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ON THE EXISTENCE OF COEXISTENCE STATES FOR AN ADVECTION-COOPERATIVE SYSTEM WITH SPATIAL HETEROGENEITIES

PABLO ÁLVAREZ-CAUDEVILLA AND TATSUYA WATANABE

Abstract. This paper is devoted to the analysis of the existence of coexistence states for cooperative systems under homogeneous Dirichlet boundary conditions. We assume rather general spatial heterogeneities for the coefficients of the non-linearities and the cooperative terms. In order to obtain sharp existence results, we add advection terms related to the cooperative terms so that the system can be transformed into a variational one. Applying then several variational methods and bifurcation theory, we arrive at necessary and sufficient condition on the parameter λ for the existence and the uniqueness of coexistence states. We also compare the conditions for the heterogeneous case and the constant case.

1. Introduction and main results

1.1. **Backgrounds.** In the last decade, a lot of works have been done for the following elliptic systems:

$$-\Delta u = \lambda u + \alpha v - p(x)f(u)u \quad \text{in } \Omega, \tag{1.1}$$

$$-\Delta v = \beta u + \lambda v - q(x)g(v)v \quad \text{in } \Omega, \tag{1.2}$$

$$(u,v) = (0,0) \quad \text{on } \partial\Omega,$$

where Ω is a bounded domain of $\mathbb{R}^N$ with boundary $\partial\Omega$ of class $\mathcal{C}^{2+\mu}$ for some $\mu \in (0,1)$, $N \geq 1$ and $\lambda \in \mathbb{R}$ is a parameter. The system (1.1)-(1.2) is said to be *cooperative* if α and β are positive; see the definition below. We are interested in the existence and the uniqueness of coexistence states of (1.1)-(1.2), that is, solutions (u,v) with $u > 0$ and $v > 0$ in Ω.

Supposing that the nonlinear terms f and g are of logistic type, it is known that conditions for the existence of coexistence states heavily depends on spatial heterogeneities of p and q. To be more precise, functions p and q are assumed to be non-negative, belong to $\mathcal{C}^\mu(\bar\Omega)$ and satisfy the following hypothesis, which will be maintained throughout this work:

(A) The open sets

$$\Omega_+^p := \{\, x \in \Omega \ : \ p(x) > 0\} \quad \text{and} \quad \Omega_+^q := \{\, x \in \Omega \ : \ q(x) > 0\},$$

are sub-domains of Ω of class $\mathcal{C}^{2+\mu}$ with $\bar\Omega_+^j \subset \Omega$ for $j \in \{p,q\}$.

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(B) The open sets

$$\Omega_0^p := \Omega \setminus \bar{\Omega}_+^p \quad \text{and} \quad \Omega_0^q := \Omega \setminus \bar{\Omega}_+^q,$$

are sub-domains of Ω of class $\mathcal{C}^{2+\mu}$, and the sets

$$K_0^p := (p)^{-1}(0) = \bar{\Omega} \setminus \Omega_+^p \quad \text{and} \quad K_0^q := (q)^{-1}(0) = \bar{\Omega} \setminus \Omega_+^q,$$

are compact. Moreover, $\partial\Omega_0^j$, with $j \in \{p, q\}$, might consist of two components Γ_1^j and Γ_2^j which are also of class $\mathcal{C}^{2+\mu}$.

For later use, we denote

$$\Omega_+ := \Omega_+^p \cap \Omega_+^q.$$

In the case $j > 0$ in Ω, one has

$$\Omega_+^j = \Omega, \quad \Omega_0^j = \emptyset \quad \text{and} \quad K_0^j = \emptyset.$$

On the other hand, if $j \equiv 0$ in Ω, we then have

$$\Omega_+^j = \emptyset, \quad \Omega_0^j = \Omega \quad \text{and} \quad K_0^j = \bar{\Omega}.$$

An important feature is that K_0^p and K_0^q might be different, that is, one of the nonlinear terms may vanish depending on the location.

If both p and q are positive everywhere in Ω, a necessary and sufficient condition for the existence of coexistence states has been obtained in [27]. (See Figure 1 below.) General spatial heterogeneities have been also considered in [3, 6, 9, 28]. However in the previous papers, the authors studied the problem under the assumptions $\Omega_+ = \Omega_+^p = \Omega_+^q$ ([6], [28]), or $\Omega_+^p \subset \Omega$, $\Omega_+^q = \Omega$ and α, β are positive constants ([3], [9]). Our aim of this paper is to extend the results into two directions: Firstly, we deal with the case $\Omega_+^p \neq \Omega_+^q$ and secondly we study the case when α and β may depend on x. Hence, in contrary to what happen in those previous works, we can consider very general spatial heterogeneities for both coefficients of each equation providing with a more realistic problem from the application point of view. As for results on the related parabolic problems, we refer to [4], [9], [30].

It is worth mentioning that when $\Omega_+^p \subset \Omega$, $\Omega_+^q = \Omega$ and α, β are positive constants, a variational approach to the existence of coexistence states of (1.1)-(1.2) has been studied in [3]. In fact in this case, the system (1.1)-(1.2) leads to

$$-\Delta u = \lambda u + \alpha v - p(x) f(u) u \quad \text{in } \Omega, \tag{1.3}$$

$$-\Delta v = \beta u + \lambda v \quad \text{in } \Omega. \tag{1.4}$$

If there exists a coexistence state (u, v) of (1.3)-(1.4), then one has from (1.4) that

$$(-\Delta - \lambda) v = \beta u > 0 \quad \text{in } \Omega.$$

Then, owing to the Maximum Principle, the following condition must hold

$$\lambda < \sigma[-\Delta, \Omega], \tag{1.5}$$

where $\sigma[-\Delta, \Omega]$ is the principal eigenvalue of $-\Delta$ in Ω under homogeneous Dirichlet boundary condition. From (1.5), it follows that $-\Delta - \lambda$ is positive and invertible. Hence one deduces from (1.4) that

$$v = \beta(-\Delta - \lambda)^{-1} u.$$

Substituting this expression into (1.3), we obtain the following *non-local* elliptic problem:

$$-\Delta u = \lambda u + \alpha\beta(-\Delta - \lambda)^{-1} u - p(x) f(u) u \quad \text{in } \Omega. \tag{1.6}$$

Regarding $\gamma = \alpha\beta$ as a continuation parameter, the author in [3] obtained sharp necessary and sufficient conditions for the existence of positive solutions of (1.6). We note that this type of non-local reductions in the study of elliptic systems has been widely considered, see e.g. [15], [22]. We also refer to [16] for related result on the generalized logistic equation.

On the other hand, when α or β is a function, the above reduction cannot be applied because α, β and the operator $(-\Delta - \lambda)^{-\frac{1}{2}}$ do not commute. Moreover as we will see in Section 2.1, the system (1.1)-(1.2) seems not to have variational structure even if we perform change of variables. Thus in order to generalize previous results, we add the advection terms related to the cooperative terms so that we are able to transform the system into a variational one. This variational formulation enables us to obtain a sharp result.

1.2. **Models and main results.** Now, we introduce our models and state the main results. We consider the following advection-cooperative elliptic system:

$$-\Delta u + \nabla(\log\alpha(x)) \cdot \nabla u = \lambda u + \alpha(x)v - p(x)f(u)u \quad \text{in } \Omega, \tag{1.7}$$

$$-\Delta v + \nabla(\log\beta(x)) \cdot \nabla v = \beta(x)u + \lambda v - q(x)g(v)v \quad \text{in } \Omega, \tag{1.8}$$

$$(u, v) = (0, 0) \quad \text{on } \partial\Omega,$$

where α and β are positive functions belonging to $\mathcal{C}^{2+\mu}(\bar{\Omega})$. We notice that the system (1.7)-(1.8) reduces to (1.1)-(1.2) when α and β are constants. Our goal is to obtain the existence and uniqueness of coexistence states of (1.7)-(1.8) depending on the parameter λ. Note that, this model might be step forward in the analysis of cooperative systems with advection terms and spatial heterogeneities; see recent works in that direction [1] and [2] for one single equation. In terms of population dynamics, if $u(x,t)$ measures the density of the population the advection term means how the population moves up the gradient of the growth rate $\alpha(x)$ toward more favorable habitats and away from less favorable habitats (respectively for $v(x,t)$ and $\beta(x)$).

With the aid of advection terms, the system (1.7)-(1.8) can be transformed into a variational one. Indeed making the change of variable

$$u = \alpha^{\frac{1}{2}}\tilde{u}, \quad v = \beta^{\frac{1}{2}}\tilde{v}, \tag{1.9}$$

and dividing (1.7)-(1.8) by $\alpha^{\frac{1}{2}}$ and $\beta^{\frac{1}{2}}$ respectively, we see that (1.7)-(1.8) is reduced to

$$\begin{cases} -\Delta\tilde{u} + \left(\frac{3}{4}\alpha^{-2}|\nabla\alpha|^2 - \frac{1}{2}\alpha^{-1}\Delta\alpha\right)\tilde{u} = \lambda\tilde{u} + \alpha^{\frac{1}{2}}\beta^{\frac{1}{2}}\tilde{v} - pf(\alpha^{\frac{1}{2}}\tilde{u})\tilde{u}, \\ -\Delta\tilde{v} + \left(\frac{3}{4}\beta^{-2}|\nabla\beta|^2 - \frac{1}{2}\beta^{-1}\Delta\beta\right)\tilde{v} = \alpha^{\frac{1}{2}}\beta^{\frac{1}{2}}\tilde{u} + \lambda\tilde{v} - qg(\beta^{\frac{1}{2}}\tilde{v})\tilde{v}. \end{cases} \tag{1.10}$$

Thus, it suffices to study the existence and the uniqueness of coexistence states of (1.10). Hereafter, we write $\tilde{u} = u$ and $\tilde{v} = v$ for simplicity. Equation (1.10) is the Euler-Lagrange equation of the functional $\mathcal{I}(u,v) : H_0^1(\Omega) \times H_0^1(\Omega) \to \mathbb{R}$ defined by

$$
\begin{aligned}
\mathcal{I}(u,v) := \; & \frac{1}{2}\int_\Omega \left[|\nabla u|^2 + \Big(\frac{3}{4}\alpha^{-2}|\nabla\alpha|^2 - \frac{1}{2}\alpha^{-1}\Delta\alpha\Big)u^2 \right] dx \\
& + \frac{1}{2}\int_\Omega \left[|\nabla v|^2 + \Big(\frac{3}{4}\beta^{-2}|\nabla\beta|^2 - \frac{1}{2}\beta^{-1}\Delta\beta\Big)v^2 \right] dx \\
& - \frac{\lambda}{2}\int_\Omega \left(u^2+v^2\right)\, dx - \int_\Omega \alpha^{\frac{1}{2}}\beta^{\frac{1}{2}} uv\, dx \\
& + \int_\Omega \alpha^{-1} F(x, \alpha^{\frac{1}{2}}u)\, dx + \int_\Omega \beta^{-1} G(x, \beta^{\frac{1}{2}}v)\, dx,
\end{aligned} \tag{1.11}
$$

where

$$F(x,u) := \int_0^u p(x) f(\xi)\xi\, d\xi \quad \text{and} \quad G(x,v) := \int_0^v q(x) g(\xi)\xi\, d\xi.$$

It is known that principal eigenvalues of linear operators play an essential role in the study of the existence of coexistence states for cooperative systems. For every V_1, $V_2 \in C^\mu(\bar{\Omega})$, we define the operator $\mathfrak{L}(V_1,V_2) : H_0^1(\Omega) \times H_0^1(\Omega) \to L^2(\Omega) \times L^2(\Omega)$ by

$$\mathfrak{L}(V_1,V_2) := \begin{pmatrix} -\Delta + V_1 & -\alpha^{\frac{1}{2}}\beta^{\frac{1}{2}} \\ -\alpha^{\frac{1}{2}}\beta^{\frac{1}{2}} & -\Delta + V_2 \end{pmatrix}. \tag{1.12}$$

Definition 1.1. *The operator* $\mathfrak{L}(V_1,V_2)$ is called *strongly cooperative* if $\alpha^{\frac{1}{2}}\beta^{\frac{1}{2}} > 0$ in $\bar{\Omega}$.

As we will see in Section 2.2, there is one-to-one correspondence between $\mathfrak{L}$ and the operator $\tilde{\mathfrak{L}}$ associated with (1.7)-(1.8) which is given by

$$
\begin{aligned}
\tilde{\mathfrak{L}}(V_1,V_2) &= \begin{pmatrix} -\Delta + \nabla(\log\alpha)\cdot\nabla + V_1 & -\alpha \\ -\beta & -\Delta + \nabla(\log\beta)\cdot\nabla + V_2 \end{pmatrix} \\
&= \begin{pmatrix} -\alpha \operatorname{div}\left(\frac{1}{\alpha}\nabla\ \right) + V_1 & -\alpha \\ -\beta & -\beta \operatorname{div}\left(\frac{1}{\beta}\nabla\ \right) + V_2 \end{pmatrix}.
\end{aligned} \tag{1.13}
$$

Thus, in the sense of Definition 1.1, the system (1.7)-(1.8) is said to be *strongly cooperative* if $\alpha > 0$ and $\beta > 0$ in $\bar{\Omega}$. We also note that $\tilde{\mathfrak{L}}$ is not self-adjoint unless $\alpha = \beta$. Next, for any smooth $D \subset \Omega$, suppose that there is a unique τ for which the linear eigenvalue problem

$$\mathfrak{L}(V_1,V_2)\begin{pmatrix}\varphi \\ \psi\end{pmatrix} = \tau\begin{pmatrix}\varphi \\ \psi\end{pmatrix} \quad \text{in} \quad D, \quad (\varphi,\psi) = (0,0) \quad \text{on} \quad \partial D, \tag{1.14}$$

possesses a solution (φ,ψ) with $\varphi > 0$ and $\psi > 0$. Such value of τ will be referred to as the *principal eigenvalue* of $\mathfrak{L}(V_1,V_2)$ in D (under homogeneous Dirichlet boundary conditions), and denoted by

$$\sigma[\mathfrak{L}(V_1,V_2), D].$$

According to [11, Theorem 12] and [26, Theorem 3.1], $\sigma[\mathfrak{L}(V_1,V_2),D]$ is simple and *dominant*, in the sense that $\operatorname{Re}\tau > \sigma[\mathfrak{L}(V_1,V_2),D]$ for any other eigenvalue τ of (1.14). Moreover, the *principal eigenfunction* (φ,ψ) is unique up to a positive multiplicative constant, and

$$\varphi \gg 0, \quad \psi \gg 0.$$

Here a function $w \in \mathcal{C}^1(\bar{\Omega})$ is said to satisfy $w \gg 0$ if it lies in the interior of the cone of non-negative functions of $\mathcal{C}^1(\bar{D})$, i.e., if $w(x) > 0$ for all $x \in D$ and $\partial w/\partial n(x) < 0$ for all

$x \in w^{-1}(0) \cap \partial D$, where $n = n(x)$ stands for the outward unit normal to D at $x \in \partial D$. (cf. Amann [11] for any further required details.)

