UNIVERSIDAD CARLOS III DE MADRID working papers

Working Paper 08-63
Statistics and Econometrics Series 21
November 2008

Departamento de Estadística
Universidad Carlos III de Madrid
Calle Madrid, 126
28903 Getafe (Spain)
Fax (34) 91 624-98-49

COPULAS IN FINANCE AND INSURANCE

Rosario Romera y Elisa M. Molanes

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Keywords: dependence structure, extremal values, copula modeling, copula review.

Universidad Carlos III de Madrid, Department of Statistics, Madrid, Spain. e-mail addresses: mrromera@est-econ.uc3m.es (Rosario Romera) and emolanes@est-econ.uc3m.es (Elisa M. Molanes).

Copulas in Finance and Insurance

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Department of Statistics, Universidad Carlos III de Madrid

Abstract

Copulas provide a potential useful modeling tool to represent the dependence structure among variables and to generate joint distributions by combining given marginal distributions. Simulations play a relevant role in finance and insurance. They are used to replicate efficient frontiers or extremal values, to price options, to estimate joint risks, and so on. Using copulas, it is easy to construct and simulate from multivariate distributions based on almost any choice of marginals and any type of dependence structure. In this paper we outline recent contributions of statistical modeling using copulas in finance and insurance. We review issues related to the notion of copulas, copula families, copula-based dynamic and static dependence structure, copulas and latent factor models and simulation of copulas. Finally, we outline hot topics in copulas with a special focus on model selection and goodness-of-fit testing.

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1 Introduction

Statistical models for portfolios in finance and insurance have been helpful in understanding the uncertainty underlying variability in claim sizes and price fluctuations of financial assets, in measuring the risk of financial/insurance portfolios, and in taking actions/decisions based on risk preferences. Pricing of financial/insurance contracts requires a previous evaluation of a measure(s) of the risk based on the fitted statistical model to the observed data of financial losses or insurance losses. New complex products in finance and insurance give rise to portfolios with complex dependence structures. Wrong dependence structure may lead to severe underestimation of the portfolio risk. Value at Risk (VaR) estimated under multivariate normality may lead to underestimation of the portfolio VaR. On the other hand, joint survivals of husband/wife pairs have been noted by the actuaries that tend to exhibit nonlinear behavior with strong tail dependence and are poorly suited for models based on normality. Many microeconometric modeling situations use discrete or limited dependent variables which cannot be easily combined into joint distributions. Financial scenarios are often multidimensional and hence require the joint modeling of several random variables. Traditionally, the pairwise dependence between variables has been described by using classical families of bivariate distributions. The main limitation of this approach is that the individual behavior of the two variables (or transformation thereof) must then be characterized by the same parametric family of univariate distributions. The knowledge of the marginal distributions and the dependence structure is usually the only information available to construct the portfolio model, the multivariate distribution. Furthermore, there are few, if any, parametric joint distributions based on marginals from different families.

The copula approach provides a general and straightforward framework for constructing joint distributions based on marginals from different families. Since linear correlation is not a satisfactory dependence measure to capture the observed advanced dependence structures, particularly in the tails of the joint distributions, modeling based on a copula parameterized by nonnormal marginals is an attractive alternative for estimating the portfolio VaR or any other univariate or multivariate measure of the portfolio risk. The main advantage that the copula approach provides is that the selection of an appropriate model for the dependence between variables, represented by the copula, can then proceed independently from the choice of the marginal distribution.

The basic idea of a copula is to *separate* the dependence and the continuous marginal distribution functions in a multivariate distribution (portfolio model), i.e., to *couple* the continuous marginal distribution functions to a joint distribution function. This is of great importance for practical work, especially in economics and finance where one always tries to find dependencies among different random variables.

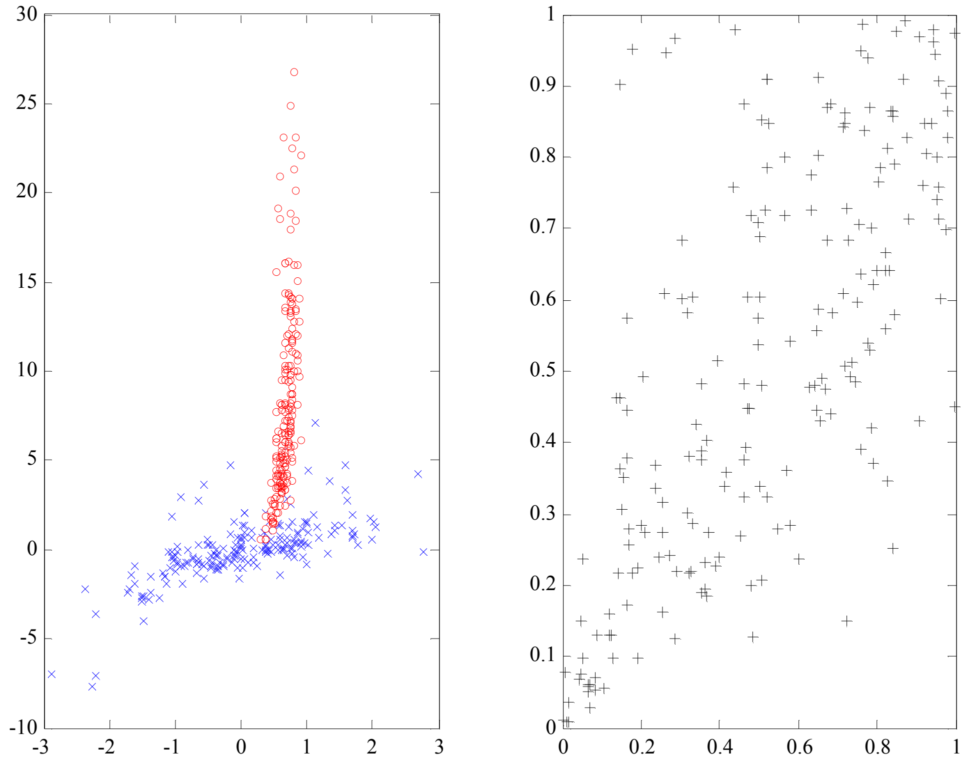

Figure 1: *Data sampled from two bivariate distributions with different marginals but linked via the same copula. Left-hand panel: 200 data drawn from each of the two bivariate joint distributions:* H_1 *(cross points) and* H_2 *(circle points) obtained via the same copula C with non-uniform marginals. Right-hand panel: 200 data drawn from copula C with uniform marginals.*

To illustrate these ideas let consider the following bivariate example. Let (X_1, Y_1) and (X_2, Y_2) be two bivariate random vectors with continuous joint distribution functions, H_1 and H_2, respectively, and such that

$$H_1(x_1, y_1) = C(F_1(x_1), G_1(y_1)) \text{ and } H_2(x_1, y_1) = C(F_2(x_1), G_2(y_1)),$$

where C denotes a mixture of two copulas with mixing probability α,

$$C(u_1, u_2) = \alpha C_C(u_1, u_2) + (1-\alpha) C_{G-H}(u_1, u_2),$$

C_C denotes a copula from the Clayton family with parameter θ_1 and C_{G-H} denotes a copula from the Gumbel-Hougaard family with parameter θ_2

$$C_C(u_1,u_2) = \left(u_1^{-\theta_1} + u_2^{-\theta_1} - 1\right)^{\frac{1}{\theta_1}},$$

$$C_{G-H}(u_1,u_2) = \exp\left\{-[(-\log u_1)^{\theta_2} + (-\log u_2)^{\theta_2}]^{\frac{1}{\theta_2}}\right\}.$$

In our simulation study we have selected $\theta_1 = 2, \theta_2 = 3$ and $\alpha = 0.7$. In Figure 1, we present the biplots obtained by sampling from the bivariate joint distributions H_1 and H_2 linked via the same copula C, under two different scenarios of the corresponding marginal distributions, i.e., F_1 and G_1 for the bivariate joint distribution H_1 and F_2 and G_2 for the bivariate joint distribution H_2. More precisely, we have first drawn 200 samples from copula C with uniform marginals, (u_1, u_2) (see right-panel of Figure 1). Then, in a second step we have obtained the corresponding 200 samples given by $\left(F_1^{-1}(u_1), G_1^{-1}(u_2)\right)$ and the corresponding 200 samples given by $\left(F_2^{-1}(u_1), G_2^{-1}(u_2)\right)$, i.e. drawn from the bivariate distributions H_1 and H_2, respectively. The results of the first step in which we have drawn 200 samples from copula C with uniform marginals are plotted in the right-hand panel of Figure 1. The left-hand panel of Figure 1 shows the results of the second step, i.e. a biplot of 400 samples: 200 of them (cross points) drawn from the joint distribution function H_1, with marginals given by F_1, a standard normal distribution and G_1, a Student t-distribution, and the other 200 (circle points) drawn from the joint distribution function H_2, with marginals given by F_2, a beta distribution and G_2, a gamma distribution. Note that, albeit we have used the same copula function, i.e., the same dependence structure, since the marginals of H_1 and H_2 are different, the two-step copula-approach for constructing bivariate distributions leads to two quite different bivariate distributions, H_1 and H_2.

