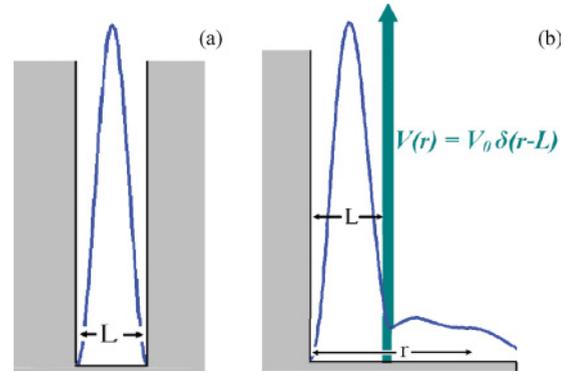


FIG. 1. (Color online) Scheme of the Winter model. (a) The initial state is the ground state of an infinite square well. (b) One of the walls is replaced by a δ barrier $V = V_0 \delta(r - L)$.

where $S(k) = f(k)^*/f(k)$, and the Jost function for this model is

$$f(k) = 1 + \frac{\alpha}{2ik} (e^{2ikL} - 1), \quad (25)$$

with $\alpha = 2mV_0/\hbar^2$. $\rho(v) = |\langle w_k | \Psi_0 \rangle|^2 m / \hbar$ can be calculated exactly and takes the form

$$\rho(v) = \frac{Lm}{\pi\hbar} \frac{k^2}{k^2 + \alpha k \sin(2kL) + \alpha^2 \sin^2(kL)} \times \left[\frac{2n\pi \sin(kL)}{k^2 L^2 - n^2 \pi^2} \right]^2. \quad (26)$$

From here the exact classical probability density is calculated numerically using Eq. (4), whereas the quantum density is given by the square modulus of Eq. (9). In the large- t region the probability density has analytical expressions in both quantum and classical cases given by Eqs. (19) and (21), respectively. The coefficient β is easy to find from Eq. (26) in the limit $v \rightarrow 0$,

$$\beta = \frac{4m^3 L^3}{(1 + \alpha L)^2 n^2 \pi^3 \hbar^3}. \quad (27)$$

Also, Eq. (24) and $S(k) = e^{2i\delta(k)}$ give for the Winter model the explicit source shift

$$a_0 = \frac{\alpha L^2}{1 + \alpha L}, \quad (28)$$

which, for $\alpha \geq 0$, lies between 0 (for $\alpha L \rightarrow 0$, no barrier) and L (for large αL , strong confinement).

Finally, the quantum and classical probability densities both take (shifting the classical point source by a_0) the postexponential form

$$P_{q,c}(r,t) \sim \frac{4}{n^2(1 + \alpha L)^2} \left(\frac{Lm}{\pi\hbar} \right)^3 \left(r - \frac{\alpha L^2}{1 + \alpha L} \right)^2 \frac{1}{t^3}. \quad (29)$$

The agreement is illustrated in Fig. 2, where the exact decay curves (numerically integrated) are plotted. The classical density (triangles) indeed reproduces the quantum behavior (solid line) if the source shift is taken into account. For comparison we also show a curve in which the shift is not applied, so that the classical source remains at $r = 0$ (circles).

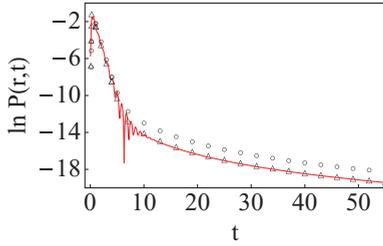


FIG. 2. (Color online) The exact probability density obtained numerically versus t : quantum result, solid (red) line; classical solution including the source shift, Eq. (28), triangles; classical solution without applying the source shift, circles. Parameter values: $\hbar = 1$, $m = 1/2$, $L = 1$, $\alpha = 5$, $n = 1$, $r = 2$, and $\tau = 0.5$.

