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A FAIR PROCEDURE IN A MARRIAGE MARKET

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Abstract

We propose a new algorithm in the two-sided marriage market wherein both sides of the market propose in each round. The algorithm always yields a stable matching. Moreover, the outcome is often a non-extremal matching, and in fact, is a Rawlsian stable matching if the matching market is "balanced." Lastly, the algorithm can be computed in polynomial time and, hence, from a practical standpoint, can be used in markets in which fairness considerations are important.

Keywords: two-sided matching, fair procedure, deferred acceptance algorithm

JEL Classification: C72, C78, D41

1. Introduction

In a classical two-sided marriage market, the celebrated Deferred Acceptance Algorithm (DAA henceforth) due to Gale and Shapley (1962) proved that there always exists a stable matching—a matching for which no man-woman pair would leave their current match to match with each other. DAA is a simple iterative algorithm in which one side of the market makes proposals to the other side. However, this asymmetry in proposals, whereby one side proposes while the other can only passively accept or reject, translates into an asymmetry of outcomes—the resulting stable matching is the most favorable outcome for the proposing side. As Knuth (1997) suggests:

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> The different algorithms considered until now favor the men, and if we interchange the roles of men and women they would become favorable to the women. Such injustice is too shocking for the present day. Can we therefore find a solution that treats both sexes fairly?

Notwithstanding the unfairness embedded in the DAA, one of its appeal is its simplicity. This raises a natural question: can we devise a simple iterative algorithm in which both sides of the market make proposals, and the resulting outcome is stable but, hopefully, non-extremal?

To this end, we propose an algorithm in which both sides propose in each round, and one that always generates a stable matching. There are three reasons why we view our algorithm as an attractive alternative. First, the algorithm is "procedurally fair"—it treats both sides of the market equally. Second, it often produces non-extremal outcomes and, in fact, a Rawlsian stable matching in "balanced markets."[1,2] Third, the algorithm is computationally efficient— i.e., can be run in polynomial time.

Our algorithm can be described as follows. In any Round k, we start with an existing match. Agents propose to either their top k agents if they are unmatched or to the set of agents weakly better than their current match. Based on these proposals, for each agent, we compute the set of mutually proposing agents—the set to whom (s)he proposes and from which (s)he also receives a proposal. Using these, we construct a graph with nodes as agents and edges given by each agent, pointing to his/her top mutually proposing agent. This graph produces cycles. We form a tentative match by breaking these cycles in favor of men or women randomly, and we remove those set of agents to redraw the graph. Once the graph is empty, we check whether the tentative match is stable with respect to the truncated (top k agents) preferences. If yes, we move to the next round; and otherwise, we construct a truncated stable matching before proceeding to the next round. We stop the algorithm if, at any point, the tentative matching is stable.

1.1. Related Literature

We are, obviously, not the first ones to address the asymmetry in DAA wherein it picks the extreme points of the set of stable matchings depending on who the proposing side is. There are, primarily, two approaches towards this problem. First is the idea of procedural fairness proposed by Klaus and Klijn (2006). They discuss

[1]A market is balanced if, for every unstable matching, we can find at least one blocking pair such that both the agents of this blocking pair are matched to someone in any stable matching. While this requirement may be harder to verify, a much easier condition to verify is the following. If all the agents are matched in any stable matching, then the market is balanced.

[2]We want to emphasize that in defining the Rawlsian criterion, we focus only on the set of matched agents. We argue that this is a more reasonable definition of the Rawlsian criterion since the set of matched agents is the same across every stable matching (Roth and Sotomayor (1990)).

how well-known randomized procedures such as Employment by Lotto proposed by Aldershof et al. (1999) or Random Order Mechanism by Ma (1996), that is based on Roth and Vate (1990), achieve ex-ante fairness. While our algorithm is also procedurally fair in the sense of Klaus and Klijn (2006), a key distinction between the earlier procedures and the one we propose is the nature of the algorithm itself; that is, due to both sides proposing in every round, the outcomes tend to be at the center of the stable matchings' lattice rather than at the extremes (Example 4). This is also bolstered by the fact that our algorithm picks one of the Rawlsian stable matchings in balanced markets unlike these algorithms.

