LINEAR AND NONLINEAR NONLOCAL OPERATORS OF ORDER NEAR ZERO

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Part I (Chapters $1-4$)

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Part II (Chapters $5-7$)

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Introducción y resumen de resultados

La presente tesis está dedicada al estudio de ciertos operadores de tipo integro-diferencial, de orden diferencial cercano a cero, de interés en aplicaciones y en teoría matemática. Los operadores integro-diferenciales aparecen con relativa frecuencia en diferentes situaciones del mundo real como por ejemplo, y por mencionar solo algunas de estas:
i) Dinámica de poblaciones en biología y modelos de relación depredador-presa en ecología; de hecho la teoría de búsqueda óptima predice que los depredadores deberían adoptar estrategias de búsqueda basadas en saltos largos donde la presa es escasa y se distribuye de manera impredecible, en lugar de siguiendo un movimiento browniano que es más eficiente solo para localizar presas abundantes, ver por ejemplo [60]. *ii*) Modelos de fluctuación de precios para activos en economía, cuyos procesos pueden tener cambios repentinos y bruscos, ver [52]. *iii*) Procesamiento de ruido de imagen, donde los algoritmos de eliminación de ruido no locales pueden detectar patrones y contornos de una manera más eficiente que los modelos clásicos, ver [68]. *iv*) Modelos de mecánica de fluidos, como la ecuación cuasi-geostrófica de superficie que se utiliza en oceanografía para modelizar la temperatura en la superficie, ver [20].

También se pueden encontrar algunos resultados teóricos clásicos sobre operadores integro-diferenciales en los trabajos de S. Bochner [9], T. Kato [42], E. Lieb [49] y libros como H. Landkof [46] y E. Stein [66] entre muchos otros. Finalmente señalamos algunos trabajos más recientes que tienen que ver con problemas clásicos aplicados, como el problema del obstáculo, la transición de fase o los materiales estratificados, ver [1, 17, 63] y otros.

Todos estos problemas comparten su propia naturaleza no local, lo que significa que para conocer el valor de la variable que nos interesa en cierto punto, es necesario conocer cierta información sobre su comportamiento en puntos distantes. Como ilustración, consideremos el siguiente modelo de población.

Un modelo de población

Consideremos una población de individuos cuya densidad en cualquier punto $x \in \mathbb{R}^N$ y tiempo $t \geq 0$ está representada por $u(x,t)$. Asumiremos que cualquier individuo ubicado en x puede saltar a cualquier lugar en $\mathbb{R}^N$ con una cierta distribución de probabilidad $\mathbb{P}(x, dy)$, que por simplicidad tiene una densidad $P_0(x, y)$. Así la probabilidad de saltar de x a un conjunto medible A viene dada por

$$\mathbb{P}(x, A) = \int_A \mathbb{P}(x, dy) = \int_A P_0(x, y)dy.$$

Si hacemos el balance en (x,t) entre las posibilidades de saltar de x a otra ubicación y las de llegar a x desde otros lugares, obtenemos una ecuación que expresa la variación instantánea de la densidad u de la siguiente manera

$$\frac{\partial u}{\partial t}(x,t) = \int_{\mathbb{R}^N} u(y,t)P_0(x,y)dy - \int_{\mathbb{R}^N} u(x,t)P_0(y,x)\, dy.$$

Una suposición natural es que $P_0(x, y) = P_0(y, x) = J(x-y)$ donde J es una función de densidad de probabilidad simétrica. En este caso podemos escribir la ecuación anterior en la forma

$$\frac{\partial u}{\partial t} = J * u - u. \tag{1}$$

Esta ecuación es no local porque la difusión de la función de densidad u en un punto x y el tiempo t no solo depende de $u(x,t)$, sino de todos los valores de u en una vecindad de x (dependiendo del soporte de J) a través del término de convolución $J * u$.

El carácter no local de estos operadores está directamente relacionado con las propiedades aleatorias y las discontinuidades de salto de los fenómenos que modelan. Este tipo de procesos se conocen como Procesos de Lévy.

