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Source enumeration in the presence of colored noise

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Abstract—In array signal processing the detection of the number of sources is an important step. Most approaches assume the signals to be embedded in white noise. However, this assumption is unrealistic in many scenarios. In this paper, we propose a strategy that can handle colored noise. We model the source detection as a regression problem and apply information-theoretic criteria to determine the model order of the regression. We show simulations of different scenarios, where our approach outperforms traditional techniques.

Index Terms—Array signal processing, colored noise, information-theoretic criteria, model-order selection, multivariate regression.

I. INTRODUCTION

Array signal processing comprises many and varied applications, such as radar, sonar, multiple-input multiple-output (MIMO) wireless communications, and electroencephalography (EEG) [1]. In many cases, they share a mathematical framework: The observed data is modeled as a superposition of a finite number of independent sources that are embedded in additive noise. One of the key processing steps for many of these applications is to estimate the number of source signals [2]. These signals and the noise are often assumed to be random processes with certain statistical properties. In this context, it is common to assume that the noise is spatially white, meaning that its covariance matrix is $\sigma^2\mathbf{I}$, and the noise eigenvalues are identical and equal to σ^2 .

Yet white noise is unrealistic in many applications. Noise can be filtered, and there are certain cases where it shows structure. For example, in undersea sonar, shipping noise is directional [1]. Also, often different sensors experience different noise power [3], and there typically exists stronger correlation between sensors that are close [4]. When the noise is arbitrarily spatially correlated, the problem may become untractable [4]. However, if the noise is sufficiently weaker than the signal, detection is still possible. There are few approaches in the literature for the case of colored noise. Furthermore, in such approaches, certain structure in the noise covariance matrix is typically assumed. In [3] the authors considered non-uniform noise, which is still spatially uncorrelated. In [5] the noise model is colored, but two arrays of sensors that have uncorrelated noise between each other are assumed. Similarly in [4], the authors modeled the noise covariance matrix as

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MATLAB code is available at <https://github.com/SSTGroup/Source-detection-in-colored-noise>

block-diagonal and assumed to know the block size. The authors in [6] considered colored noise in their model, but required a different data set with noise-only observations.

We design a new strategy to detect the number of sources, which does not require an independent noise-only observation set or uncorrelated arrays, and which allows an unknown noise covariance matrix. The only requirement is that the noise is sufficiently weaker than the signal, which is also an implicit requirement of existing techniques. Our approach is inspired by statistical shape modeling [7] used in computer vision. We model the signal using multivariate linear regression and then separate signal from noise by regarding everything that looks non-representative as noise.

In this paper we have extended and validated the formulation for shape models that we proposed in [8], such that the technique is theoretically justified and generalized to array processing applications.

II. PROBLEM FORMULATION

We consider an array of N sensors and the observed N -dimensional vector $\mathbf{x}(m)$, which corresponds to the m th observation. We observe M observations according to model

$$\mathbf{x}(m) = \mathbf{A}\mathbf{s}(m) + \mathbf{n}(m), \quad (1)$$

where $\mathbf{A} = [\mathbf{a}_1, \dots, \mathbf{a}_q]$ is the array mixture matrix with full column rank, and $\mathbf{s}(m) = [s_1(m), \dots, s_q(m)]^T$ is the vector of sources. The number of sources, q , is unknown, and we wish to determine it. These sources are modeled as unknown Gaussian random variables [2]. The q source signals are assumed to be zero-mean and independent. Typically, the noise vector $\mathbf{n}(m)$ is assumed to be independent of the sources, Gaussian and white, i.e., with covariance matrix $\sigma^2\mathbf{I}$. In that case the $N - q$ smallest eigenvalues of the covariance matrix $\mathbf{R}_{xx} = \mathbf{E}[\mathbf{x}\mathbf{x}^H]$ correspond to the noise subspace and are equal to σ^2 . The separation of signal from noise is typically based on the assumption that the noise eigenvalues are identical [2], [9]. This does not work if the noise is not white.

Let us now assume the noise $\mathbf{n}(m)$ is colored, i.e., a zero-mean random vector with an arbitrary covariance matrix Σ . If the source vector $\mathbf{s}(m)$ has covariance matrix \mathbf{R}_{ss} , the covariance matrix of the observed data $\mathbf{x}(m)$ is

$$\mathbf{R}_{xx} = \mathbf{A}\mathbf{R}_{ss}\mathbf{A}^H + \Sigma. \quad (2)$$

If the signal-to-noise ratio (SNR) is sufficiently high, the signal subspace is at least approximately spanned by the q eigenvectors corresponding to the q largest eigenvalues of \mathbf{R}_{xx} . However, the problem is that, in order to separate signal

from noise eigenvalues, the approaches in [2] and [9] need to assume that the $N - q$ smallest eigenvalues (due to noise) are equal. If this is not the case, they will fail. We now propose a strategy that can estimate q accurately under these conditions.