The second approach regarding fairness is based on the outcomes. For example, Romero-Medina (2005) define an *Equitable Set* to capture fairness.[3] Also, Masarani and Gokturk (1989) show the impossibilities of obtaining a fair stable matching, based on a Rawlsian notion of fairness. This would seem to contradict our algorithm. However, in their paper, the Rawlsian criterion is defined over the set of all the matchings, not just stable matchings. More importantly, while the outcome we obtain is Rawlsian in the case of balanced matching markets, that is not our primary objective. Our goal is to suitably modify the most appealing feature of the DAA —simplicity due to its iterative nature— while allowing both sides to make proposals in each round to tackle the asymmetry.

In a contemporaneous paper, Dworczak (2016) proposes an iterative algorithm, Deferred Acceptance with Compensation Chains (DACC), in which both men and women make proposals. While similar in motivation, the key difference from our algorithm is that his procedure makes the agents propose in a pre-determined order (as opposed to simultaneously in our algorithm). Moreover, by varying the initial order, DACC can obtain all the stable matchings, while, in general, we obtain a strict subset of stable matchings.

Ma (1996) proposes a procedure wherein we start with a *Random Priority* over agents. Following the priority, we start with an empty match and add agents one by one to the matching, satisfying all the blocking pairs *within the match*. That is, blocking pairs outside the agents not added so far are ignored. While this procedure does not distinguish between sexes, it can result in situations in which it can select only the extremes but not the middle outcomes. In contrast, the algorithm we propose has narrower support and, often, will pick a strict subset of stable matchings that do not involve any of the extremal matchings (See Section 4.1).

Using another approach, Teo and Sethuraman (1998) and Sethuraman et al. (2006) establish the existence of '*the median stable matching(s)*' which is appealing

[3]The equitable set also focuses on the Rawlsian criterion, but one that is defined over the set of all the agents, unlike ours. This seemingly minor distinction is important in markets in which merely one single agent can render all the matchings equally "unfair" according to his criterion. Moreover, his proposed procedure is closer to the Random Priority mechanism due to Ma (1996). Lastly, it is unclear where this algorithm can be executed in polynomial time. See Romero-Medina (2001) as well.

from a fairness perspective. However, the main issue with median stable matching is that, if computing it can be done efficiently, then computing the number of stable matchings can also be done efficiently; and the latter is known to be a computationally hard problem (Cheng (2010)).

Cheng et al. (2011) propose a concept of *Center Stable Matchings* that are stable matchings whose maximum distance to any stable matching is as small as possible. These matchings are close to median stable matchings and can also be computed efficiently. However, primarily median stable matchings or center stable matchings are fair due to the properties of the stable matchings themselves. Our approach to fairness through the proposed algorithm focuses on the procedural fairness of the mechanism itself. In that sense, our algorithm is more similar to the mechanism due to Ma (1996).

For the problem of school choice, when schools carry out their matching independently, Manjunath and Turhan (2016) propose a new mechanism to avoid wasted seats: a *Matching and Rematching* mechanism that yields a solution different from the two extremes. The outcome of their procedure varies depending on the number of iterations and is not gender-neutral.

2. Model

Let M and W denote two finite sets of men and women. Let $A := M \cup W$ stand for the set of all agents. A generic agent in $M(W)$ will be denoted by $m(w)$. When the distinction is immaterial, we will denote an agent by i. Let $O(i) = W \cup \{i\}$ if $i \in M$ and $O(i) = M \cup \{i\}$ if $i \in W$; denote the *opposite side* for agent i.

Each agent $i \in A$ is endowed with a preference relation, $\succeq_i$ over $O(i)$. $\succeq_i$ is a binary relation that is reflexive, transitive and antisymmetric (strict).[4] Moreover, since $\succeq_i$ is antisymmetric, $j \succeq_i k \implies \neg k \succeq_i j$ if $j \neq k$, where $\neg$ denotes negation. $\succ$ denotes the strict part of $\succeq$. Finally, $\succeq := (\succeq_i)_{i \in A}$ is called a preference profile. An *instance of a marriage market* is a tuple $(M, W, (\succ_i)_{i \in A})$.

