# $\epsilon - NTU$ relationships in parallel-series arrangements: Application to plate and tubular heat exchangers

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**Abstract**

The $\epsilon - NTU$ method has advantages over other methods because it does not require assuming an outlet temperature to determine the duty of a heat exchanger. In this work, the $\epsilon - NTU$ method was used to obtain a general algebraic expression of the effectiveness of PHEs in the $N_{pa}$ pass-1 pass configuration as a function of the heat capacity ratio ($C^*$) and the number of transfer units ($NTU$). This expression is valid for PHEs with a high number of plates ($N_p$ >40). In addition, 1 pass-1 pass and 2 pass-1 pass PHEs were numerically analyzed to obtain an algebraic expression of the effectiveness of PHEs with a low number of thermal plates ($N_p$ <40) as a function of $C^*$, $NTU$ and the number of plates. Finally, the expression obtained for $N_{pa}$ pass-1 pass PHEs with a high number of plates was generalized for use in any heat exchanger with series-parallel configurations, such as series-parallel arrangements of double-pipe or shell-and-tube heat exchangers.

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## 1. Introduction

Plate heat exchangers are widely used in industry for different applications. They were introduced at the beginning of the XX century in the food industry, although they further matured during the 1960s with the development of new materials and plate geometries [1]. These types of heat exchangers have multiple advantages over tubular ones, such as compactness, flexibility and the impossibility of fluid mixing. This last advantage was the reason for their initial development for different processes in the food industry.

PHEs can easily adapt two streams with very different flow rates to achieve similar velocities between plates and, consequently, similar heat transfer coefficients and pressure drops in both fluids. When the flow rate of one stream is much lower than that of the other or if the pressure drop is much lower than permissible, this stream can be channeled through multiple passes. For example, if one stream circulates in two passes along the PHE, its velocity is doubled and its pressure drop is multiplied by approximately eight relative to when flow occurs via a one-pass configuration.

When the heat capacities ($C = \dot{m}\, c_p$) of both fluids are similar, the typical configuration of the PHE is 1 pass-1 pass in counterflow to achieve the highest possible efficiency. Therefore, the $\epsilon - NTU$ relation is

$$\epsilon = \begin{cases} \dfrac{1 - \exp\left[-NTU\,(1 - C^*)\right]}{1 - C^* \exp\left[-NTU\,(1 - C^*)\right]} & \text{if } 0 < C^* < 1 \\ \dfrac{NTU}{1 + NTU} & \text{if } C^* = 1 \end{cases} \tag{1}$$

where $C^*$ is the heat capacity ratio ($C^* = \frac{C_{min}}{C_{max}}$) and $NTU = \frac{U \cdot A}{C_{min}}$. Equation (1) is valid if the number of plates is sufficiently high ($N_p > 40$ according to [2]) because the influence of the first and last plates, which exchange heat from one side only, is negligible. However, end effects become important as the number of thermal plates decreases and/or the value of $NTU$ increases, particularly when the $NTU$ of the one-pass side exceeds 3; hence, the efficiency of the PHE is reduced. Kandlikar and Shah [3] obtained different tables of data using different pass arrangements and numbers of plates. These tables show the correction factors of the logarithmic mean temperature difference $F$ and temperature effectiveness $P_1$ for the fluid stream with a lower number of passes, which is defined as

$$P_1 = \frac{\Delta T_1}{T_{hi} - T_{ci}} \tag{2}$$

as a function of the $NTU_1$ and $R_1$, which are defined as $NTU_1 = UA/C_1; R_1 = C_1/C_2$. In another work, Kandlikar and Shah [2] obtained algebraic expressions for an infinite number of plates with the same flow configurations. These expressions allow the temperature effectiveness of one fluid $P_1$ to be determined using different expressions depending on the number of passes of the PHE. In this work, we propose a general algebraic expression of the effectiveness of a PHE with a high number of plates, which is defined as follows

$$\epsilon = \frac{\dot{Q}}{\dot{Q}_{max}} = \frac{\dot{Q}}{C_{min}\left(T_{hi} - T_{ci}\right)} \tag{3}$$

as a function of $NTU$, $C^*$ and the number of passes $N_{pa}$. In addition, for PHEs with 1 pass-1 pass and 2 pass-1 pass configurations with a smaller number of plates, a correction factor of the algebraic solution for an infinite number of plates is presented as a function of the number of plates.

Multiple-pass PHEs are a specific class of heat exchangers that combine series and parallel arrangements: the 1-pass stream of the PHEs is considered to be in parallel, and the multiple-pass stream is considered to be in series. Combined series-parallel arrangement heat exchangers can be used as multiple-pass PHEs in some applications in which the difference between the flow rates of the streams is substantial to achieve similar heat transfer coefficients and pressure drops. In addition to PHEs, other types of heat exchangers can be used in arrangements combining series and parallel couplings, including double-pipe and shell-and-tube heat exchangers.

Regarding tubular heat exchangers, Kern [4] obtained the expressions for a temperature correction factor of series-parallel arrangements of countercurrent heat exchangers. This correction factor is defined as follows:

$$\gamma = \frac{\dot{Q}}{U\,A\,(T_{hi} - T_{ci})} \tag{4}$$

The expression of $\gamma$ when the cold fluid flows in parallel is

$$\frac{1-\hat{P}}{\gamma} = \frac{N_{pa}\,\hat{R}}{\hat{R}-1} \ln\left(\frac{\hat{R}-1}{\hat{R}}\left(\frac{1}{\hat{P}}\right)^{1/N_{pa}} + \frac{1}{\hat{R}}\right) \tag{5}$$

where

$$\hat{P} = \frac{T_{ho} - T_{ci}}{T_{hi} - T_{ci}} \quad \text{and} \quad \hat{R} = \frac{T_{hi} - T_{ho}}{N_{pa}\,(T_{co} - T_{ci})} \tag{6}$$

If the fluid flowing in parallel is the hot one, the expression of $\gamma$ is

$$\frac{1-\hat{P}}{\gamma} = \frac{N_{pa}}{1-\hat{R}} \ln\left(\left(1-\hat{R}\right)\left(\frac{1}{\hat{P}}\right)^{1/N_{pa}} + \hat{R}\right) \tag{7}$$

with

$$\hat{P} = \frac{T_{ci} - T_{ho}}{T_{ci} - T_{hi}} \quad \text{and} \quad \hat{R} = \frac{N_{pa}\,(T_{ci} - T_{co})}{T_{ho} - T_{hi}} \tag{8}$$

In Equations (5)-(8), $N_{pa}$ indicates the number of passes in series. Equations (5) and (7) have been reported elsewhere, such as [1, 5, 6], expressed in terms of other non-dimensional parameters.

The method proposed by Taborek and Guy [7], who reported the correction factor of the logarithmic mean temperature difference $F$ for series-parallel arrangements of countercurrent heat exchangers with 2 and 3 passes graphically, represents an alternative method.

The $\epsilon - NTU$ method permits calculating the outlet temperatures of a heat exchanger without requiring an iterative process. However, the general relations $\epsilon - NTU$ have not been reported in the literature for series-parallel arrangements of heat exchangers. Using the $\epsilon - NTU$ method, Domingos [9] obtained the total effectiveness of an assembly of heat exchangers in terms of the effectiveness of the individual heat exchangers. Therefore, to calculate the total effectiveness, it is necessary to know the intermediate temperatures to obtain the individual effectiveness of each heat exchanger. In addition, the overall effectiveness expression for exchangers in parallel in one stream obtained by Domingos [9] can only be used when the stream that flows in parallel has the lower heat capacity.

This work presents a general algebraic expression of the effectiveness of heat exchangers combining series and parallel arrangements ($N_{pa}$-1 arrangements) as a function of $N_{pa}$ and $NTU$ and $C^*$ of the overall heat exchanger. The expression was determined by considering the overall heat exchanger as a set of as many individual exchangers as passes in series. First, plate heat exchangers were analyzed for both an infinite number of plates (analytical solution) and a reduced number of plates (numerical solution). In this type of heat exchanger, the flow arrangement changes countercurrent and parallel in each pass. Second, the analytical $\epsilon - NTU$ relation for PHEs with $N_{p \to \infty}$ is extended to other heat exchangers combining series and parallel arrangements, such as double-pipe and multitubular or shell-and-tube heat exchangers.

## 2. Expression of the effectiveness of N$_{pa}$-1 PHEs

### *2.1. Analytical expression of the effectiveness of $N_{pa}$-1 PHEs for an infinite number of plates*

This section explains the procedure to obtain the analytical expression of the effectiveness of a PHE with an $N_{pa}$-1 pass configuration for an infinite number of plates. First, the method is presented for 2 pass-1 pass exchangers, and then, it is extended for $N_{pa}$-1 configurations.

#### *2.1.1. Analytical expression of the effectiveness of a 2 pass-1 pass configuration for an infinite number of plates*

The method involves considering the global PHE as a set of two PHEs: the first one (which is the first half of the PHE) in counterflow and the second one in parallel flow. The following assumptions were made: the number of plates is high, and thus, the influence of the first and last plates, which exchange heat from one side only, is negligible; the flow is uniformly distributed among parallel flow channels, and hence,the mass flow rate of the one-pass fluid is divided into two equal streams ($\dot{m}_1 = \dot{m}_2$); the global heat transfer coefficient is the same for each pass ($U_1 = U_2$); and the exchange area is the same for each pass ($A_1 = A_2$), where subindex 1 refers to the first pass (counterflow), and subindex 2 refers to the second pass (parallel flow). According to these assumptions, $C_1^* = C_2^*$ and $NTU_1 = NTU_2$. Then, the following expressions can be derived to relate these parameters of the counterflow and parallel flow heat exchangers with the parameters $NTU$

and $C^*$ of the overall exchanger.