III. PROPOSED SOLUTION

Inspired by statistical shape models [7], we propose an interpretation of the problem as a multivariate regression. In [7], the eigenvectors of the sample covariance matrix obtained from the observed (training) data determine the regression parameters. Furthermore, [7] places constraints on the regression in order to ensure that the values of the explained variables are sufficiently likely considering the examples seen in the training data.

A. Regression interpretation

We separate the observed data in two subsets of sizes M_1 and M_2 , with $M_1 + M_2 = M$, in order to obtain M_1 training and M_2 test observations. In Section II of the supplementary material, we show that our method is fairly robust with respect to the precise split into M_1 and M_2 , and choosing M_1 of similar size as M_2 works best. Thus, these subsets are $\mathbf{X}_1 = [\mathbf{x}(1), \dots, \mathbf{x}(M_1)] \in \mathbb{C}^{N \times M_1}$, and $\mathbf{X}_2 = [\mathbf{x}(M_1 + 1), \dots, \mathbf{x}(M_1 + M_2)] \in \mathbb{C}^{N \times M_2}$. Within the first set we obtain the matrix of regressors, for which we estimate the sample covariance matrix of the observed data, $\hat{\mathbf{R}}_{xx} = \frac{1}{M_1} \mathbf{X}_1 \mathbf{X}_1^H$, and compute its eigenvalue decomposition as $\hat{\mathbf{R}}_{xx} = \mathbf{P} \mathbf{\Lambda} \mathbf{P}^H$. We also define $\boldsymbol{\lambda} = [\lambda_1, \dots, \lambda_q, \dots, \lambda_N]$ as the diagonal of $\mathbf{\Lambda}$, with eigenvalues in decreasing order. These will determine constraints on the regression. We assume the first q values in $\boldsymbol{\lambda}$ correspond to signal and the remaining $N - q$ to noise.

Following the model in [7], let us assume the following interpretation of (1) for a fixed q :

$$\mathbf{x}(m) = \mathbf{P}_q \mathbf{b}_q(m) + \mathbf{e}(m), \quad (3)$$

where \mathbf{P}_q is the matrix of regressors containing the eigenvectors corresponding to the q largest eigenvalues of $\hat{\mathbf{R}}_{xx}$, $\mathbf{b}_q(m) \in \mathbb{C}^q$ is a deterministic vector of regression parameters, and $\mathbf{e}(m) \sim \mathcal{CN}(\mathbf{0}, \boldsymbol{\Sigma}_q)$ is the residual error. Let $\boldsymbol{\epsilon} = [\epsilon_1, \dots, \epsilon_N]$ denote the eigenvalues of $\boldsymbol{\Sigma}_q$, arranged in descending order. In order to make sure that $\mathbf{x}(m)$ generated by (3) have statistics sufficiently similar to the observations from the first split \mathbf{X}_1 , the following structure is assumed in the regression [10]:

- 1) An energy constraint is placed on the vector of unknown parameters $\mathbf{b}_q(m) = [b_1, \dots, b_q]^T$, such that $|b_n|^2 \leq \lambda_n, \forall n = 1 \dots q$ [7].
- 2) $\boldsymbol{\Sigma}_q$ is assumed to be full rank. In order to enforce this, its smallest eigenvalue ϵ_N is constrained to be $\epsilon_N \geq \lambda_N$.

Considering the samples in \mathbf{X}_2 , we define the following multivariate linear regression:

$$\mathbf{X}_2 = \mathbf{P}_q \mathbf{B}_q + \mathbf{E}, \quad (4)$$

where $\mathbf{B}_q = [\mathbf{b}_q(1), \dots, \mathbf{b}_q(M_2)] \in \mathbb{C}^{q \times M_2}$ is the matrix of unknown regression parameters, and the columns of the error

matrix $\mathbf{E} = [\mathbf{e}(1), \dots, \mathbf{e}(M_2)] \in \mathbb{C}^{N \times M_2}$ are samples of the residual error $\mathbf{e}(m)$.