A bijection $\mu : A \to A$ is called a matching if $\mu(i) \in O(i)$ for all $i \in A$, and $\mu(i) = j \implies \mu(j) = i$. The set of all possible matchings is denoted by $\mathcal{M}$. Also, $\mu(i) = i$ means that i is single or unmatched. Say that a matching $\mu \in \mathcal{M}$ is *individual rational* if $\mu(i) \neq i \implies \neg(i \succ_i \mu(i))$ for all i. We define the following notions of stability that are adapted from their analogues in the environment with complete preferences.

Definition: *A matching μ is stable if it is individually rational and there is no $(m, w) \in M \times W$, such that $w \succ_m \mu(m)$ and $m \succ_w \mu(w)$. If there exists such a (m, w), then we say that (m, w) is a "blocking pair" for μ.*

[4] We assume away indifferences purely for convenience, and no results are affected by allowing the agents to have weak preferences.

As is well known, Gale and Shapley (1962) show that a stable matching exists using the DAA. In fact, the set of stable matchings is a complete lattice.[5] In particular, this means that a men- (women)-proposing DAA produces a stable matching most preferred by men (women). Roughly, in a men-proposing DAA, men propose in each round, and women can choose only from the men who propose to them. This passive role played by women, where they cannot initiate proposals themselves, leads to an extreme outcome—men-proposing DAA selects the stable matching most preferred by men. However, given the appeal of the DAA—a simple iterative procedure that produces a stable matching—the natural question that motivates us the following: Can we have an iterative algorithm, along the lines of the DAA, in which both sides propose in each round and we obtain something non-extremal?[6] With this motivation, we provide an algorithm in which both sides propose in each round and the result is a stable matching—a "Rawlsian" one, in fact (if the market is balanced).

A few simple definitions will be useful in presenting the algorithm. First, let us define the rank of agent $j \neq i$ for agent i to mean the position that j occupies in i's preference ordering. Formally, $\rho_i(j) := |\{k : k \succeq_i j\}|$.

Definition: *$k-$truncated preferences of $\succ$, denoted by $\succ^k$, are preferences with the following two properties:*

1. *For each $i \in A$ and $j, l \neq i$ such that $\max\{\rho_i(j), \rho_i(l)\} \leq k$, $j \succeq_i l \Leftrightarrow j \succeq'_i l$.*
2. *For each $i \in A$ and $j \neq i$ such that $\rho_i(j) > k$, $i \succ'_i j$.*

For a matching μ, we say that (m, w) is a "$\succ^k$ blocking pair" if $w \succ^k_m \mu(m)$ and $m \succ^k_w \mu(w)$.

Simply speaking, $\succ^k$ is a truncation of $\succ$ that looks only at the top k agents for each $i \in A$.

Now, we define a $k-$stable matching and a $k-$stable submatching.

Definition 1: *Say that an individually rational matching $\mu \in \mathcal{M}$ is "$k-$stable" if it is a stable matching with respect to $\succ^k$. Also, say that a matching $\mu \in \mathcal{M}$ is "$k-$stable with active agents in $V \subset A$", if, for every $\succ^k$ blocking pair (m, w) of μ, $\{m, w\} \not\subset V$*

Definition 2: *Given an individually rational matching μ and a number $k \in \mathbb{N}$, such that $\mu(i) \neq i \implies \rho_i(\mu(i)) \leq k$, μ' is called "a submatching" of μ if, $\mu'(i) \neq i \implies \mu'(i) = \mu(i)$. μ' is called $k-$stable submatching of μ with active agents in $V \subset A$, if μ' is a submatching of μ that is $k-$stable with active agents in V.*

Since the algorithm will produce a Rawlsian stable matching for balanced mar-

[5] To be precise, it is a complete lattice under two natural orders: $\succ_M$ and $\succ_W$. We say that $\mu \succ_M \mu'$ iff $\mu \neq \mu'$ and $\mu(m) \succeq_m \mu'(m)$ for all $m \in M$. A similar definition applies for $\succ_W$. See Roth and Sotomayor (1990) for more details.