$$C_1^* = \frac{C_{min,1}}{C_{max,1}} = \frac{\hat{C}_{max}}{\hat{C}_{min}} C^* \quad (9)$$

$$C_2^* = \frac{C_{min,2}}{C_{max,2}} = \frac{\hat{C}_{max}}{\hat{C}_{min}} C^* \quad (10)$$

$$NTU_1 = \frac{U_1 A_1}{C_{min,1}} = \frac{\hat{C}_{min}}{2} NTU \quad (11)$$

$$NTU_2 = \frac{U_2 A_2}{C_{min,2}} = \frac{\hat{C}_{min}}{2} NTU \quad (12)$$

where

$$\hat{C}_{min} = \frac{C_{min}}{C_{min,1}} \quad (13)$$

$$\hat{C}_{max} = \frac{C_{max}}{C_{max,1}} \quad (14)$$

The effectiveness relationships for counterflow and parallel flow heat exchangers are as follows

$$\epsilon_c = \begin{cases} \dfrac{1 - \exp\left[-NTU_1\left(1 - C_1^*\right)\right]}{1 - C_1^* \exp\left[-NTU_1\left(1 - C_1^*\right)\right]} & \text{for } C_1^* < 1 \\ \dfrac{NTU_1}{1 + NTU_1} & \text{for } C_1^* = 1 \end{cases} \quad (15)$$

$$\epsilon_p = \frac{1 - \exp\left[-NTU_2 (1 + C_2^*)\right]}{1 + C_2^*} \quad (16)$$

To analyze a 2 pass-1 pass heat exchanger, the heat exchanger can be considered as a series coupling a counterflow heat exchanger, with effectiveness $\epsilon_1 = \epsilon_c$, and a parallel flow heat exchanger with effectiveness $\epsilon_2 = \epsilon_p$. Then, introducing Equations (9)-(12) into Equations (15) and (16), the effectiveness $\epsilon_1$ and $\epsilon_2$ can be expressed in terms of the overall parameters $NTU$ and

$C^*$ as follows

$$\epsilon_1 = \begin{cases} \dfrac{1-\exp\left[-\frac{\hat{C}_{min}}{2}NTU\left(1-\frac{\hat{C}_{max}}{\hat{C}_{min}}C^*\right)\right]}{1-\frac{\hat{C}_{max}}{\hat{C}_{min}}C^*\exp\left[-\frac{\hat{C}_{min}}{2}NTU\left(1-\frac{\hat{C}_{max}}{\hat{C}_{min}}C^*\right)\right]} & \text{for } C_1^* < 1 \\ \dfrac{\frac{\hat{C}_{min}}{2}NTU}{1+\frac{\hat{C}_{min}}{2}NTU} & \text{for } C_1^* = 1 \end{cases} \quad (17)$$

$$\epsilon_2 = \frac{1-exp\left[-\frac{\hat{C}_{min}}{2}NTU(1+\frac{\hat{C}_{max}}{\hat{C}_{min}}C^*)\right]}{1+\frac{\hat{C}_{max}}{\hat{C}_{min}}C^*} \quad (18)$$

Additionally, the effectiveness of the overall PHE can be calculated as follows

$$\epsilon = \frac{\dot{Q}_1+\dot{Q}_2}{\dot{Q}_{max}} = \frac{\dot{Q}_1+\dot{Q}_2}{C_{min}\left(T_{hi}-T_{ci}\right)} \quad (19)$$

When the hot fluid makes two passes, the heat transfer rates in the first and second passes are defined by Equations (20) and (21), respectively:

$$\dot{Q}_1 = \epsilon_1 \cdot \dot{Q}_{max,1} = \epsilon_1 \cdot C_{min,1} \cdot (T_{hi} - T_{ci}) \quad (20)$$

$$\dot{Q}_2 = \epsilon_2 \cdot \dot{Q}_{max,2} = \epsilon_2 \cdot C_{min,2} \cdot (T_{h,int} - T_{ci}) \quad (21)$$

where $T_{h,int}$ is the temperature of the hot fluid at the exit of the first pass, as shown in Figure 1.

[Figure 1 about here.]

Introducing Equations (20) and (21) into Equation (19)

$$\epsilon = \frac{\epsilon_1}{\hat{C}_{min}} + \frac{\epsilon_2}{\hat{C}_{min}}\left[1-\frac{\Delta T_{1,2pa}}{T_{hi}-T_{ci}}\right] \quad (22)$$

where $\Delta T_{1,2pa}$ is the temperature difference of the 2-pass stream in the first pass (counterflow). Although Equation (22) was obtained for the cold fluid circulating in parallel, it is also valid when the cold fluid makes two passes.

The value of this temperature difference changes depending on the flow that has the lower heat capacity in the first pass

$$\frac{\Delta T_{1,2pa}}{T_{hi}-T_{ci}} = \begin{cases} \dfrac{C_{1,2pa}\cdot\Delta T_{1,2pa}}{C_{1,2pa}\cdot(T_{hi}-T_{ci})} = \dfrac{\dot{Q}_1}{\dot{Q}_{1,max}} = \epsilon_1 & \text{if } C_{min,1}=C_{1,2pa} \\ \dfrac{C_{1,2pa}\cdot\Delta T_{1,2pa}}{C_{1,1pa}\cdot(T_{hi}-T_{ci})}\dfrac{C_{1,1pa}}{C_{1,2pa}} = \dfrac{\dot{Q}_1}{\dot{Q}_{1,max}}C_1^* = \epsilon_1\dfrac{\hat{C}_{max}}{\hat{C}_{min}}\cdot C^* & \text{if } C_{min,1}=C_{1,1pa} \end{cases} \tag{23}$$

where $C_{1,2pa}$ is the heat capacity of the 2-pass fluid in the first pass and $C_{1,1pa}$ is the heat capacity of the 1-pass fluid in the first pass. Introducing Equation (23) into Equation (22), the overall heat exchanger effectiveness becomes

$$\epsilon = \frac{\epsilon_1}{\hat{C}_{min}} + \frac{\epsilon_2}{\hat{C}_{min}}(1-\epsilon_1) \qquad \text{if } C_{min,1}=C_{1,2pa} \tag{24}$$

$$\epsilon = \frac{\epsilon_1}{\hat{C}_{min}} + \frac{\epsilon_2}{\hat{C}_{min}}\left(1-\epsilon_1\frac{\hat{C}_{min}}{\hat{C}_{max}}C^*\right) \qquad \text{if } C_{min,1}=C_{1,1pa} \tag{25}$$

Thus, there are different expressions of the effectiveness depending on which flow (1-pass or 2-pass flow) has the lower heat capacity. In addition, when 2-pass flow has the lower heat capacity, the effectiveness expression also depends on the value of $C^*$, and as a result, there are different expressions for $C^* \leq 1/N_{pa}$ and $C^* > 1/N_{pa}$. The expressions of the effectiveness for all cases are summarized in Table 1.

[Table 1 about here.]

The effectiveness of each pass ($\epsilon_1$,$\epsilon_2$) can be calculated by replacing the values of $\hat{C}_{min}$ and $\hat{C}_{max}$ of each case in Equations (17) and (18) to obtain the expressions shown in Table 2 for passes in counterflow and Table 3 for passes in parallel flow by replacing $N_{pa}$ with 2.

To calculate the effectiveness for PHEs with phase changes ($C^* = 0$), which depend only on $NTU$, the following expression can be used:

$$\epsilon = 1 - \exp[-NTU] \quad (26)$$

#### *2.1.2. Analytical expression of the effectiveness of a $N_{pa}$ pass-1 pass configuration for an infinite number of plates*

The previous method can be generalized to obtain the following expression of the effectiveness of a $N_{pa}$ pass-1 pass configuration with an infinite number of plates:

$$\epsilon_{N_{pa}} = \frac{1}{\alpha}\sum_{i_1=1}^{N_{pa}} \epsilon_{i_1} - \frac{\beta}{2!\,\alpha^2}\sum_{i_1=1}^{N_{pa}}\sum_{i_2=1}^{N_{pa}} \epsilon_{i_1 i_2}\,\delta_{i_1 i_2} + \frac{\beta^2}{3!\,\alpha^3}\sum_{i_1=1}^{N_{pa}}\sum_{i_2=1}^{N_{pa}}\sum_{i_3=1}^{N_{pa}} \epsilon_{i_1 i_2 i_3}\,\delta_{i_1 i_2 i_3} - \frac{\beta^3}{4!\,\alpha^4}\sum_{i_1=1}^{N_{pa}}\sum_{i_2=1}^{N_{pa}}\sum_{i_3=1}^{N_{pa}}\sum_{i_4=1}^{N_{pa}} \epsilon_{i_1 i_2 i_3 i_4}\,\delta_{i_1 i_2 i_3 i_4} + \dots \quad (27)$$

where

$$\delta_{i_1 \dots i_{N_{pa}}} = \begin{cases} 1 \text{ if } i_1 \dots i_{N_{pa}} \text{ are all distinct integers} \\ 0 \text{ in all other cases} \end{cases}$$

$$\epsilon_{i_1 \dots i_{N_{pa}}} = \prod_{x=i_1}^{i_{N_{pa}}} \epsilon_x$$

$\epsilon_{i_1 \dots i_{N_{pa}}}$ are the effectiveness of each pass, which can be calculated using the expressions shown in Table 2 for passes in counterflow and Table 3 for passes in parallel flow, and $\alpha$ and $\beta$ are constants with different values for the various cases described in the previous section, which are shown in Table 4.

Equation (27) can be expressed in compact form as follows

$$\epsilon_{N_{pa}} = \frac{1}{\alpha}\sum_{i_1=1}^{N_{pa}} \epsilon_{i_1} + \sum_{a=2}^{N_{pa}} \left( \frac{(-\beta)^{a-1}}{a!\,\alpha^a} \sum_{i_1=1}^{N_{pa}} \cdots \sum_{i_a=1}^{N_{pa}} \epsilon_{i_1 \ldots i_a} \delta_{i_1 \ldots i_a} \right) \tag{28}$$

Note that the efficiencies of the odd passes are equal ($\epsilon_1 = \epsilon_3 = \epsilon_5 = \cdots = \epsilon_{cc}$), as are the efficiencies of the even passes ($\epsilon_2 = \epsilon_4 = \epsilon_6 = \cdots = \epsilon_p$).

[Table 2 about here.]

[Table 3 about here.]

[Table 4 about here.]