To state the main results, we denote

$$W_1 = \frac{3}{4}\alpha^{-2}|\nabla\alpha|^2 - \frac{1}{2}\alpha^{-1}\Delta\alpha, \quad W_2 = \frac{3}{4}\beta^{-2}|\nabla\beta|^2 - \frac{1}{2}\beta^{-1}\Delta\beta, \tag{1.15}$$

$\mathfrak{L}(\lambda) = \mathfrak{L}(W_1 - \lambda, W_2 - \lambda)$ and

$$\mathfrak{L} = \mathfrak{L}(0) = \mathfrak{L}(W_1, W_2). \tag{1.16}$$

Let $\sigma[\mathfrak{L}, \Omega]$ be the corresponding principal eigenvalue. Since $\mathfrak{L}$ is self-adjoint, $\sigma[\mathfrak{L}, \Omega]$ can be characterized as

$$\sigma[\mathfrak{L}, \Omega] = \inf_{(u,v)} \frac{\int_\Omega (|\nabla u|^2 + W_1 u^2)\, dx + \int_\Omega (|\nabla v|^2 + W_2 v^2)\, dx - 2\int_\Omega \alpha^{\frac{1}{2}}\beta^{\frac{1}{2}} uv\, dx}{\int_\Omega (u^2 + v^2)\, dx}, \tag{1.17}$$

where the infimum is taken over $(u, v) \in H_0^1(\Omega) \times H_0^1(\Omega)$ with $(u, v) \neq (0, 0)$. Moreover we denote by σ_ω the principal eigenvalue of $\mathfrak{L}$ assuming that the associated eigenfunction satisfies

$$(\varphi_\omega, \psi_\omega) \in H_0^1(\Omega_0^p) \times H_0^1(\Omega_0^q),$$

and the following system:

$$\begin{cases} -\Delta\varphi_\omega + W_1\varphi_\omega - \alpha^{\frac{1}{2}}\beta^{\frac{1}{2}}\psi_\omega = \lambda\varphi_\omega & \text{in } \Omega_0^p, \\ -\Delta\psi_\omega - \alpha^{\frac{1}{2}}\beta^{\frac{1}{2}}\varphi_\omega + W_2\psi_\omega = \lambda\psi_\omega & \text{in } \Omega_0^q. \end{cases} \tag{1.18}$$

It is worth pointing out that σ_ω can be characterized as

$$\sigma_\omega = \inf \Big\{ B[u, v]\, ;\, (u, v) \in H_0^1(\Omega_0^p) \times H_0^1(\Omega_0^q) \setminus \{(0, 0)\} \Big\}, \tag{1.19}$$

where

$$B[u, v] = \frac{\int_{\Omega_0^p} (|\nabla u|^2 + W_1 u^2)\, dx + \int_{\Omega_0^q} (|\nabla v|^2 + W_2 v^2)\, dx - 2\int_{\Omega_0^p \cap \Omega_0^q} \alpha^{\frac{1}{2}}\beta^{\frac{1}{2}} uv\, dx}{\int_{\Omega_0^p} u^2\, dx + \int_{\Omega_0^q} v^2\, dx}.$$

We note that since $p \equiv 0$ in Ω_0^p and $q \equiv 0$ in Ω_0^q, any coexistence state (u, v) of (1.10) satisfies (1.18). Thus, it is natural to describe the condition for the existence of coexistence states by σ_ω. In the case

$$\Omega_0^p = \emptyset \quad \text{and} \quad \Omega_0^q \neq \emptyset,$$

we denote

$$\varphi_\omega \equiv 0 \quad \text{and} \quad \sigma_\omega = \sigma[-\Delta + W_2, \Omega_0^q].$$

We perform the same for the case

$$\Omega_0^p \neq \emptyset \quad \text{and} \quad \Omega_0^q = \emptyset.$$

Furthermore, we set

$$\sigma_\omega = +\infty \quad \text{if} \quad \Omega_0^p = \Omega_0^q = \emptyset.$$

Moreover, by the monotonicity of the principal eigenvalue with respect to the domain, it is clear that

$$\sigma[\mathfrak{L}, \Omega] < \sigma_\omega \quad \text{if} \quad \Omega_0^q \subset \Omega \quad \text{or} \quad \Omega_0^q \subset \Omega.$$

For more properties of $\mathfrak{L}$, see Section 2.2.

Now according to the cases where each of the functions p and q vanishes everywhere in Ω, somewhere in Ω or nowhere in Ω, the problem is distinguished into four types.

- (Type 1): $p \equiv q \equiv 0$. Equivalently $\Omega_0^p = \Omega_0^q = \Omega$.
- (Type 2): $p \equiv 0$, $q \geq 0$, $q \not\equiv 0$ or $p \geq 0$, $p \not\equiv 0$, $q \equiv 0$ or $p \geq 0$, $p \not\equiv 0$, $q \geq 0$, $q \not\equiv 0$.
  Equivalently $\Omega_0^p = \Omega$, $\Omega_0^q \subset \Omega$ or $\Omega_0^p \subset \Omega$, $\Omega_0^q = \Omega$ or $\Omega_0^p \subset \Omega$, $\Omega_0^q \subset \Omega$.
- (Type 3): $p > 0$, $q \equiv 0$ or $p \equiv 0$, $q > 0$ or $p > 0$, $q \geq 0$ or $p \geq 0$, $q > 0$.
  Equivalently $\Omega_0^p = \emptyset$, $\Omega_0^q = \Omega$ or $\Omega_0^p = \Omega$, $\Omega_0^q = \emptyset$ or $\Omega_0^p = \emptyset$, $\Omega_0^q \subset \Omega$ or $\Omega_0^p \subset \Omega$, $\Omega_0^q = \emptyset$.
- (Type 4): $p > 0$, $q > 0$. Equivalently $\Omega_0^p = \Omega_0^q = \emptyset$.

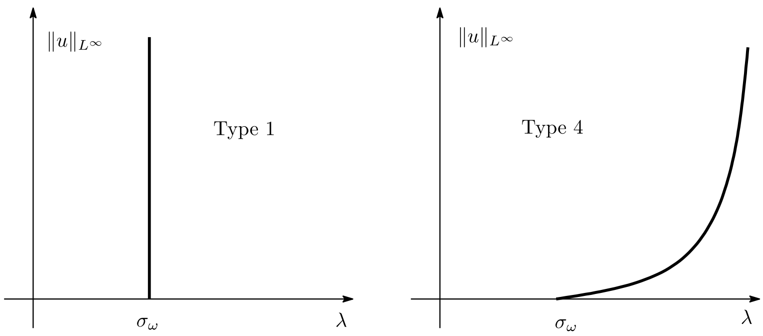

FIGURE 1. Bifurcation diagrams for Types 1 and 4.

In the case Type 1, one has $\sigma[\mathfrak{L}, \Omega] = \sigma_\omega$. Moreover since $p \equiv q \equiv 0$, (1.7)-(1.8) is a linear cooperative system. Then, the coexistence state exists if and only if $\lambda = \sigma[\mathfrak{L}, \Omega] = \sigma_\omega$ and the system (1.7)-(1.8) admits a straight line of coexistence states which consists of all constant multiples of $(\varphi_\omega, \psi_\omega)$. On the other hand, in the case Type 4, (1.7)-(1.8) is a classical cooperative system, and the structure of coexistence states has been widely studied. Especially a coexistence state of (1.7)-(1.8) exists if and only if $\lambda > \sigma[\mathcal{L}, \Omega]$; see Figure 1; further details in this direction in [27]. We also refer to [24] for related results on scalar equations. Thus we are interested in the cases Type 2 or Type 3, namely we further assume that:

(C) Either $p(x)$ or $q(x)$ must be positive and vanish somewhere in Ω.

As for the nonlinear terms f and g, we impose the following conditions:

(D) $f, g \in \mathcal{C}^{1+\mu}[0,\infty)$ for some $\mu \in (0,1)$ and satisfy

$$f(0) = 0, \quad f_s(s) > 0 \quad \text{for all } s > 0.$$

$$g(0) = 0, \quad g_s(s) > 0 \quad \text{for all } s > 0,$$

such that $\frac{dh}{ds} = h_s$, with $h \in \{f, g\}$.

(E) There exists $h \in C^{1+\mu}[0,\infty)$ such that

$$h(0) = 0, \quad h_s(s) > 0 \quad \text{for all } s > 0,$$

$$\lim_{s\to\infty} h(s) = \infty \quad \text{and} \quad \min\{f(s), g(s)\} \geq h(s) \quad \text{for all } s \geq 0.$$

A typical example of the nonlinear term is the logistic nonlinearity $f(s) = g(s) = s$. Moreover, we extend $f(s)$ and $g(s)$ as even functions for $s \leq 0$. For any pair of positive constants Λ_1 and Λ_2, we consider the auxiliary function:

$$H_{[\Lambda_1,\Lambda_2]}(s) := \Lambda_1 h(s)s - \Lambda_2 s \quad \text{for } s \geq 0.$$

Thanks to (E), one can see that the function $H_{[\Lambda_1,\Lambda_2]}$ has a unique positive zero $z_{[\Lambda_1,\Lambda_2]}$ for every Λ_1 and Λ_2. We note that since $f(s)$ and $g(s)$ are even functions, $F(x,s)$ and $G(x,s)$ are also even functions in s. Thus, from (E) it holds that

$$\frac{F(x,s)}{s^2} \to \infty, \ \frac{G(x,s)}{s^2} \to \infty \quad \text{as } |s| \to \infty \quad \text{uniformly in } x.$$

The next assumption guarantees uniform a priori L^∞-estimates for all solutions of (1.7)-(1.8) on any compact subsets $K \subset \Omega_+^j$.

(F) For any Λ_1, Λ_2 and $s > z_{[\Lambda_1,\Lambda_2]}$, it holds

$$I(s) := \int_s^\infty \left[\int_s^t H_{[\Lambda_1,\Lambda_2]}(z)dz \right]^{-\frac{1}{2}} dt < \infty \quad \text{and} \quad \lim_{s\to\infty} I(s) = 0.$$

This condition refines the classical one of Keller [21] and Osserman [29] (cf. [24, Section 4]).

In this setting, we can state the following result.

Theorem 1.1. *Assume (A)-(F). Then the system* (1.7)-(1.8) *possesses a coexistence state if and only if*

$$\sigma[\mathfrak{L}, \Omega] < \lambda < \sigma_\omega, \tag{1.20}$$

and it is unique if it exists. Here $\sigma[\mathfrak{L},\Omega]$ *and* σ_ω *are principal eigenvalues defined in* (1.17) *and* (1.19) *respectively. Let* $\big(u(\lambda), v(\lambda)\big) \in \mathcal{C}^{2+\mu}(\bar\Omega) \times \mathcal{C}^{2+\mu}(\bar\Omega)$ *be the unique coexistence state. Then it follows that*

$$\lim_{\lambda \downarrow \sigma[\mathfrak{L},\Omega]} \|\big(u(\lambda), v(\lambda)\big)\|_{\mathcal{C}(\bar\Omega)\times\mathcal{C}(\bar\Omega)} = 0,$$

$$\lim_{\lambda \uparrow \sigma_\omega} \|\big(u(\lambda), v(\lambda)\big)\|_{\mathcal{C}(\bar\Omega)\times\mathcal{C}(\bar\Omega)} = \infty,$$

$$\begin{cases} \lim_{\lambda\uparrow\sigma_\omega} u(\lambda) = \infty \\ \lim_{\lambda\uparrow\sigma_\omega} v(\lambda) = \infty \end{cases} \textit{uniformly in compact subsets of } \begin{cases} \Omega_0^p & \textit{if} \quad \Omega_0^p \neq \emptyset, \\ \Omega_0^q & \textit{if} \quad \Omega_0^q \neq \emptyset. \end{cases}$$

Furthermore, the map

$$\begin{array}{ccc} (\sigma[\mathfrak{L},\Omega], \sigma_\omega) & \longrightarrow & \mathcal{C}(\bar\Omega)\times\mathcal{C}(\bar\Omega) \\ \lambda & \mapsto & \big(u(\lambda), v(\lambda)\big) \end{array}$$

is point-wise increasing and of class $\mathcal{C}^1$.

Although we may obtain similar results by using sub-supersolution method, it will definitely not be easy. Especially it seems to be difficult to construct suitable super-solutions. We emphasize that our variational approach is simple and enables us to obtain sharp existence and non-existence results. It is also worth mentioning that if we drop the advection terms in (1.7)-(1.8), it is rather hard to show the existence of coexistence states even by applying the sub-supersolution method.

This paper is organized as follows. We study the variational structure of the system (1.7)-(1.8) in Section 2.1, and introduce several spectral properties for the associated linear operators in Section 2.2. In Section 3, we consider the necessary conditions for the existence of coexistence states. In Section 4, we adopt the variational method to obtain the existence and the uniqueness of coexistence states of (1.10). We study the parameter dependence of the unique coexistence state in Section 5. Finally in Section 6, we compare our result with that of the case where α and β are constants, and give further remarks.