[6] As mentioned in Section 1.1, a number of papers address a similar question.

kets, we define it formally below. For any stable matching μ, define,

$$f(\mu) := \max\{\rho(\mu(i)) : i \in A \text{ such that } \mu(i) \neq i\}$$

to be the rank of the agent who has a partner with the highest rank (higher rank means worse). Notice that we focus only on agents who are matched in defining the Rawlsian score. Since the set of single agents is identical across all stable matchings (Theorem 2.2, Roth and Sotomayor (1990)), to consider the well-being of the single agents in comparing two stable matchings is unappealing. A Rawlsian stable matching μ is one, such that $f(\mu) \leq f(\mu')$ for all stable matchings μ'.

Definition: *For an instance of a marriage market, let S be the set of agents who are matched in any stable matching. We say that an instance of a matching market is balanced if, for every unstable matching, there is at least one blocking pair, such that both the agents of the blocking pair belong to S.*

This is admittedly a hard condition to verify. But, one simple (but perhaps too demanding) sufficient condition of a balanced market is that no agent is unmatched in any stable matching.

3. The Algorithm

3.1. An example

Example 1: *Below is a simple example of a market in which one could expect to obtain an outcome different than the DAA outcome should the two sides be allowed to make proposals in each round.*

Table 1: Preferences for Example 1

m_1	m_2	m_3	m_4	w_1	w_2	w_3	w_4
w_1	w_2	w_3	w_4	m_2	m_3	m_4	m_1
w_2	w_3	w_4	w_1	m_3	m_4	m_1	m_2
w_3	w_4	w_1	w_2	m_4	m_1	m_2	m_3
w_4	w_1	w_2	w_3	m_1	m_2	m_3	m_4

■

The DAA outcomes are:

$$\{(m_1, w_1), (m_2, w_2), (m_3, w_3), (m_4, w_4)\} \text{ and } \{(m_1, w_4), (m_2, w_1), (m_3, w_2), (m_4, w_3)\}.$$

Imagine a procedure whereby the agents from both sides of the market make proposals to a set of agents in every round. If there is no agent to whom they are proposing to and who is also proposing to them, then they expand their *proposing* *to* set of agents in the following round. They start by proposing to the top agent according to their preferences and then expand the set of agents they propose to progressively.

In the first round, all the agents propose to their top choice, and no pair of agents is "mutually proposing." For example, m_1 proposes to w_1, while w_1 proposes to w_2, and so on. In the following round, for example, m_1 proposes to w_1 and w_2. However, w_1 proposes to m_2 and m_3, while w_2 proposes to m_3 and m_4. Therefore, no woman is proposing to m_1 or receiving a proposal from m_1. It is easy to see that there are no two agents who propose to each other in this round, either.

However, in round 3, consider agent m_1. He proposes to w_1,w_2 and w_3. Observe that w_2 proposes to m_3, m_4 and m_1, while w_3 proposes to m_4, m_1 and m_2. That is, m_1 is interested in being matched two agents who are also interested in being matched with m_1. Those two agents are w_2 and w_3. m_1 would prefer being matched to w_2 over w_3. But observe that w_2 also has two mutually proposing agents (m_1 and m_4), and she prefers m_4 over m_1. Continuing this way, if we look at the top agent for each person from the list of their mutually acceptable agents in this round, we obtain a cycle as below.

$$m_1 \to w_2 \to m_4 \to w_1 \to m_3 \to w_4 \to m_2 \to w_3 \to m_1$$

If we break this cycle in favor of either men or women, we obtain the following two matches respectively:

$$\{(m_1, w_2), (m_2, w_3), (m_3, w_4), (m_4, w_1)\} \text{ or } \{(m_1, w_3), (m_2, w_4), (m_3, w_1), (m_4, w_2)\}$$

The algorithm we propose makes this simple idea from the above example formal and solves a complicated cycling problem that can often arise in markets with arbitrary preferences. However, at the core, the motivation and the essence are embodied in this simple example.

3.2. The Algorithm

We now present our algorithm. Let $N := \max\{|M|, |W|\}$.