The PHE effectiveness is plotted in Figure 2 as a function of $NTU$ for different values of $C^*$ and $N_{pa}$. Figure 2(a) shows the effectiveness for a heat exchanger in which the single-pass fluid has the lower heat capacity, whereas Figure 2(b) shows the effectiveness of a heat exchanger where the multiple-pass fluid has the lower heat capacity. Figure 2(a) shows that the effectiveness for $NTU$ values of approximately 10 reaches an asymptotic value below 1 (except for $C^*$=0), while Figure 2(b) shows that the effectiveness when $C^* < 1/\mathrm{N}_{pa}$ can be 1 for high $NTU$ values. Therefore, it can be concluded that the effectiveness of a PHE where $C_{min} = C_{1pa}$ is lower than that of a PHE where $C_{min} = C_{Npa}$ and $C^* < 1/\mathrm{N}_{pa}$ for high $NTU$ values. In all cases, for fixed values of $C^*$ and $NTU$, the effectiveness with $N_{pa}$=3 is always higher than that with $N_{pa}$=2, whereas that with $N_{pa}$=4 is lower. This is because PHEs with $N_{pa}$=3 have two passes in countercurrent (one more than $N_{pa}$=2) and only one in parallel (one fewer than $N_{pa}$=4).

[Figure 2 about here.]

### *2.2. Effectiveness of PHEs with a low number of plates for different pass arrangements*

The heat transfer rate from the end channel of a PHE is different than that from an interior channel of the same fluid because the heat is transferred to only one side of the channel. This end effect reduces the exchanger effectiveness relative to that with an infinite number of plates and is important only for $N_p$ <40 [3]. In this work, an algebraic expression to calculate the effectiveness of 1 pass-1 pass and 2 pass-1 pass PHEs with a low number of plates was obtained numerically using a finite difference method. The temperature variation along the channel is expressed in terms of backward difference formulas, and the finite difference equations are then solved using Jacobi's method. Details of the numerical method can be found in Appendix A (4).

In this work, 1 pass-1 pass and 2 pass-1 pass PHEs were classified according to the classification method presented by Kandlikar and Shah [3], which considers the flow directions of each stream in the channels resulting in counterflow or parallel flow across any thermal plate, the total number of channels for the two streams, and the end plate effects. Therefore, for 1 pass-1 pass heat exchangers, four different configurations are considered, which are shown in Figure 3. In addition, for heat exchangers with an even number of plates (configurations 112 and 114), it is necessary to distinguish between two different cases depending on whether the fluid with the lower or higher heat capacity flows through the outer channels. Thus, 6 different cases must be analyzed.

[Figure 3 about here.]

[Figure 4 about here.]

The numerical effectiveness results are represented as a function of the number of plates for different values of $NTU$ and $C^*$. The results for the case of a 1 pass-1 pass counterflow heat exchanger with an odd number of plates (111 configuration) are shown in Figure 4 as an example. The various configurations show that the highest effectiveness, which is obtained with a large number of thermal plates, is achieved with 1 or 2 thermal plates and is minimized before subsequently increasing with the number of thermal plates to approach the asymptotic value at $N_p \to \infty$.

[Figure 5 about here.]

[Figure 6 about here.]

For 2 pass-1 pass heat exchangers, four different configurations are considered. Figure 5 shows four basic (Nos 211, 212, 213, and 214) and four additional (Nos 211 a, 212 a, 213 a, and 214 a) arrangements of 2 pass-1 pass configurations. In all these arrangements, the number of channels in each pass of the 2-pass fluid is assumed to be the same. According to the flow reversibility principle of Pignotti [8], when the two fluid streams in any heat exchanger are reversed simultaneously, the $\epsilon - NTU$ relations remain the same. Thus, by reversing the flow directions of both streams in the additional arrangements, it can be seen that 211 a, 212 a, 213 a and 214 a arrangements are equivalent to the 211, 212, 213, and 214 arrangements, respectively. Therefore, the four basic flow arrangements were analyzed in this work. In addition, the results depend on whether the 1-pass fluid has the lower heat capacity or vice versa. As a result, 8 different cases must be analyzed.

The curves obtained for each case of 2 pass-1 pass PHEs show that the effectiveness asymptotically increases as the number of plates increases, reaching the value of the effectiveness calculated in the previous section when the number of plates tended to infinity. As an example, Figure (6) presents the numerical result for a 212 configuration (see Figure 5) when the 1-pass fluid has the lower heat capacity; the curves obtained for configurations 213 and 214 follow the same trend. However, a different behavior was observed for configuration 211 for some values of $NTU$ that present reduced effectiveness as the number of plates is increased. This result can be explained because configuration 211 always has one more channel in counterflow than in parallel flow, which counterbalances the end effect and increases the effectiveness.

To obtain an algebraic expression of the PHE effectiveness for a low number of plates, the curves $\epsilon - N_p$, as shown in Figures 4 or 6, were fitted using an exponential expression, and the effectiveness can thus be calculated as a function of $C^*$, $NTU$ and $N_p$.

$$\frac{\epsilon - \epsilon_{N_p \to \infty}}{\epsilon_{max} - \epsilon_{min}} = F_1 \cdot \exp(F_2 \cdot N_p) \tag{29}$$

where

$$\epsilon_{max} - \epsilon_{min} = F_3 + F_4 \cdot C^* + F_5 \cdot C^{*2} \tag{30}$$

Then, Equation (29) can also be expressed as follows

$$\epsilon = \epsilon_{N_p \to \infty} + \Delta\epsilon, \tag{31}$$

where

$$\Delta\epsilon = \left[F_3 + F_4 \cdot C^* + F_5 \cdot C^{*2}\right] \cdot F_1 \cdot \exp\left[F_2 \cdot N_p\right] \tag{32}$$

and each function $F_i$ is defined as follows

$$F_i = a_{i0} + a_{i1} \cdot NTU + a_{i2} \cdot NTU^2 \tag{33}$$

In previous expressions, $N_p$ was the number of plates, $C^*$ was the heat capacity ratio, $NTU = U \cdot A/C_{min}$, the parameters $a_i$ were constants fitted with numerical results shown in Tables 5-7 and $\epsilon_{N_p \to \infty}$ could be calculated using Equation (28). When phase change occurs ($C^*$=0) and the stream that changes its phase flows through both outer channels (cases 112 and 114 with $C_{max}$=$C_{o,c}$, 213 with $C_{min}$=$C_{1p}$ and 214 with $C_{min}$=$C_{2p}$), the effectiveness only depends on NTU; therefore, Equation (26) must be used instead of the expressions proposed in this work.

[Table 5 about here.]

[Table 6 about here.]

[Table 7 about here.]

For 1 pass-1 pass configurations, the highest relative error between the numerical results and the fitting curves is 2%, while for 2 pass-1 pass configurations, the highest relative error is 4%. However, the relative error decreases when the number of plates increases, and as a result, the maximum error for PHEs with at least ten plates is approximately 1% for 1 pass-1 pass configurations and 2% for 1 pass-2 pass configurations.

## 3. Expression of the effectiveness of tubular heat exchangers combining series and parallel arrangements

Multiple-pass PHEs are a particular case of heat exchangers combining series and parallel arrangements, and thus, their effectiveness can be obtained by generalizing Equation (28). To obtain the overall effectiveness of any heat exchanger in a series-parallel configuration, the overall heat exchanger is considered to be a combination of a number $N_{pa}$ of individual heat exchangers, which all have the same effectiveness. Therefore, Equation (27) reduces to

$$\epsilon_{N_{pa}} = \frac{N_{pa}\,\epsilon_1}{\alpha} - \frac{\beta}{2!\,\alpha^2}\,\epsilon_1^2 \sum_{i_1=1}^{N_{pa}} \sum_{i_2=1}^{N_{pa}} \delta_{i_1 i_2} + \frac{\beta^2}{3!\,\alpha^3}\,\epsilon_1^3 \sum_{i_1=1}^{N_{pa}} \sum_{i_2=1}^{N_{pa}} \sum_{i_3=1}^{N_{pa}} \delta_{i_1 i_2 i_3} - \frac{\beta^3}{4!\,\alpha^4}\,\epsilon_1^4 \sum_{i_1=1}^{N_{pa}} \sum_{i_2=1}^{N_{pa}} \sum_{i_3=1}^{N_{pa}} \sum_{i_4=1}^{N_{pa}} \delta_{i_1 i_2 i_3 i_4} + \ldots \tag{34}$$

where

$$\delta_{i_1 \ldots i_{N_{pa}}} = \begin{cases} 1 \text{ if } i_1 \ldots i_{N_{pa}} \text{ are all distinct integers} \\ 0 \text{ in all other cases} \end{cases}$$

and $\epsilon_1$ is the effectiveness of each heat exchanger. Equation (34) can be written in compact form as follows

$$\epsilon_{N_{pa}} = \sum_{i=1}^{N_{pa}} \frac{(-\beta)^{i-1}}{\alpha^i}\,\epsilon_1^i \frac{N_{pa}!}{i!(N_{pa}-i)!} \tag{35}$$

Equation (35) results in the following expressions for $N_{pa} = 2, 3, 4$:

$$\epsilon_{N_{pa}=2} = \frac{2}{\alpha}\epsilon_1 - \frac{\beta}{\alpha^2}\epsilon_1^2 \tag{36}$$

$$\epsilon_{N_{pa}=3} = \frac{3}{\alpha}\epsilon_1 - \frac{3\beta}{\alpha^2}\epsilon_1^2 + \frac{\beta^2}{\alpha^3}\epsilon_1^3 \tag{37}$$

$$\epsilon_{N_{pa}=4} = \frac{4}{\alpha}\epsilon_1 - \frac{6\beta}{\alpha^2}\epsilon_1^2 + \frac{4\beta^2}{\alpha^3}\epsilon_1^3 - \frac{\beta^3}{\alpha^4}\epsilon_1^4 \tag{38}$$

The individual effectiveness $\epsilon_1$ of each heat exchanger can be calculated in the same way as the effectiveness of a basic heat exchanger. In addition to PHEs, other types of heat exchangers can be found in arrangements combining series and parallel couplings: double-pipe, multitubular and shell-and-tube heat exchangers. For counterflow or parallel flow heat exchangers, such as multitubular or double-pipe heat exchangers, the individual effectiveness of each pass can be calculated using the expressions shown in Tables 2 and 3, respectively, because the stream that flows in series is equivalent to the stream that flows through $N_{pa}$ passes, whereas the stream that flows in parallel is equivalent to the stream that flows through 1 pass. Figures 7 and 8 show the graphical representations of the overall effectiveness of these counterflow and parallel flow heat exchangers as a function of $NTU$ for $N_{pa}$=2,3,4 and different values of $C^*$. For shell-and-tube heat exchangers, the effectiveness of each pass can be calculated using the expressions shown in Table 8. Figure 9 presents the overall effectiveness of type E shell-and-tube heat exchangers as a function of $NTU$, for $N_{pa}$=2,3,4 and different values of $C^*$.