As a brief historical background of copulas we start with Hoeffding's studies from 1940's regarding some properties of multivariate distributions. The word copula appears for the first time in Sklar (1959) written in French; a similar article written in English followed in 1973 (Sklar, 1973). Academic literature on how to use copulas in risk management started in the later 1990's. From 2004 some financial institutions and insurance companies have started to use copulas as a risk management tool. Following Paul Embrechts's recommendations (see Embrechts, 2009), the first and second must-read papers on copulas are Genest and Nelehová (2007), and Genest and Favre (2007). Nevertheless, we also recommend as a primer the personal view of Embrechts (2009). Two interesting references in copulas: Patton (2006b) reviews the use of copulas in econometric modeling and Genest et al. (2008) provides an interesting bibliometric overview. Standard monographs are Joe (1997) and Nelsen (2006).

In this paper we give an overview of the basics and most recent advances on copula functions and provide the reader with an extensive bibliography of the most recent applications based on copula functions. As it will be clearer along the paper, copula functions will be useful when dealing with multivariate applications, for modeling dependence and for quantifying and analyzing the behavior at the tail of a multidimensional distribution. We expect that this overview helps those interested in financial and actuarial applications use the copula approach in their daily activity.

The rest of the paper is organized as follows. In Section 2, we present a more formal introduction to the concept of copula function; we collect different copula-based measures of dependence, refer to different interesting applications on modeling dynamic dependence, extreme values and measuring financial and actuarial risks and conclude with a subsection devoted to sampling mechanisms, mainly for the well-known Archimedean copula family. Section 3 is devoted to some interesting hot topics on copula-based applications, some of them more general and others more specific in finance and insurance, such as model selection and goodness-of-fit testing, new copula families and copula approximations, and copula-based modeling of competing risks survival times. We end this overview with a final section where we collect some personal thoughts regarding the copula approach and the controversial and critical point of view of Mikosch (2006).

2 Measuring Dependence with Copulas

Copulas provide a natural way to model dependence between random variables. After a brief and general introduction to copulas, we present different copula-based measures of dependence and some interesting applications and related issues in finance and insurance.

2.1 Introduction to Copulas

During the last decades, copula functions have been used in different fields such as mathematical finance, statistics (survival analysis, clinical trials), extreme value theory (hydrology), risk management (risk measure or pricing), resource management, fuzzy set theory (preference modeling and similarities), etc. Due to regulatory requirements set in Basel II for banks and Solvency 2 for insurance business, copula functions have become a standard tool for modeling dependence in risk management.

Copulas are functions that join or couple multivariate distribution functions to their one-dimensional marginals. Besides, they are multivariate distribution functions with uniform marginals. Below, we rewrite the easy version of Sklar's theorem that relates a multivariate distribution function with a copula and vice versa.

Theorem 1 *Suppose* $X_{1,} X_2, \ldots, X_d$ *are random variables with continuous distribution functions,* $F_1, F_2, \ldots, F_d$ *, and joint distribution function* F*, then there exists a unique* $d-$*dimensional copula* C *such that* $\forall x \in \Re^d$ *:*

$$F(x_1, \ldots, x_d) = C(F_1(x_1), \ldots, F_d(x_d)).$$

Conversely, if C *is a* $d-$*dimensional copula and* $F_1, F_2, \ldots, F_d$*, are continuous univariate distribution functions, then* F *is a multivariate distribution function with margins,* $F_1, F_2, \ldots, F_d$.

Based on Sklar's theorem, copula functions allow the construction of a joint distribution function, F, in two steps. First, the marginals need to be fixed and secondly they are coupled through a copula function with a certain interdependence structure. The other way around, from any joint distribution with marginals $F_1, F_2, \ldots, F_d$, one can construct a copula function. The reader should be aware of the problems arising when the assumption of continuity is vanished (see McNeil et al., 2005, for more details).

An interesting result that goes back to Fréchet (1951) and Hoeffding (1940), is that any bivariate distribution F with marginals F_1, F_2 satisfies the relationship

$$F_L(x_1, x_2) \leq F(x_1, x_2) \leq F_U(x_1, x_2),$$

where

$$F_L(x_1,x_2) = \max\{F_1(x_1)+F_2(x_2)-1,0\}$$

and

$$F_U(x_1,x_2) = \min\{F_1(x_1),F_2(x_2)\}$$

are both bivariate distribution functions with marginals F_1, F_2. In the statistical literature, the corresponding copulas,

$$C_L(u_1,u_2) = \max\{u_1+u_2-1,0\}$$
$$C_U(u_1,u_2) = \min\{u_1,u_2\}$$

are referred to as the Fréchet-Hoeffding bounds. From Sklar's theorem, it follows that every 2-dimensional copula yields between these bounds, $C_L(u_1,u_2)$ and $C_U(u_1,u_2)$. An n-dimensional version of the Fréchet-Hoeffding inequality can be found in Nelsen (2006).

Several methods for constructing copulas have been described in the literature. The Archimedean copula family (Nelsen, 2006), for instance, is an important class of dependence functions that can be used to generate multivariate joint distributions with specified marginals. This family of copulas arose in the context of associative functions and probabilistic metric spaces (for more details, the reader may refer to Schweizer and Sklar, 1983 and Alsina et al., 2006). In fact, multivariate exchangeable Archimedean copulas are one of the most popular copula families widely used in applications due to their simple structure and nice properties, specially, in actuarial science and finance for modeling risk dependencies and quantifying tail dependencies, and in hydrology for modeling multivariate extremes. Archimedean copulas are relatively easy to implement and quite flexible in fitting a real dataset due to their ability to cover many distributional shapes.

Given φ, a single-valued function mapping from [0,1] to [0,1], strictly decreasing, convex and such that $\varphi(1)=0$, the copula C below is a member of the Archimedean family,

$$C(u_1,\ldots,u_d) = \varphi^{-1}(\varphi(u_1)+\cdots+\varphi(u_d)),$$

where φ is referred to as the generator of the copula and φ^{-1} denotes the inverse of φ and it is a completely monotone function (see Nelsen, 2006, for a definition of "completely monotonic").

Some of the methods introduced in the literature for the construction of copulas make use of purely geometric information on the copula, such as a description of the support or the shape of the graphs of horizontal, vertical and diagonal sections. In the framework of triangular norms (t-norms), constructions of this type have been considered several times. Other approaches are based on copula mixing. The convex sum method is indeed a particular mixing approach which requires the specification of an entire family of copulas from which the desired copula function is finally obtained.

2.2 Measures of Dependence

Different measures of dependence have been introduced in the literature to quantify the dependence between two random variables, such as the linear correlation coefficient, and measures of association such as the Kendall's tau or the Spearman's rho. Since the copula of a multivariate distribution function describes its dependence structure, the most appropriate measures of dependence should be copula-based measures and therefore invariant under strictly increasing transformations. The linear correlation coefficient, however, does not satisfy this condition since it can not be expressed in terms of the underlying copula alone, as opposed to the Kendall's tau and the Spearman's rho. The invariance property has practical value since the majority of financial data are non-stationary time series and require some variance-stabilizing transformation such as the log transformation. We can be assured that the dependence structure is preserved because the transformation has not changed the copula. Given the importance of copula-based measures, we collect in the following the definition of the main copula-based measures used in the literature.