The problem is therefore to determine the model order q of the regression in (4), which is equal to the number of source signals. In this interpretation, \mathbf{P}_q is assumed to be a known matrix of regressors and \mathbf{B}_q is modeled as a deterministic matrix of unknown parameters. In order to determine the best trade-off between over- and underfitting, we propose an information-theoretic criterion for multivariate regression [11], which we modify in order to include the constraints and the proposed colored noise model. An information-theoretic criterion is composed of a likelihood and a penalty term, and the best model order q^* is obtained as

$$q^* = \arg \min_q \left(\underbrace{-\log p(\mathbf{X}_2 | \hat{\mathbf{B}}_q, \hat{\boldsymbol{\Sigma}}_q)}_{\text{likelihood term}} + \underbrace{\eta(q)}_{\text{penalty}} \right), \quad (5)$$

where $\hat{\mathbf{B}}_q$ and $\hat{\boldsymbol{\Sigma}}_q$ are the constrained Maximum Likelihood (ML) estimates of the model parameters for model order q , and $p(\cdot)$ stands for the likelihood function. The penalty $\eta(q)$ depends on the selected criterion (Akaike, Bayesian, etc. [12]). We assume a frequentist interpretation of the regression. Thus, the only random variable in the regression is the residual error matrix $\mathbf{E} = \mathbf{X}_2 - \mathbf{P}_q \mathbf{B}_q$, which we have defined as samples of a complex Gaussian distribution with covariance matrix $\boldsymbol{\Sigma}_q$. Consequently, the log-likelihood expression in (5), ignoring the constant terms, becomes

$$\begin{aligned} \log p(\mathbf{X}_2 | \hat{\mathbf{B}}_q, \hat{\boldsymbol{\Sigma}}_q) &= -M_2 \log |\hat{\boldsymbol{\Sigma}}_q| \\ &\quad - \text{Tr}\{(\mathbf{X}_2 - \mathbf{P}_q \hat{\mathbf{B}}_q)^H \hat{\boldsymbol{\Sigma}}_q^{-1} (\mathbf{X}_2 - \mathbf{P}_q \hat{\mathbf{B}}_q)\}, \end{aligned} \quad (6)$$

where $\text{Tr}\{\cdot\}$ stands for trace of a matrix.

B. ML estimates of the model parameters

The ML estimates of parameters $\boldsymbol{\Sigma}_q$ and \mathbf{B}_q are mutually dependent. Therefore we propose an alternating optimization (AO), in which the optimal solution for each subproblem is obtained to ensure the convergence.

1) *Estimate of $\boldsymbol{\Sigma}_q$* : This is obtained from the expression

$$\begin{aligned} \hat{\boldsymbol{\Sigma}}_q &= \arg \min_{\boldsymbol{\Sigma}_q \succeq \lambda_N \mathbf{I}} \left[M_2 \log |\boldsymbol{\Sigma}_q| + \right. \\ &\quad \left. \text{Tr}\{(\mathbf{X}_2 - \mathbf{P}_q \hat{\mathbf{B}}_q)^H \boldsymbol{\Sigma}_q^{-1} (\mathbf{X}_2 - \mathbf{P}_q \hat{\mathbf{B}}_q)\} \right], \end{aligned} \quad (7)$$

where $\boldsymbol{\Sigma}_q \succeq \lambda_N \mathbf{I}$ denotes that $\boldsymbol{\Sigma}_q - \lambda_N \mathbf{I}$ is positive semi-definite. The optimal solution to (7) is derived from the unconstrained ML estimate of $\boldsymbol{\Sigma}_q$ [13], which is the sample covariance matrix, that is $\hat{\boldsymbol{\Sigma}}_q^u = \frac{1}{M_2} (\mathbf{X}_2 - \mathbf{P}_q \hat{\mathbf{B}}_q) (\mathbf{X}_2 - \mathbf{P}_q \hat{\mathbf{B}}_q)^H$. Let its eigenvalue decomposition be $\hat{\boldsymbol{\Sigma}}_q^u = \mathbf{V} \boldsymbol{\Psi} \mathbf{V}^H$. The eigenvalues in $\boldsymbol{\Psi}$ that are smaller than λ_N are replaced with λ_N . As shown in Section I-A of the supplementary material, based on results in [13], this leads to the solution

$$\hat{\boldsymbol{\Sigma}}_q = \mathbf{V} \tilde{\boldsymbol{\Psi}} \mathbf{V}^H, \quad [\tilde{\boldsymbol{\Psi}}]_{nn} = \max([\boldsymbol{\Psi}]_{nn}, \lambda_N), \quad \forall n = 1 \dots N, \quad (8)$$

where $[\boldsymbol{\Psi}]_{nn}$ denotes the element on the diagonal of $\boldsymbol{\Psi}$. This $\hat{\boldsymbol{\Sigma}}_q$ is the optimal solution to (7).