[Table 8 about here.]

[Figure 7 about here.]

[Figure 8 about here.]

[Figure 9 about here.]

## 4. Conclusions

In this work, a general algebraic expression to calculate the effectiveness of $N_{pa}$-1 pass PHEs was obtained as a function of $NTU$, $C^*$ and $N_{pa}$. This expression can be used for PHEs with a high number of plates ($N_p$ >40) and is valid for any number of passes. The results obtained in this work show that the exchanger effectiveness when the 1-pass stream has the lower heat capacity for high $NTU$ values approaches an asymptotic value below 1, whereas when a multiple-pass stream has the lower heat capacity and $C^* < \frac{1}{N_{pa}}$, the effectiveness can reach 1 for high $NTU$ values.

In addition, in this work, 1 pass-1 pass and 2 pass-1 pass PHEs were numerically solved to obtain an effectiveness expression for PHEs with a low number of plates ($N_p$ <40), and the results were fitted to a mathematical expression. The maximum relative error between the numerical results and the fitting curves is approximately 4% for a 2 pass-1 pass configuration with seven plates. Normally, PHEs have more than ten plates, and in this case, the relative error between the numerical results and the fitting curves is less than 2%.

Finally, as multiple-pass PHEs represent an specific case of series-parallel configurations, the obtained $\epsilon - NTU$ relation for PHEs was generalized to derive the $\epsilon - NTU$ relations for series-parallel heat exchangers, including double-pipe, multitubular and shell-and-tube heat exchangers. The $\epsilon - NTU$ relations were also represented graphically. These graphs facilitate calculating or designing different heat exchanger configurations with multiple passes by eliminating the necessity of solving each pass separately.

## Appendix A. Development of the numerical model used to obtain the effectiveness expression for PHEs with a low number of plates

This section presents a more detailed explanation of the numerical model used to obtain the expressions of the effectiveness for 1 pass-1 pass and 2 pass-1 pass PHEs with a low number of plates.

- 1 pass-1 pass arrangement

[Figure 10 about here.]

Figure 10 shows the grid structure for the finite difference method of a PHE in a 1 pass-1 pass configuration. Each channel, $(j)$, is divided into $N$ number of steps of equal area, (i), and the first step $(i = 1)$ corresponds to the inlet section of each channel. For any step $i$ in each channel $j$, the energy equation, assuming steady state and negligible changes in kinetic and potential energies, can be expressed as follows

$$0 = \dot{Q} + \dot{m}_c(i_i - i_o) \tag{39}$$

where $\dot{m}_c$ is the mass flow rate per channel, and the enthalpy variation can be calculated as

$$i_i - i_o = c_p\left(T_{i-1,j} - T_{i,j}\right) \tag{40}$$

The heat transfer rate for an inner channel in counterflow is defined as follows

$$\dot{Q} = -\frac{U \cdot A_p}{N}\left[(T_{i,j} - T_{N+1-i,j-1}) + (T_{i,j} - T_{N+1-i,j+1})\right] \tag{41}$$

and for an inner channel in parallel flow as

$$\dot{Q} = -\frac{U \cdot A_p}{N}\left[(T_{i,j} - T_{i,j-1}) + (T_{i,j} - T_{i,j+1})\right] \tag{42}$$

The derivation of the finite difference equation will be explained in the following for a counterflow heat exchanger. A similar procedure can be applied for a parallel flow exchanger by replacing the adjacent channels subindex "$N+1-i$" with "$i$".

Introducing Equations (41) and (40) into Equation (39)

$$0 = -\frac{U \cdot A_p}{N} \left[ (T_{i,j} - T_{N+1-i,j-1}) + (T_{i,j} - T_{N+1-i,j+1}) \right] + \frac{\dot{m}}{N_c} c_p \left( T_{i-1,j} - T_{i,j} \right) \tag{43}$$

where $U$ is the global heat transfer coefficient, $A_p$ is the heat transfer surface area on one side of a thermal plate, $N$ is the number of steps of equal area, $T_{i,j}$ is the temperature in each node, $\dot{m}$ is the mass flow rate of a fluid, $N_c$ is the number of channels per fluid pass, and $c_p$ is the specific heat of the fluid. If the temperatures are non-dimensionalized as follows

$$\theta = \frac{T - T_{ci}}{T_{hi} - T_{ci}} \tag{44}$$

Equation (43) results in the following non-dimensional equation:

$$0 = -\frac{NTU}{N \cdot N_p} \left[ (\theta_{i,j} - \theta_{N+1-i,j-1}) + (\theta_{i,j} - \theta_{N+1-i,j+1}) \right] + \frac{\dot{m} \cdot c_p}{N_c \cdot C_{min}} \left( \theta_{i-1,j} - \theta_{i,j} \right) \tag{45}$$

To obtain an expression that can be used to calculate the temperatures in each node $T_{i,j}$, two different cases must be analyzed, depending on which fluid has the lower heat capacity. The expression for the temperature of the fluid that has the lower heat capacity in an inner channel of a 1 pass-1 pass counterflow PHE is

$$\theta_{i,j} = \frac{NTU \cdot N_{c,min} \left( \theta_{N+1-i,j-1} + \theta_{N+1-i,j+1} \right) + N_p \cdot N \cdot \theta_{i-1,j}}{2 \cdot NTU \cdot N_{c,min} + N_p \cdot N} \tag{46}$$

and the expression for the temperature of the fluid that has the higher heat capacity is

$$\theta_{i,j} = \frac{NTU \cdot C^* \cdot N_{c,max} \left( \theta_{N+1-i,j-1} + \theta_{N+1-i,j+1} \right) + N_p \cdot N \cdot \theta_{i-1,j}}{2 \cdot NTU \cdot C^* \cdot N_{c,max} + N_p \cdot N} \tag{47}$$

where $N_{c,min}$ is the number of channels per fluid with the lower heat capacity, and $N_{c,max}$ is the number of channels per fluid with the higher heat capacity. For parallel flow channels, the temperature expression for the fluid that has the lower heat capacity is

$$\theta_{i,j} = \frac{NTU \cdot N_{c,min} \left(\theta_{i,j-1} + \theta_{i,j+1}\right) + N_p \cdot N \cdot \theta_{i-1,j}}{2 \cdot NTU \cdot N_{c,min} + N_p \cdot N} \tag{48}$$

and the expression for the temperature of the fluid that has the higher heat capacity is

$$\theta_{i,j} = \frac{NTU \cdot C^{*} \cdot N_{c,max} \left(\theta_{i,j-1} + \theta_{i,j+1}\right) + N_p \cdot N \cdot \theta_{i-1,j}}{2 \cdot NTU \cdot C^{*} \cdot N_{c,max} + N_p \cdot N} \tag{49}$$

For an outer channel, because heat is transferred via only one side, the heat transfer rate for a counterflow channel is

$$\dot{Q} = -\frac{U \cdot A_p}{N} \left(T_{i,j} - T_{N+1-i,j+1}\right) \tag{50}$$

and for a parallel flow channel, it is:

$$\dot{Q} = -\frac{U \cdot A_p}{N} \left(T_{i,j} - T_{i,j+1}\right) \tag{51}$$

Thus, the non-dimensional energy equation for the fluid temperature for an outer channel in counterflow results in

$$0 = -\frac{NTU}{N \cdot N_p} \left(\theta_{i,j} - \theta_{N+1-i,j+1}\right) + \frac{\dot{m} \cdot c_p}{N_c \cdot \dot{C}_{min}} \left(\theta_{i-1,j} - \theta_{i,j}\right) \tag{52}$$

Then, the temperature of the fluid that has the lower heat capacity in an outer channel in counterflow can be calculated as

$$\theta_{i,j} = \frac{NTU \cdot N_{c,min} \cdot \theta_{N+1-i,j+1} + N_p \cdot N \cdot \theta_{i-1,j}}{NTU \cdot N_{c,min} + N_p \cdot N} \tag{53}$$

and the temperature of the fluid that has the higher heat capacity can be calculated as

$$\theta_{i,j} = \frac{NTU \cdot C^{*} \cdot N_{c,max} \cdot \theta_{N+1-i,j+1} + N_p \cdot N \cdot \theta_{i-1,j}}{NTU \cdot C^{*} \cdot N_{c,max} + N_p \cdot N} \tag{54}$$

Similarly, the temperature of the fluid that has the lower heat capacity in an outer channel in parallel flow can be calculated as

$$\theta_{i,j} = \frac{NTU \cdot N_{c,min} \cdot \theta_{i,j+1} + N_p \cdot N \cdot \theta_{i-1,j}}{NTU \cdot N_{c,min} + N_p \cdot N} \tag{55}$$

and the temperature of the fluid that has the higher heat capacity can be calculated as

$$\theta_{i,j} = \frac{NTU \cdot C^* \cdot N_{c,max} \cdot \theta_{i,j+1} + N_p \cdot N \cdot \theta_{i-1,j}}{NTU \cdot C^* \cdot N_{c,max} + N_p \cdot N} \tag{56}$$

Equations (46)- (49) and (53)- (56) with finite differences were solved using Jacobi's algorithm with the following boundary conditions:

$$\theta_{hi} = \theta_{1,j} = 1 \tag{57}$$

$$\theta_{ci} = \theta_{1,j^*} = 0 \tag{58}$$

where $j$ refers to hot fluid channels, and $j^*$ refers to cold fluid channels. Finally, the effectiveness of the heat exchanger can be calculated as

$$\epsilon = 1 - \frac{\sum_{j=1}^{j=N_p+1} \theta_{N,j}}{N_{c,h}} \qquad \text{for } C_{min}=C_h \tag{59}$$

$$\epsilon = \frac{\sum_{j=1}^{j=N_p+1} \theta_{N,j}}{N_{c,c}} \qquad \text{for } C_{min}=C_c \tag{60}$$

where the outlet temperature of each fluid was calculated as the average of the outlet temperatures of the different channels.