The Kendall's tau and the Spearman's rho both measure a form of dependence known as concordance. A pair of random variables are said to be concordant if large (small) values of one tend to be associated with large (small) values of the other. More formally, two observations, (x_{11}, x_{21}) and (x_{12}, x_{22}), from a vector (X_1, X_2) are concordant (discordant) if $(x_{11} - x_{12})(x_{21} - x_{22}) > 0$ (< 0). On one hand, the population Kendall's tau is defined as the difference between the probabilities of concordance and discordance of (X_1, X_2) and (Y_1, Y_2) ,

$$\tau = P((X_1 - Y_1)(X_2 - Y_2) > 0) - P((X_1 - Y_1)(X_2 - Y_2) < 0)$$
$$= 4\iint_{I^2} C_2(u_1,u_2)\mathrm{d}C_1(u_1,u_2) - 1,$$

where (X_1, X_2) and (Y_1,Y_2) are independent vectors of continuous random variables with joint distributions, $H_1(x_1,x_2) = C_1(F_1(x_1), F_2(x_2))$ and $H_2(y_1,y_2) = C_2(F_1(y_1), F_2(y_2))$, respectively. On the other hand, the Spearman's rho is defined to be proportional to the probability of concordance minus the probability of discordance of (X_1, X_2) and (Y_1, Z_2),

$$\rho = P((X_1 - Y_1)(X_2 - Z_2) > 0) - P((X_1 - Y_1)(X_2 - Z_2) < 0)$$
$$= 12\iint_{I^2} C(u_1,u_2)\mathrm{d}u_1\mathrm{d}u_2 - 3,$$

where (X_1, X_2), (Y_1,Y_2) and (Z_1,Z_2) are three independent vectors of continuous random variables with a common joint distribution function,

$$H(x_1,x_2) = C(F_1(x_1),F_2(x_2)).$$

Other measures of association have been defined in the literature, based on a distance between the copula of a pair of random variables (X,Y) and the independence copula $\Pi(u_1,u_2) = u_1u_2$ or based on a distance between the copula and the Fréchet bounds, F_L and F_U , introduced above.

Other interesting measures of dependence are the quadrant and tail dependencies. Two random variables, X and Y, with joint distribution function $H(x,y) = C(F_1(x), F_2(y))$ are said to be positively quadrant dependent (PQD) if for all $(x,y) \in \Re^2$, it is satisfied that

$$H(x,y) = P(X \le x, Y \le y) \ge P(X \le x)P(Y \le y) = F_1(x)F_2(y)$$

or equivalently,

$$P((X > x, Y > y)) \ge P(X > x)P(Y > y).$$

Analogously, we say that X and Y are negatively quadrant dependent (NQD) if for all $(x,y) \in \Re^2$, it is satisfied that

$$H(x,y) = P(X \le x, Y \le y) \le P(X \le x)P(Y \le y) = F_1(x)F_2(y)$$

or equivalently,

$$P((X > x, Y > y)) \leq P(X > x)P(Y > y).$$

Summarized in a few words, X and Y are PQD (NQD) if the probability that they are simultaneously small/simultaneously large is at least (at the most) as great as it would be if they were independent. Note that the condition

$$H(x,y) \geq F_1(x)F_2(y), \forall (x,y) \in \Re^2$$

can be rewritten in terms of the copula C as follows

$$C(u_1,u_2) \geq u_1 u_2, \forall (u_1,u_2) \in \mathbb{I}^2,$$

which means that PQ and NP dependence are based-copula properties and consequently invariant under strictly increasing transformations of the random variables.

Taking into account that an alternative formulation of the Spearman's rho is given by

$$\rho = 12 \int\int_{I^2} [C(u_1,u_2) - u_1 u_2] \mathrm{d}u_1 \mathrm{d}u_2,$$

$\frac{\rho}{12}$ can be interpreted as the average quadrant dependence (both positive and negative) for random variables whose copula is C. Tail dependence is another concept that measures the dependence between variables in the upper-right and lower-left quadrants of $\mathbb{I}^2$. Specifically, the lower and upper tail dependence coefficients are defined, respectively by

$$\lambda_L = \lim_{t \to 0^+} P(Y \leq F_2^{-1}(t) | X \leq F_1^{-1}(t)),$$
$$\lambda_U = \lim_{t \to 1^-} P(Y > F_2^{-1}(t) | X > F_1^{-1}(t)).$$

There exist alternative expressions for λ_L and λ_U that only depend on the copula function,

$$\lambda_L = \lim_{t \to 0^+} \frac{C(t,t)}{t},$$
$$\lambda_U = 2 - \lim_{t \to 1^-} \frac{1 - C(t,t)}{1 - t}.$$

For the well-known Archimedean copula, these coefficients can be expressed in terms of the generator and its inverse. In Table 1, we collect some of the most popular copula families with their associated Kendall's tau, Spearman's rho and lower and upper tail dependence coefficients.

So far, we have considered only the bivariate case of measuring dependence between two random variates. Although understanding dependence in the multivariate case is more complex and difficult, there exist natural extensions to the multivariate case of many of the concepts introduced for measuring dependence between two random variates (see Nelsen, 2006, for more details).

Depending on each application, it is important to select a copula family that covers or appropriately describes the type of dependence observed in our data. Although a great number of copula families have been proposed with different dependence structures, how to select the family is still an open problem in real applications. See Section 3.1 for more details regarding model selection and goodness-of-fit testing.

2.3 Measuring Financial/Actuarial Risk

For an extensive treatment on modeling dependence with copulas and applications to Risk Management we refer the reader to Embrechts et al. (2003). In this section we focus on the most recent trends that have appeared in the literature concerning this issue.

Real-life financial and actuarial data often turn out to be highly non-Gaussian. Returns (increments of log-prices over successive periods of time, say daily) have heavy-tailed distributions, are dependent through time and may be non-stationary. Because there is not a simple alternative in the non-Gaussian world, one needs multivariate models for portfolios with different marginal distributions (including different tail behavior) and a dependence structure which is determined not only by covariances. Therefore, copulas seem to be the right tools to overcome these difficulties.

The goal of an integrated risk management in a financial institution is to both measure and manage risk and capital across a diverse range of activities in the banking, securities and insurance sectors. It requires an approach for aggregating different risk distributions, a problem found in many applications in finance including risk management and portfolio choice. Regardless of the sector in which a financial institution participates, all of them are subject to three types of risk: market, credit and operational risk. The distributional shape of each risk varies notably; market risk typically generates portfolio value distributions nearly symmetric whereas credit and specially operational risk generate more skewed distributions because of occasional extreme losses. These losses might be due to large lending exposures in the case of credit risk, or large catastrophes such as 9/11 in the case of operational risk.

Market risk is more easily characterized and measured than others, such as operational risk, but much less is known about the relations among the three risks. In Rosenberg and Schuermann (2006), a practical copula-based approach for integrated risk management with skewed, fat-tailed risks is presented.

Copulas offer financial risk managers a powerful tool to model the dependence between the different elements of a portfolio and are preferable to the traditional, correlation-based approach. In Koedijk et al. (2007) the importance of selecting an accurate copula for risk management is analyzed and an extension of the standard goodness-of-fit tests to copulas is presented. Contrary to existing, indirect tests, these tests can be applied to any copula of any dimension and are based on empirical analysis. The authors show that for a portfolio consisting of stocks, bonds and real estate, these tests provide clear evidence in favour of the Student's *t* copula, and reject both the correlation-based Gaussian copula and the extreme value-based Gumbel copula. In comparison with the Student's *t* copula, they find that the Gaussian copula underestimates the probability of joint extreme downward movements, while the Gumbel copula overestimates this risk. Similarly they establish that the Gaussian copula is too optimistic on diversification benefit, while the Gumbel copula is too pessimistic.

2.4 Dynamic Dependence Modeling using Copulas

Copulas are powerful tools in statistical modeling because the modeling problem can be split into two stages: the first stage deals with the identification of the marginal distribution and the second stage involves defining the suitable copula for the dependence structure. Although this two-stage principle may provide estimates with limited efficiency, it is noted that methods based on this two-stage approach simplify computational difficulties (see Joe, 1997 and 2005). They allow modeling and inference for multivariate models by exploiting rich parametric statistical univariate methods, besides they can be used for finding starting values when maximum likelihood estimation is possible, and provide a convenient approach for the comparison of different copulas. For instance, the two-stage estimation method known as inference function for margins (IFM) has been applied in estimation of Markov-type dependence structures (see Abegaz and Naik-Nimbalkar, 2008). Shih and Louis (1995) proposed a parametric version of the two-stage approach under a bivariate independent and identically distributed setup whereas Chen and Fan (2006) introduce a new class of semiparametric copula-based multivariate dynamic (SCOMDY) models. While a SCOMDY model specifies the multivariate conditional mean and conditional variance parametrically using common specifications such as ARCH, GARCH, VAR, among others, a SCOMDY model, however, specifies the distribution of the standardized innovations semiparametrically as a parametric copula evaluated at the nonparametric univariate marginals. Chen and Fan show different examples of possible combinations that lead to SCOMDY models: GARCH(1,1) + normal copula, GARCH(1,1) + Student's *t*-copula, etc. They propose simple estimators of the parameters involved in a SCOMDY model and establish their large sample properties under misspecification of the copula.