2. Preliminaries

2.1. **Variational alternative approach.** In this subsection, we study variational structures of the system (1.7)-(1.8) and the converted system (1.10). First, we note that the original system (1.7)-(1.8) cannot be obtained as Euler-Lagrange equation directly. Indeed, let us consider a function

$$\Phi(x, u, v, a_1, \cdots a_N, b_1, \cdots, b_N) : \Omega \times \mathbb{R}^{2+2N} \to \mathbb{R},$$

and the associated functional

$$\mathcal{E}(u, v) := \int_\Omega \Phi\left(x, u, v, \frac{\partial u}{\partial x_1}, \cdots, \frac{\partial u}{\partial x_N}, \frac{\partial v}{\partial x_1}, \cdots, \frac{\partial v}{\partial x_N}\right) dx.$$

Then, the classical theory of calculus of variations shows that the corresponding Euler-Lagrange equation is given by

$$\frac{\partial \Phi}{\partial u} - \sum_{i=1}^{N} \frac{\partial}{\partial x_i}\left(\frac{\partial \Phi}{\partial u_{x_i}}\right) = 0 \quad \text{and} \quad \frac{\partial \Phi}{\partial v} - \sum_{i=1}^{N} \frac{\partial}{\partial x_i}\left(\frac{\partial \Phi}{\partial v_{x_i}}\right) = 0. \tag{2.1}$$

Thus, if the system (1.7)-(1.8) coincides with (2.1), we must have

$$\frac{\partial \Phi}{\partial u} = \lambda u + \alpha v - p f(u) u \quad \text{and} \quad \frac{\partial \Phi}{\partial v} = \beta u + \lambda v - q g(v) v,$$

which is impossible unless $\alpha = \beta$. Moreover, even if $\alpha = \beta$, (2.1) is still far from (1.7)-(1.8) because $-\Delta u + \nabla(\log \alpha) \cdot \nabla u$ and $-\Delta v + \nabla(\log \beta) \cdot \nabla v$ on the left hand side of (1.7)-(1.8) are not of divergence form unless α and β are constants. In other words, when α and β are functions, we have to perform a suitable change of variables in order to transform the problem into a variational one. These calculations also show that without advection terms, the system (1.1)-(1.2) for spatially heterogeneous α and β cannot be variational.

Our change of variables is a simple multiplicative type. Among this type of change of variables, (1.9) seems to be the best one. In fact, let us consider a change of variables:

$$u = L\tilde{u} \quad \text{and} \quad v = M\tilde{v} \quad \text{for some positive functions } L, M.$$

Substituting (u, v) into (1.7)-(1.8), one has

$$\begin{aligned} &- L\Delta\tilde{u} - 2\nabla L \cdot \nabla\tilde{u} - \Delta L\tilde{u} + \nabla(\log \alpha) \cdot \nabla L\tilde{u} + L\nabla(\log \alpha) \cdot \nabla\tilde{u} \\ &\quad = \lambda L\tilde{u} + \alpha M\tilde{v} - p f(L\tilde{u})L\tilde{u}, \end{aligned} \tag{2.2}$$

$$\begin{aligned} &- M\Delta\tilde{v} - 2\nabla M \cdot \nabla\tilde{v} - \Delta M\tilde{v} + \nabla(\log \beta) \cdot \nabla M\tilde{v} + M\nabla(\log \beta) \cdot \nabla\tilde{v} \\ &\quad = \lambda M\tilde{v} + \beta L\tilde{u} - q g(M\tilde{v})M\tilde{v}. \end{aligned} \tag{2.3}$$

Then, one can see that the reduced system (2.2)-(2.3) is variational if and only if

$$\frac{\alpha M}{L} = \frac{\beta L}{M} \quad \text{or} \quad \frac{L}{M} = \sqrt{\frac{\alpha}{\beta}}. \tag{2.4}$$

Furthermore, the advection terms in (2.2) and (2.3) vanish if and only if

$$2\frac{\nabla L}{L} = \frac{\nabla \alpha}{\alpha} \quad \text{and} \quad 2\frac{\nabla M}{M} = \frac{\nabla \beta}{\beta},$$

from which we deduce that

$$L = C\sqrt{\alpha}, \quad M = C'\sqrt{\beta}, \quad \text{for some} \quad C,\ C' > 0.$$

Then, from (2.4), it holds that $C = C'$. This implies that up to a positive constant, the change of variables (1.9) is the only multiplicative type which enables us to transform (1.7)-(1.8) into a variational system with no advection term. In this paper, we do not pursue a use of nonlinear change of variables. As for change of variables through a nonlinear transform in the study of elliptic PDEs, we refer to e.g. [17], [20], [33].

Next we observe that (1.10) has another variational formulation. Indeed, we set

$$\log\alpha(x) = A(x) \quad \text{and} \quad \log\beta(x) = B(x).$$

Then, (1.10) is rewritten by

$$\begin{cases} -\Delta u + \nabla A\cdot\nabla u = \lambda u + e^{A}v - pf(u)u, \\ -\Delta v + \nabla B\cdot\nabla v = e^{B}u + \lambda v - qg(v)v. \end{cases}$$

Multiplying this system by e^{-A} and e^{-B} respectively, we get

$$\begin{cases} -e^{-A}\Delta u + e^{-A}\nabla A\cdot\nabla u = \lambda e^{-A}u + v - e^{-A}pf(u)u, \\ -e^{-B}\Delta v + e^{-B}\nabla B\cdot\nabla v = u + \lambda e^{-B}v - e^{-B}qg(v)v. \end{cases} \tag{2.5}$$

Noticing that

$$-e^{-A}\Delta u + e^{-A}\nabla A\cdot\nabla u = -\mathrm{div}(e^{-A}\nabla u),$$

we can see that equation (2.5) is the Euler-Lagrange equation of the functional $\mathcal{J}(u,v)$ defined by

$$\begin{aligned} \mathcal{J}(u,v) := {} & \frac{1}{2}\int_\Omega \left(e^{-A}|\nabla u|^2 + e^{-B}|\nabla v|^2\right)dx - \frac{\lambda}{2}\int_\Omega \left(e^{-A}u^2 + e^{-B}v^2\right)dx \\ & - \int_\Omega uv\,dx + \int_\Omega e^{-A}F(x,u)\,dx + \int_\Omega e^{-B}G(x,v)\,dx. \end{aligned} \tag{2.6}$$

Next putting $\alpha = e^A$ and $\beta = e^B$ in (1.11), we obtain a third functional $\mathcal{E}(u,v)$ which is given by

$$\begin{aligned} \mathcal{K}(u,v) := {} & \frac{1}{2}\int_\Omega \Big[|\nabla u|^2 + \Big(\frac{1}{4}|\nabla A|^2 - \frac{1}{2}\Delta A\Big)u^2\Big]dx \\ & + \frac{1}{2}\int_\Omega \Big[|\nabla v|^2 + \Big(\frac{1}{4}|\nabla B|^2 - \frac{1}{2}\Delta B\Big)v^2\Big]dx \\ & - \frac{\lambda}{2}\int_\Omega \left(u^2+v^2\right)dx - \int_\Omega e^{\frac{A+B}{2}}uv\,dx \\ & + \int_\Omega e^{-A}F(x, e^{\frac{A}{2}}u)\,dx + \int_\Omega e^{-B}G(x, e^{\frac{B}{2}}v)\,dx. \end{aligned} \tag{2.7}$$

We remark that the three functionals (1.11), (2.6) and (2.7) are related by

$$\mathcal{I}(u,v) = \mathcal{J}(e^{\frac{A}{2}}u, e^{\frac{B}{2}}v) = \mathcal{K}(u,v), \quad \alpha = e^A,\ \beta = e^B.$$

Since these functionals give us the same result, we only use $\mathcal{I}$ to get to the results of this paper.

Finally, if our cooperative system has another type of advection term and the problem still can be transformed into a variational one, then the conditions for the existence of coexistence states will be expressed by principal eigenvalues of different operators. In this direction, see Section 6 for details.

2.2. **Spectral properties for the associated linear problems.** In this subsection, we show that the operators defined in (1.12) and (1.13) for the systems (1.10) and (1.7)-(1.8) respectively, admit the principal eigenvalues and the corresponding principal eigenfunctions (φ, ψ) satisfying $\varphi \gg 0$, $\psi \gg 0$. Indeed, this is established by the following result.

Proposition 2.1. *For any potential $V_1, V_2 \in C(\bar{\Omega})$, the linear operators $\mathfrak{L}(V_1, V_2)$ and $\tilde{\mathfrak{L}}(V_1, V_2)$ respectively, admit a discrete set of eigenvalues that tend to $+\infty$. Moreover there exist a principal eigenvalue σ and the corresponding principal eigenfunction $(\varphi, \psi) \in H_0^1(\Omega) \times H_0^1(\Omega)$ with $\varphi > 0$ and $\psi > 0$ in Ω.*

Proof. It is standard to show that for sufficiently large $\mu > 0$, the operator $\Big(\mathfrak{L}(V_1, V_2) + \mu \mathrm{Id}\Big)^{-1}$ is positive and compact in $\mathcal{L}\Big(C(\bar{\Omega}) \times C(\bar{\Omega}) \to C_0(\bar{\Omega}) \times C_0(\bar{\Omega})\Big)$; see [32, Lemma 1.2], [25, Theorem 7.6]. Then, owing to [12, Theorem VI.8], the spectrum might contain either infinitely many isolated eigenvalues or a finite number of isolated eigenvalues. However in this case, we can see that the spectrum consists of infinitely many isolated eigenvalues which tends to $+\infty$. To prove that there are infinitely many eigenvalues, one can apply the method shown in [19]. The same is true for $\tilde{\mathfrak{L}}(V_1, V_2)$.

Next since $\mathcal{L}(V_1, V_2)$ is self-adjoint, the existence of a principal eigenvalue follows by the classical spectral theory. On the other hand, even though $\tilde{\mathcal{L}}(V_1, V_2)$ is not self-adjoint, it is strongly cooperative. Hence, according to [11, Theorem 11], the existence of a principal eigenvalue for $\tilde{\mathcal{L}}(V_1, V_2)$ holds. Finally, once we have a principal eigenvalue, the positivity of the corresponding principal eigenfunction can be obtained by the standard argument. □

Remark 2.1. *It is known that the principal eigenvalue is real, simple, isolated and dominant in the sense that, for any other eigenvalue σ_k of the operator $\mathfrak{L}(V_1, V_2)$, it follows that Re $\sigma_k > \sigma_1$ for any $k > 1$. Moreover, the associated eigenfunction (φ, ψ) is unique up to a positive multiplicative constant and satisfies $\varphi \gg 0$, $\psi \gg 0$. Finally the principal eigenvalue is the only one with a positive eigenfunction. We refer to* [11, Theorem 12] *for the proof.*

Now, thanks to Proposition 2.1, the operator $\mathfrak{L} = \mathfrak{L}(W_1, W_2)$ defined by (1.16) has the principal eigenvalue denoted by $\sigma[\mathfrak{L}, \Omega]$. Moreover, let (φ, ψ) be the corresponding principal eigenfunction. We observe that the existence of a principal eigenvalue and the corresponding eigenfunction of $\mathfrak{L}(W_1, W_2)$ guarantee that of $\tilde{\mathfrak{L}}(0, 0)$. Indeed, putting

$$\varphi = \alpha^{-\frac{1}{2}} \tilde{\varphi} \quad \text{and} \quad \psi = \beta^{-\frac{1}{2}} \tilde{\psi},$$

one has from (1.12) and (1.13) that

$$\begin{aligned}
\mathfrak{L}(W_1, W_2)\begin{pmatrix}\varphi\\ \psi\end{pmatrix} &= \begin{pmatrix}-\Delta\varphi + \frac{3}{4}\alpha^{-2}|\nabla\alpha|^2\varphi - \frac{1}{2}\alpha^{-1}\Delta\alpha\varphi - \alpha^{\frac{1}{2}}\beta^{\frac{1}{2}}\psi\\ -\Delta\psi + \frac{3}{4}\beta^{-2}|\nabla\beta|^2\psi - \frac{1}{2}\beta^{-1}\Delta\beta\psi - \alpha^{\frac{1}{2}}\beta^{\frac{1}{2}}\varphi\end{pmatrix}\\
&= \begin{pmatrix}-\alpha^{-\frac{1}{2}}\Delta\tilde{\varphi} + \alpha^{-\frac{3}{2}}\nabla\alpha\cdot\nabla\tilde{\varphi} - \alpha^{\frac{1}{2}}\tilde{\psi}\\ -\beta^{-\frac{1}{2}}\Delta\tilde{\psi} + \beta^{-\frac{3}{2}}\nabla\beta\cdot\nabla\tilde{\psi} - \beta^{\frac{1}{2}}\tilde{\varphi}\end{pmatrix}\\
&= \begin{pmatrix}\alpha^{-\frac{1}{2}}\big[-\Delta\tilde{\varphi} + \nabla(\log\alpha)\cdot\nabla\tilde{\varphi} - \alpha\tilde{\psi}\big]\\ \beta^{-\frac{1}{2}}\big[-\Delta\tilde{\psi} + \nabla(\log\beta)\cdot\nabla\tilde{\psi} - \beta\tilde{\varphi}\big]\end{pmatrix}\\
&= \begin{pmatrix}\alpha^{-\frac{1}{2}} & 0\\ 0 & \beta^{-\frac{1}{2}}\end{pmatrix}\tilde{\mathfrak{L}}(0,0)\begin{pmatrix}\tilde{\varphi}\\ \tilde{\psi}\end{pmatrix},
\end{aligned}$$

$$\sigma[\mathfrak{L},\Omega]\begin{pmatrix}\varphi\\ \psi\end{pmatrix} = \sigma[\mathfrak{L},\Omega]\begin{pmatrix}\alpha^{-\frac{1}{2}}\tilde{\varphi}\\ \beta^{-\frac{1}{2}}\tilde{\psi}\end{pmatrix} = \begin{pmatrix}\alpha^{-\frac{1}{2}} & 0\\ 0 & \beta^{-\frac{1}{2}}\end{pmatrix}\sigma[\mathfrak{L},\Omega]\begin{pmatrix}\tilde{\varphi}\\ \tilde{\psi}\end{pmatrix}.$$

This implies that

$$\tilde{\mathfrak{L}}(0,0)\begin{pmatrix}\tilde{\varphi}\\ \tilde{\psi}\end{pmatrix} = \sigma[\mathfrak{L},\Omega]\begin{pmatrix}\tilde{\varphi}\\ \tilde{\psi}\end{pmatrix},$$

yielding that $\sigma[\mathfrak{L},\Omega]$ is the principal eigenvalue of $\tilde{\mathfrak{L}}(0,0)$ and $(\tilde{\varphi},\tilde{\psi}) = (\alpha^{-\frac{1}{2}}\varphi, \beta^{-\frac{1}{2}}\psi)$ is the corresponding principal eigenfunction.