- 1 pass-2 pass arrangement

The process used to analyze the various 2 pass-1 pass configurations was the same as that used for the 1 pass-1 pass configuration, taking into account that 2 pass-1 pass configurations exhibit channels in both counterflow and parallel flow and an intermediate channel, which is located between a counterflow channel and a parallel flow channel. Therefore, Equations (46) and (47) can be used to calculate the temperature of the fluid that flows through the inner counterflow channels using the first equation if the fluid has the lower heat capacity and the second one in the opposite case. Similarly, Equations (48) and (49) can be used to calculate the temperature of the fluid that flows through the inner parallel flow channels. For intermediate channels, the temperature equation for the fluid that has the lower heat capacity is

$$\theta_{i,j} = \frac{NTU \cdot N_{c,min} \left(\theta_{N+1-i,j-1} + \theta_{i,j+1}\right) + N_p \cdot N \cdot \theta_{i-1,j}}{2 \cdot NTU \cdot N_{c,min} + N_p \cdot N} \tag{61}$$

and for the fluid that has the higher heat capacity, it is

$$\theta_{i,j} = \frac{NTU \cdot N_{c,max} \cdot C^* \left(\theta_{N+1-i,j-1} + \theta_{i,j+1}\right) + N_p \cdot N \cdot \theta_{i-1,j}}{2 \cdot NTU \cdot C^* \cdot N_{c,max} + N_p \cdot N} \tag{62}$$

For outer channels, Equations (53) and (54) were used to obtain the temperature of the counterflow channels, and Equations (55) and (56) were used for parallel flow channels.

## 5. Notation

$A$ Area of the heat exchanger [$m^2$]

$A_p$ Area of a plate [$m^2$]

$C$ Heat capacity [W/K]

$C^*$ Ratio between the lower and higher heat capacities [-]

$\hat{C}$ Ratio between the overall heat capacity and the 1-pass heat capacity [-]

$c_p$ Specific heat [J/(kg K)]

$F$ Factor of the logarithmic mean temperature difference [-]

$i$ Specific enthalpy [J/kg]

$\dot{m}$ Mass flow rate [kg/s]

$N$ Number of nodes used in Jacobi's algorithm [-]

$N_c$ Number of channels for each pass [-]

$N_p$ Number of plates [-]

$N_{pa}$ Number of passes in series [-]

$NTU$ Number of heat transfer units $NTU = UA/C_{min}$ [-]

$NTU_1$ Number of heat transfer units defined using the thermal capacity of the stream with the lower number of passes $NTU_1 = UA/C_1$ [-]

$P_1$ Temperature effectiveness of the stream with the lower number of passes $P_1 = \Delta T_1/(T_{hi} - T_{ci})$ [-]

$\dot{Q}$ Heat transfer rate [W]

$R_1$ Ratio between the thermal capacities $R_1 = C_1/C_2$ [-]

$T$ Temperature [°C]

### *5.1. Greek symbols*

$\epsilon$ Effectiveness [-]

### *5.2. Subscripts*

$c$ Countercurrent

$int$ Intermediate

$i$ Node number along the flow length

$ci$ Cold inlet

$co$ Cold outlet

$hi$ Hot inlet

$ho$ Hot outlet

$j$ Channel number

$max$ Maximum

$min$ Minimum

1 First pass of a PHE

2 Second pass of a PHE

$p$ Parallel

$1pa$ 1-pass stream

$2pa$ 2-pass stream

$o, c$ Outer channels

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## List of Figures

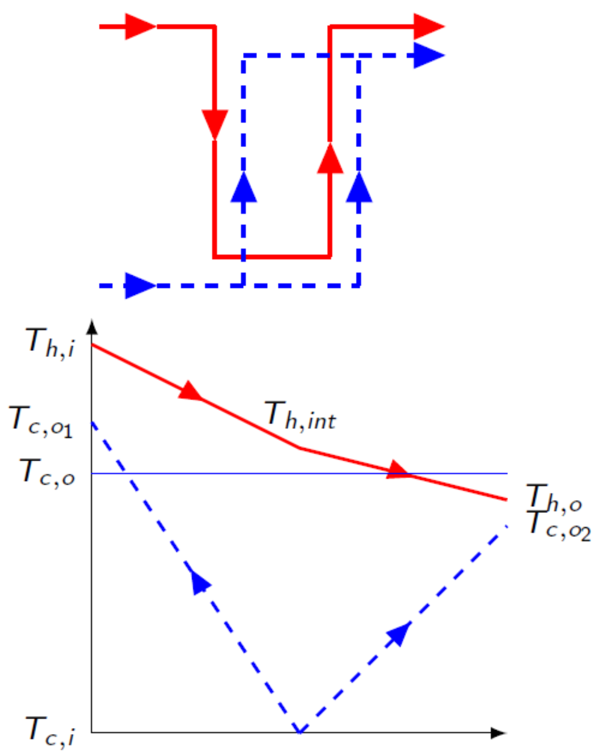
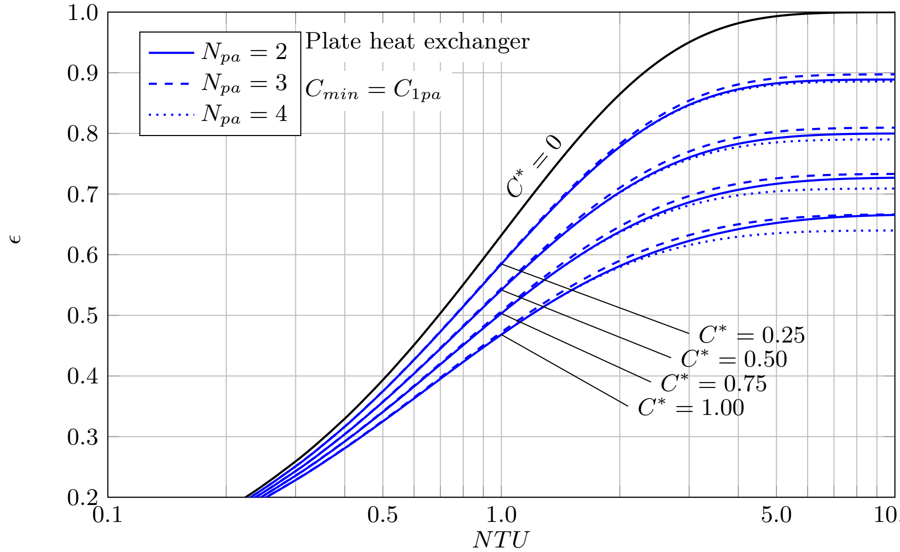
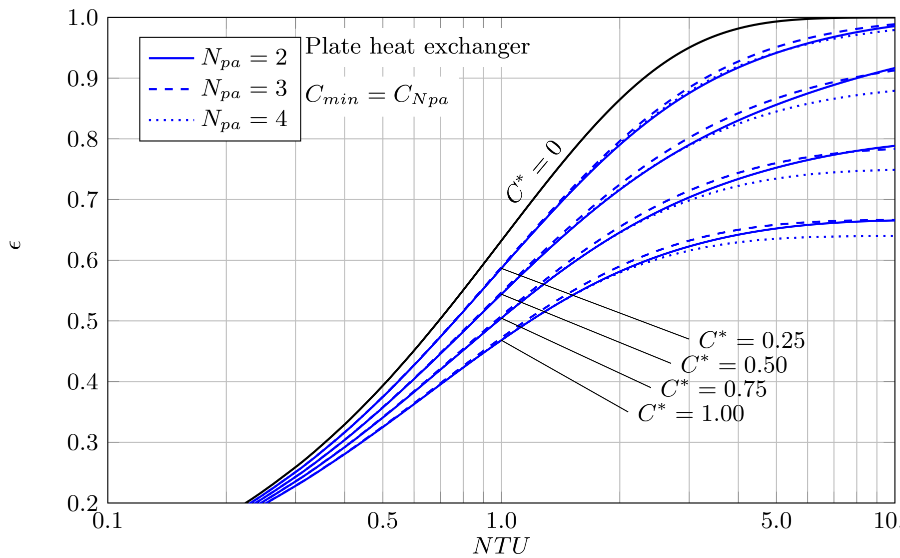



Figure 1: Flow scheme and temperature profile of a PHE in a 2 pass-1 pass configuration, with the hot fluid flowing in two passes.

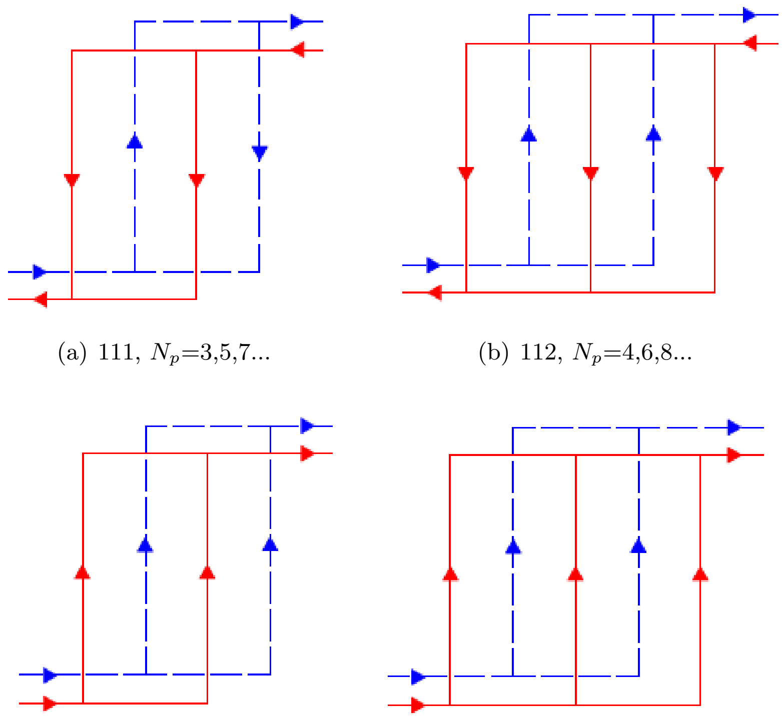



(a) $C_{min}$=$C_{1pa}$




(b) $C_{min}$=$C_{Npa}$

Figure 2: Effectiveness of PHEs as a function of $NTU$ for $N_{pa}$=2,3,4 with different values of $C^*$.



(a) 111, $N_p$=3,5,7...