On copula-based time series modeling there are two different lines of research. The first one is the application to multivariate time series and it is focused on modeling the joint distribution of some random vector, $X_t = [X_{1t},...,X_{it},...,X_{dt}]^T$, conditional on the information set, usually $\mathsf{F}_{t-1} = \sigma(X_{t-s}, s \geq 1)$. This scenario requires considering the conditional distribution of X_t given F_{t-1}. The key idea is the "conditional copula" (see Patton, 2006a and 2006b) defined as a multivariate distribution of (possibly correlated) variables that are each distributed as a *Uniform* (0,1) conditional on F_{t-1}, i.e.

$$F_t(x_1,\ldots,x_d / \mathsf{F}_{t-1}) = C_t(F_{1,t}(x_1 / \mathsf{F}_{t-1}),\ldots,F_{d,t}(x_d / \mathsf{F}_{t-1}) / \mathsf{F}_{t-1}),\ x \in \Re^d$$

where $X_{it} / \mathsf{F}_{t-1} \sim F_{i,t}$, and C_t is the conditional copula of X_t given F_{t-1}. The main drawback is that the information set F_{t-1} must be the same for all marginal distributions and the copula, and this will generally imply that the function $F_t(x_1,\ldots,x_d / \mathsf{F}_{t-1})$ is not a valid conditional distribution function (see Patton, 2006a). The use of conditional copulas leads naturally to the question of whether these exhibit significant changes through time. Conditional correlations between financial assets returns are known to fluctuate through time, and so it is important to allow for time-varying conditional copulas. Variation in the conditional copula by allowing the parameter(s) of a given copula to vary through time in a manner analogous to a GARCH model for conditional variance is employed by Patton (2006a) and applied to an international stock market by Jondeau and Rockinger (2006). See also Patton (2006b) and references therein.

The second line of research is different from the previous one since it considers cross-sectional dependencies between two or more time series, and uses the copula approach for describing the dependence between observations from a given univariate time series, i.e, for analyzing the joint distribution of $[X_t, X_{t+1},..., X_{t+s}]^T$. If the copula is invariant through time and satisfies some constraint on its multivariate marginals, and the marginal distributions are identical and also invariant through time, then it describes a stationary Markov process. The main benefit of this approach is that the researcher is able to specify the unconditional (marginal) distribution of X_t separately from the time series dependence of X_t. Ibragimov (2005, 2006) present useful results to high-order Markov chains based on copulas and Beare (2007) studies weak dependence properties of Markov chains through the properties of their copulas.

An application of the concept of conditional copula due to Patton (2006a), in order to price bivariate options under GARCH processes in a dynamic way, is in Zhang and Guégan (2008). The proposed methodology is an alternative to the Goorbergh's method (see Goorbergh et al., 2005) which describes the dynamics of a copula by allowing the dependence parameter (specifically say, Kendall's tau) to evolve according to a particular regression equation where the forcing variables are the conditional volatilities of the underlying asset. Then, the dynamic Kendall's tau is computed and used to decide the parameters of several one-parameter copulas. Note that with this methodology the dynamics are only reflected in the rolling-window and time series regression and therefore, some drawbacks appear: the size of the rolling-window may influence the result for the option prices; the specified time series regression equation may fail to represent the dynamic dependence correctly; and the copula families are restricted to one-parameter ones. In Zhang and Guégan (2008), for the innovations obtained from GARCH filtering of the underlying assets, a series of the best copulas are selected for different subsamples divided by moving windows under AIC criterion (Akaike, 1974). Thus, the copula changes can be observed clearly and used for specifying the dynamics in the dependence structure proposed instead of transforming the joint distribution. An innovative feature of this paper is investigating the dynamic evolution of the copula's parameter as a time-varying function of some predetermined variables, which gives a tractable dynamic expression to the changes of the copula. Multivariate-parameter copulas are considered in the paper.

With the appearance of ultra-high frequency data that are mainly characterized by irregular time intervals between two consecutive observations, many researches have pushed further the development of the Autoregressive Conditional Duration framework in order to describe limit order book activities more accurately. Usual improvements often have strong parameterization and involve computational burdens due to the large size of the data set and the recursive procedure in the estimation. A modeling of duration clusters based on dynamic copulas is provided by Ng, W. L. (2008), not only for the *degree of dependence* of consecutive durations, but also for the *structure of (temporal) dependence* of the duration process.

2.5 Copulas and Multivariate Extreme Values Analysis

We start this section with a couple of real situations in finance and insurance borrowed from Embrechts et al. (2003), in which copulas will help in modeling extreme events.

The consideration of different lines of business for an insurance company poses the problem of seeking protection against simultaneous big losses in the business lines. These potential losses may be represented by a portfolio consisting of n risks $X_1, X_2, \dots, X_n$. One suitable reinsurance contract might be one which pays the possible excess losses $X_i - c_i$, for $i \in I \subset \{1,2,\dots,n\}$, where $I = \{1,2,...d\}$ is a prespecified set of business lines, and $X_i \geq c_i$ for all $i \in I$. In order to price this contract the reinsurer (the seller) would typically need to estimate $E(f((X_i, c_i); i \in I)$, where f is a suitable payout function. If the joint distribution H of risks $(X_1, X_2,..., X_d)$ could be accurately estimated, the expected value would not be difficult to evaluate (probably by using numerical methods). Due to the lack of reliable data it is more realistic to assume that the data available allow for estimation of the margins, $F_1, F_2,.., F_d$, and pairwise rank correlations. Thus, the probability of payout is given by

$$\bar{H}(c_1, c_2, \dots, c_d) = \bar{C}(\bar{F}_1(c_1), \bar{F}_2(c_2), \dots, \bar{F}_d(c_d)),$$

where $\overline{H}$, $\overline{C}$, and $\overline{F}_1, \overline{F}_2,.., \overline{F}_d$ denote the joint survival function, survival copula and the marginal survival functions of $X_1, X_2,..., X_d$. The notion of the bivariate survival copula is analyzed in detail in section 3.3. Note that if the thresholds are chosen to be quantiles of the risks, i.e., $c_i = F_i^{-1}(\alpha_i), 1 \le i \le d$, usually given in a reinsurance context as return periods, then the copula expression simplifies to $\overline{C}(1-\alpha_1, 1-\alpha_2, ..., 1-\alpha_d)$. Once a copula family is decided (Gaussian, Gumbel, etc), calculating the probability of payout of the expected value of the contract is direct. Note that the Kendall's tau estimates can typically be transformed into estimates of the copula parameters. Nevertheless, there is much uncertainty in choosing a copula family representing the dependence between potential losses for the d lines of business.

Let consider now the problem of measuring the risk of holding equity portfolio over a short time horizon (one day, say) without the possibility of rebalancing, i.e., we assume that the current value of the portfolio of d equities is given by

$$V_t = \sum_{i=1}^{d} \beta_i \, S_{i,t},$$

where β_i accounts the units of equity i in the portfolio and $S_{i,t}$ is the current price of equity i. Let consider the relative loss over time period $(t, t+1]$ of equity i given by

$$D_{t+1} = \sum_{i=1}^{d} \gamma_{i,t} \, \delta_{i,t+1}$$

where $\gamma_{i,t} = \beta_i S_{i,t} / V_t$ is the proportion of the current portfolio value corresponding to equity i, and $\delta_{i,t+1} = -(S_{i,t+1} - S_{i,t}) / S_{i,t}$ is the (negative) relative loss corresponding to equity i. Different distributional assumptions for $\delta := (\delta_{1,t+1}, ..., \delta_{d,t+1})^T$ will result in different effects on the aggregate risk D_{t+1}. Although the classical distributional paradigm on δ is the multivariate normality, there is a very critical problem in this context, even worst than the fact of δ has marginals which are heavier tailed than normal distributions. Extreme falls in equity prices are often joint extremes, in the sense that if a big fall in one equity price happens simultaneously, big falls in other equity prices happen too. Thus, daily equity return data often indicate that the underlying dependence structure has the property of tails dependence, a property which Gaussian copulas lack.