Algorithm 1: Main

```
   input  : An instance of a marriage market: M, W, ≻.
   output: A stable matching.
1  initialization: Set μ(i) = i for all i, k = 1 and V = A.;
2  while k ≤ N do
3      Set V := A;
4      repeat
5          (μ', V') = ℍ(μ, V, k);
6          μ = μ', V = V';
7      until V ≠ ∅;
8      if μ' is k−stable, then
9          μ = μ', V = A, k = k + 1 ;
10     else
11         Let μ' be some k−stable submatching of μ with active agents in
             V := {i : μ'(i) ≠ i};
12         μ = RO(μ', M, W, ≻^k);
13     end
14     if μ is stable, then
15         Terminate with output μ
16     end
17 end
```

We call steps 3-16 above a "Round k."

Step 12 above is the "Random Order Mechanism" (RO mechanism henceforth) due to Ma (1996) that we use to generate a $k-$stable matching in case the map $\mathbb{H}$ (specified below) fails to produce a $k-$stable matching. We will now describe the map $\mathbb{H}$ and then, for the sake of completeness, will describe the RO mechanism.

Map $\mathbb{H}(\mu, V, k)$

We now describe the map $\mathbb{H}$. Let us first define a few objects to this end. Given an arbitrary matching μ, a subset of agents $V \subset A$, and a number $k \in \mathbb{N}$, define, for each $i \in V$,

$$S_i := \{j \in V : \rho_i(j) \leq k, j \succeq_i \mu(i)\}$$
$$T_i := \{j \in V : j \in S_i, i \in S_j\}.$$

For each $i \in V$ such that $T_i \neq \emptyset$, define,

$$\alpha_i := \{j : j \succ_i k, \forall k \in T_i, k \neq j\}.$$

We will make use of some simple graphs in $\mathbb{H}$. Consider a directed bipartite graph $G = (V, E)$ with the following properties:

(i) $V \subset A$ is the set of nodes, and E is the set of directed edges, such that if $(i, j) \in E$, then exactly one of (i, j) is a man and the other is a woman.

(ii) The out-degree (number of outgoing edges) of any node is, at most, 1, and, any node that has an incoming edge also has an outgoing edge.

It is obvious that such a graph will exhibit disjoint cycles if E is nonempty.[7]

Definition: *Consider a cycle $C = \{m_1 \to w_1 \to m_2 \dots w_k \to m_1\}$ and a matching μ. We refer to the following operation as: "resolve a cycle in favor of men given a matching μ."*

Input: μ, C.

Output: μ'.

1. *Set $\mu' = \mu$.*
2. *First, for each $i \in C$, set $\mu'(i) = i$ and $\mu'(\mu(i)) = \mu(i)$.*
3. *Then, for each $(m, w) \in C$ such that $m \to w$, set $\mu'(m) = w, \mu'(w) = m$. In words, if there is an edge from m to w, we match m and w.*

Similarly, if, for every edge (w, m) in C, if we match m and w, we refer to that as "resolving a cycle in favor of women given a matching μ."

Algorithm 2: Map $\mathbb{H}$

```
input  : μ, V, k
output: μ', V'
initialization: Let μ' = μ. Draw a graph G = (V, E) where (i, j) ∈ E iff
  j = α_i.
if E = ∅, then
    V' = ∅;
else
    for Each cycle C in G, given a matching μ' do
        Resolve C in favor of either men or women with probability 1/2
          each.;
        Let the output matching be μ' ;
        V' = V\C;
    end
end
```

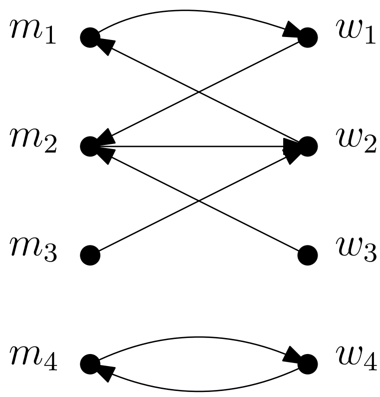
Example 2 (Example of map $\mathbb{H}$)**:** *As a simple example, suppose that, for some k and an empty matching μ (all agents are single), we obtain the graph in Figure*

[7] To see why there will be cycles if E is nonempty, start with some node, say m_1, with an outgoing edge to w_1. Since w_1 must have exactly one outgoing edge, if it is m_1, we have found a cycle. Otherwise, let it be m_2. Since the number of agents is finite, we must eventually form a cycle. That the cycles will be disjoint is immediate because any node has an outdegree of, at most, 1.