We note that the positivity of the principal eigenvalue is crucially related to the strong Maximum Principle. Indeed, thanks to the characterization theorem due to López-Gómez & Molina-Meyer [26, Theorem 2.1], there is an equivalence between the strong Maximum Principle, the positivity of the principal eigenvalue and the existence of a strict positive supersolution of (1.14).

Similarly we consider the problem:

$$\text{(2.8)}\qquad \begin{cases} -\Delta\varphi + W_1\varphi - \alpha^{\frac{1}{2}}\beta^{\frac{1}{2}}\psi = \lambda\varphi,\\ -\Delta\psi - \alpha^{\frac{1}{2}}\beta^{\frac{1}{2}}\varphi + W_2\psi = \lambda\psi,\end{cases} \quad ; \quad (\varphi,\psi)\in H_0^1(\Omega_0^p)\times H_0^1(\Omega_0^q).$$

We say that (φ,ψ) is a solution for the system (2.8) when each equation is satisfied. Moreover, thanks to Proposition 2.1, we also know that the spectrum for the problem (2.8) is a discrete set of eigenvalues that tends to $+\infty$ and the resolvent of the operator $(\mathfrak{L}+\mu\mathrm{Id})^{-1}$ is compact for a sufficiently large $\mu > 0$. In particular, we denote by σ_ω the principal eigenvalue of $\mathfrak{L}$ assuming that the associated eigenfunction $(\varphi_\omega,\psi_\omega)$ satisfies

$$(\varphi_\omega,\psi_\omega)\in H_0^1(\Omega_0^p)\times H_0^1(\Omega_0^q).$$

Remark 2.2. *We must notice that depending on the different geometrical cases for the spatial distribution of the the domains Ω_0^p and Ω_0^q, the value of σ_ω will change, as analyzed in* [5, Proposition 61]*. (cf. that work for further details, several situations and proofs.)*

3. Necessary conditions for the existence

In this section, we study the necessary conditions on the parameter λ in order to have the existence of coexistence states of (1.10). Next result provides us with the first part of the proof of Theorem 1.1.

Proposition 3.1. *Suppose that (A), (B), (C), (D) hold and assume that the system* (1.10) *possesses a classical solution* (u, v) *with* $u > 0$, $v > 0$ *in* Ω. *Then, it holds*

$$\sigma[\mathfrak{L}, \Omega] < \lambda < \sigma_\omega.$$

Proof. We observe that (1.10) can be written as

$$\begin{pmatrix} -\Delta + W_1 + pf(\alpha^{\frac{1}{2}}u) & -\alpha^{\frac{1}{2}}\beta^{\frac{1}{2}} \\ -\alpha^{\frac{1}{2}}\beta^{\frac{1}{2}} & -\Delta + W_2 + qg(\beta^{\frac{1}{2}}v) \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \lambda \begin{pmatrix} u \\ v \end{pmatrix},$$

and, hence, by the uniqueness of the principal eigenvalue as noted in Remark 2.1, we have

$$\lambda = \sigma[\mathfrak{L}\big(W_1 + pf(\alpha^{\frac{1}{2}}u), W_2 + qg(\beta^{\frac{1}{2}}v)\big), \Omega]. \tag{3.1}$$

From (C) and (D), it follows that $pf(\alpha^{\frac{1}{2}}u)$ and $qg(\beta^{\frac{1}{2}}v)$ must be positive somewhere in Ω. Thus by the monotonicity of the principal eigenvalue with respect to the potential, we find that

$$\lambda > \sigma[\mathfrak{L}(W_1, W_2), \Omega] = \sigma[\mathfrak{L}, \Omega].$$

On the other hand, from (A), (B), (C), (3.1) and by monotonicity of the principal eigenvalue with respect to the domain, we also have

$$\lambda < \sigma[\mathfrak{L}\big(W_1 + pf(\alpha^{\frac{1}{2}}u), W_2 + qg(\beta^{\frac{1}{2}}v)\big), \Omega_0^p \cup \Omega_0^q] \leq \sigma_\omega,$$

completing the proof. □

4. Sufficient conditions for the existence

In this section, we study the sufficient conditions on the parameter λ in order to have the existence of coexistence states of (1.11). To this end, we let

$$X := H_0^1(\Omega) \times H_0^1(\Omega), \ \|(u, v)\|_X := \|u\|_{H_0^1(\Omega)} + \|v\|_{H_0^1(\Omega)},$$

such that,

$$\|u\|_{H_0^1(\Omega)} = \left(\int_\Omega (|\nabla u|^2 + u^2) \, dx \right)^{\frac{1}{2}},$$

and show that the functional $\mathcal{I}(u, v) : X \to (-\infty, \infty]$ defined by (1.11) has a global minimizer on X. First, we observe that $\mathcal{I}$ is of class C^1. Moreover one can see that a critical point (u, v) of $\mathcal{I}$ satisfies

$$\begin{aligned} &\int_\Omega \nabla u \cdot \nabla \nu_1 \, dx + \int_\Omega \nabla v \cdot \nabla \nu_2 \, dx + \int_\Omega pf(\alpha^{\frac{1}{2}}u) u \nu_1 \, dx + \int_\Omega qg(\beta^{\frac{1}{2}}v) v \nu_2 \, dx \\ &- \int_\Omega (\lambda - W_1) u \nu_1 \, dx - \int_\Omega (\lambda - W_2) v \nu_2 \, dx - \int_\Omega \alpha^{\frac{1}{2}}\beta^{\frac{1}{2}}(\nu_1 v + u \nu_2) \, dx = 0, \end{aligned}$$

for any $(\nu_1, \nu_2) \in X$, which implies that (u, v) is a weak solution of (1.10). We say that $(u_0, v_0) \in X$ is a global minimizer of $\mathcal{I}$ if it holds

$$\mathcal{I}(u_0, v_0) \leq \mathcal{I}(u, v) \quad \text{for every } (u, v) \in X.$$

Next we recall the notion of the weak lower semi-continuity.

Definition 4.1. Let X be a Banach space. The functional $\mathcal{I} : X \to \mathbb{R}$ is said to be weakly (sequentially) lower semi-continuous (wls) if for any weakly convergent sequence $\{(u_n, v_n)\} \subset X$ satisfying $(u_n, v_n) \rightharpoonup (u, v)$ as $n \to \infty$, it holds

$$\mathcal{I}(u, v) \le \liminf_{n\to\infty} \mathcal{I}(u_n, v_n).$$

Thus, we establish the weak lower semi-continuity of the functional $\mathcal{I}$ defined in (1.11) by the following lemma.

Lemma 4.1. *Assume (D). Then, the functional $\mathcal{I}$ defined in* (1.11) *is weakly lower semi-continuous in X.*

Proof. Suppose that $u_n \rightharpoonup u$, $v_n \rightharpoonup v$ in $H_0^1(\Omega)$. Then, it follows that

$$\|u_n\|_{H_0^1(\Omega)} \le C, \quad \|v_n\|_{H_0^1(\Omega)} \le C, \quad \text{for some } C > 0.$$

Now by the definition of $\mathcal{I}$, we can write $\mathcal{I}$ in the following form:

$$\begin{aligned}
&\mathcal{I}(u, v) = \mathcal{I}_1(u, v) + \mathcal{I}_2(u, v) + \mathcal{I}_3(u, v),\\
&\mathcal{I}_1(u, v) := \frac{1}{2}\int_\Omega (|\nabla u|^2 + |\nabla v|^2)\, dx,\\
&\mathcal{I}_2(u, v) := -\frac{\lambda}{2}\int_\Omega (u^2 + v^2)\, dx - \int_\Omega \alpha^{\frac{1}{2}}\beta^{\frac{1}{2}} uv\, dx + \frac{1}{2}\int_\Omega (W_1 u^2 + W_2 v^2)\, dx,\\
&\mathcal{I}_3(u, v) := \int_\Omega \alpha^{-1} F(x, \alpha^{\frac{1}{2}} u)\, dx + \int_\Omega \beta^{-1} G(x, \beta^{\frac{1}{2}} v)\, dx.
\end{aligned}$$

First, from (D) it follows that $\alpha^{-1}F(x, \alpha^{\frac{1}{2}}u) \ge 0$ and $\beta^{-1}G(x, \beta^{\frac{1}{2}}v) \ge 0$. Thus, due to Fatou's lemma, one has

$$\liminf_{n\to\infty} \mathcal{I}_3(u_n, v_n) \ge \mathcal{I}_3(u, v). \tag{4.1}$$

Next, if $\lambda > 0$ we have from $\|u_n\|_{L^2(\Omega)} \le C$, $\|v_n\|_{L^2(\Omega)} \le C$ and, by the reverse Fatou's lemma, that

$$\liminf_{n\to\infty} \int_\Omega (u_n^2 + v_n^2)\, dx \le \limsup_{n\to\infty} \int_\Omega (u_n^2 + v_n^2)\, dx \le \int_\Omega (u^2 + v^2)\, dx,$$

from which we deduce that

$$\liminf_{n\to\infty} \left\{ -\lambda \int_\Omega (u_n^2 + v_n^2)\, dx \right\} \ge -\lambda \int_\Omega (u^2 + v^2)\, dx. \tag{4.2}$$

On the other hand, when $\lambda \le 0$ it is clear that (4.2) holds by the weak lower semi-continuity of L^2-norm. Arguing similarly, we obtain

$$\liminf_{n\to\infty} \mathcal{I}_2(u_n, v_n) \ge \mathcal{I}_2(u, v). \tag{4.3}$$

Finally, it is well-known that $\|\nabla u\|_{L^2(\Omega)}$ is equivalent to $H_0^1(\Omega)$-norm. Thus, by the weak lower semi-continuity of norm in Hilbert space, we obtain

$$\liminf_{n\to\infty} \int_\Omega |\nabla u_n|^2\, dx \ge \int_\Omega |\nabla u|^2\, dx,$$

from which we conclude that

$$\liminf_{n\to\infty} \mathcal{I}_1(u_n, v_n) \geq \mathcal{I}_1(u, v). \tag{4.4}$$

Consequently, thanks to (4.1), (4.3) and (4.4), $\mathcal{I}$ is weakly lower semi-continuous in X. □

The next result is pivotal in ascertaining the characterization of the existence coexistence states of (1.10). It provides us with the coercivity of the functional (1.11).

Lemma 4.2. *Suppose that (D), (E) hold and $\lambda < \sigma_\omega$. Then, the functional $\mathcal{I}(u, v)$ defined by (1.11) is coercive in X, that is, $\mathcal{I}(u, v) \to \infty$ as $\|(u, v)\|_X \to \infty$.*

Proof. We argue by contradiction. Then, there exists $C > 0$ and a sequence $\{(u_n, v_n)\} \subset X \setminus \{(0, 0)\}$ such that

$$\mathcal{I}(u_n, v_n) \leq C \quad \text{for all } n \in \mathbb{N}, \tag{4.5}$$

but

$$\|(u_n, v_n)\|_X \to \infty \quad \text{as } n \to \infty. \tag{4.6}$$

First we claim that we also have

$$\|(u_n, v_n)\|_Y \to \infty \quad \text{as } n \to \infty, \tag{4.7}$$

where $Y := L^2(\Omega) \times L^2(\Omega)$. To this end, we suppose by contradiction that $\|(u_n, v_n)\|_Y$ is uniformly bounded for any $n \in \mathbb{N}$. Then from (1.11) and (4.5), one has

$$\begin{aligned}
&\int_\Omega |\nabla u_n|^2\,dx + \int_\Omega |\nabla v_n|^2\,dx + 2\int_\Omega \alpha^{-1} F(x, \alpha^{\frac{1}{2}} u_n)\,dx + 2\int_\Omega \beta^{-1} G(x, \beta^{\frac{1}{2}} v_n)\,dx \\
&\leq \int_\Omega (\lambda - W_1) u_n^2\,dx + \int_\Omega (\lambda - W_2) v_n^2\,dx + 2\int_\Omega \alpha^{\frac{1}{2}} \beta^{\frac{1}{2}} u_n v_n\,dx + 2C.
\end{aligned}$$

Moreover, from (D) and by the Hölder inequality, we find that

$$\int_\Omega |\nabla u_n|^2 \leq C \quad \text{and} \quad \int_\Omega |\nabla v_n|^2 \leq C,$$

for some constant $C > 0$. This contradicts our assumption (4.6), showing that (4.7) must hold.