Procesos de Lévy

Los procesos de Lévy pueden considerarse como paseos aleatorios en tiempo continuo, es decir, son procesos estocásticos con incrementos independientes y estacionarios. Los ejemplos más importantes son los procesos de Poisson, el movimiento browniano, los procesos de Cauchy y procesos estables más generales. En particular, son prototipos de procesos de Markov. Buenas referencias para estos tipos de procesos y sus aplicaciones son los libros [4, 7, 16] y los artículos [11, 39], entre otros.

Decimos que $X = (X_t)_{t\geq 0}$ es un proceso de Lévy para un espacio de probabilidad $(\Omega, \mathfrak{A}, \mathbb{P})$ si por cada $s, t \geq 0$ el incremento $X_{t+s} - X_t$ es independiente del proceso $(X_v, 0 \leq v \leq t)$ y tiene la misma ley de distribución que X_s. Intuitivamente, un proceso de Lévy representa el movimiento de un punto cuyos desplazamientos sucesivos son aleatorios e independientes, y estadísticamente tienen la misma distribución en diferentes intervalos de tiempo de la misma longitud.

Dado X un proceso de Lévy, su función característica ϕ_X está definida por

$$\phi_X(\xi) := \mathbb{E}\Big[e^{i\xi\cdot X_t}\Big].$$

A partir de los trabajos de Kolmogorov [44], Lévy [47] y Khintchine [43] se obtiene la forma exacta de la función característica de un proceso de Lévy a través de la fórmula, conocida como fórmula de Lévy-Khintchine,

$$-\log \mathbb{E}\Big[e^{i\xi\cdot X_t}\Big] = -ia\cdot\xi + \frac{1}{2}Q(\xi) + \int_{\mathbb{R}^N} \big(1 - e^{i\xi\cdot y} + i\xi\cdot y\mathbb{1}_{\{|y|<1\}}\big)d\mu(y),$$

donde $a \in \mathbb{R}^N$, Q es una forma cuadrática semi-definida positiva en $\mathbb{R}^N$, $\mathbb{1}$ es la función indicatriz y μ es una medida de Lévy sigma-finita de X, es decir, que satisface la propiedad $\displaystyle\int_{\mathbb{R}^N} (1 \wedge |y|^2)\,\mu(dy) < \infty$. Cuando $\mu(dy) = J(y)\,dy$, decimos que J es un núcleo de Lévy. Como veremos hay dos casos extremos de núcleos de Lévy que conducen a modelos completamente diferentes: $J \in L^1(\mathbb{R}^N)$ (o incluso J acotado), y J con un comportamiento de tipo potencia muy singular $J(z) \sim |z|^{-N-\alpha}$.

Por otro lado se puede probar que la función característica ϕ_X viene dada por una función continua $\psi : \mathbb{R}^N \to \mathbb{C}$, llamada exponente característico del proceso de Lévy X, de manera que

$$\mathbb{E}\Big[e^{i\xi\cdot X_t}\Big] = e^{-t\psi(\xi)}, \quad t \geq 0, \quad \xi \in \mathbb{R}^N.$$

Así el generador del semigrupo correspondiente al proceso de Lévy $\mathfrak{L}$ está caracterizado por la fórmula

$$\mathfrak{L}u(x) = -a\cdot\nabla u(x) + \frac{1}{2}\nabla\cdot Q\nabla u(x) + \text{V.P.}\int_{\mathbb{R}^N} \big(u(x-y) - u(x) - \nabla u(x)\cdot y\mathbb{1}_{\{|y|<1\}}\big)\,d\mu(y).$$

Equivalentemente, $\mathfrak{L}$ es un operador pseudo-diferencial

$$\widehat{\mathfrak{L}u}(\xi) = \widehat{u}(\xi)\psi(\xi),$$

cuyo símbolo (el exponente característico $-\psi$) viene dado por la fórmula de Lévy-Khintchine anterior.

Si nos concentramos en la parte no local del operador, es decir, asumimos que no hay difusión ($Q \equiv 0$), ni convección ($a \equiv 0$), y además la medida de Lévy viene dada por un núcleo simétrico, nos encontramos con operadores de la forma

$$\mathfrak{L}u(x) = \text{V.P.} \int_{\mathbb{R}^N} (u(x) - u(x-y))\, J(x,y)\, dy. \tag{2}$$

El comportamiento matemático de estos operadores depende de las propiedades de la medida de Lévy. En el caso de los operadores definidos por núcleos de Lévy integrables $J \in L^1(\mathbb{R}^N)$, digamos por ejemplo, $\int_{\mathbb{R}^N} J = 1$, entonces $\mathfrak{L}$ esta dado por

$$\mathfrak{L}u = u - J * u, \tag{3}$$

y nos encontramos con el operador de difusión del modelo de población introducido antes. En este caso $\mathfrak{L}$ es un operador de orden cero.