1 in running $\mathbb{H}(\mu, A, k)$*.* ■

There are two cycles here: $(m_1 \to w_1 \to m_2 \to w_2 \to m_1)$ and $(m_4 \to w_4 \to w_4)$. We first unmatch everyone involved in any cycle. Thereafter, we resolve each cycle randomly in favor of either the men or the women. For the cycle involving m_4 and w_4, regardless of how we resolve it, we obtain the match $\mu'(m_4) = w_4$ and $\mu'(w_4) = m_4$. For the first cycle, say we resolve it in favor of the women. Then, we obtain, $\{(m_1, w_2), (m_2, w_1)\}$ as the two new matches in μ'. Finally, since m_3 and w_3 are not involved in any cycle, we set $V' = \{m_3, w_3\}$ and the algorithm moves to Step 5 with $V = V'$.

Figure 1: Example of an instance of $\mathbb{H}$



Now, we will describe Step 12, the RO mechanism due to Ma (1996).

Algorithm 3: The RO mechanism

input : $\mu', M, W, \succ^k$
output: μ
initialization: Set $V = \{i : \mu'(i) \neq i\}$, $t = 1, R = A \backslash V$;
repeat
  Pick an agent $i \in R$ randomly and set $V := V \cup \{i\}$;
  repeat
    $V' := V$;
    repeat
      $(\mu'', V') = \mathbb{H}(\mu', V', k)$;
      $\mu' := \mu''$;
    until $V' \neq \emptyset$;
  until μ' *is* $k-$*stable with active agents in* V;
until $R \neq \emptyset$;
$\mu := \mu'$

The original version of the RO mechanism starts with an empty matching, and agents are "made active" in random order. From among the active agents, we form a stable matching. After doing so, we make the next agent active. In the above algorithm, rather than starting with an empty matching, we start with a

$k-$stable submatching. Thereafter, we make agents active in random order and apply $\mathbb{H}$ repeatedly to obtain a $k-$stable matching with an updated set of active agents.[8]

Theorem 1: *For every instance of the stable marriage problem, Algorithm 1 terminates to produce a stable matching. Moreover, if the matching market is balanced, then the output is a Rawlsian stable matching.*

Proof. To prove that the algorithm terminates amounts to proving that steps $4-7$ in Algorithm 1 do not cycle for any k. This is proved in Lemma 1 below.

Lemma 1: *For every instance of a marriage market, Algorithm 1 reaches Step 8 for every k. That is, steps $4-7$ in Algorithm 1 do not cycle for any $k \in \mathbb{N}$.*

Proof. Notice that we start with $V = A$. If E is empty, then we set $V = V' = \emptyset$ and proceed. If not, then any iteration of $\mathbb{H}$ resolves all the cycles. Moreover, the output V' is a *strict subset* of V. Therefore, eventually, we must reach a situation where E is empty, and therefore, V is empty. Hence, we go to Step 8. □

If the algorithm terminates with $k = N$, then we have a stable matching because $N-$stable matchings and stable matchings are equivalent by definition. If we terminate for $k < N$, then the stopping condition itself establishes that μ is stable.

Therefore, we need to only prove that μ is a Rawlsian stable matching if the marriage market is balanced. To this end, suppose that μ is not a Rawlsian stable matching. Thus, $\exists$ a stable matching μ' such that $r' := f(\mu') < f(\mu)$. Let us consider what happens in Round r'. Since the algorithm did not terminate in Round r', the output in this round, say μ'' is $r'-$stable but is not stable. Since μ' is stable, it is also $r'-$stable. The following claim will help us complete the proof.

Claim 1: *There is at least one blocking pair in μ'', (m, w), such that both m and w are matched to someone in μ''.*

Proof. By Theorem 2.22 of Roth and Sotomayor (1990), the set of agents who are single is the same across all the stable matchings. Therefore, if we view the truncated market with preferences given by $\succ^{r'}$, then we know that the set of matched agents in μ' and μ'' is the same as both are r'-stable. At the same time, since μ' is a stable matching, we also have, from the same theorem, that the set of matched agents in μ' and any other stable matching is the same. Therefore, the set of matched agents in μ''—an unstable matching—and any other stable matching is the same. Therefore, since the market is balanced, there is a blocking pair, (m, w), in μ'' such that both the agents are matched to someone in μ''.