2) *Estimate of \mathbf{B}_q* : The constrained ML estimation of \mathbf{B}_q is obtained by maximizing the likelihood given in (6), or, equivalently, solving the generalized least-squares problem

$$\hat{\mathbf{B}}_q = \arg \min_{\mathbf{B}_q \in \mathbb{B}(\boldsymbol{\lambda})} \text{Tr}\{(\mathbf{X}_2 - \mathbf{P}_q \mathbf{B}_q)^H \hat{\boldsymbol{\Sigma}}_q^{-1} (\mathbf{X}_2 - \mathbf{P}_q \mathbf{B}_q)\}, \quad (9)$$

where $\mathbb{B}(\boldsymbol{\lambda}) = \{\mathbf{y}(1), \dots, \mathbf{y}(M_2)\} \in \mathbb{C}^{q \times M_2} : |y_n(m)|^2 \leq \lambda_n, \forall n = 1 \dots q, \forall m = 1 \dots M_2\}$. Even though the foregoing problem is convex and hence its optimal solution can be found by standard numerical methods [14], the computational cost is expensive since it does not admit a closed-form solution. As an alternative, we propose a more focused algorithm based on majorization-minimization (MM) [15] that is faster and converges to the optimal solution. Considering the least-squares form in (9), and $\mathbf{E} = \mathbf{X}_2 - \mathbf{P}_q \mathbf{B}_q$, we construct the surrogate function $g(\mathbf{E}|\mathbf{E}_{(t)})$ that is derived from the second-order Taylor expansion of (9). Thus, the cost function in (9) is upper-bounded by $g(\mathbf{E}|\mathbf{E}_{(t)})$, which is derived in [15] as

$$g(\mathbf{E}|\mathbf{E}_{(t)}) = \text{Tr}\{\mathbf{E}^H \boldsymbol{\Sigma}_{\min}^{-1} \mathbf{E} + 2\text{Re}[\mathbf{E}(\hat{\boldsymbol{\Sigma}}_q^{-1} - \boldsymbol{\Sigma}_{\min}^{-1})\mathbf{E}_{(t)}] + \mathbf{E}_{(t)}^H (\boldsymbol{\Sigma}_{\min}^{-1} - \hat{\boldsymbol{\Sigma}}_q^{-1}) \mathbf{E}_{(t)}\}, \quad (10)$$

where $\boldsymbol{\Sigma}_{\min}^{-1} = \frac{1}{\epsilon_N} \mathbf{I}$. The original cost function in (9) is equal to (10) only when $\mathbf{E} = \mathbf{E}_{(t)}$. At each iteration of the MM algorithm, the feasible points $\mathbf{E}_{(t)}$ are updated as

$$\mathbf{E}_{(t+1)} = \arg \min_{\mathbf{B}_q \in \mathbb{B}(\boldsymbol{\lambda})} g(\mathbf{E}|\mathbf{E}_{(t)}). \quad (11)$$

The above problem is convex and satisfies Slater's condition, i.e., the feasible set has a non-empty interior. Therefore, the Karush-Kuhn-Tucker (KKT) conditions are necessary and sufficient for optimality [14]. Through these conditions it is shown, with the details provided in Section I-B of the supplementary material, that the optimal solution to (11) can be obtained by scaling the unconstrained solution. This unconstrained solution can be obtained straightforwardly as

$$\begin{aligned} \mathbf{B}_{q(t+1)}^u &= \epsilon_N \mathbf{P}_q^H (\boldsymbol{\Sigma}_{\min}^{-1} \mathbf{X}_2 + (\hat{\boldsymbol{\Sigma}}_q^{-1} - \boldsymbol{\Sigma}_{\min}^{-1}) \mathbf{E}_{(t)}) \\ &= \epsilon_N (\mathbf{P}_q^H \hat{\boldsymbol{\Sigma}}_q^{-1} \mathbf{X}_2 - \mathbf{P}_q^H (\boldsymbol{\Sigma}_{\min}^{-1} - \hat{\boldsymbol{\Sigma}}_q^{-1}) \mathbf{P}_q \mathbf{B}_{q(t)}). \end{aligned} \quad (12)$$

The optimal solution $\mathbf{B}_{q(t+1)}$ is therefore obtained by scaling each element of $\mathbf{B}_{q(t+1)}^u$ such that the constraints are fulfilled. That is, each column $\mathbf{b}_{q(t+1)}(m)$ of $\mathbf{B}_{q(t+1)}$ is

$$\mathbf{b}_{q(t+1)}(m) = \tilde{\mathbf{b}}, \quad \tilde{b}_n = \frac{b_n^u(m)}{|b_n^u(m)|} \times \min(|b_n^u(m)|, \sqrt{\lambda_n}) \quad \forall n = 1, \dots, q \quad \forall m = 1, \dots, M_2, \quad (13)$$

where \tilde{b}_n is the n th element of $\tilde{\mathbf{b}}$, and $b_n^u(m)$ is the n th element of the m th column of $\mathbf{B}_{q(t+1)}^u$. The resulting ML estimate $\hat{\mathbf{B}}_q$ is then given by the value of $\mathbf{B}_{q(t+1)}$ upon convergence.