Next, we define

$$\hat{u}_n(x) := \frac{u_n(x)}{\|(u_n, v_n)\|_Y} \quad \text{and} \quad \hat{v}_n(x) := \frac{v_n(x)}{\|(u_n, v_n)\|_Y}.$$

Then, it follows that $\|(\hat{u}_n, \hat{v}_n)\|_Y = 1$. Moreover, from (4.5) and (4.7), we also have

$$\begin{aligned}
&\frac{1}{2}\int_\Omega |\nabla \hat{u}_n|^2\,dx + \frac{1}{2}\int_\Omega |\nabla \hat{v}_n|^2\,dx + \int_\Omega \alpha^{-1} \frac{F(x, \alpha^{\frac{1}{2}} u_n)}{\|(u_n, v_n)\|_Y^2}\,dx + \int_\Omega \beta^{-1} \frac{G(x, \beta^{\frac{1}{2}} v_n)}{\|(u_n, v_n)\|_Y^2}\,dx \\
&\leq \frac{1}{2}\int_\Omega (\lambda - W_1) \hat{u}_n^2\,dx + \frac{1}{2}\int_\Omega (\lambda - W_2) \hat{v}_n^2\,dx + \int_\Omega \alpha^{\frac{1}{2}} \beta^{\frac{1}{2}} \hat{u}_n \hat{v}_n\,dx + o(1),
\end{aligned} \tag{4.8}$$

where W_1 and W_2 are denoted by (1.15). Since $\|\hat{u}_n\|_{L^2(\Omega)} \le 1$ and $\|\hat{v}_n\|_{L^2(\Omega)} \le 1$, it holds

$$\frac{1}{2}\int_\Omega (\lambda - W_1)\hat{u}_n^2\,dx + \frac{1}{2}\int_\Omega (\lambda - W_2)\hat{v}_n^2\,dx + \int_\Omega \alpha^{\frac{1}{2}}\beta^{\frac{1}{2}}\hat{u}_n\hat{v}_n\,dx$$
$$\le \frac{1}{2}(\lambda + \|W_1\|_{L^\infty})\|\hat{u}_n\|_{L^2}^2 + \frac{1}{2}(\lambda + \|W_2\|_{L^\infty})\|\hat{v}_n\|_{L^2}^2 + \|\alpha\|_{L^\infty}^{\frac{1}{2}}\|\beta\|_{L^\infty}^{\frac{1}{2}}\|\hat{u}_n\|_{L^2}\|\hat{v}_n\|_{L^2} \le C.$$

Thus, from (D) and (4.8), one gets

$$\int_\Omega |\nabla \hat{u}_n|^2\,dx \le K, \quad \int_\Omega |\nabla \hat{v}_n|^2\,dx \le K, \tag{4.9}$$

$$\int_\Omega \alpha^{-1}\frac{F(x,\alpha^{\frac{1}{2}}u_n)}{\|(u_n,v_n)\|_Y^2}\,dx \le K, \quad \int_\Omega \beta^{-1}\frac{G(x,\beta^{\frac{1}{2}}v_n)}{\|(u_n,v_n)\|_Y^2}\,dx \le K, \tag{4.10}$$

for some $K > 0$. Hence, expression (4.9) implies that $\{(\hat{u}_n, \hat{v}_n)\}$ is uniformly bounded in X. Then, due to the compact embedding $H_0^1(\Omega) \hookrightarrow L^2(\Omega)$, there exists a subsequence, again labeled by n, and $(u_\omega, v_\omega) \in X$ such that

$$\hat{u}_n \rightharpoonup u_\omega \text{ in } H_0^1(\Omega), \quad \hat{u}_n \to u_\omega \text{ in } L^2(\Omega), \quad \hat{u}_n \to u_\omega \text{ a.e. in } \Omega,$$
$$\hat{v}_n \rightharpoonup v_\omega \text{ in } H_0^1(\Omega), \quad \hat{v}_n \to v_\omega \text{ in } L^2(\Omega), \quad \hat{v}_n \to v_\omega \text{ a.e. in } \Omega.$$

Next we claim that

$$u_\omega \equiv 0 \text{ in } \Omega_+^p \quad \text{and} \quad v_\omega \equiv 0 \text{ in } \Omega_+^q.$$

In fact, we suppose by contradiction that $|\{x \in \Omega_+^p\ ;\ u_\omega(x) \ne 0\}| > 0$. We recall from (E) that $\frac{F(x,s)}{s^2} \to \infty$ as $|s| \to \infty$ uniformly in x. Thus, one has

$$\alpha^{-1}\frac{F\big(x,\alpha^{\frac{1}{2}}u_n(x)\big)}{\|(u_n,v_n)\|_Y^2} = \frac{F\big(x,\alpha^{\frac{1}{2}}u_n(x)\big)}{\big(\alpha^{\frac{1}{2}}u_n(x)\big)^2}\hat{u}_n(x)^2 \to \infty \quad \text{a.e. in } \{x \in \Omega_+^p\ ;\ u_\omega(x) \ne 0\}.$$

Then, due to Fatou's lemma, it follows that

$$\liminf_{n\to\infty}\int_\Omega \alpha^{-1}\frac{F\big(x,\alpha^{\frac{1}{2}}u_n(x)\big)}{\|(u_n,v_n)\|_Y^2}\,dx \ge \liminf_{n\to\infty}\int_{\Omega_+^p} \alpha^{-1}\frac{F\big(x,\alpha^{\frac{1}{2}}u_n(x)\big)}{\|(u_n,v_n)\|_Y^2}\,dx$$
$$\ge \liminf_{n\to\infty}\int_{\{x\in\Omega_+^p\,;\,u_\omega(x)\ne 0\}} \alpha^{-1}\frac{F\big(x,\alpha^{\frac{1}{2}}u_n(x)\big)}{\|(u_n,v_n)\|_Y^2}\,dx = \infty,$$

contradicting to (4.10). Therefore, we arrive at

$$u_\omega \equiv 0 \quad \text{in} \quad \Omega_+^p.$$

Similarly it holds that

$$v_\omega \equiv 0 \quad \text{in} \quad \Omega_+^q.$$

Especially one has $(u_\omega, v_\omega) \in H_0^1(\Omega_0^p) \times H_0^1(\Omega_0^q)$.

Now from (4.8), the strong convergence of $\hat{u}_n \to u_\omega$, $\hat{v}_n \to v_\omega$ in $L^2(\Omega)$ and thanks to the weak lower semi-continuity of H_0^1-norm, it follows that

$$\int_\Omega |\nabla u_\omega|^2\,dx + \int_\Omega |\nabla v_\omega|^2\,dx \le \int_\Omega (\lambda - W_1)u_\omega^2\,dx + \int_\Omega (\lambda - W_2)v_\omega^2\,dx + 2\int_\Omega \alpha^{\frac{1}{2}}\beta^{\frac{1}{2}}u_\omega v_\omega\,dx.$$

Since $\|(\hat{u}_n, \hat{v}_n)\|_Y = 1$, one has

$$\|(u_\omega, v_\omega)\|_Y = \|u_\omega\|_{L^2(\Omega_0^p)} + \|v_\omega\|_{L^2(\Omega_0^q)} = 1.$$

Hence, we find that

$$\lambda \geq \int_\Omega (|\nabla u_\omega|^2 + W_1 u_\omega^2)\, dx + \int_\Omega (|\nabla v_\omega|^2 + W_2 v_\omega^2)\, dx - 2\int_\Omega \alpha^{\frac{1}{2}} \beta^{\frac{1}{2}} u_\omega v_\omega\, dx. \tag{4.11}$$

On the other hand from $\lambda < \sigma_\omega$ and by the characterization of σ_ω, we also have

$$\lambda < \inf_{(u,v)} \Big\{ \int_\Omega (|\nabla u|^2 + W_1 u^2)\, dx + \int_\Omega (|\nabla v|^2 + W_2 v^2)\, dx - 2\int_\Omega \alpha^{\frac{1}{2}} \beta^{\frac{1}{2}} uv\, dx \Big\},$$

where the infimum is taken over $(u, v) \in H_0^1(\Omega_0^p) \times H_0^1(\Omega_0^q)$ with $\|u\|_{L^2(\Omega_0^p)} + \|v\|_{L^2(\Omega_0^q)} = 1$. Note that the pair (u, v) is extended trivially to $H_0^1(\Omega) \times H_0^1(\Omega)$ since outside Ω_0^p and Ω_0^q, respectively, they are zero. This contradicts (4.11) and, therefore, the functional $\mathcal{I}$ is coercive in X. $\square$

Now we are ready to prove the existence of a global minimizer of $\mathcal{I}$ on X, which gives us the existence of a weak solution of (1.10).

Proposition 4.1. *Suppose that (D)-(F) hold and λ satisfies* (1.20)*. Then, the functional $\mathcal{I}(u, v)$ defined by* (1.11) *has a global minimizer $\big(u(\lambda), v(\lambda)\big) \neq (0, 0)$.*

Proof. From Lemmas 4.1-4.2 and thanks to the classical direct method of calculus of variations ([31, Theorem 1.2]), there exists $\big(u(\lambda), v(\lambda)\big) \in X$ such that

$$\mathcal{I}\big(u(\lambda), v(\lambda)\big) = \inf_{(u,v)\in X} \mathcal{I}(u, v) =: d.$$

Thus, it suffices to show that $\big(u(\lambda), v(\lambda)\big) \neq (0, 0)$. To this end, we prove that $d < 0$. Now, let $(\varphi, \psi) \in X$ be a principal eigenfunction associated with the principal eigenvalue $\sigma[\mathfrak{L}, \Omega]$. Then, since $\lambda > \sigma[\mathfrak{L}, \Omega]$ (1.20) and by the characterization of $\sigma[\mathfrak{L}, \Omega]$, one has

$$\lambda \int_\Omega (\varphi^2 + \psi^2)\, dx > \int_\Omega (|\nabla \varphi|^2 + W_1 \varphi^2)\, dx + \int_\Omega (|\nabla \psi|^2 + W_2 \psi^2)\, dx - 2\int_\Omega \alpha^{\frac{1}{2}} \beta^{\frac{1}{2}} \varphi\psi\, dx. \tag{4.12}$$

Next, by the definition of $\mathcal{I}$, it follows that for $\varepsilon > 0$,

$$\begin{aligned} \frac{\mathcal{I}(\varepsilon\varphi, \varepsilon\psi)}{\varepsilon^2} &= \frac{1}{2}\int_\Omega (|\nabla \varphi|^2 + W_1 \varphi^2)\, dx + \frac{1}{2}\int_\Omega (|\nabla \psi|^2 + W_2 \psi^2)\, dx - \int_\Omega \alpha^{\frac{1}{2}} \beta^{\frac{1}{2}} \varphi\psi\, dx \\ &\quad - \frac{\lambda}{2}\int_\Omega (\varphi^2 + \psi^2)\, dx + \int_\Omega \alpha^{-1} \frac{F(x, \alpha^{\frac{1}{2}}\varepsilon\varphi)}{\varepsilon^2}\, dx + \int_\Omega \beta^{-1} \frac{G(x, \beta^{\frac{1}{2}}\varepsilon\psi)}{\varepsilon^2}\, dx. \end{aligned} \tag{4.13}$$

Moreover, from (D), one can easily see that

$$\frac{F(x, s)}{s^2} \to 0, \ \frac{G(x, s)}{s^2} \to 0 \quad \text{as } s \to 0 \quad \text{uniformly in } x.$$

Therefore, thanks to (4.12) and (4.13), we find that

$$d \leq \mathcal{I}(\varepsilon\varphi, \varepsilon\psi) < 0 \quad \text{for sufficiently small } \varepsilon > 0.$$

Finally since $\mathcal{I}(0, 0) = 0$, it follows that $\big(u(\lambda), v(\lambda)\big) \neq (0, 0)$, completing the proof. $\square$

Now by the elliptic regularity theory, we have that $\big(u(\lambda), v(\lambda)\big) \in C^{2+\mu}(\bar{\Omega}) \times C^{2+\mu}(\bar{\Omega})$. Indeed from (F), arguing as in [24, Section 4], one can show that there exists a constant $M > 0$ such that $\|u\|_{L^\infty(\Omega_+^p)} \leq M$ and $\|v\|_{L^\infty(\Omega_+^q)} \leq M$ for any solution (u, v) of (1.13)-(1.14). Since $p \equiv 0$ on Ω_0^p and $q \equiv 0$ on Ω_0^q, it follows that

$$(\lambda - W_1)u + \alpha^{\frac{1}{2}}\beta^{\frac{1}{2}}v - pf(\alpha^{\frac{1}{2}}u)u \in L^2(\Omega), \quad (\lambda - W_2)v + \alpha^{\frac{1}{2}}\beta^{\frac{1}{2}}u - qf(\beta^{\frac{1}{2}}v)v \in L^2(\Omega).$$

Thus, due to L^2-estimate ([18, Theorem 8.12]), one has $(u, v) \in H^2(\Omega) \times H^2(\Omega)$. Hence, after applying a bootstrap argument and embedding theorem, we get $(u, v) \in \Big(C^\mu(\bar{\Omega}) \cap C^2(\Omega)\Big)^2$. Finally, thanks to Schauder estimate [18, Theorem 6.19, Theorem 9.19], we arrive at $\big(u(\lambda), v(\lambda)\big) \in C^{2+\mu}(\bar{\Omega}) \times C^{2+\mu}(\bar{\Omega})$ as required.

Next since $F(x, s)$ and $G(x, s)$ are even in s, it follows that $\mathcal{I}\big(|u(\lambda)|, |v(\lambda)|\big) \leq \mathcal{I}\big(u(\lambda), v(\lambda)\big)$. This implies that $\big(|u(\lambda)|, |v(\lambda)|\big)$ is also a global minimizer of $\mathcal{I}$, from which we find that

$$u(\lambda) \geq 0 \quad \text{and} \quad v(\lambda) \geq 0 \quad \text{in} \quad \Omega.$$

We show that $\big(u(\lambda), v(\lambda)\big)$ is a coexistence state, that is,

$$u(\lambda) > 0 \quad \text{and} \quad v(\lambda) > 0 \quad \text{in} \quad \Omega.$$

First we observe that

$$u(\lambda) = 0 \quad \text{iff} \quad v(\lambda) = 0. \tag{4.14}$$

Indeed if $u(\lambda) \equiv 0$, then from (1.10), one has $0 = \alpha^{\frac{1}{2}}\beta^{\frac{1}{2}}v(\lambda)$ in Ω. Since $\alpha > 0$ and $\beta > 0$ in $\bar{\Omega}$, we deduce that $v(\lambda) \equiv 0$ in Ω, and vice versa. We next claim that

$$u(\lambda) \geq 0,\ v(\lambda) \geq 0 \ \text{ and } \ \big(u(\lambda), v(\lambda)\big) \not\equiv (0, 0) \ \Rightarrow \ u(\lambda) > 0 \ \text{ and } \ v(\lambda) > 0 \text{ in } \Omega. \tag{4.15}$$

As noted in Remark 2.1, we know that the principal eigenvalue is the only one with a positive eigenfunction. Then we can apply the characterization theorem due to López-Gómez & Molina-Meyer [26, Theorem 2.1] and [6, Lemma 3.6] to obtain (4.15), from which we conclude that $\big(u(\lambda), v(\lambda)\big)$ is a coexistence state of (1.10).