(b) 112, $N_p$=4,6,8...

(c) 113, $N_p$=3,5,7...

(d) 114, $N_p$=4,6,8...

Figure 3: 1 pass-1 pass configurations.

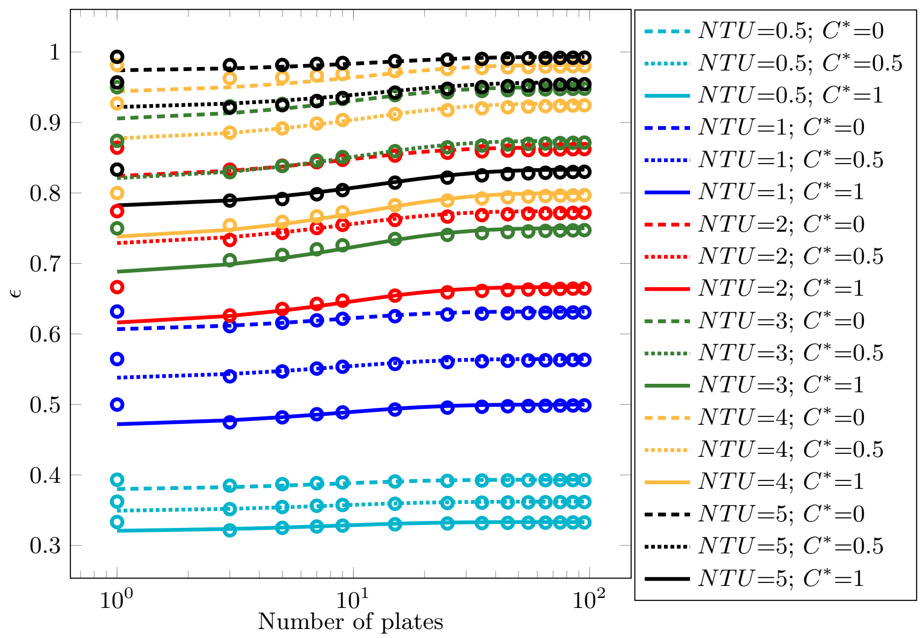
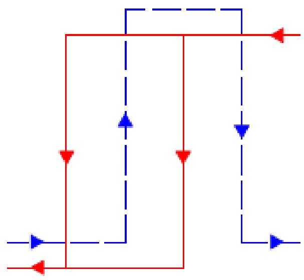
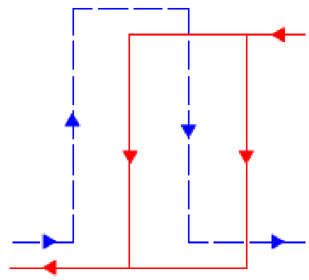


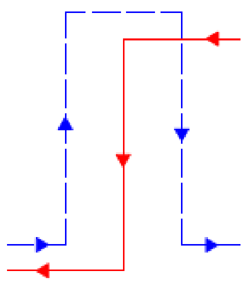

Figure 4: Numerical results (o) and fitting curves (Equation (29)) for PHEs in a 111 configuration



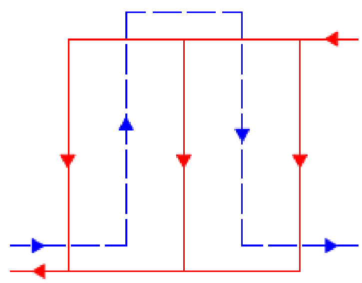
(a) 211, $N_p$=3,7,11...



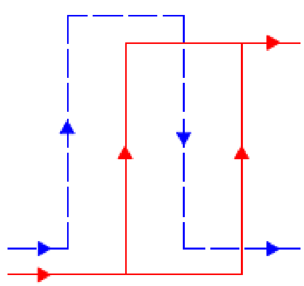
(b) 212, $N_p$=3,7,11...



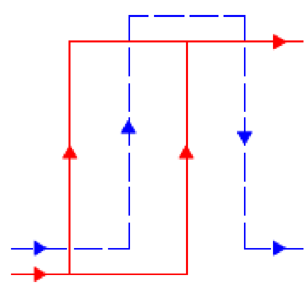
(c) 213, $N_p$=2,6,10...



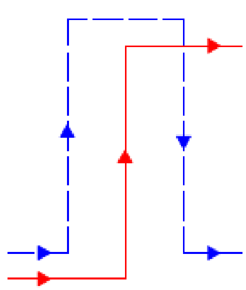
(d) 214, $N_p$=4,8,12...



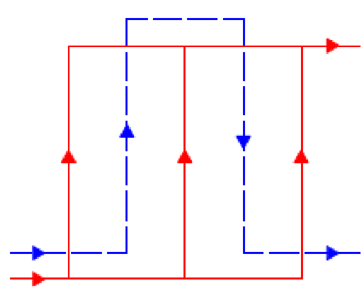
(e) 211 a, $N_p$=3,7,11...



(f) 212 a, $N_p$=3,7,11...



(g) 213 a, $N_p$=2,6,10...



(h) 214 a, $N_p$=4,8,12...

Figure 5: 2 pass-1 pass configurations.

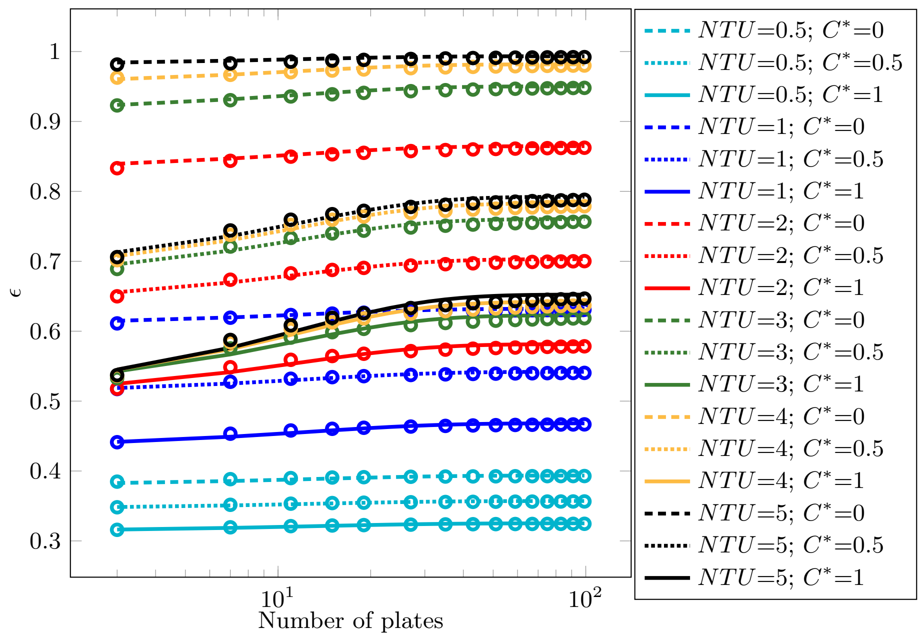
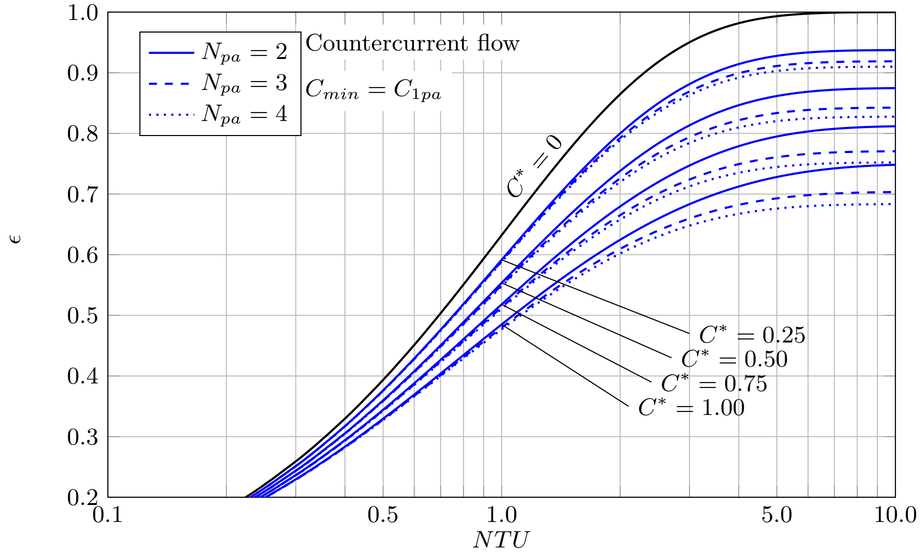
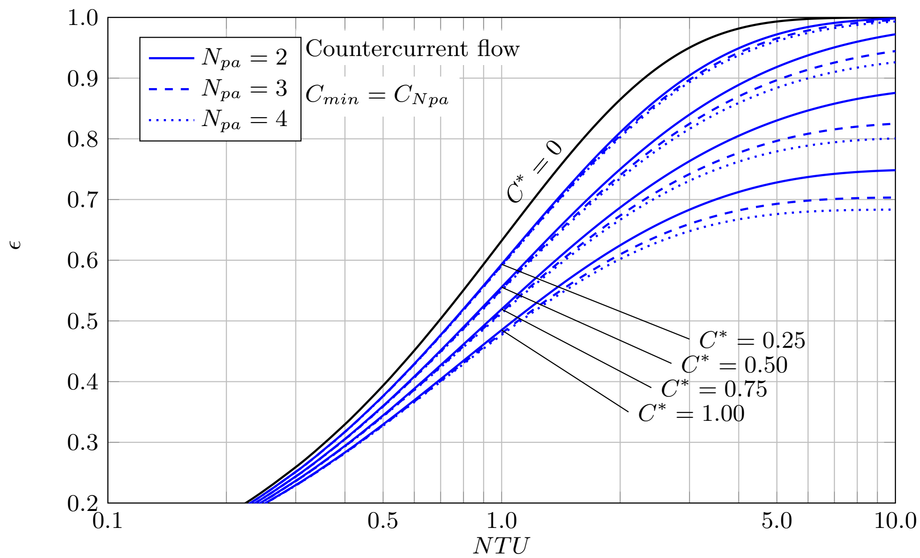



Figure 6: Numerical results (o) and fitting curves (Equation (29)) for PHEs in a 212 configuration with $C_{min}$=$C_{1pa}$.