C. Determining the model order

As shown in (7) and (9), the ML estimates $\hat{\mathbf{B}}_q$ and $\hat{\boldsymbol{\Sigma}}_q$ are mutually dependent, which prohibits finding a closed-form solution. We propose an AO algorithm (see Algorithm 1) to

Data: Inputs are \mathbf{P}_q , $\boldsymbol{\lambda}$, \mathbf{X}_2 , i_{\max} and c_{\min}

Result: $\hat{\mathbf{B}}_q$ and $\hat{\boldsymbol{\Sigma}}_q$

initialization $i = 1$, $\hat{\boldsymbol{\Sigma}}_q^{(0)} = \mathbf{I}$, $c > c_{\min}$

while $c > c_{\min}$ **and** $i < i_{\max}$ **do**

1) Obtain $\hat{\mathbf{B}}_q^{(i)}$ as in (9) with MM.

2) Obtain $\hat{\boldsymbol{\Sigma}}_q^{(i)}$ as in (7) and (8).

3) $c = \|\hat{\boldsymbol{\Sigma}}_q^{(i)} - \hat{\boldsymbol{\Sigma}}_q^{(i-1)}\|_F$.

4) $i = i + 1$.

end

Algorithm 1: Alternating optimization to obtain constrained ML estimates of \mathbf{B}_q and $\boldsymbol{\Sigma}_q$.

maximize the log-likelihood in (5). Since $\hat{\mathbf{B}}_q$ and $\hat{\boldsymbol{\Sigma}}_q$ are obtained optimally, the algorithm is guaranteed to converge. In summary, the procedure to estimate q is as follows. For every possible model order q (with $q = 1, \dots, \min(N, M_1) - 1$), we compute the ML estimates $\hat{\mathbf{B}}_q$ and $\hat{\boldsymbol{\Sigma}}_q$ by means of Algorithm 1. Then, following (5), we estimate the model order as

$$q^* = \arg \min_q \left[M_2 \log |\hat{\boldsymbol{\Sigma}}_q| + \text{Tr}(\hat{\mathbf{E}}^H \hat{\boldsymbol{\Sigma}}_q^{-1} \hat{\mathbf{E}}) + \eta(q) \right], \quad (14)$$

where $\hat{\mathbf{E}} = \mathbf{X}_2 - \mathbf{P}_q \hat{\mathbf{B}}_q$.

D. Sample-poor case

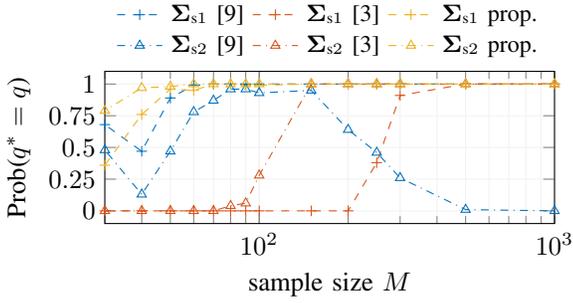
Sometimes we have access to only a small number M of snapshots relative to the number N of sensors. Under these circumstances, the covariance matrix of residuals in (7) may be ill-conditioned. In order to address this situation and reduce the number of parameters to estimate in (7), we propose the following solution. We model $\boldsymbol{\Sigma}_q$ as block-diagonal (even though it has an arbitrary structure) with blocks of size $s \times s$. This results in $d = \lceil \frac{N}{s} \rceil$ blocks, that is $\boldsymbol{\Sigma}_q = \text{blkdiag}\{\boldsymbol{\Sigma}_q^1, \dots, \boldsymbol{\Sigma}_q^d\}$, where the size of the last block $\boldsymbol{\Sigma}_q^d$ differs from the other blocks if $\frac{N}{s}$ is not an integer. The size s of the blocks gives a trade-off between a well-conditioned and a biased estimate of the true $\boldsymbol{\Sigma}_q$. Considering this, the ML estimate $\hat{\boldsymbol{\Sigma}}_q$ is simply the block-diagonal version of the matrix defined in (7). We reformulate (14) such that the model order q^* is obtained as

$$q_{\text{sp}}^* = \arg \min_q \left[\sum_{n=1}^d (M_2 \log(|\hat{\boldsymbol{\Sigma}}_q^n|) + \text{Tr}(\hat{\mathbf{E}}^H (\hat{\boldsymbol{\Sigma}}_q^n)^{-1} \hat{\mathbf{E}})) + \eta_{\text{sp}}(q) \right], \quad (15)$$

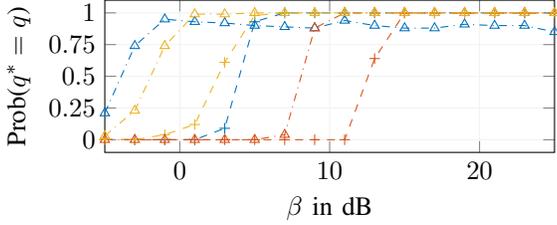
where the subscript of q_{sp}^* stands for sample poor.