Finally, to prove the uniqueness of coexistence states of (1.10), we go back to [6, Lemma 3.9].

Proposition 4.2. *Assume that λ satisfies* (1.20)*. Then, the system* (1.10) *possesses a coexistence state which is actually unique.*

Proof. We have proved the existence of a coexistence state by Proposition 4.1. To prove the uniqueness, we proceed by contradiction similarly to [6, Theorem 3.7]. Suppose that (1.10) has at least two coexistence states (u_1, v_1), (u_2, v_2) with $u_1 \not\equiv u_2$ and $v_1 \not\equiv v_2$. Adapting the abstract theory of Amann [10] or [6, Theorem 3.2], there exist a minimal coexistence state (u_*, v_*) and a maximal coexistence state (u^*, v^*) of (1.10) such that

$$u_* \leq \min\{u_1, u_2\} < \max\{u_1, u_2\} \leq u^*, \quad v_* \leq \min\{v_1, v_2\} < \max\{v_1, v_2\} \leq v^*.$$

Note that the existence of semi-trivial solutions is not allowed due to [6, Lemma 3.6] and, hence, to (4.14). Therefore, problem (1.10) possesses two ordered coexistence states, since $u_* < u^*$ and $v_* < v^*$. Set

$$w_1 := u^* - u_* \quad \text{and} \quad w_2 := v^* - v_*.$$

Then, it follows, by construction, that $w_1 > 0$, $w_2 > 0$ and

$$
\left\{
\begin{array}{ll}
(-\Delta + W_1 - \lambda + Z_1)w_1 - \alpha^{\frac{1}{2}}\beta^{\frac{1}{2}} w_2 = 0 & \text{in} \quad \Omega, \\
(-\Delta + W_2 - \lambda + Z_2)w_2 - \alpha^{\frac{1}{2}}\beta^{\frac{1}{2}} w_1 = 0 & \\
(w_1, w_2) = (0,0) & \text{on} \quad \partial\Omega.
\end{array}
\right. \tag{4.16}
$$

where Z_1 and Z_2 are given through

$$Z_1 := p\alpha^{\frac{1}{2}} \int_0^1 f_s\Big(\alpha^{\frac{1}{2}}(\tau u^* + (1-\tau)u_*)\Big)(\tau u^* + (1-\tau)u_*)\, d\tau + p\int_0^1 f\Big(\alpha^{\frac{1}{2}}(\tau u^* + (1-\tau)u_*)\Big)\, d\tau.$$

$$Z_2 := q\beta^{\frac{1}{2}} \int_0^1 g_s\Big(\beta^{\frac{1}{2}}(\tau v^* + (1-\tau)v_*)\Big)(\tau v^* + (1-\tau)v_*)\, d\tau + q\int_0^1 g\Big(\beta^{\frac{1}{2}}(\tau v^* + (1-\tau)v_*)\Big)\, d\tau.$$

Since $u^* > u_*$ and $v^* > v_*$, it follows from (D) that

$$Z_1 > p\int_0^1 f\Big(\alpha^{\frac{1}{2}}(\tau u^* + (1-\tau)u_*)\Big)\, d\tau \geq pf(\alpha^{\frac{1}{2}}u_*),$$

$$Z_2 > q\int_0^1 g\Big(\beta^{\frac{1}{2}}(\tau v^* + (1-\tau)v_*)\Big)\, d\tau \geq qg(\beta^{\frac{1}{2}}v_*).$$

Thus, from (1.10) and due to the monotonicity of the principal eigenvalue with respect to the potential, we find that

$$\begin{aligned}
&\sigma\left[\mathfrak{L}(W_1 - \lambda + Z_1, W_2 - \lambda + Z_2), \Omega\right] \\
&> \sigma\Big[\mathfrak{L}\big(W_1 - \lambda + pf(\alpha^{\frac{1}{2}}u_*), W_2 - \lambda + qg(\beta^{\frac{1}{2}}v_*)\big), \Omega\Big] = 0.
\end{aligned}$$

On the other hand, thanks to (4.16), (w_1, w_2) provides us with a positive eigenfunction of the operator

$$\mathfrak{L}(W_1 - \lambda + Z_1, W_2 - \lambda + Z_2),$$

associated with the eigenvalue 0. This implies that

$$\sigma\left[\mathfrak{L}(W_1 - \lambda + Z_1, W_2 - \lambda + Z_2), \Omega\right] = 0,$$

leading to a contradiction which ends the proof of Proposition 4.2. □

Remark 4.1. *We note that thanks to* [6, Lemma 3.5]*, the unique coexistence state* (u, v) *is a strong positive solutions in the sense that* $u \gg 0$ *and* $v \gg 0$.

5. Structure of the set of coexistence states with respect to the parameter λ

In this section, we analyze the structure of the set of coexistence states with respect to the parameter λ. To this end, we study the limiting behavior of the unique coexistence state of the problem (1.10) when the parameter λ approaches the limiting values.

Lemma 5.1. *Let* λ *be the continuation parameter satisfying* (1.20) *and denote by* $\big(u(\lambda), v(\lambda)\big)$ *the unique coexistence state of* (1.10)*. Then* $(u(\lambda), v(\lambda))$ *can be regarded as a zero of the operator*

$$\mathfrak{F} : E := \mathcal{C}_0^{2+\mu}(\bar{\Omega}) \times \mathcal{C}_0^{2+\mu}(\bar{\Omega}) \times \mathbb{R} \longrightarrow \mathcal{C}^{\mu}(\bar{\Omega}) \times \mathcal{C}^{\mu}(\bar{\Omega})$$

defined by

$$\mathfrak{F}(u,v,\lambda) := \mathfrak{L}\left(W_1 + pf(\alpha^{\frac{1}{2}}u) - \lambda, W_2 + qg(\beta^{\frac{1}{2}}v) - \lambda\right)\binom{u}{v}$$

$$= \begin{pmatrix} -\Delta + W_1 + pf(\alpha^{\frac{1}{2}}u) - \lambda & -\alpha^{\frac{1}{2}}\beta^{\frac{1}{2}} \\ -\alpha^{\frac{1}{2}}\beta^{\frac{1}{2}} & -\Delta + W_2 + qg(\beta^{\frac{1}{2}}v) - \lambda \end{pmatrix}\binom{u}{v}. \tag{5.1}$$

Moreover, as long as (1.10) *admits a coexistence state, the map* $\lambda \to \big(u(\lambda), v(\lambda)\big)$ *is increasing and of class* $\mathcal{C}^1$*.*

Proof. For simplicity, we write $\big(u(\lambda), v(\lambda)\big) = (u_\lambda, v_\lambda)$. First by the definition of $\mathfrak{F}$, one can see that (u_λ, v_λ) is a zero of $\mathfrak{F}(u,v,\lambda)$ for each λ. Next, due to the Implicit Function Theorem, we can differentiate the identity

$$\mathfrak{F}(u_\lambda, v_\lambda, \lambda) = \binom{0}{0} \tag{5.2}$$

with respect to λ. Then setting

$$\mathfrak{H}_\lambda := \mathfrak{L}\left(W_1 + pf(\alpha^{\frac{1}{2}}u_\lambda) + p\alpha^{\frac{1}{2}}f_s(\alpha^{\frac{1}{2}}u_\lambda)u_\lambda - \lambda,\ W_2 + qg(\beta^{\frac{1}{2}}v_\lambda) + q\beta^{\frac{1}{2}}g_s(\beta^{\frac{1}{2}}v_\lambda)v_\lambda - \lambda\right),$$

it follows that

$$\mathfrak{H}_\lambda \begin{pmatrix} \frac{du_\lambda}{d\lambda} \\ \frac{dv_\lambda}{d\lambda} \end{pmatrix} = \binom{u_\lambda}{v_\lambda}. \tag{5.3}$$

Indeed, differentiating (5.2) with respect to λ, we obtain that

$$D_{(u,v)}\mathfrak{F}(u_\lambda, v_\lambda, \lambda)\begin{pmatrix} \frac{du_\lambda}{d\lambda} \\ \frac{dv_\lambda}{d\lambda} \end{pmatrix} + D_\lambda \mathfrak{F}(u_\lambda, v_\lambda, \lambda) = \binom{0}{0},$$

where

$$D_{(u,v)}\mathfrak{F}(u,v,\lambda)$$
$$= \begin{pmatrix} -\Delta + W_1 + pf(\alpha^{\frac{1}{2}}u) + p\alpha^{\frac{1}{2}}f_s(\alpha^{\frac{1}{2}}u)u - \lambda & -\alpha^{\frac{1}{2}}\beta^{\frac{1}{2}} \\ -\alpha^{\frac{1}{2}}\beta^{\frac{1}{2}} & -\Delta + W_2 + qg(\beta^{\frac{1}{2}}v) + q\beta^{\frac{1}{2}}g_s(\beta^{\frac{1}{2}}v)v - \lambda \end{pmatrix}.$$

But by the definition of $\mathfrak{L}$ in (1.12), one has

$$D_{(u,v)}\mathfrak{F}(u_\lambda, v_\lambda, \lambda) = \mathfrak{H}_\lambda.$$

Moreover, differentiating (5.1) with respect to λ, we also have

$$D_\lambda \mathfrak{F}(u_\lambda, v_\lambda, \lambda) = -\binom{u_\lambda}{v_\lambda},$$

from which we deduce that (5.3) holds.

Next, from (C) and (D), it follows that $p\alpha^{\frac{1}{2}}f_s(\alpha^{\frac{1}{2}}u)u$ and $q\beta^{\frac{1}{2}}g_s(\beta^{\frac{1}{2}}v)v$ must be positive somewhere in Ω. Hence, by the monotonicity of the principal eigenvalue with respect to the potential, we get

$$\sigma[\mathfrak{H}_\lambda,\Omega]$$
$$= \sigma\left[\mathfrak{L}\left(W_1 + pf(\alpha^{\frac{1}{2}}u_\lambda) + p\alpha^{\frac{1}{2}}f_s(\alpha^{\frac{1}{2}}u_\lambda)u_\lambda - \lambda, W_2 + qg(\beta^{\frac{1}{2}}v_\lambda) + q\beta^{\frac{1}{2}}g_s(\beta^{\frac{1}{2}}v_\lambda)v_\lambda - \lambda\right), \Omega\right]$$
$$> \sigma\left[\mathfrak{L}\left(W_1 + pf(\alpha^{\frac{1}{2}}u_\lambda) - \lambda, W_2 + qg(\beta^{\frac{1}{2}}v_\lambda) - \lambda\right), \Omega\right].$$

Moreover, since (u_λ, v_λ) satisfies

$$\begin{pmatrix}0\\0\end{pmatrix} = \mathfrak{F}(u_\lambda, v_\lambda, \lambda) = \mathfrak{L}\left(W_1 + pf(\alpha^{\frac{1}{2}}u_\lambda) - \lambda, W_2 + qg(\beta^{\frac{1}{2}}v_\lambda) - \lambda\right)\begin{pmatrix}u_\lambda\\v_\lambda\end{pmatrix},$$

it follows by the definition of the principal eigenvalue that

$$\sigma\left[\mathfrak{L}\left(W_1 + pf(\alpha^{\frac{1}{2}}u_\lambda) - \lambda, W_2 + qg(\beta^{\frac{1}{2}}v_\lambda) - \lambda\right), \Omega\right] = 0$$

and, hence, $\sigma[\mathfrak{H}_\lambda,\Omega] > 0$. This implies that the operator $\mathfrak{H}_\lambda$ is an isomorphism and, hence, it is invertible in the sense that its inverse $\mathfrak{H}_\lambda^{-1}$ is strongly positive. Finally, since $(u(\lambda), v(\lambda))$ is a coexistence state, one gets from (5.3) that

$$\frac{du_\lambda}{d\lambda} = \mathfrak{H}_\lambda^{-1}u_\lambda \gg 0, \quad \frac{dv_\lambda}{d\lambda} = \mathfrak{H}_\lambda^{-1}v_\lambda \gg 0.$$

In particular, regarding λ as the continuation parameter, the structure of the coexistence states of (1.10) consists of a curve:

$$\lambda \mapsto (u_\lambda, v_\lambda) \qquad \text{with} \quad \sigma[\mathfrak{L},\Omega] < \lambda < \sigma_\omega,$$

which is increasing and of class C^1. □

The next result provides us with the limiting behavior of the coexistence state when the parameter λ approaches to the end points of of the existence interval (1.20). In particular, we prove that the principal eigenvalue $\sigma[\mathfrak{L},\Omega]$ is a bifurcation point where a branch of coexistence states emanates from trivial solutions $(0,0)$. We also show that such a branch of coexistence states is monotone increasing and blows up when the parameter λ approaches to the upper value σ_ω of (1.20).