□

Consider a blocking pair, (m, w), in μ''. The fact that m, w are matched and

[8] The original RO mechanism is not stated in terms of $\mathbb{H}$. However, that the operation is the same is straightforward to see, and we omit it. This is illustrated more formally in Kuvalekar (2014).

form a blocking pair means that $\rho_w(m) \leq r'$ and $\rho_m(w) \leq r'$. Therefore, μ'' is not r'-stable, a contradiction.

□

4. Some examples

Example 3: *Here is a slightly more complicated example than Example 1.*

Table 2: Preferences

m_1	m_2	m_3	m_4	w_1	w_2	w_3	w_4
w_2	w_3	w_1	w_4	m_4	m_3	m_4	m_3
w_3	w_4	w_3	w_4	m_1	m_4	m_1	m_2
w_1	w_1	w_4	w_2	m_3	m_1	m_2	m_4
w_4	w_2	w_2	w_3	m_2	m_2	m_3	m_1

■

Notice that m_4 and w_1 appear as each other's top preference. Therefore, they will be matched in the first round, and that match will never be broken. We will, therefore, exclude them in writing the S_i's and the T_i's for each round.

Round 1 In the first round,

$$T_i = \emptyset, \forall i \in \{m_1, m_2, m_3, w_2, w_3, w_4\}, T_{m_4} = \{w_1\}, T_{w_1} = \{m_4\}$$
$$\text{and } \mu = \{(m_4, w_1)\}.$$

At this point, we set $k = 2$.

Round 2 Let us present a table for S_i, T_i and μ_i for each agent in this round.

Table 3: Table for S_i, T_i and μ_i for Round 2

Agent	S_i	T_i	μ_i
m_1	w_2, w_3	w_3	w_3
m_2	w_3, w_4	w_4	w_4
m_3	w_1, w_3		
w_2	m_3, m_4		
w_3	m_4, m_1	m_1	m_1
w_4	m_3, m_2	m_2	m_2

Therefore, at the end of this round, agents m_3 and w_2 are unmatched. The allocation

$$\mu = \{(m_1, w_3), (m_2, w_4), (m_4, w_1), (m_3, m_3), (w_2, w_2)\}$$

is 2-stable but it is not stable; it can be blocked by (m_1, w_2). Hence, we proceed to $k = 3$.

Round 3 Let us look at a similar table as above in this round.

Table 4: Table for S_i, T_i and μ_i for Round 3

Agent	S_i	T_i	μ_i
m_1	w_2, w_3	w_2, w_3	w_2
m_2	w_3, w_4	w_4	
m_3	w_1, w_3, w_4	w_4	w_4
w_2	m_3, m_4, m_1	m_1	m_1
w_3	m_4, m_1	m_1	
w_4	m_3, m_2	m_3, m_2	m_3

Therefore, at the end of this round, agents m_2 and w_3 are unmatched. The allocation

$$\mu = \{(m_1, w_2), (m_3, w_4), (m_4, w_1), (m_2, m_2), (w_3, w_3)\}$$

is not 3-stable, as it can be blocked by (m_2, w_3). At this point, we apply the RO mechanism due to Ma (1996). We start with a 3-stable submatching. Notice that μ is a 3-stable submatching (of itself, in fact; see Definition 2), as there are no blocking pairs involving any agent who is matched. Hence, we are now ready to start the RO mechanism.

We have, $V = \{m_1, m_3, m_4, w_1, w_2, w_3\}$ in the RO mechanism given in Algorithm 3, and $R = \{m_2, w_3\}$. We pick one agent randomly—say, m_2—and set $V' = V \cup \{m_2\}$. Notice that μ is 3-stable with active agents in V'. Therefore, $\mathbb{H}(\mu, V', 3)$ produces $\mu' = \mu$. We go back to Step 3 in the RO mechanism (Algorithm 3). The only remaining agent is w_3. Therefore, we now have $V = A$. It is easy to check that $\mathbb{H}(\mu, V, 3) = \mu' := \mu \cup \{(m_2, w_3)\}$. This is a 3-stable and a stable matching. Therefore, the algorithm terminates with the output.

$$\mu = \{(m_1, w_2), (m_2, w_3), (m_3, w_4), (m_4, w_1)\}$$

It can be checked that this is also a Rawlsian matching.