Multivariate extreme value theory is studied by mathematicians (see Coles, 2001, for a review of the topic) but theoretical results are often hard to use in practice. Standard multivariate extreme value theory analyses the vector of *component-wise maxima* of a sample of independent and identically distributed random vectors and its limit distribution under affine transformations. Another point of view is the theory which provides *maximum domain of attraction* conditions which justify the assignment of probabilities to multivariate rare events which, possibly, have not happened before. Suppose that the marginals come from different distributions and that, in the required normalization to obtain the limit distribution, one chooses the normalization from a dominating component, thus, the remaining components might vanish in the limit, and the limit distribution of this vector of component-wise maxima can be rather unpleasant. To avoid this problem it is common to transform the components to a standard distribution, i.e. the uniform distribution (see Resnick 1987 and 2004). Mikosch (2006) discusses advantages of other selections including Pareto, unit Fréchet distribution or Gumbel distribution, in terms that they allow for an interpretation of the spectral measure. More technically, he suggests that the distribution of the directions of multivariate extremes leads one believes that, after the transformation to these alternative standard marginal distributions, the vector has a distribution in the maximum domain of attraction of a suitable multivariate extreme value distribution.

Increasing transformations of the marginals make sense in the context of multivariate extreme value theory; it can be advantageous if one gets simpler representations and more accessible formulas for certain probalilistic quantities (quantiles, VaR). Thus, in this multivariate extremes context, copulas have recently received particular attention and the selection of the copula should be clearly related to multivariate extreme value theory.

Let review briefly extreme value copulas. An extreme value copula is derived from multivariate extreme value distribution by transforming its marginals to the unit cube $[0,1]^d$. Multivariate extreme value distributions occurs as weak limits of affinely transformed vectors of component-wise maxima of independent and identically distributed random variables. The marginal distributions are necessarily one-dimensional extreme value distributions of Fréchet, Gumbel or Weibull family. The occurrence of events on the extreme tails of a marginal distribution may be related to the behavior of the maximum (or minimum) of this distribution. A main result is given by the fundamental Fisher-Tippet theorem (Fisher and Tippet, 1928). According to this result, the (nondegenerated) limit distribution of the approximately normalized maxima of a sequence of d independent and identically distributed random variables is the standard generalized extreme value distribution. The corresponding location (μ) scale (σ) family is given by

$$H_{\xi,\sigma,\mu}(y) = \begin{cases} \exp\{-(1+\xi\frac{y-\mu}{\sigma})^{\frac{-1}{\xi}} \text{ if } \xi \neq 0 \\ \exp\{-\exp\{-\frac{y-\mu}{\sigma}\}\} \text{ if } \xi = 0 \end{cases}$$

where $1+\xi(y-\mu/\sigma)>0$. The parameter ξ is related to the shape of the distribution. If $\xi=0$, we obtain the Gumbel distribution, while if $\xi<0$ or $\xi>0$, we obtain the Weibull or the Fréchet distributions, respectively.

The most popular extreme value copula is the Gumbel copula,

$$C(x_1,x_2,...,x_d) = \exp\{-(\sum_{i=1}^{d}(-\log x_i)^{1/\alpha})^{\alpha}\},$$

for some $\alpha \in (0,1)$. Alternative parametric extreme value copulas are studied by Kotz and Nadarajah (2000). Nevertheless, this approach assumes that the underlying data are generated by some extremal mechanism, partially questionable for financial or insurance data. The fundamentals of the alternative approach based on distributions in the *maximum domain of attraction* of an extreme value distribution are analyzed by Hult and Lindskog (2005). Another approach related with modeling extreme values is the *copula domain of attraction* (see McNeil et al., 2005). The relation between this approach and the concept of maximum domain of attraction has been pointed out by Mikosch (2006).

2.6 Copulas and Latent Factor Models

As Trivedi and Zimmer (2005) pointed out, another interesting dependence structure related to copulas appears in the setting of nonlinear regression models that arise in the context of analysis of cross section survival data, event counts, jointly dependent continuous and discrete variables and so forth. Latent factors have been used to model conditional dependence, but they can also be observed as a general approach to joint modeling. This approach has an interesting similarity with the copula approach in that the joint model is built using marginal (regression) models in which either common or correlated latent factors enter the models in the same way as regressors. Their simultaneous presence in different marginals generates dependence between variables. Such latent factor models have appeared in statistics under a variety of names such as the shared frailty model in survival analysis, trivariate reduction model and latent factor models (see Janssen and Duchateau, 2007, for a recent monography on *the frailty model)*. They are all mixture models and can also be interpreted as random effects models, where the differences are placed on the way the random effects are introduced into the model and the implicit restrictions that are imposed on the dependence structure.

In the context of regression, common observable variables in regression functions usually account for some dependence. However, models may specify additional dependence through common unobserved factors, usually referred to as "frailty" in demography and "unobserved heterogeneity" in econometrics. Dependence induced by such factors can follow a variety of structures. The Maximum Simulated Likelihood (MSL) estimation of a latent factor model has a theoretical advantage that it can be generalized to higher dimensions, although at additional and nontrivial computational cost. Like the copula approach it is based on marginals, which is also a potential advantage. Zimmer and Trivedi (2006) showed that MSL produces similar results to copula models.

2.7 Simulating Copulas

According to Nelsen (2006), `one of the primary applications of copulas is in simulation and Monte Carlo studies'. Sampling from multivariate copulas has indeed become a crucial issue in applied work, for instance, in model selection or goodness of fit testing to approximate the statistics null distribution, and in financial applications for estimating the Value at Risk and other measures of tail dependence.

Due to the simplicity and great flexibility showed by the Archimedean family of copulas, the reader may refer to Wu et al. (2007), where a sampling algorithm to draw values from this family is designed, as an extension of a bivariate sampling scheme previously introduced in the literature. Wu et al. (2007) consider Archimedean generators that are continuous and whose higher derivatives exist. The theoretical result from where the sampling algorithm derives is reproduced below. See Wu et al. (2007) for more details on the sketch of the sampling algorithm itself.

Theorem 2 *Let* $\left(U_1,\ldots,U_d\right)$ *be a d-dimensional random vector with uniform marginals* *and joint distribution function defined by the Archimedean copula*

$$C(u_1,\ldots,u_d) = \varphi^{-1}(\varphi(u_1)+\cdots+\varphi(u_d))$$

for some continuous generator φ *whose higher derivatives exist. Define the* $d-1$ *transformed random variables,* $S_1,\ldots,S_{d-1}$*, as follows*

$$S_k = \frac{\sum_{i=1}^{k}\varphi(u_i)}{\sum_{i=1}^{k+1}\varphi(u_i)} \text{ for } k = 1,\ldots,d-1,$$

and the random variable T *via the copula function* C *as given below*

$$T = C(U_1,\ldots,U_d) = \varphi^{-1}(\varphi(U_1)+\cdots+\varphi(U_d)).$$

The joint distribution function of $(S_1,\ldots,S_{d-1},T)$ *is characterized by*

$$\begin{aligned} h(s_1,\ldots,s_{d-1},t) &= \frac{\partial^d \Pr(S_1 \le s_1,\ldots,S_{d-1} \le s_{d-1}, T \le t)}{\partial s_1 \cdots \partial s_{d-1}\partial t} \\ &= s_1^0 s_2^1 \cdots s_{d-1}^{d-2}\varphi^{-1(d)}(t)(\varphi(t))^{d-1}\varphi^{(1)}(t) \end{aligned}$$

where $\varphi^{-1(d)}$ *denotes the* d *-th derivative of* φ^{-1} *.*

Using the method of Jacobian transform, Wu et al. (2007) were able to characterize the joint density, $h(s_1,\ldots,s_{d-1},t)$, of the transformed random vector, $(S_1,\ldots,S_{d-1},T)$, via the closed expression shown in Theorem 2. Straightforwardly from that expression it becomes obvious the fact that $S_1,\ldots,S_{d-1}$ and T are independent random variates.

It is worthy to mention here that in their article, Wu et al. (2007) include a simulation example regarding the problem of estimating the distribution of aggregation loss and the total capital required for an insurance company with multiple lines of business. In that simulation example, their proposed sampling algorithm is used to draw 4-dimensional observations from the Gumbel-Hougaard copula and the Frank copula, two popular members of the Archimedean copula family. With this choice of the copula, and assuming that every line of business is well described by a log-normal marginal, Wu et al. (2007) approximate the $VaR_p(S) = \inf\{s \in \Re \mid F_S(s) \ge p\}$, i.e. the p th percentile of the total loss,

$S = \sum_{i=1}^{4} X_i$, where $0 < p < 1$ and F_S denotes the cumulative distribution function of S and X_i, with $i = 1,\ldots,4$, denotes the loss derived from every line of business. They also approximate the conditional tail expectation at level p, i.e. $CTE_p(S) = E\lfloor S \mid S > VaR_p(S)\rfloor$. The VaR of the aggregate loss can be interpreted as the amount for which there is a probability of $(1-p)$ of losing beyond that amount. On the other hand, the CTE can be interpreted as the average of the top $(1-p)$ losses. The VaR and the CTE are both useful tools to measure dependence in the extremes of a multivariate distribution.