Proposition 5.1. *For $\lambda \in (\sigma[\mathfrak{L},\Omega], \sigma_\omega)$, let $\big(u(\lambda), v(\lambda)\big)$ be the unique coexistence state of* (1.10). *Then, it holds*

$$\lim_{\lambda\downarrow\sigma[\mathfrak{L},\Omega]} \|\big(u(\lambda), v(\lambda)\big)\|_{\mathcal{C}(\bar\Omega)\times\mathcal{C}(\bar\Omega)} = 0, \tag{5.4}$$

$$\lim_{\lambda\uparrow\sigma_\omega} \|\big(u(\lambda), v(\lambda)\big)\|_{\mathcal{C}(\bar\Omega)\times\mathcal{C}(\bar\Omega)} = \infty, \tag{5.5}$$

$$\left\{\begin{array}{l}\lim_{\lambda\uparrow\sigma_\omega} u(\lambda) = \infty\\ \lim_{\lambda\uparrow\sigma_\omega} v(\lambda) = \infty\end{array}\right. \text{ uniformly in compact subsets of } \left\{\begin{array}{lll}\Omega_0^p & \text{if} & \Omega_0^p \neq \emptyset,\\ \Omega_0^q & \text{if} & \Omega_0^q \neq \emptyset.\end{array}\right. \tag{5.6}$$

Proof. We argue as in [23, Chapter 7]. First we observe that $(u(\lambda), v(\lambda))$ is a zero of the nonlinear operator

$$\mathfrak{G} \;:\; \mathcal{C}_0(\bar\Omega)\times\mathcal{C}_0(\bar\Omega)\times\mathbb{R} \longrightarrow \mathcal{C}_0(\bar\Omega)\times\mathcal{C}_0(\bar\Omega),$$

defined by

$$\mathfrak{G}(u,v,\lambda) := \begin{pmatrix} u \\ v \end{pmatrix} - (-\Delta)^{-1} \begin{pmatrix} (\lambda - W_1)u + \alpha^{\frac{1}{2}}\beta^{\frac{1}{2}}v - pf(\alpha^{\frac{1}{2}}u)u \\ \alpha^{\frac{1}{2}}\beta^{\frac{1}{2}}u + (\lambda - W_2)v - qg(\beta^{\frac{1}{2}}v)v \end{pmatrix},$$

where the operator $(-\Delta)^{-1}$ stands for the inverse of $-\Delta$ in Ω under the homogeneous Dirichlet boundary condition (cf. [23, Chapter 7]). Then, one can see that $\mathfrak{G}$ is of class $\mathcal{C}^1$ and, by the elliptic regularity theory, $\mathfrak{G}(\cdot,\cdot,\lambda)$ is a compact perturbation of the identity, for every $\lambda \in \mathbb{R}$. Moreover, one has

$$\mathfrak{G}(0,0,\lambda) = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \text{for all } \lambda \in \mathbb{R},$$

and from (D),

$$D_{(u,v)}\mathfrak{G}(0,0,\lambda)(u,v) = \begin{pmatrix} u \\ v \end{pmatrix} - (-\Delta)^{-1} \begin{pmatrix} (\lambda - W_1)u + \alpha^{\frac{1}{2}}\beta^{\frac{1}{2}}v \\ \alpha^{\frac{1}{2}}\beta^{\frac{1}{2}}u + (\lambda - W_2)v \end{pmatrix} \quad \text{for } \lambda \in \mathbb{R} \text{ and } u,v \in \mathcal{C}_0(\bar{\Omega}).$$

Thus, the linear operator denoted by

$$\mathfrak{M}(\lambda) := D_{(u,v)}\mathfrak{G}(0,0,\lambda) \quad \text{for } \lambda \in \mathbb{R},$$

is Fredholm of index zero and it is real analytic in λ, yielding that $\mathfrak{M}(\lambda)$ is a compact perturbation of the identity of linear type with respect to λ. Moreover, by elliptic regularity, λ_0 is a singular value of $\mathfrak{M}(\lambda)$ if and only if there exists $(u,v) \neq (0,0)$ such that

$$(5.7) \qquad \begin{cases} -\Delta u = (\lambda_0 - W_1)u + \alpha^{\frac{1}{2}}\beta^{\frac{1}{2}}v & \text{in } \Omega, \\ -\Delta v = \alpha^{\frac{1}{2}}\beta^{\frac{1}{2}}u + (\lambda_0 - W_2)v & \\ (u,v) = (0,0) & \text{on } \partial\Omega. \end{cases}$$

Hence, $\lambda_0 := \sigma[\mathfrak{L},\Omega]$ is a singular value of $\mathfrak{M}(\lambda)$ and there exists a solution (φ_0,ψ_0) of (5.7) which satisfies $\varphi_0 \gg 0$, $\psi_0 \gg 0$ and is unique up to a multiplicative constant. Next, we define two linear operators

$$\mathfrak{M}_0 := \mathfrak{M}(\sigma[\mathfrak{L},\Omega]) \quad \text{and} \quad \mathfrak{M}_1 := \frac{d\mathfrak{M}}{d\lambda}(\sigma[\mathfrak{L},\Omega]).$$

Then, as discussed in [26], it follows that $N[\mathfrak{M}_0] = \text{span}[(\varphi_0,\psi_0)]$. Moreover the following transversality condition (see Crandall and Rabinowitz [13], [14]) holds:

$$(5.8) \qquad \mathfrak{M}_1 \begin{pmatrix} \varphi_0 \\ \psi_0 \end{pmatrix} \notin R[\mathfrak{M}_0].$$

Indeed, to prove (5.8), we suppose by contradiction that

$$\mathfrak{M}_1 \begin{pmatrix} \varphi_0 \\ \psi_0 \end{pmatrix} = -(-\Delta)^{-1} \begin{pmatrix} \varphi_0 \\ \psi_0 \end{pmatrix} \in R[\mathfrak{M}_0].$$

Then, there exist $u,v \in \mathcal{C}_0(\bar{\Omega})$ such that

$$(5.9) \qquad \begin{pmatrix} u - (-\Delta)^{-1}\left[(\lambda_0 - W_1)u + \alpha^{\frac{1}{2}}\beta^{\frac{1}{2}}v\right] \\ v - (-\Delta)^{-1}\left[\alpha^{\frac{1}{2}}\beta^{\frac{1}{2}}u + (\lambda_0 - W_2)v\right] \end{pmatrix} = -(-\Delta)^{-1} \begin{pmatrix} \varphi_0 \\ \psi_0 \end{pmatrix}.$$

Using the elliptic regularity theory again, one has $u,\, v \in \mathcal{C}_0^{2+\mu}(\bar{\Omega})$. Hence, applying $-\Delta$ to both sides of (5.9), we get

$$\mathfrak{L}(W_1 - \lambda_0, W_2 - \lambda_0)\begin{pmatrix} u \\ v \end{pmatrix} = -\begin{pmatrix} \varphi_0 \\ \psi_0 \end{pmatrix},$$

which is equivalent to

$$\begin{cases} -\Delta u - (\lambda_0 - W_1)u - \alpha^{\frac{1}{2}}\beta^{\frac{1}{2}} v = -\varphi_0, \\ -\Delta v - \alpha^{\frac{1}{2}}\beta^{\frac{1}{2}} u - (\lambda_0 - W_2)v = -\psi_0. \end{cases} \tag{5.10}$$

Now we multiply (5.10) by (φ_0, ψ_0), integrate over Ω and apply the integration by parts. Then, we find that

$$\int_\Omega \nabla u \cdot \nabla \varphi_0 - (\lambda_0 - W_1)u\varphi_0 - \alpha^{\frac{1}{2}}\beta^{\frac{1}{2}} v\varphi_0 \, dx = -\int_\Omega \varphi_0^2 \, dx,$$
$$\int_\Omega \nabla v \cdot \nabla \psi_0 - \alpha^{\frac{1}{2}}\beta^{\frac{1}{2}} u\psi_0 - (\lambda_0 - W_2)v\psi_0 \, dx = -\int_\Omega \psi_0^2 \, dx.$$

On the other hand, since (φ_0, ψ_0) satisfies (5.7), we see that the left hand side of the above equations are both zero, showing that

$$\int_\Omega (\varphi_0^2 + \psi_0^2) \, dx = 0.$$

This contradicts to $\varphi_0 \gg 0$, $\psi_0 \gg 0$. Therefore, (5.8) is actually true.

Consequently, according to the bifurcation theory due to Crandall–Rabinowitz [13], $(u, v, \lambda) = (0, 0, \sigma[\mathfrak{L}, \Omega])$ is a bifurcation point and a branch of coexistence states emanates from trivial solutions $(u, v, \lambda) = (0, 0, \lambda)$. Moreover, thanks to Lemma 5.1, the continuous curve of coexistence states emanating from the trivial solution is of class $\mathcal{C}^1$ and increasing point-wise with respect to the parameter λ. We also note that thanks to Proposition 3.1, there is not coexistence state if $\lambda \le \sigma[\mathfrak{L}, \Omega]$. Consequently, due to the uniqueness of the coexistence state, (5.4) holds.

Subsequently, we prove (5.5). To this end, we apply a compactness argument in [23, Chapter 7]. Now arguing by contradiction, there exists a constant $C > 0$ such that

$$u(\lambda)(x) \le C, \;\; v(\lambda)(x) \le C \quad \text{for all } x \in \Omega \text{ and } \lambda \in (\sigma[\mathfrak{L}_1, \Omega], \sigma_\omega). \tag{5.11}$$

Let $\{\lambda_n\}_{n\ge1}$ be a increasing sequence such that

$$\lim_{n\to\infty} \lambda_n = \sigma_\omega.$$

We write $(u_n, v_n) = \big(u(\lambda_n), v(\lambda_n)\big)$ for simplicity. Next multiplying (1.10) by (u_n, v_n), integrating in Ω and applying the formula of integration by parts gives

$$\int_\Omega |\nabla u_n|^2 \, dx + \int_\Omega |\nabla v_n|^2 \, dx + \int_\Omega pf(\alpha^{\frac{1}{2}} u_n)u_n^2 \, dx + \int_\Omega qg(\beta^{\frac{1}{2}} v_n)v_n^2 \, dx.$$
$$= \int_\Omega (\lambda_n - W_1)u_n^2 \, dx + \int_\Omega (\lambda_n - W_2)v_n^2 \, dx + 2\int_\Omega \alpha^{\frac{1}{2}}\beta^{\frac{1}{2}} u_n v_n \, dx.$$

From (D) and (5.11), we find that

$$\int_\Omega |\nabla u_n|^2 \, dx \le K \quad \text{and} \quad \int_\Omega |\nabla u_n|^2 \, dx \le K,$$

for some positive constant K. Then, according to the Agmon–Douglis–Nirenberg elliptic estimates [25, Theorem 5.3], one gets

$$\|\nabla u_n\|_{L^\infty(\Omega)} \le K_1 \quad \text{and} \quad \|\nabla u_n\|_{L^\infty(\Omega)} \le K_1$$

for some $K_1 > 0$ and any $n \in \mathbb{N}$. Let $\varepsilon > 0$ be arbitrarily small and take $x, y \in \Omega$ with $|x-y| < \frac{\varepsilon}{K_1}$. Then we have

$$\begin{aligned} |u_n(x) - u_n(y)| &= \left| \int_0^1 \frac{d}{d\tau} u_n\big(\tau x + (1-\tau)y\big)\, d\tau \right| \le \int_0^1 \left| \big\langle \nabla u_n\big(\tau x + (1-\tau)y\big), x-y \big\rangle \right| d\tau \\ &\le \|\nabla u_n\|_{L^\infty(\Omega)} |x-y| \int_0^1 d\tau \le K_1 |x-y| < \varepsilon, \end{aligned}$$

where $\langle \cdot, \cdot \rangle$ is the standard inner product in $\mathbb{R}^N$. Similarly we get $|v_n(x) - v_n(y)| < \varepsilon$. This implies that the sequence $\{(u_n, v_n)\}_{n \ge 1}$ is uniformly bounded and equi-continuous in $\mathcal{C}_0(\bar\Omega) \times \mathcal{C}_0(\bar\Omega)$. Hence, the Ascoli–Arzelá theorem, there exists a convergent subsequence, relabeled again by $\{(u_n, v_n)\}$, such that

$$(u_n, v_n) \to (u_*, v_*) \quad \text{in} \quad \mathcal{C}_0(\bar\Omega) \times \mathcal{C}_0(\bar\Omega)$$

and $\|u_*\|_{L^\infty(\Omega)} \le C$, $\|v_*\|_{L^\infty(\Omega)} \le C$. On the other hand (u_n, v_n) satisfies

$$\left\{ \begin{array}{l} u_n = (-\Delta)^{-1} \left((\lambda_n - W_1) u_n + \alpha^{\frac12} \beta^{\frac12} v_n - p f(\alpha^{\frac12} u_n) u_n \right), \\ v_n = (-\Delta)^{-1} \left(\alpha^{\frac12} \beta^{\frac12} u_n + (\lambda_n - W_2) v_n - q g(\beta^{\frac12} v_n) v_n \right). \end{array} \right. \tag{5.12}$$

Passing to the limit in (5.12) actually shows that a curve of coexistence states can be extended beyond σ_ω by the Implicit Function Theorem. This contradicts to Proposition 3.1 and, therefore, (5.5) must hold.

Finally we show that (5.6) holds. The proof can be done in a similar argument as in the proof of Lemma 4.2. For simplicity, we suppose that $\Omega_0^p \ne \emptyset$ and $\Omega_0^q \ne \emptyset$. In this case, there exists a principal eigenfunction $(\varphi_\omega, \psi_\omega)$ for (2.8) associated with the principal eigenvalue σ_ω. Now let $Y = L^2(\Omega) \times L^2(\Omega)$ and set

$$\hat u_n := \frac{u_n}{\|(u_n, v_n)\|_Y}, \quad \hat v_n := \frac{v_n}{\|(u_n, v_n)\|_Y}.$$

Then, one actually has that

$$\|\hat u_n\|_{L^2(\Omega)} \le 1 \quad \text{and} \quad \|\hat v_n\|_{L^2(\Omega)} \le 1.$$

Moreover, dividing the system (1.10) by $\|(u_n, v_n)\|_Y$, multiplying by $(\hat u_n, \hat v_n)$ and integrating by parts in Ω, we find that

$$\begin{aligned} &\int_\Omega |\nabla \hat u_n|^2\, dx + \int_\Omega |\nabla \hat v_n|^2\, dx + \int_\Omega p f(\alpha^{\frac12} u_n) \hat u_n^2\, dx + \int_\Omega q g(\beta^{\frac12} v_n) \hat v_n^2\, dx \\ &= \int_\Omega (\lambda_n - W_1) \hat u_n^2\, dx + \int_\Omega (\lambda_n - W_2) \hat v_n^2\, dx + 2 \int_\Omega \alpha^{\frac12} \beta^{\frac12} \hat u_n \hat v_n\, dx. \end{aligned} \tag{5.13}$$

Since $\lambda_n \to \sigma_\omega$, one has

$$\int_\Omega (\lambda_n - W_1)\hat{u}_n^2\,dx + \int_\Omega (\lambda_n - W_2)\hat{v}_n^2\,dx + 2\int_\Omega \alpha^{\frac{1}{2}}\beta^{\frac{1}{2}}\hat{u}_n\hat{v}_n\,dx$$
$$\leq \big(\sigma_\omega + o(1) + \|W_1\|_{L^\infty}\big)\|\hat{u}_n\|^2_{L^2(\Omega)} + \big(\sigma_\omega + o(1) + \|W_2\|_{L^\infty}\big)\|\hat{v}_n\|^2_{L^2(\Omega)}$$
$$+ 2\|\alpha\|^{\frac{1}{2}}_{L^\infty}\|\beta\|^{\frac{1}{2}}_{L^\infty}\|\hat{u}_n\|_{L^2(\Omega)}\|\hat{v}_n\|_{L^2(\Omega)} \leq C. \tag{5.14}$$