4.1. Difference between the output of Algorithm 1 and the median matching and the RO Mechanism

Example 4: *The following example is due to Knuth as reproduced in Roth and Sotomayor (1990).*

Table 5: Preferences

m_1	m_2	m_3	m_4	w_1	w_2	w_3	w_4
w_1	w_2	w_3	w_4	m_4	m_3	m_2	m_1
w_2	w_1	w_4	w_3	m_3	m_4	m_1	m_2
w_3	w_4	w_1	w_2	m_2	m_1	m_4	m_3
w_4	w_3	w_2	w_1	m_1	m_2	m_3	m_4

We enumerate all the stable matchings below.

Table 6: All the stable matchings

	w_1	w_2	w_3	w_4
1	m_1	m_2	m_3	m_4
2	m_2	m_1	m_3	m_4
3	m_1	m_2	m_4	m_3
4	m_2	m_1	m_4	m_3
5	m_3	m_1	m_4	m_2
6	m_2	m_4	m_1	m_3
7	m_3	m_4	m_1	m_2
8	m_4	m_3	m_1	m_2
9	m_3	m_4	m_2	m_1
10	m_4	m_3	m_2	m_1

The median stable matching of Teo and Sethuraman (1998) is matching either 4 or 7. The RO Mechanism due to Ma (1996) generates any matching *except* for $4, 5, 6$, and 7. Our algorithm can produce any matching out of $4, 5, 6$, and 7.

5. Computational Complexity

The *Median Stable Matching* of Teo and Sethuraman (1998) is a very appealing solution concept when fairness concerns matter. However, a practical problem is that, to compute the median matching, one has to compute the entire set of stable matchings, which is computationally difficult. On the other hand, Theorem 2 below shows that Algorithm 1 can be executed in polynomial time. Therefore, in markets where fairness considerations are important, we view our algorithm as a compelling alternative.

Theorem 2: *Algorithm 1 can be executed in polynomial time.*

Proof. We need, at most, N rounds to end the algorithm. Each round has an execution of map $\mathbb{H}$, potentially N number of times. $\mathbb{H}$ involves computing T_i to construct the graph G. T_i can be computed in $\mathcal{O}(N^3)$. Following the construction of the graph, we need to enumerate the cycles, which is the same as enumerating all the strongly connected components. This can be done using Tarjan's algorithm in $\mathcal{O}(|V|+|E|) = \mathcal{O}(N)$. The process of eliminating cycles can be repeated within a particular execution of $\mathbb{H}$. Every iteration removes at least two agents by forming a match between them. Therefore, steps $4-7$ have a complexity of $\mathcal{O}(n^4)$. Checking for stability or k-stability has a complexity of $\mathcal{O}(N^2)$. Moreover, if the output is not k-stable, we run the RO mechanism that uses a k-stable submatching as an input. Such a submatching can be obtained by removing any agent involved in a blocking pair, which has a complexity of $\mathcal{O}(N^2)$. Thereafter, the RO mechanism itself can be run in polynomial time as mentioned in Cheng (2016). □

6. Conclusion

Motivated by the asymmetry in DAA, we have proposed an alternative algorithm in the classical two-sided marriage market. The algorithm has the flavor of a DAA but it attempts to correct the asymmetry in the outcomes by allowing both sides of the market to make proposals. We show that we can still retain the simplicity of the DAA in constructing such a procedure, and also obtain a stable matching. More importantly, allowing both sides to make proposals results in a Rawlsian stable matching—a well-established criterion of fairness in numerous settings—if the market is balanced. Finally, practical considerations are of paramount importance when one proposes a new algorithm in these contexts. On this matter, the fact that the algorithm can be executed in polynomial time makes it a compelling alternative in markets where the designer considers fairness of the procedure and the outcomes as important.

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