Apart from this recent sampling algorithm, we cite below three other algorithms previously proposed in the literature to sample data from a more specific copula. The conditional sampling, which is appropriate for simulating from the Clayton, Frank and Farlie-Gumbel-Morgenstern copulas; the elliptical sampling, appropriate for the Gaussian copula; and the alternative algorithm proposed by Marshall and Olkin (1988) based on mixture of powers and appropriate for the Gumbel copula, for which the conditional sampling method is no longer efficient since it requires iterative solutions that make it computationally expensive.

3 Recent Hot Topics in Copulas with Applications in Finance and Insurance

In this section we present recent and open problems in copula modeling. We pay special attention to the problem of copula selection and we review recent proposals of copula families. We outline risk survivals, and aging characterizations that have been recently proposed in connection to copula modelling. After mentioning a couple of open problems, we finish this contribution with some brief personal rejoinders to some recently sceptical opinions about “the copula fashion”.

3.1 Model Selection and Goodness-of-fit Tests

In practical applications, it is very important to know how to estimate and test a copula function. On one hand, fitting a parametric class of copulas can be viewed as an effective way to reduce dimension. On the other hand, nonparametric estimation of copulas provides a robust guidance for fitting a parametric class of copulas and plays an important role in goodness-of-fit testing for copulas. Chen et al. (2006) and Molanes et al. (2008) present two approaches for nonparametric estimation of copulas based on the empirical likelihood methodology, with and without smoothing. Shen et al. (2008) propose a new class of copulas, called linear B-spline copulas. They show that empirical linear B-spline copulas perform better than empirical copulas when estimating either perfectly or non-perfectly dependent copulas. The linear B-spline copula can be seen as a semiparametric approach for copula estimation. While it is still defined in terms of a parametric form, it shares the same flexibility as that exhibited by a nonparametric approach.

In the case of fitting a parametric class of copulas, $C \in C_\theta$, where C_θ is a copula family indexed by a real-valued or vector-valued parameter, we need to find the θ that best fit our data. The problem of estimating θ under the assumption $H_0 : C \in C_\theta$ has already been the object of much work. In empirical applications of copulas, it is common to use several parametric copulas to fit the data and compare the results obtained from different models, however this is not a formal goodness-of-fit test for model selection. Other similar approaches have been introduced in the literature. Genest and Rivest (1993), for instance, propose a graphical method to select an Archimedean family of copulas and Vandenhende and Lambert (2002), in a longitudinal study, consider different parametric generators and select the one yielding the smallest Akaike information criterion for the fitted models. More recently, the issue of formally testing H_0 has started to draw attention and as Genest et al. (2007) point out, the different proposals can be classified in three different groups:

- statistics developed for testing specific dependence structures such as the Normal copula or the Clayton family, which is also referred to as the gamma frailty model in survival analysis (see Duchateau and Janssen, 2008);
- procedures developed for testing the goodness-of-fit of any class of copulas but whose use involves either an arbitrary parameter, or weight functions and associated smoothing parameters or ad hoc categorization of the data in a multiway contingency table to apply an analogue of the standard chi-squared test;
- `blanket tests' which are applicable to all copula structures and do not require any strategic choice for their implementation.

In this setting, bootstrap procedures will be required to circumvent problems in the determination of the limiting distribution of the test statistics under composite null hypothesis. Genest et al. (2007) propose new blanket test statistics, and design indeed a double parametric bootstrap procedure to approximate the statistics null distribution. The reader should be aware of the fact that the implementation of a bootstrap procedure will entail the knowledge of how to draw data from a given copula pertaining to a parametric family of copulas. The sampling methods listed in Section 2.6 may be helpful for that purpose.

In order to overcome the problem of arbitrary model selection, Vandenhende and Lambert (2006) propose a non-parametric Archimedean family of copula, defined in terms of a continuous piecewise log-linear combination of existing Archimedean generators. Vandenhende and Lambert propose as well an efficient constrained least-squares method to estimate the coefficients involved in the non-parametric Archimedean copula which leads to a data-driven model selection procedure.

3.2 New Copula Families and Copula Approximations

Sancetta and Satchell (2001, 2004) introduce a new family of d-dimensional copulas based on Bernstein polynomials. As they mention it is expected that their scope of application can reach many problems concerning the aggregation of asset returns. This new class of copulas, referred to as the Bernstein copula family, belongs to the family of polynomial copulas (see Nelsen, 2006) and can be used as an approximation to any copula. When the Bernstein copula family is used to approximate a known copula function, the authors refer to it as the Bernstein representation of the given copula. This new representation of a known parametric copula leads to a simplified and general estimation procedure and it is especially useful when the parametric copula is available but in a very complicated form. When the Bernstein copula family is used to approximate an unknown copula function, then the approximation leads to what the authors call the empirical Bernstein copula.

The formal definition of the Bernstein copula is as follows. Let $\alpha\left(\frac{v_1}{m_1},\ldots,\frac{v_d}{m_d}\right)$ be a real-valued constant indexed by $(v_1,\ldots,v_d)$ such that $0 \le v_j \le m_j \in \mathrm{N}$, $\forall j$,

$$P_{v_j,m_j}(u_j) = \binom{m_j}{v_j} u_j^{v_j}(1-u_j)^{m_j-v_j}$$

and let C_B be defined as the following mapping from the d-dimensional unit hypercube, $[0,1]^d$, to $[0,1]$

$$C_B(u_1,\ldots,u_d) = \sum_{v_1=0}^{m_1}\cdots\sum_{v_d=0}^{m_d}\alpha\left(\frac{v_1}{m_1},\ldots,\frac{v_d}{m_d}\right)P_{v_1,m_1}(u_1)\cdots P_{v_d,m_d}(u_d).$$

As Sancetta and Satchell (2004) specify, C_B turns out to be a copula function whenever specific conditions on $\alpha\left(\frac{v_1}{m_1},\ldots,\frac{v_d}{m_d}\right)$ are satisfied. Besides, from the definition given for the Bernstein copula, it follows that it is defined in terms of d-dimensional Bernstein polynomials. It is interesting to notice that some simple families of copulas are multivariate polynomials of quadratic or cubic order, and therefore their structure resemble very much the Bernstein copula. The bivariate Farlie-Gumbel-Morgensten copula, for instance, is a specific case of this fact:

$$C(u_1,u_2) = u_1u_2 + \theta u_1u_2(1-u_1)(1-u_2)$$

where $-1<\theta<1$ measures either positive or negative dependence. Some other families are identical to a Bernstein polynomial plus a known additional term.

Related to the Bernstein copula, Sancetta and Satchell (2004) present a closed-formula for computing the Spearman's rho coefficient as follows

$$\rho_S = 12\sum_{v_1=0}^{m}\sum_{v_2=0}^{m}\gamma\left(\frac{v_1}{m},\frac{v_2}{m},1,\ldots,1\right)\cdot\prod_{j=1,2}\binom{m}{v_j}B(v_j+1,m+1-v_j),$$

where

$$\gamma\left(\frac{v_1}{m_1},\ldots,\frac{v_k}{m_k}\right) = \alpha\left(\frac{v_1}{m_1},\ldots,\frac{v_k}{m_k}\right) - \frac{v_1}{m_1}\cdot\cdots\cdot\frac{v_k}{m_k}$$

and $B(a,b)$ is the beta function. Provided that enough terms are included in the Bernstein approximation, the Spearman's rho of the Bernstein copula can yield an approximation to the true Spearman's rho of any copula with any degree of accuracy and with the advantage of avoiding the evaluation of complicated integrals.

In the last years, three important correlations - comonotonicity, countermonotonicity and independence - have played significant roles in finance and insurance (see Dhaene et al. 2002a,b). The term `comonotonic' comes from `common monotonic' and is related with the extreme positive correlation between two random variables, i.e. a deterministic correlation. Countermonotonic case is the exact opposite of the comonotonic situation, i.e. X and Y are countermonotonic if X and $-Y$ are monotonic. Finally, describing dependence structures of risks, independency is a very important case. Recently, the existence and uniqueness of a bivariate copula decomposition into four parts: a comonotonic, an independent, a countermonotonic, and a indecomposable part is investigated in Yang et al. (2006). Moreover, for the indecomposable part, an optimal approximation, as a convex combination of a comonotonic, an independent and a countermonotonic part is provided and analyzed in the paper, jointly with an approximation error bound. Applications in finance and insurance of this convex decomposition are given: in the variance's decomposition to find the mean-variance optimal investment portfolio in finance; in the stop-loss premium's decomposition when ordering risk in insurance; or in finance for hedging the risk by finding a suitable asset existing in the financial market.