Taking the same value for τ , we see that the classical model also agrees with the quantum one in the pre-exponential and exponential zones ($0 < t < 5$ in the drawing). The classical model differs only in the absence of oscillations which occur at the onset of postexponential behavior, due to quantum interference. The asymptotic behavior is indistinguishable on the scale of the figure from the analytical expression Eq. (29).

The exceptional case of a zero energy resonance corresponds to an attractive δ with $\alpha = -1/L$. In this case from Eqs. (7) and (22) we get

$$P_{q,c}(x,t) \sim \frac{4Lm}{\hbar n^2 \pi^3 t} \quad (30)$$

for both the classical and quantum cases (see Fig. 3). If α is close to the critical value, say $\alpha = -1/L + \epsilon$, the decay follows a t^{-1} decay law for some substantial period of time until the t^{-3} decay eventually dominates (see Fig. 4). The smaller ϵ is, the longer the t^{-1} behavior persists.

IV. DISCUSSION

To summarize, the foregoing results provide an intuitive physical picture and quantitative description of postexponential decay of the probability density at points distant from the source. We have developed the classical model suggested by Newton so as to achieve an accurate match between classical and quantum decays. Purely exponential decay from

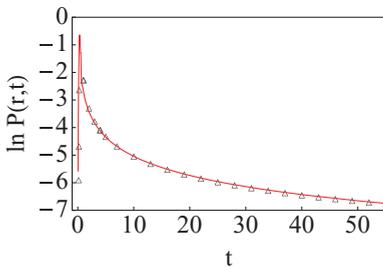


FIG. 3. (Color online) Logarithm of the probability density versus t , for an attractive δ potential having a zero energy resonance: exact quantum numerical solution, solid red line; classical model solution, triangles. The long-time behavior is indistinguishable from Eq. (30). $\alpha = -1$, $\tau = 0.2$, and other parameters are as in Fig. 2.

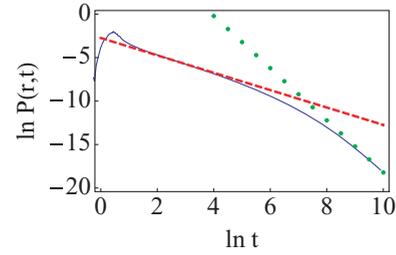


FIG. 4. (Color online) Logarithm of the probability density versus t for a δ potential slightly less attractive than required to produce a zero energy resonance: exact quantum numerical solution (solid blue line) and approximate power-law decays proportional to t^{-1} [Eq. (30), dashed red line] and t^{-3} [Eq. (29), dotted green line]. $\alpha = -0.98$, $r = 10$, $\tau = 0.2$, and other parameters are as in Fig. 2.

a source leads naturally, because of dispersion associated with a velocity distribution of the emitted particles, to the same power-law decay in quantum and classical scenarios. Quantum mechanics is required to provide the emission characteristics, but initial-state reconstruction (ISR) plays no role in the classical, purely outgoing dynamics. We have checked with the methodology of [25], that ISR terms are negligible in the postexponential range of times, in the quantum calculation of Fig. 2. This contrasts with their relevance to the survival probability [25] and indicates different mechanisms for the transition to postexponential decay inside and outside the source. Indeed, the survival probability (calculated either as $|\langle \Psi(0) | \Psi(t) \rangle|^2$ or $|\langle r | \Psi(t) \rangle|^2$ with $r < L$) is still in its exponential regime when the transition shown in Fig. 2 (at $r = 2$) takes place; that is, the purely exponential decay hypothesis (1) for the classical source is justified, and the onset of the postexponential regime of survival within the trap cannot causally affect the transition observed in the density outside the source.

Due to recent advances in lasers, semiconductors, nanoscience, and cold atoms, microscopic interactions are now relatively easy to manipulate, decay parameters have become controllable, and postexponential decay has become more accessible to experimental scrutiny and/or applications [8]. Under appropriate conditions it could become the dominant regime and be used to speedup decay *via* an anti-Zeno effect [26]. Moreover, recent experiments on periodic waveguide arrays provide a classical, electric field analog of a quantum system with exponential decay [27,28] where the postexponential region could be studied in a particularly direct way.

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