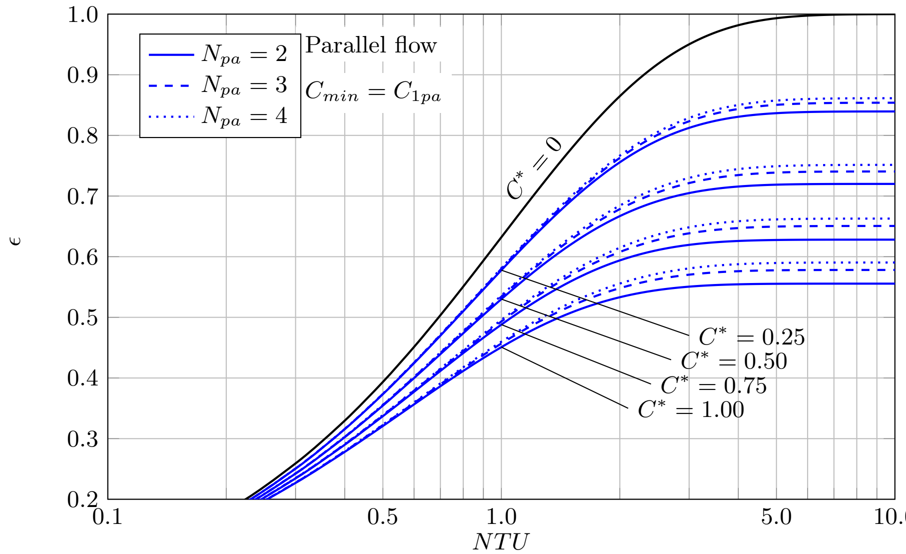



(a) $C_{min}$=$C_{1pa}$

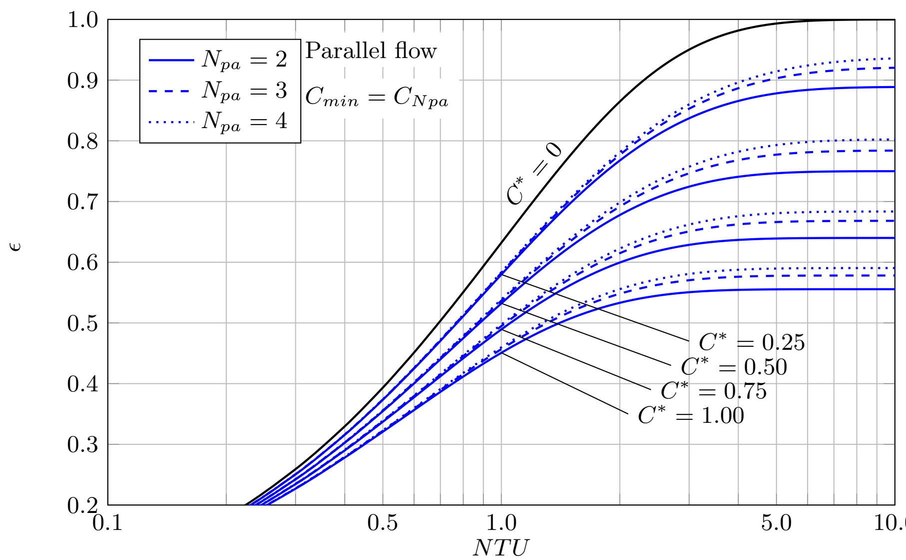



(b) $C_{min}$=$C_{Npa}$

Figure 7: Effectiveness of counterflow heat exchangers as a function of $NTU$ for $N_{pa}$=2,3,4 and different values of $C^*$.

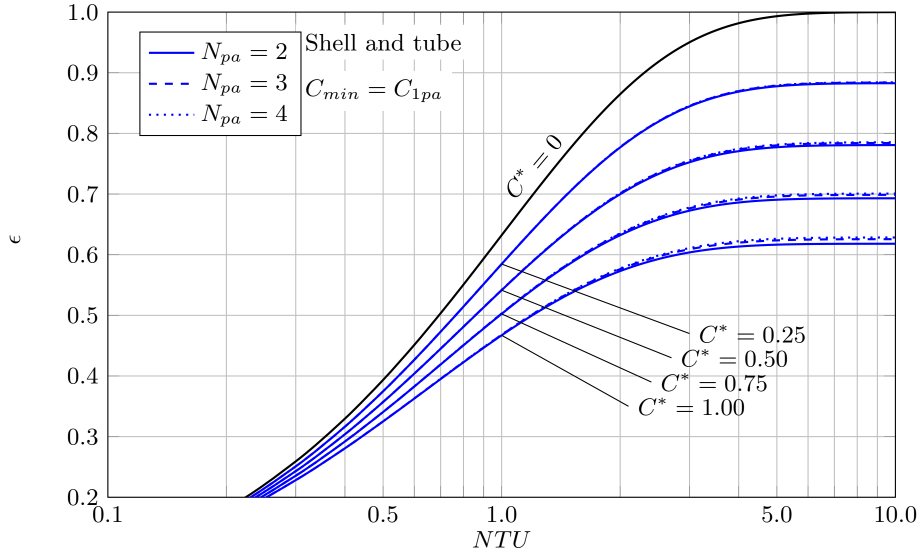



(a) $C_{min}$=C$_{1pa}$

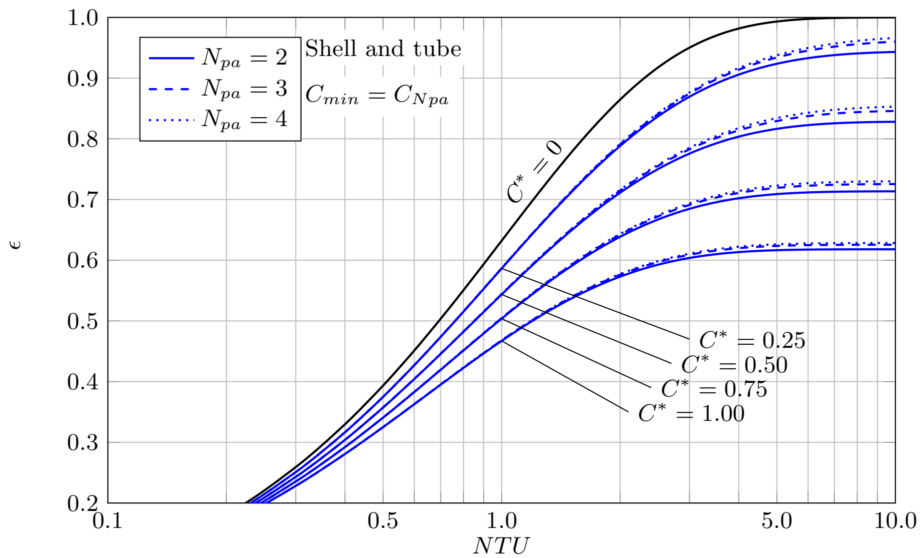



(b) $C_{min}$=C$_{Npa}$

Figure 8: Effectiveness of parallel flow heat exchangers as a function of $NTU$ for $N_{pa}$=2,3,4 and different values of $C^*$.

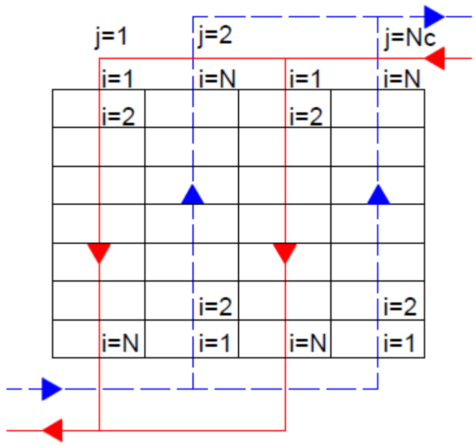



(a) $C_{min}$=C$_{1pa}$

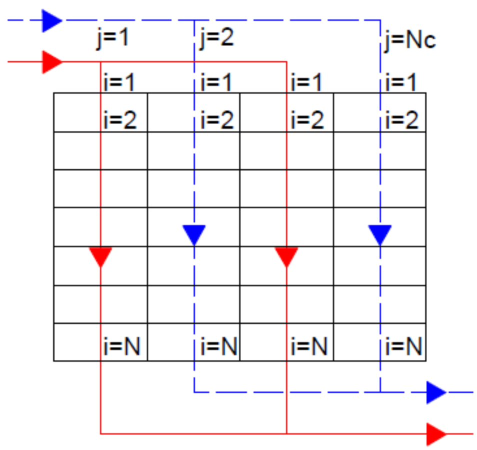



(b) $C_{min}$=C$_{Npa}$

Figure 9: Effectiveness of shell-and-tube heat exchangers (shell type E and 2, 4, 6... passes per tube) as a function of $NTU$ for $N_{pa}$=2,3,4 and different values of $C^*$.




(a) Counterflow




(b) Parallel flow

Figure 10: Grid structure for the finite difference method of PHEs in a 1 pass-1 pass configuration.