Durante et al. (2007) introduce a new family of bivariate copulas useful for describing aggregate processes and that extend the Archimedean family of copulas. This new family is defined in terms of two univariate functions instead of one as in the bivariate Archimedean family. Along their paper several examples of the new family are provided and results regarding the concordance order between two copulas are proved. Specifically, it is proved that the concordance order between two copulas pertaining to this new class, is determined by specific properties of their generators. From this result, it is easy to find conditions on the generators of the new copula family that lead to positively (or negatively) quadrant dependent copulas. Note that the concordance order, also known as the positive orthant dependence order is referred to as the quadrant dependence order when $d=2$ and it is equivalent to the supermodular order for $d=2$. A stochastic order of great interest in actuarial science and reliability is, for instance, the increasing convex order, and their multidimensional generalizations, which are focused on either studying stochastic comparisons between two random variables or between two random vectors, respectively.

We refer the reader to Belzunce et al. (2006) for more details on stochastic orders and some positive dependence orders that have been introduced in the literature for comparing the strength of positive dependence of two underlying multivariate distributions. Durante et al. (2007) plan to extend in a future paper, the results regarding the new family of bivariate copulas to the d-dimensional case with $d > 2$. This seems too attractive for real applications because stochastic orders of random sums given by

$$Z_i = \sum_{k=1}^{N_i} X_{ik}, \text{ for } i = 1, \ldots, d,$$

where $N = (N_1, \ldots, N_d)$ is a vector of integer-valued random variables and $\{X_{1k}\}_{k=1}^{N_1}, \ldots, \{X_{dk}\}_{k=1}^{N_d}$ are sequences of positive random variables, have recently received special attention in several fields, such as reliability, economics, insurance and queuing theory. For example, in actuarial sciences random sums of this type model the total claim amount paid by an insurer for the N_i claims associated to the i-th class of business. On the other hand, dependencies among the claims X_{ik} by means of geographical or economic environment are typical in actuarial problems, for example, when we are interested in representing the effects of catastrophes, such as epidemics, hurricanes or earthquakes, that hit several risks simultaneously. Catastrophic events can cause indeed insurance and reinsurance losses of increasing frequency and severity.

Yu and Voit (2006) extend, via the copula approach, univariate S-distributions to bivariate S-distributions whose marginals are S-distributions. As Yu and Voit show, they cover a variety of marginals and a wide range of dependences between the variates and facilitate the formulation of relationships between measures of dependence and model parameters. Yu and Voit show that the bivariate S-distributions retain the great flexibility in shape and the relative computational simplicity of their univariate counterparts. The resulting distributions cover the full range of dependence, from -1 to 1, and different degrees of tail dependence (stronger dependence in the lower tail, upper tail or both). This is interesting because although the popular Archimedean family presents many advantages, it also bears some limitations, for instance, some one-parameter Archimedean families permit only a limited degree of dependence; other are asymmetric in the sense that they lead to different tails; or they are monotonic and cannot accommodate negative dependence. Bivariate S-distributions overcome these limitations.

3.3 Modeling Competing Risks Survival Times via Copulas

Dependence is an important characteristic in survival analysis and reliability due to its significantly effect on the decision people make with regard to maintenance, repair or replacement, warranties, price setting, and medical studies, among others. Below we refer to three real data applications that can be seen as three primer candidates for survival-copula modeling.

In pricing life insurance products, it is important to study the survival of the insured party. When modeling times to event data, it is common that we can distinguish more than a single type of event causing individual failure. The competing risks model, also known as the multiple-decrement model is useful for modeling the joint distribution of this type of survival data. The concept of competing risks comes from the interpretation that an individual faces different risks of leaving the state in which is now, each risk giving rise to its own future destination. In other words, we could say that every individual is exposed along his/her life to different competing causes of death (or to different causes of withdrawal from the study). Since most real life applications are truly multivariate, there are more than two dependent competing causes of decrement or failure. Kaishev et al. (2007) develop a copula-based methodology for dealing with scenarios of this type and explore its applicability with a real data set where four competing risks are acting simultaneously on the individual survival: heart diseases, cancer, respiratory diseases, and other causes of death. They explore the effect of simultaneously removing one, two or three of these causes of death on the overall survival, on the life expectancy at birth, at 65 years old, and on the value of a life annuity, which are all important issues in pricing life insurance products.

When modeling survival applications, it is more intuitive to talk in terms of the joint survival function,

$$\bar{F}(t_1, \ldots, t_d) = P(T_1 > t_1, \ldots, T_d > t_d),$$

rather than in terms of its equivalent, the joint distribution function,

$$F(t_1,\dots,t_d) = P(T_1 \le t_1,\dots,T_d \le t_d).$$

This seems to be the right place to briefly introduce the bivariate survival copula (see Nelsen 2006). The dependence structure of the bivariate lifetime variable (X_1, X_2) can be usefully described by its survival copula defined by

$$\bar{C}(u,v) = \bar{F}(\bar{F}_1^{-1}(u),\bar{F}_2^{-1}(v)),(u,v) \in [0,1]^2,$$

where $\overline{F}$, $\overline{F}_1$ and $\overline{F}_2$ are the corresponding joint survival function and the univariate marginal survival functions, respectively. Given a bivariate copula function, $C(u_1,u_2)$, its survival copula is defined in terms of $C(u_1,u_2)$ as follows

$$\overline{C}(u_1,u_2) = u_1 + u_2 - 1 + C(1-u_1, 1-u_2)$$

And satisfies the property of linking the univariate marginal survival functions, $\overline{F}_i(t_i) = 1 - F_i(t_i)$, $i = 1,2$, and the joint survival function, $\overline{F}$, as given below,

$$\bar{F}(t_1,t_2) = \bar{C}(\bar{F}_1(t_1),\bar{F}_2(t_2)).$$

In actuarial science, it has been noted that the decrements or causes of withdrawal tend to be dependent. Consequently, it is normal to consider that $T_1,\dots,T_d$, are stochastically dependent survival times and non-defective, i.e. satisfying $P(T_i < \infty) = 1$. Here, we denote by T_i, the potential or latent lifetime of an individual until the i-cause of death bursts in. In fact, only one cause of death will finally and actually burst in on every individual. Therefore the observable lifetime, T, of that individual will be given by the minimum of the d potential lifetimes, i.e. $T = \min\{T_1,\dots,T_d\}$. The overall survival function is defined in this setting as $\overline{\boldsymbol{F}}(t) = P(T > t)$, with $t \ge 0$, i.e. as the probability that the observed survival time, $T = \min\{T_1,\dots,T_d\}$, exceeds the time t. If the objective of interest is to know how the lifetime of an individual is affected with the removal of one of the d potential causes of death, say the j cause, we need to model the marginal overall survival given below

$$\overline{\boldsymbol{F}}^{(-j)}(t) = P(\min\{T_1,\dots,T_{j-1},T_{j+1},\dots,T_d\} > t) \text{ for } t \ge 0.$$

Through the use of a suitable survival copula, Kaishev et al. (2007) find a representation of this marginal survival.

Kaishev et al. (2007) introduce other two interesting concepts, the crude survival functions and the net survival functions. The crude survival function $\bar{F}^{(j)}(t_j)$ is defined as the survival function with respect to the j-th cause of death, when death actually occurs by that cause. Formally, we write

$$\bar{F}^{(j)}(t_j) = P(\min\{T_1,\dots,T_d\} > t, \min\{T_1,\dots,T_d\} = T_j).$$

The crude survival function can be estimated from the observed mortality of a population because it reflects the actual or observed individual mortality. This is not the case with the (marginal) overall survivals. For $j = 1,\dots,d$, the net survival functions are defined as $\bar{F}_j(t_j) = P(T_j > t_j)$, i.e. as the marginal survival functions due to only the each cause of death alone. The survival copula-approach considered by Kaishev et al. (2007) is focused on first estimating these marginals, secondly coping with the joint survival function and finally evaluating the overall survival function of interest.