Thus, from (D) and (5.13), we get

$$\|\nabla\hat{u}_n\|^2_{L^2(\Omega)} + \|\nabla\hat{v}_n\|^2_{L^2(\Omega)} \leq C,$$

showing that $\|\hat{u}_n\|_{H^1(\Omega)}$ and $\|\hat{v}_n\|_{H^1(\Omega)}$ is uniformly bounded. By the compact embedding $H_0^1(\Omega) \hookrightarrow L^2(\Omega)$, there exists a subsequence, still denoted by the $\{(u_n, v_n)\}$, and $(\hat{u},\hat{v}) \in H_0^1(\Omega)\times H_0^1(\Omega)$ such that $\hat{u} \geq 0$, $\hat{v} \geq 0$ a.e. in Ω and

$$\hat{u}_n \rightharpoonup \hat{u} \text{ in } H_0^1(\Omega), \quad \hat{u}_n \to \hat{u} \text{ in } L^2(\Omega), \quad \hat{u}_n \to \hat{u} \text{ a.e. in } \Omega,$$
$$\hat{v}_n \rightharpoonup \hat{v} \text{ in } H_0^1(\Omega), \quad \hat{v}_n \to \hat{v} \text{ in } L^2(\Omega), \quad \hat{v}_n \to \hat{v} \text{ a.e. in } \Omega.$$

Next we claim that

$$\hat{u} \equiv 0 \text{ in } \Omega_+^p \quad \text{and} \quad \hat{v} \equiv 0 \text{ in } \Omega_+^q.$$

Indeed, suppose by contradiction that $\big|\{x \in \Omega_+^p\ ;\ \hat{u}(x) > 0\}\big| > 0$. Then, from (D) and (5.5), it follows that

$$p(x)f\big(\alpha^{\frac{1}{2}}u_n(x)\big)\hat{u}_n(x)^2 \to \infty \quad \text{in } \{x \in \Omega_+^p\ ;\ \hat{u}(x) > 0\}.$$

Thanks to Fatou's lemma, one gets

$$\liminf_{n\to\infty}\int_\Omega pf(\alpha^{\frac{1}{2}}u_n)\hat{u}_n^2\,dx = \liminf_{n\to\infty}\int_{\Omega_+^p} pf(\alpha^{\frac{1}{2}}u_n)\hat{u}_n^2\,dx$$
$$\geq \liminf_{n\to\infty}\int_{\{x\in\Omega_+^p\ ;\ \hat{u}(x)>0\}} pf(\alpha^{\frac{1}{2}}u_n)\hat{u}_n^2\,dx = \infty,$$

contradicting (5.14). Therefore, we obtain that $\hat{u} \equiv 0$ in Ω_+^p. Similarly it holds that $\hat{v} \equiv 0$ in Ω_+^q. Especially one has $(\hat{u},\hat{v}) \in H_0^1(\Omega_0^p)\times H_0^1(\Omega_0^q)$.

Now using (5.13), the strong convergences of $\hat{u}_n \to \hat{u}$, $\hat{v}_n \to \hat{v}$ in $L^2(\Omega)$ and the weak lower semi-continuity of H^1-norm, it follows that

$$\int_\Omega |\nabla\hat{u}|^2\,dx + \int_\Omega |\nabla\hat{v}|^2\,dx \leq \int_\Omega (\sigma_\omega - W_1)\hat{u}^2\,dx + \int_\Omega (\sigma_\omega - W_2)\hat{v}^2\,dx + 2\int_\Omega \alpha^{\frac{1}{2}}\beta^{\frac{1}{2}}\hat{u}\hat{v}\,dx.$$

Moreover, since $\|\hat{u}_n\|_{L^2(\Omega)} + \|\hat{v}_n\|_{L^2(\Omega)} = 1$ and $\hat{u}_n \to \hat{u}$, $\hat{v}_n \to \hat{v}$ in $L^2(\Omega)$, we find that $(\hat{u},\hat{v}) \neq (0,0)$. Hence, we obtain

$$\sigma_\omega \geq \frac{\int_\Omega \big(|\nabla\hat{u}|^2 + W_1\hat{u}^2\big)\,dx + \int_\Omega \big(|\nabla\hat{v}|^2 + W_2\hat{v}^2\big)\,dx - 2\int_\Omega \alpha^{\frac{1}{2}}\beta^{\frac{1}{2}}\hat{u}\hat{v}\,dx}{\int_\Omega \big(\hat{u}^2 + \hat{v}^2\big)\,dx}. \tag{5.15}$$

On the other hand, since σ_ω is the principal eigenvalue, we have

$$\sigma_\omega = \inf_{(u,v)} \frac{\int_\Omega \big(|\nabla u|^2 + W_1u^2\big)\,dx + \int_\Omega \big(|\nabla v|^2 + W_2v^2\big)\,dx - 2\int_\Omega \alpha^{\frac{1}{2}}\beta^{\frac{1}{2}}uv\,dx}{\int_\Omega (u^2 + v^2)\,dx},$$

where the infimum is taken over $(u,v) \in H_0^1(\Omega_0^p) \times H_0^1(\Omega_0^q)$ with $(u,v) \neq (0,0)$. Due to (5.15) and the uniqueness of the principal eigenfunction, $(\hat{u},\hat{v})$ is a constant multiple of $(\varphi_\omega, \psi_\omega)$ and, hence, $\hat{u}$, $\hat{v}$ are strictly positive on compact subsets of Ω_0^p, Ω_0^q respectively. Consequently, thanks to (5.5), we conclude that (5.6) holds. We can argue similarly even if one of Ω_0^p or Ω_0^q is empty. This completes the proof. □

Now Theorem 1.1 follows by Propositions 3.1, 4.1, 4.2 and 5.1.

6. Concluding remarks

In this section, we give several observations and concluding remarks. By Theorem 1.1, we have shown that the (unique) coexistence state of (1.7)-(1.8) exists if and only if

$$\sigma[\mathfrak{L},\Omega] < \lambda < \sigma_\omega. \tag{6.1}$$

As a first remark, we investigate how the spatial heterogeneity of α, β can affect the existence interval (6.1) of λ. Especially we are interested in the value of $\sigma[\mathfrak{L},\Omega] = \sigma[\mathfrak{L}(W_1,W_2),\Omega]$, where we recall that

$$W_1 = \frac{3}{4}\alpha^{-2}|\nabla\alpha|^2 - \frac{1}{2}\alpha^{-1}\Delta\alpha \quad \text{and} \quad W_2 = \frac{3}{4}\beta^{-2}|\nabla\beta|^2 - \frac{1}{2}\beta^{-1}\Delta\beta.$$

Biologically, λ describes the growth rate of the cooperative species (u,v). This implies that if $\sigma[\mathfrak{L},\Omega]$ becomes smaller for some non-constant α and β, the coexistence state of (1.7)-(1.8) may exist with smaller growth rate. On the other hand, if $\sigma[\mathfrak{L},\Omega]$ gets larger, more growth rate is necessary in order to have the coexistence state.

To simplify the observation, we suppose that $\alpha \equiv \beta$. In this case, it follows that $W \equiv W_1 \equiv W_2$ and

$$\sigma[\mathfrak{L}(W,W),\Omega] = \sigma[-\Delta + W - \alpha, \Omega]$$

by the symmetry of $\mathfrak{L}(W,W)$. Then, due to the monotonicity of the principal eigenvalue with respect to the potential, one has

$$\sigma[-\Delta + W - \alpha,\Omega] > \sigma[-\Delta - \alpha,\Omega] \quad \text{if } W > 0 \text{ in } \Omega,$$
$$\sigma[-\Delta + W - \alpha,\Omega] < \sigma[-\Delta - \alpha,\Omega] \quad \text{if } W < 0 \text{ in } \Omega.$$

Thus, our question is the following: Is there a positive function $\alpha \in C^{2,\mu}(\bar{\Omega})$ such that $W > 0$ or $W < 0$ in Ω? The answer is "both situations actually happen". Indeed by the definition of W, it is clear that every positive sub-harmonic functions α, that is $\Delta\alpha < 0$, satisfy $W > 0$ in Ω. On the other hand, for any fixed super-harmonic function Ψ and a positive constant δ, we put $\alpha = \Psi + \delta$. Then, one has $W < 0$ in Ω for sufficiently large δ, showing that both cases occur. This observation together with Remark 2.2 concludes that the existence interval (6.1) changes variously depending on the spatial heterogeneity of α, β and the spatial distribution of Ω_0^p, Ω_0^q.

Next, we observe that another advection cooperative system gives us a similar result. In fact, let us consider the following system:

$$\begin{cases} -\Delta u - \nabla(\log\beta)\cdot\nabla u = \lambda u + \alpha v - pf(u)u & \text{in } \Omega, \\ -\Delta v - \nabla(\log\alpha)\cdot\nabla v = \beta u + \lambda v - qg(v)v & \\ (u,v) = (0,0) & \text{on } \partial\Omega, \end{cases} \tag{6.2}$$

namely advections terms of u, v are derived by cooperative coefficients of v, u respectively. Performing a change of variables

$$u = \beta^{-\frac{1}{2}}\hat{u}, \quad v = \alpha^{-\frac{1}{2}}\hat{v},$$

and multiplying (6.2) by $\beta^{\frac{1}{2}}$ and $\alpha^{\frac{1}{2}}$ respectively, we obtain

$$\begin{cases} -\Delta\hat{u} + \left(\frac{1}{2}\beta^{-1}\Delta\beta - \frac{1}{4}\beta^{-2}|\nabla\beta|^2\right)\hat{u} = \lambda\hat{u} + \alpha^{\frac{1}{2}}\beta^{\frac{1}{2}}\hat{v} - pf(\beta^{-\frac{1}{2}}\hat{u})\hat{u}, \\ -\Delta\hat{v} + \left(\frac{1}{2}\alpha^{-1}\Delta\alpha - \frac{1}{4}\alpha^{-2}|\nabla\alpha|^2\right)\hat{v} = \alpha^{\frac{1}{2}}\beta^{\frac{1}{2}}\hat{u} + \lambda\hat{v} - qg(\alpha^{-\frac{1}{2}}\hat{v})\hat{v}. \end{cases}$$

We define the associated linear operator $\hat{\mathfrak{L}}$ by

$$\hat{\mathfrak{L}} := \begin{pmatrix} -\Delta + \hat{W}_1 & -\alpha^{\frac{1}{2}}\beta^{\frac{1}{2}} \\ -\alpha^{\frac{1}{2}}\beta^{\frac{1}{2}} & -\Delta + \hat{W}_2 \end{pmatrix},$$

$$\hat{W}_1 = \frac{1}{2}\beta^{-1}\Delta\beta - \frac{1}{4}\beta^{-2}|\nabla\beta|^2, \quad \hat{W}_2 = \frac{1}{2}\alpha^{-1}\Delta\alpha - \frac{1}{4}\alpha^{-2}|\nabla\alpha|^2,$$

and denote the principal eigenvalues by $\sigma[\hat{\mathfrak{L}}, \Omega]$, $\hat{\sigma}_\omega$ in the same manner as $\sigma[\mathfrak{L}, \Omega]$ and σ_ω. Then, similarly as Theorem 1.1, the necessary and sufficient conditions for the existence of the (unique) coexistence state of (6.2) is given by

$$\sigma[\hat{\mathfrak{L}}, \Omega] < \lambda < \hat{\sigma}_\omega.$$

Again suppose that $\alpha \equiv \beta$, $\hat{W} \equiv \hat{W}_1 \equiv \hat{W}_2$ and compare $\sigma[\hat{\mathfrak{L}}, \Omega]$ with $\sigma[\mathfrak{L}, \Omega]$. By the definitions of W and $\hat{W}$, one has

$$W - \hat{W} = \alpha^{-2}|\nabla\alpha|^2 - \alpha^{-1}\Delta\alpha.$$

Hence, a simple calculation shows that the cases $0 > \hat{W} > W$ and $0 > W > \hat{W}$ never happen. Next in a similar reason as above, we find that

$$\begin{aligned} &W > 0 > \hat{W} \text{ in } \Omega \quad \text{if } \alpha \text{ is positive and sub-harmonic,} \\ &\hat{W} > 0 > W \text{ in } \Omega \quad \text{if } \alpha = \Psi + \delta,\ \Psi \text{ is super-harmonic and } \delta \text{ is sufficiently large.} \end{aligned}$$

Moreover, let $N = 1$, $\ell > 0$, $\Omega = (0, \ell)$ and define a super-harmonic function α by $\alpha(x) = (x - \ell - 2)^2 + 1$. Then, by elementary computations, one has $W > \hat{W} > 0$ in Ω. On the other hand, defining another super-harmonic function $\alpha(x) = (x - 4\ell)^2 + 17\ell^2$, we also have $\hat{W} > W > 0$ in Ω. Therefore, we can conclude that the order of $\sigma[\mathfrak{L}, \Omega]$, $\sigma[\hat{\mathfrak{L}}, \Omega]$ and $\sigma[-\Delta - \alpha, \Omega]$ varies according to the shape of α and β.

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(P. Álvarez-Caudevilla) Universidad Carlos III de Madrid,
Av. Universidad 30, 28911-Leganés, Spain,
& Instituto de Ciencias Matemáticas, ICMAT (CSIC-UAM- UC3M-UCM),
C/Nicolás Cabrera 15, 28049 Madrid, Spain
E-mail address: pacaudev@math.uc3m.es, pablo.caudevilla@icmat.es

(T. Watanabe) Department of Mathematics,
Faculty of Science, Kyoto Sangyo University,
Motoyama, Kamigamo, Kita-ku, Kyoto-City, 603-8555, Japan
E-mail address: tatsuw@cc.kyoto-su.ac.jp