Por otro lado, cuando el núcleo de Lévy es una potencia no integrable, $J(y) = |y|^{-N-\alpha}$ con $0 < \alpha < 2$, entonces el operador $\mathfrak{L}$ es un múltiplo del llamado laplaciano fraccionario

$$(-\Delta)^{\alpha/2}u(x) = C_{N,\alpha}\text{V.P.} \int_{\mathbb{R}^N} \frac{u(x) - u(x-y)}{|y|^{N+\alpha}}\, dy, \tag{4}$$

donde $C_{N,\alpha}$ es una constante de normalización. Este es un operador pseudo-diferencial de orden α y se comporta como α derivadas. De hecho, $\lim_{\alpha\to 2}(-\Delta)^{\alpha/2}u = -\Delta u$ para toda $u \in C_0^2(\mathbb{R}^N)$.

Estos dos tipos de operadores (3) y (4) dan lugar a dos líneas de investigación muy distintas, a menudo desconectadas. El umbral entre esos dos tipos de operadores es lo que motiva este trabajo, caracterizar las propiedades de los operadores de la forma (2) en el límite entre ambos rangos, lo que denominamos de *orden casi cero*. Notemos que al hacer tender $\alpha \to 0^+$ en (4) obtenemos el operador identidad, precisamente debido a la constante de normalización, que verifica $C_{N,\alpha} \sim \alpha \to 0^+$. Nuestro propósito en este trabajo es pues, de una manera muy informal, estudiar el caso límite $\alpha \sim 0^+$ en la singularidad del núcleo cerca del origen, pero sin la constante de normalización, ver [12].

Para estudiar el operador (2) establecemos primero el marco funcional adecuado, describiendo algunas propiedades de los espacios de tipo Sobolev asociados en un dominio acotado, como estimaciones de simetrización, desigualdades de Hardy, inclusiones compactas en L^2 o la inclusión en algún espacio de tipo Lorentz. Estudiamos entonces el efecto de aplicar $\mathfrak{L}$ a funciones continuas, explicando la denominación de operador de orden casi cero. También estudiamos cuándo $\mathfrak{L}\mathbb{1}_\Omega$ es integrable en Ω, lo que lleva al concepto de J-perímetro. Luego aplicamos las propiedades descritas para estudiar los problemas de Dirichlet y Neumann relacionados con la ecuación $\mathfrak{L}u = f$ en un dominio acotado Ω, junto con la condición $u \equiv 0$ en $\Omega^c = \mathbb{R}^N \setminus \Omega$; consideramos los casos $f = f(x)$ y $f = f(u)$, incluido el problema de valores propios $f(u) = \lambda u$.

Aunque los problemas que inicialmente motivaron el estudio de los operadores no locales fueron en su mayoría operadores lineales, en los últimos años ha habido un aumento en el interés por los modelos no lineales. Por ejemplo, los modelos que tienen que ver con la difusión fraccionaria que involucra operadores no locales de la forma del p-laplaciano fraccionario

$$(-\Delta)_p^{s/2} u(x) = \int_{\mathbb{R}^{\mathbb{N}}} \frac{\Phi(u(x) - u(x-y))}{|y|^{N+\alpha p/2}}\, dy, \qquad \Phi(z) = |z|^{p-2}z, \tag{5}$$

con $1 < p < \infty$ y $0 < \alpha < 2$, ver por ejemplo [35, 37, 45]. Es claro que para $p = 2$ obtenemos un múltiplo del operador laplaciano fraccionario estándar $(-\Delta)^{\alpha/2}u$; por otro lado se puede probar que en el límite $\alpha \to 2$ con $p > 1$, con una constante de normalización, se obtiene el conocido operador p-laplaciano $\Delta_p u = \nabla \cdot (|\nabla u|^{p-2}\nabla u)$.