On the other hand, actuaries have noted the existence of a broken heart syndrome in which an individual's death substantially increases the probability that the person's spouse will also experience death within a fixed period of time. The possibility to model accurately the joint survivals of husband/wife pairs is of great important for actuaries who are interested in annuity pricing models in which the relationship between two individuals' death is jointly related (Clayton, 1978). Since joint survivals of husband/wife pairs tend to exhibit nonlinear behavior with strong tail dependence, this is another real application where survival copula modeling can properly deal with.

In the literature one can find several characterizations of *aging notions* by means of stochastic comparisons between the residual lifetimes X_t given by

$$X_t = [X - t / X > t]$$

where X is a random variable representing a lifetime and consequently X_t can be considered as the survival lifetime at time t, given that the individual/device is alive at time t. A description of a pair of exchangeable lifetimes whose dependence structure is described by an Archimedian copula is in Pellerey (2008). This work is mainly related among others to recent papers by Charpentier (2006) and Oakes (2005).

3.4 Some Open Problems

There are very few results in the literature relating order statistics and associated copulas (see Nelsen 2006 and references therein). In risk analysis, life modeling or reliability, given the multivariate continuous random variable $(X_1,...,X_n)$ one could be interested in the range $X_{n\,:\,n}-X_{1\,:\,n}$ or subranges $X_{r_1\,:\,n}-X_{r_2\,:\,n}$ for $r_1>r_2$ where $X_{r\,:\,n}$ is the rth order statistic $(1\le r\le n)$. However, in order to derive explicit formulas, we need the joint distribution of $X_{r_1\,:\,n}$ and $X_{r_2\,:\,n}$. Nelsen found the copula $C_{1\,:\,n}$ of $X_{1\,:\,n}$ and $X_{n\,:\,n}$

$$C_{1\,:\,n}(u,v)=v-[\max\{(1-u)^{1/n}+v^{1/n}-1,0\}]^n.$$

The general case is still an open problem. Nevertheless one solution is to use Monte Carlo methods. Anjos et al. (2005) provide a copula representation of the joint distribution function of the rth and the sth order statistics corresponding to two random multivariate variables X and Y, given the associated copula C. The asymptotic copula is obtained and the corresponding approximation is used to evaluate the joint distribution function of order statistics $(X_{r\,:\,n},Y_{s\,:\,n})$.

A review of recent developments in copula theory, namely: order statistics copula, copulas with given multivariate marginals, copula representation via a local dependence measure and applications of extreme value copulas is in Kolev et al. (2006).

Another recent line of research relating copulas and financial applications deals with volatility and dependence parameters. An axiomatization of premium functionals using average absolute deviation or, more general, distorted probabilities can be found in Wang et al. (1997). In Dennenberg and Leufer (2008) the second order dual volatility parameters, average absolute deviation and Gini index are surveyed and new rank based dependence parameters are proposed. Dual is used in this context that not the outcomes of the random variables are transformed, but their distributions. The authors argue that the dual dependence parameters are better suited than the classical parameters for applications in finance and insurance. More technically, the authors extend the notion of copula by perceiving distribution functions as interval valued functions if they are not continuous, or by attributing the midpoint of the interval. They use methods of non-additive measure and integration. Because this dual view to distorted probabilities leads to Choquet integral, then, comonotonicity of random variables plays a central role in both theories, for the copula and for the Choquet integral. For the copula increasing distribution functions dominate, whereas for the Choquet integral decreasing distribution functions are the natural tool, and the authors employ both in the paper.

3.5 The Copula Fashion-The Emperor's New Clothes?

All the world's problems related to stochastic dependence and multivariate distributions can be solved via copulas? Are there statistical problems in handling copulas, one of them being the curse of dimensionality?

Although there are some sceptical people that have questioned the potential of the copula approach (see Mikosch, 2006), we still expect that over the years to come the copula approach can convince the most sceptical ones. In fact, in the last two years the copula approach has started to fill out some of the gaps pointed out by Mikosch (2006). For instance, one of the main statements given by this author against copulas was that they do not fit into the existing framework of stochastic processes and time series analysis, essentially because they are static models and therefore not useful for modeling dependence over time. However, as we have mentioned in Section 2.2, Chen and Fan (2006) introduce a new class of semiparametric copula-based multivariate dynamic (SCOMDY) models.

Kallsen and Tankov (2006) define Lévy copulas and construct parametric families of them to characterize the dependence among components of multidimensional Lévy processes. In this new setting, they prove an analog of Sklar's theorem that basically states that the law of a general multivariate Lévy process can be obtained by combining arbitrary univariate Lévy processes with an arbitrary Lévy copula. This is another example of how the scope of the copula approach is recently reaching unexplored areas such as modeling dependence in the dynamic context of stochastic processes. Jointly with the paper by Chen and Fan (2006), mentioned above, it seems that more and more efforts start to be made to partially fill out one of the gaps mentioned by Mikosch (2006).

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Table 1: Closed-expressions for the population values of Kendall's tau, Spearman's rho and lower and upper tail dependence indexes for different copula families

Copula function	Kendall's τ	Spearman's ρ	λ_L	λ_U
Bivariate copula	$4E[C(u_1,u_2)]-1$	$12\iint_{I^2} C(u_1,u_2)\mathrm{d}u_1\mathrm{d}u_2-3$	$\lim_{t\longrightarrow 0^+}(C(t,t))/t$	$2-\lim_{t\longrightarrow 1^-}(1-C(t,t))/(1-t)$
Archimedean $(\varphi)^a$	$1+4\int_I \frac{\varphi(t)}{\varphi^{(1)}(t)}\mathrm{d}t$		$\lim_{x\to\infty}\frac{\varphi^{-1}(2x)}{\varphi^{-1}(x)}$	$2-\lim_{x\to 0}\frac{1-\varphi^{-1}(2x)}{1-\varphi^{-1}(x)}$
Clayton	$\frac{\theta}{\theta+2}$		$2^{-\frac{1}{\theta}}$, with $\theta>0$	0
Gumbel-Hougaard	$\frac{\theta-1}{\theta}$	no-closed form	0	$2-2^{\frac{1}{\theta}}$, with $\theta>1$
Frank^b	$1-\frac{4}{\theta}[1-D_1(\theta)]$	$1-\frac{12}{\theta}[D_1(\theta)-D_2(\theta)]$	0	0
Ali-Mikhail-Haq^c	$\frac{3\theta-2}{3\theta}-\frac{2(1-\theta)^2}{3\theta^2}\ln(1-\theta)$	$\frac{12(1+\theta)}{\theta^2}\mathrm{dilog}(1-\theta)$ $-\frac{24(1-\theta)}{\theta^2}\ln(1-\theta)-\frac{3(\theta+12)}{\theta}$	?	?
Gaussian copula $(r)^d$	$\frac{2}{\pi}\arcsin(r)$	$\frac{6}{\pi}\arcsin(\frac{r}{2})$	0	0
t-copula $(r)^d$ $(t_\nu)^e$	$\frac{2}{\pi}\arcsin(r)$	$\frac{6}{\pi}\arcsin(\frac{r}{2})$	$2t_{\nu+1}\left(-\frac{\sqrt{(\nu+1)(1-r)}}{\sqrt{1+r}}\right)$, $r>-1$	$\lambda_U=\lambda_L$
Marshall-Olkin	$\frac{\alpha\beta}{\alpha+\beta-\alpha\beta}$	$\frac{3\alpha\beta}{2\alpha+2\beta-\alpha\beta}$	0	$\min\{\alpha,\beta\}$
Farlie-Gumbel-Morgensten	$\frac{2\theta}{9}$	$\theta/3$		
Fréchet	$\frac{(\alpha-\beta)(\alpha+\beta+2)}{3}$	$\alpha-\beta$		
S-copula: $[1+((u_1^{-\theta}-1)^\lambda+(u_2^{-\theta}-1)^\lambda)^{\frac{1}{\lambda}}]^{-\frac{1}{\theta}}$	$1-\frac{2}{(\theta+2)\lambda}$			
Copula product	0	0		
Mixture of 2 copulas			$\lambda\lambda_L^1+(1-\lambda)\lambda_L^2$	$\lambda\lambda_U^1+(1-\lambda)\lambda_U^2$

[a] φ is the generator

[b] $D_k(x)=\frac{k}{x^k}\int_0^x\frac{t^k}{e^t-1}\mathrm{d}t$ for any positive integer k, is the kth Debye function

[c] $\mathrm{dilog}(x)=\int_1^x\frac{\ln(t)}{1-t}\mathrm{d}t$

[d] r is the correlation coefficient

[e] t_ν is the cumulative distribution function of a univariate t distribution with ν degrees of freedom