IV. RESULTS AND DISCUSSION

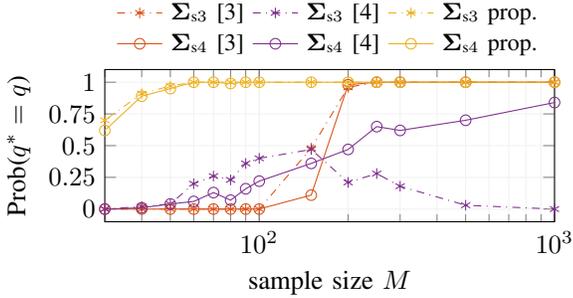
We consider the model in (1) to generate simulated observations. We assume the matrix \mathbf{A} to be unitary (in order to control the SNR) and choose it as uniformly distributed on the unitary group [16]. We fix the number of sources at $q = 10$. We generate random values uniformly distributed between 1 and 10 as the signal eigenvalues in \mathbf{R}_{ss} . We propose relatively sample-poor scenarios, so our estimate in



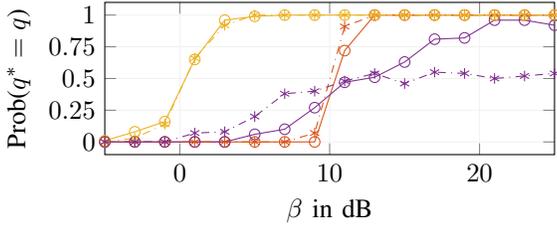
(a) Sample size variation with noise covariance matrices Σ_{s1} and Σ_{s2} .



(b) β variation with noise covariance matrices Σ_{s1} and Σ_{s2} .



(c) Sample size variation with noise covariance matrices Σ_{s3} and Σ_{s4} .



(d) β variation with noise covariance matrices Σ_{s3} and Σ_{s4} .

Fig. 1. Comparison of the performance of the proposed (prop.) estimator with the ones in [9], [3], and [4], depicted in different colors. We show results averaged over 100 simulations with parameter setting $N = 50$ and $q = 10$. In (a) and (c): $\beta = 10$ dB and we vary M . In (b) and (d): $M = 150$ and we vary β . We generate four different noise structures.

the presented simulations is based on q_{sp}^* in (15) with block size $s = 2$. Consequently, the number of parameters to estimate in Σ_q is reduced, and its estimate is less biased than a diagonal estimate. We choose the Akaike information criterion [17], as in [9], so $\eta_{sp}(q) = 2M_2q + \frac{3}{2}N$, which is the number of degrees of freedom in the likelihood expression. To simulate the noise, we randomly choose $\mathbf{R}_{nn} = N \frac{\mathbf{U}\mathbf{\Omega}\mathbf{U}^H}{\text{Tr}\{\mathbf{\Omega}\}}$. The trace of \mathbf{R}_{nn} is N , \mathbf{U} is uniformly distributed on the unitary group, and $\mathbf{\Omega}$ is a diagonal matrix of N eigenvalues

uniformly distributed between 0.1 and 1. We consider different structures of covariance matrices Σ to address four models of noise: white noise, $\Sigma_{s1} = \mathbf{I}$; a mixture of white and colored noise, $\Sigma_{s2} = \mathbf{I} + \mathbf{R}_{nn}$; colored noise, $\Sigma_{s3} = \mathbf{R}_{nn}$; and colored noise with block-diagonal structure, Σ_{s4} , whose blocks are equal to those along the block-diagonal of \mathbf{R}_{nn} . Finally, we scale the noise observations in order to accomplish the SNR settings. We define the SNR by a parameter β that is the ratio of the smallest signal eigenvalue in \mathbf{R}_{ss} and the largest noise eigenvalue in Σ .