Nuestro interés en esta parte de la tesis es estudiar operadores integrales del tipo p-laplaciano fraccionario para funciones Φ más generales que solo potencias, además de considerar también núcleos en el límite de integrabilidad. Esto nos obliga a estudiar las propiedades de los espacios de Orlicz y de Sobolev-Orlicz correspondientes. En particular, mostramos una desigualdad de Poincaré y una desigualdad de Sobolev, dependiendo de la singularidad en el origen del núcleo J considerado, que como hemos mencionado puede ser muy débil. Ambas desigualdades conducen a inclusiones compactas. A continuación usamos esas propiedades para estudiar los problemas elípticos asociados de la forma $\mathcal{L}u = f$, (con reacción lineal o no) incluido el problema de valor propio generalizado $f(u) = \lambda\psi(u)$.

El trabajo se divide en dos partes principales. En la primera parte estudiamos operadores lineales definidos por núcleos de Lévy no integrables poniendo el énfasis en núcleos con singularidad muy débil, en el límite de integrabilidad; en la segunda parte estudiamos operadores no lineales de tipo p-laplaciano fraccionario, incluyendo también núcleos débilmente no integrables. A continuación describiremos los resultados de cada parte con más detalle.

Parte I. Operadores tipo Lévy en el límite de integrabilidad

El objetivo es estudiar las propiedades del operador lineal $\mathfrak{L}$ definido por

$$\mathfrak{L}u(x) = \text{V.P.} \int_{\mathbb{R}^N} (u(x) - u(y))\mu(x, dy), \tag{6}$$

donde μ es una medida de Lévy, es decir, satisface la siguiente condición de manera uniforme en $x \in \mathbb{R}^N$

$$\int_{\mathbb{R}^N} (1 \wedge |y|^2)\mu(x, dy) < \infty. \tag{7}$$

Por simplicidad suponemos que $\mu(x, dy) = J(x, y)\, dy$, donde el núcleo J se encuentra en el llamado rango no integrable, es decir

$$J(x, y) \geq \mathcal{K}(x - y) \geq 0, \qquad \mathcal{K} \notin L^1(B_\varepsilon) \quad \forall\, \varepsilon > 0, \tag{8}$$

donde $B_\varepsilon = \{z \in \mathbb{R}^N,\ |z| < \varepsilon\}$. Más concretamente, escribimos

$$\mathcal{K}(z) = |z|^{-N}\ell(|z|) \qquad \text{para } 0 < |z| < 1, \tag{9}$$

para alguna función $\ell : (0, 1) \to (0, \infty)$, satisfaciendo $\displaystyle\lim_{r\to 0^+} \int_r^1 \frac{\ell(s)}{s}\, ds = \infty$. Como además estamos interesados principalmente en el caso débilmente singular que separa el rango del laplaciano fraccionario del rango integrable, imponemos la condición

$$\lim_{z\to 0} |z|^\alpha \ell(z) = 0 \qquad \forall\, \alpha > 0, \tag{10}$$

y suponemos que la función ℓ es de variación lenta en el origen, es decir

$$\lim_{s\to 0} \frac{\ell(\lambda s)}{\ell(s)} = 1 \qquad \forall\, \lambda > 0. \tag{11}$$

Obsérvese que además del caso de proceso estable $\mathcal{K}(z) = |z|^{-N}$ para $|z| < 1$, también incluimos posibles perturbaciones logarítmicas de esos núcleos.

En primer lugar mostramos algunas propiedades de regularidad que caracterizan al operador $\mathfrak{L}$.

Teorema 1. *Sea $J(x,y)=\mathcal{K}(x-y)$ verificando* (9).

i) *Si $u\in C^{\nu}(\mathbb{R}^N)$ para algún $\nu\in(0,1)$ entonces $\mathfrak{L}u\in C(\mathbb{R}^N)$, con algún módulo de continuidad dependiendo de $\mathcal{K}$ y ν.*

ii) *Si $u\in C(\mathbb{R}^N)$ con módulo de continuidad ϖ_0, entonces $\mathfrak{L}u\in C(\mathbb{R}^N)$ siempre que*
$$\lim_{R\to 0}\int_0^R \frac{\varpi_0(s)\ell(s)}{s}\,ds=0.$$