## List of Tables

| | |
|---|---|
| $C_{1pa} \leq C_{2pa}$ | $\epsilon = \frac{\epsilon_1}{2} + \frac{\epsilon_2}{2}(1 - C^* \frac{\epsilon_1}{2})$ |
| $C_{1pa} > C_{2pa}$, $C^* \leq \frac{1}{2}$ | $\epsilon = \epsilon_1 + \epsilon_2(1 - \epsilon_1)$ |
| $C_{1pa} > C_{2pa}$, $C^* > \frac{1}{2}$ | $\epsilon = \frac{\epsilon_1}{2C^*} + \frac{\epsilon_2}{2C^*}\left(1 - \frac{\epsilon_1}{2C^*}\right)$ |

Table 1: Expressions used to calculate the effectiveness of 1 pass-2 pass PHEs.

| | |
|---|---|
| $C_{1pa} \leq C_{Npa}$ | $\epsilon_1 = \dfrac{1 - \exp\left[-NTU\left(1 - \frac{C^*}{N_{pa}}\right)\right]}{1 - \frac{C^*}{N_{pa}} \exp\left[-NTU\left(1 - \frac{C^*}{N_{pa}}\right)\right]}$ |
| $C_{1pa} > C_{Npa},\, C^* < \frac{1}{N_{pa}}$ | $\epsilon_1 = \dfrac{1 - \exp\left[-\frac{NTU}{N_{pa}}\left(1 - C^* \cdot N_{pa}\right)\right]}{1 - C^* \cdot N_{pa} \exp\left[-\frac{NTU}{N_{pa}}\left(1 - C^* \cdot N_{pa}\right)\right]}$ |
| $C_{1pa} > C_{Npa},\, C^* > \frac{1}{N_{pa}}$ | $\epsilon_1 = \dfrac{1 - \exp\left[-NTU \cdot C^*\left(1 - \frac{1}{C^* \cdot N_{pa}}\right)\right]}{1 - \frac{1}{C^* \cdot N_{pa}} \exp\left[-NTU \cdot C^*\left(1 - \frac{1}{C^* \cdot N_{pa}}\right)\right]}$ |
| $C_{1pa} > C_{Npa},\, C^* = \frac{1}{N_{pa}}$ | $\epsilon_1 = \dfrac{NTU}{NTU + N_{pa}}$ |

Table 2: Expressions used to calculate the effectiveness of heat exchangers in counterflow.

| | |
|---|---|
| $C_{1pa} \leq C_{Npa}$ | $\epsilon_2 = \dfrac{1 - \exp\left[-NTU\left(1 + \frac{C^*}{N_{pa}}\right)\right]}{1 + \frac{C^*}{N_{pa}}}$ |
| $C_{1pa} > C_{Npa},\, C^* \leq \frac{1}{N_{pa}}$ | $\epsilon_2 = \dfrac{1 - \exp\left[-\frac{NTU}{N_{pa}}\left(1 + C^* \cdot N_{pa}\right)\right]}{1 + C^* \cdot N_{pa}}$ |
| $C_{1pa} > C_{Npa},\, C^* > \frac{1}{N_{pa}}$ | $\epsilon_2 = \dfrac{1 - \exp\left[-NTU \cdot C^*\left(1 + \frac{1}{C^* \cdot N_{pa}}\right)\right]}{1 + \frac{1}{C^* \cdot N_{pa}}}$ |

Table 3: Expressions used to calculate the effectiveness of heat exchangers in parallel flow.

| | | |
|---|---|---|
| $C_{1pa} \leq C_{Npa}$ | $\alpha = N_{pa}$ | $\beta = C^*$ |
| $C_{1pa} > C_{Npa}$, $C^* \leq \frac{1}{N_{pa}}$ | $\alpha = 1$ | $\beta = 1$ |
| $C_{1pa} > C_{Npa}$, $C^* > \frac{1}{N_{pa}}$ | $\alpha = N_{pa} \cdot C^*$ | $\beta = 1$ |

Table 4: Constant values of Equation (27).

| | 111 | 112 | | 113 | 114 | |
|---|---|---|---|---|---|---|
| | | $C_{o,c}$=$C_{min}$ | $C_{o,c}$=$C_{max}$ | | $C_{o,c}$=$C_{min}$ | $C_{o,c}$=$C_{max}$ |
| $a_{10} \cdot 10^4$ | -13327.045 | -13819.011 | -13678.157 | -13152.716 | -13969.267 | -13728.978 |
| $a_{11} \cdot 10^4$ | 119.794 | 105.365 | -83.045 | 52.087 | 357.385 | 140.180 |
| $a_{12} \cdot 10^4$ | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 |
| $a_{20} \cdot 10^4$ | -1200.880 | -672.853 | -688.627 | -1162.600 | -685.335 | -704.852 |
| $a_{21} \cdot 10^4$ | 89.238 | 77.426 | 48.382 | 74.746 | 105.281 | 86.543 |
| $a_{22} \cdot 10^4$ | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 |
| $a_{30} \cdot 10^4$ | -11.150 | 8.747 | 8.849 | -19.905 | 56.267 | -27.996 |
| $a_{31} \cdot 10^4$ | 269.747 | 221.760 | 9.040 | 300.320 | 140.312 | 96.956 |
| $a_{32} \cdot 10^4$ | -47.026 | -37.580 | -3.682 | -47.790 | -26.958 | -13.926 |
| $a_{40} \cdot 10^4$ | -37.211 | 14.585 | 15.748 | 153.435 | 28.508 | 104.250 |
| $a_{41} \cdot 10^4$ | 61.140 | -132.550 | 83.626 | -192.212 | -126.471 | -68.869 |
| $a_{42} \cdot 10^4$ | 0.000 | 26.342 | -9.229 | 24.895 | 22.022 | 6.511 |

Table 5: Constant values of Equation (33) for 1 pass-1 pass arrangements ($F_5$=0 for all cases)

| | 211 | | 212 | |
|---|---|---|---|---|
| | $C_{min}$=$C_{1p}$ | $C_{min}$=$C_{2p}$ | $C_{min}$=$C_{1p}$ | $C_{min}$=$C_{2p}$ |
| $a_{10} \cdot 10^4$ | 108679.639 | -14792.323 | -11358.445 | -10815.047 |
| $a_{11} \cdot 10^4$ | -20895.596 | 2729.366 | -377.143 | -854.775 |
| $a_{12} \cdot 10^4$ | 0.000 | -556.332 | 53.378 | 103.163 |
| $a_{20} \cdot 10^4$ | -716.134 | -613.099 | -717.570 | -462.797 |
| $a_{21} \cdot 10^4$ | 523.174 | 326.136 | -109.784 | -313.276 |
| $a_{22} \cdot 10^4$ | -143.486 | -108.209 | 16.277 | 40.754 |
| $a_{30} \cdot 10^4$ | 9.474 | 8.364 | 26.447 | 3.751 |
| $a_{31} \cdot 10^4$ | -31.825 | 33.039 | 198.412 | 35.903 |
| $a_{32} \cdot 10^4$ | 6.302 | -5.197 | -36.828 | -5.137 |
| $a_{40} \cdot 10^4$ | -52.972 | -316.046 | -299.596 | -536.027 |
| $a_{41} \cdot 10^4$ | 16.883 | 328.455 | 458.541 | 1011.663 |
| $a_{42} \cdot 10^4$ | 41.589 | 0.000 | 0.000 | 0.000 |
| $a_{50} \cdot 10^4$ | 41.195 | 346.585 | 160.059 | 579.564 |
| $a_{51} \cdot 10^4$ | 3.402 | -349.183 | -217.271 | -789.387 |
| $a_{52} \cdot 10^4$ | -43.106 | 0.000 | 0.000 | 0.000 |

Table 6: Constant value of Equation (33) for 1 pass-2 pass arrangements in configurations 211 and 212.

| | 213 | | 214 | |
|---|---|---|---|---|
| | $C_{min}$=$C_{1p}$ | $C_{min}$=$C_{2p}$ | $C_{min}$=$C_{1p}$ | $C_{min}$=$C_{2p}$ |
| $a_{10} \cdot 10^4$ | -5919.204 | -5818.611 | -11793.17 | -11522.997 |
| $a_{11} \cdot 10^4$ | -2667.144 | -2752.492 | -5.108 | -253.678 |
| $a_{12} \cdot 10^4$ | 333.402 | 322.214 | 29.101 | 0.000 |
| $a_{20} \cdot 10^4$ | -110.620 | -87.648 | -522.148 | -493.851 |
| $a_{21} \cdot 10^4$ | -189.871 | -240.995 | 45.141 | -72.971 |
| $a_{22} \cdot 10^4$ | 0.000 | 0.000 | -6.155 | 0.000 |
| $a_{30} \cdot 10^4$ | 11.264 | 20.314 | 47.511 | -1.435 |
| $a_{31} \cdot 10^4$ | -8.694 | 88.232 | 174.976 | 5.685 |
| $a_{32} \cdot 10^4$ | 0.000 | -15.573 | -33.369 | 0.000 |
| $a_{40} \cdot 10^4$ | -58.018 | -704.836 | -13.978 | -190.573 |
| $a_{41} \cdot 10^4$ | 157.824 | 669.103 | -113.784 | 513.253 |
| $a_{42} \cdot 10^4$ | 0.000 | 0.000 | 29.882 | 0.000 |
| $a_{50} \cdot 10^4$ | 0.000 | 570.938 | -42.923 | 259.463 |
| $a_{51} \cdot 10^4$ | 0.000 | -528.134 | 111.787 | -435.792 |
| $a_{52} \cdot 10^4$ | 0.000 | 0.000 | -11.526 | 0.000 |

Table 7: Constant value of Equation (33) for 1 pass-2 pass arrangements in configurations 213 and 214.

| | |
|---|---|
| $C_{1pa} < C_{Npa}$ | $\epsilon_1 = 2\left[1 + \frac{C}{N_{pa}} + \sqrt{1 + \left(\frac{C}{N_{pa}}\right)^2} \cdot \frac{1+\exp\left(-NTU\sqrt{1+\left(\frac{C}{N_{pa}}\right)^2}\right)}{1-\exp\left(-NTU\sqrt{1+\left(\frac{C}{N_{pa}}\right)^2}\right)}\right]^{-1}$ |
| $C_{1pa} > C_{Npa}$, $C^* \leq \frac{1}{N_{pa}}$ | $\epsilon_1 = 2\left[1 + N_{pa} \cdot C + \sqrt{1 + (N_{pa} \cdot C)^2} \cdot \frac{1+\exp\left(-\frac{NTU}{N_{pa}}\sqrt{1+(N_{pa} \cdot C)^2}\right)}{1-\exp\left(-\frac{NTU}{N_{pa}}\sqrt{1+(N_{pa} \cdot C)^2}\right)}\right]^{-1}$ |
| $C_{1pa} > C_{Npa}$, $C^* > \frac{1}{N_{pa}}$ | $\epsilon_1 = 2\left[1 + \frac{1}{N_{pa} \cdot C} + \sqrt{1 + \left(\frac{1}{N_{pa} \cdot C}\right)^2} \cdot \frac{1+\exp\left(-C \cdot NTU\sqrt{1+\left(\frac{1}{N_{pa} \cdot C}\right)^2}\right)}{1-\exp\left(-C \cdot NTU\sqrt{1+\left(\frac{1}{N_{pa} \cdot C}\right)^2}\right)}\right]^{-1}$ |

Table 8: Expressions used to calculate the effectiveness of shell-and-tube heat exchangers (shell type E and 2, 4, and 6... passes per tube).