We compare the performance of our technique with three other model-order selection strategies: [9], which assumes additive white Gaussian noise in sample-poor scenarios; [3], which considers non-uniform, i.e., spatially uncorrelated noise with diagonal (but not identity) covariance matrix; and [4], which allows colored noise, but assumes that the noise is block-diagonal with known block size. In Fig. 1 we illustrate the performance with respect to the number of samples for a fixed SNR parameter β , as well as with respect to β for a fixed number of samples. In Figs. 1(a) and 1(b) the noise structures are Σ_{s1} (white noise, optimal conditions for the strategy in [9]) and Σ_{s2} . In Figs. 1(c) and 1(d) we consider Σ_{s3} (noise covariance without structure) and Σ_{s4} (block-diagonal, which constitutes the optimal conditions for the strategy in [4]; we choose block size $\frac{N}{2}$). In Figs. 1(a) and 1(b) we see that the strategy based on white noise [9] fails when the noise is colored. While having similar performance to our strategy when the noise is white, in the case of colored noise its performance is poor even when the noise is considerably weaker than the signal. Similarly, in Figs. 1(c) and 1(d), the block-diagonal strategy [4] fails when the covariance matrix of the noise is not block-diagonal. Even when the noise is block-diagonal, unlike our strategy, [4] needs to know the actual block-size. It also requires more samples and larger β for a performance similar to our proposed technique. The strategy [3], which assumes non-uniform noise, delivers good results for large β and large number of samples. However, our strategy requires fewer samples and smaller β to achieve the same probability of correct model-order choice.

V. CONCLUSION

Colored noise is more realistic than white noise. However, it is not typically considered in source detection. We provided a technique for determining the number of sources when the noise observations have an unknown covariance matrix. Inspired by statistical shape models, we considered a multivariate regression and the order was inferred from the covariance matrix of the residual error. Simulations showed that our technique outperforms competing strategies in a variety of scenarios.

VI. ACKNOWLEDGMENT

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I. SUPPLEMENTARY PROOFS

A. Solution to (7)

We address the optimization problem given in (7). Notice that $\Sigma_q \succeq \lambda_N \mathbf{I} \Leftrightarrow \Sigma_q^{-1} \preceq \lambda_N^{-1} \mathbf{I}$. Thus, the optimization problem in (7) is equivalent to

$$\underset{0 \preceq \tilde{\Sigma}_q \preceq \lambda_N^{-1} \mathbf{I}}{\text{minimize}} \quad -\log |\tilde{\Sigma}_q| + \text{Tr}(\mathbf{S} \tilde{\Sigma}_q), \quad (16)$$

where $\tilde{\Sigma}_q = \Sigma_q^{-1}$ and $\mathbf{S} = M_2^{-1}(\mathbf{X}_2 - \mathbf{P}_q \hat{\mathbf{B}}_q)(\mathbf{X}_2 - \mathbf{P}_q \hat{\mathbf{B}}_q)^H$. The above problem is similar to the one considered in Lemma 3.2 of [13]. Following the proof provided in [13, Appendix B], we readily obtain

$$\tilde{\Sigma}_q = \mathbf{V} \tilde{\Psi}' \mathbf{V}^H, \quad [\tilde{\Psi}']_{nn} = \min\{([\Psi]_{nn})^{-1}, \lambda_N^{-1}\}, \quad \forall n, \quad (17)$$

where \mathbf{V} and Ψ contain the eigenvectors and eigenvalues, respectively, of \mathbf{S} . Therefore, the optimal solution of (7) is

$$\Sigma_q = \tilde{\Sigma}_q^{-1} = \mathbf{V} \tilde{\Psi} \mathbf{V}^H, \quad [\tilde{\Psi}]_{nn} = \max\{([\Psi]_{nn}), \lambda_N\}, \quad \forall n. \quad (18)$$

B. Solution to (11)

We address the optimization problem given in (11). Its objective function, $g(\mathbf{E}|\mathbf{E}_{(t)})$, is convex since Σ_{\min} is positive definite. Additionally, the feasible set is also convex, which makes (11) a convex optimization problem. Furthermore, it satisfies Slater's condition, i.e., the feasible set has a non-empty interior, which implies that the Karush-Kuhn-Tucker (KKT) conditions are necessary and sufficient for optimality [14]. Hence, we start with the Lagrangian function, which is given by

$$\mathcal{L} = g(\mathbf{E}|\mathbf{E}_{(t)}) + \sum_{m=1}^{M_2} \text{Tr} \left[\Phi_m \left(\mathbf{b}_q(m) \mathbf{b}_q(m)^H - \tilde{\lambda} \tilde{\lambda}^H \right) \right], \quad (19)$$

where $\tilde{\lambda} = [\sqrt{\lambda_1}, \dots, \sqrt{\lambda_q}]^T$, and Φ_m is a diagonal matrix with diagonal $\phi(m)$, whose n th element $[\phi(m)]_n$ is the Lagrange multiplier associated with the constraint $|\mathbf{b}_q(m)_n|^2 \leq \lambda_n$. The derivative of the Lagrangian with respect to $\mathbf{b}_q^*(m)$ can be shown to be given by

$$\frac{\partial \mathcal{L}}{\partial \mathbf{b}_q^*(m)} = -\mathbf{P}_q^H \Sigma_{\min}^{-1} (\mathbf{x}_2(m) - \mathbf{P}_q \mathbf{b}_q(m)) - \mathbf{P}_q^H \mathbf{v}(m) + \Phi_m \mathbf{b}_q(m), \quad (20)$$

where $\mathbf{v}(m)$ and $\mathbf{x}_2(m)$ are the m th columns of $(\hat{\Sigma}_q^{-1} - \Sigma_{\min}^{-1})\mathbf{E}_{(t)}$ and \mathbf{X}_2 , respectively. Equating (20) to zero yields

$$\mathbf{b}_q(m) = (\mathbf{P}_q^H \Sigma_{\min}^{-1} \mathbf{P}_q + \Phi_m)^{-1} \mathbf{P}_q^H (\Sigma_{\min}^{-1} \mathbf{x}_2(m) + \mathbf{v}(m)). \quad (21)$$

Furthermore, since $\Sigma_{\min}^{-1} = \frac{1}{\epsilon_N} \mathbf{I}$ and $\mathbf{P}_q^H \mathbf{P}_q = \mathbf{I}$, (21) can be further simplified to

$$\mathbf{b}_q(m) = \left(\frac{1}{\epsilon_N} \mathbf{I} + \Phi_m \right)^{-1} \mathbf{P}_q^H \left(\frac{1}{\epsilon_N} \mathbf{x}_2(m) + \mathbf{v}(m) \right). \quad (22)$$

The foregoing expression is the optimal value of the m th column of \mathbf{B}_q in terms of the Lagrange multipliers. Now we observe the following. First, since Φ_m is a diagonal matrix, we have that $(\frac{1}{\epsilon_N} \mathbf{I} + \Phi_m)^{-1}$ is also a diagonal matrix with elements

$$\left[\left(\frac{1}{\epsilon_N} \mathbf{I} + \Phi_m \right)^{-1} \right]_{nn} = \frac{\epsilon_N}{[\phi(m)]_n \epsilon_N + 1}, \quad (23)$$

where $[\cdot]_{nn}$ indicates the n th element along the diagonal and $[\phi(m)]_n$ is the n th element of $\phi(m)$. Thus, the Lagrange multiplier $[\phi(m)]_n$ only affects the n th element of $\mathbf{b}_q(m)$. Second, by the KKT conditions $[\phi(m)]_n \neq 0$ when $|\mathbf{b}_q(m)_n|^2 = \lambda_n$, and $[\phi(m)]_n = 0$ otherwise. Therefore, if the n th element of the unconstrained solution (obtained by taking $\Phi_m = \mathbf{0}$ in (22)) fulfills $|\mathbf{b}_q(m)_n|^2 \leq \lambda_n$, this is also the optimal solution of the original problem. Otherwise, the n th element of the unconstrained solution has to be scaled such that $|\mathbf{b}_q(m)_n|^2 = \lambda_n$, which yields (13).

II. SPLITTING DATA INTO TRAINING AND TEST DATA

We present an empirical study of the impact of sizes M_1 and M_2 to show that the precise split of data into training and test data does not strongly affect the results of the estimate in the observed scenarios. As we can see in Fig. 2, the precise split was not critical, thus there is no good reason for deviating much from an even split with $M_1 = M_2$. This choice can be further justified by looking at how many parameters we need to estimate. Within \mathbf{X}_1 , the calculation of \mathbf{P}_q and λ involves $q(2N - q) + 1$ parameters [2], and the expression $\log p(\mathbf{X}|\hat{\mathbf{B}}_q, \hat{\Sigma}_q)$, in (6), has $2(Mq) + \frac{(s+1)}{2}N$ degrees of freedom. In many scenarios, M and N are values of similar order, and q is not very high. Thus, the degrees of freedom are similar for both training and test, so a similar size for both sets seems reasonable. Therefore, in the simulations in Fig. 1 of the manuscript, we chose $M_1 = M_2$.

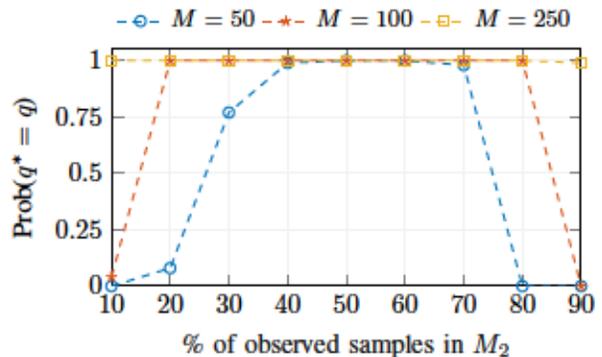


Fig. 2. Probability of correctly choosing the model order q . The settings are $\beta = 10$ dB, $N = 50$ sensors, $q = 10$ sources, and simulated noise follows the covariance matrix $\Sigma_{s2} = \mathbf{I} + \mathbf{R}_{nn}$. We averaged 100 simulations.