La forma bilineal de Dirichlet asociada al operador $\mathfrak{L}$ (de hecho asociada al núcleo J) está definida por

$$\mathcal{E}(u,v)=\frac{1}{2}\iint_{Q_\Omega}(u(x)-u(y))(v(x)-v(y))J(x,y)\,dxdy,$$

donde, $\Omega\subset\mathbb{R}^N$ es un conjunto acotado y $Q_\Omega=(\Omega^c\times\Omega^c)^c$. El correspondiente espacio de Sobolev está definido por

$$\mathcal{H}_J(\Omega)=\left\{u:\mathbb{R}^N\to\mathbb{R},\ u|_\Omega\in L^2(\Omega),\ \mathcal{E}(u,u)<\infty\right\},$$

con la norma
$$\|u\|_{\mathcal{H}_J}=\left(\int_\Omega u^2+\mathcal{E}(u,u)\right)^{1/2}.$$

Cuando se trata de problemas definidos en dominios acotados, ya que las condiciones de Dirichlet deben definirse en el complemento Ω^c, en lugar de solo en la frontera, precisamente por el carácter no local del operador, es conveniente considerar el espacio
$$\mathcal{H}_{J,0}(\Omega)=\{u\in\mathcal{H}_J(\Omega),\ u\equiv 0 \text{ in } \Omega^c\}.$$

La condición de Lévy (7) implica

$$H_0^1(\Omega)\subset\mathcal{H}_{J,0}(\Omega)\subset\mathcal{H}_J(\Omega)\subset L^2(\Omega),$$

si consideramos las funciones en $H_0^1(\Omega)$ extendidas por cero fuera de Ω. En el caso del laplaciano fraccionario se tiene $\mathcal{K}(z)=|z|^{-N-\alpha}$ para algún $0<\alpha<2$, y por tanto, si $N>\alpha$ se cumple

$$\mathcal{H}_J(\Omega)\subset H^{\alpha/2}(\Omega)\subset L^{\frac{2N}{N-\alpha}}(\Omega),$$

gracias a la desigualdad de Hardy-Littlewood-Sobolev, donde $H^{\alpha/2}(\Omega)$ es el espacio de Sobolev fraccionario usual. Tenemos en ese caso $\mathcal{H}_J(\Omega)\hookrightarrow L^2(\Omega)$ de manera compacta.

Por otro lado, en el caso de núcleos integrables, $\sup_{x\in\mathbb{R}^N}\int_{\mathbb{R}^N} J(x,y)\,dy = B < \infty$, (por lo tanto no satisfacen (8)), se tiene $\mathcal{E}(u,u) \le B\|u\|_2^2$ lo que implicaría $\mathcal{H}_{J,0}(\Omega) \equiv L^2(\Omega)$.

Uno de los principales objetivos de esta parte del trabajo consiste en establecer el lugar exacto donde los espacios de Sobolev $\mathcal{H}_{J,0}(\Omega)$ y $\mathcal{H}_J(\Omega)$ se encuentran respecto a $L^2(\Omega)$.

Teorema 2. *Bajo las hipótesis* (8)–(11)*,*

i) *la inclusión* $\mathcal{H}_{J,0}(\Omega) \hookrightarrow L^2(\Omega)$ *es compacta;*

ii) *si además* $\ell(0^+) = \infty$*, entonces la inclusión* $\mathcal{H}_J(\Omega) \hookrightarrow L^2(\Omega)$ *es compacta.*

La compacidad de la inclusión $\mathcal{H}_{J,0}(\Omega) \hookrightarrow L^2(\Omega)$ se puede explicar por una inclusión más estricta en un espacio de tipo Lorentz $\mathcal{H}_{J,0}(\Omega) \hookrightarrow \mathcal{L}_{\mathcal{A},2}(\Omega)$, para alguna función $\mathcal{A}$ dependiendo de J; ver la definición de estos espacios en la Sección 3.2. Como herramienta para probar este resultado establecemos algunas desigualdades de tipo Hardy, de interés en sí mismas, además de un resultado de simetrización.

Pasamos entonces a estudiar los problemas elípticos asociados al operador $\mathfrak{L}$, comenzando con el problema lineal

$$\begin{cases} \mathfrak{L}u = f(x), & \text{en } \Omega, \\ u = 0, & \text{en } \Omega^c. \end{cases}$$

La existencia y unicidad se establecen fácilmente para $f \in H^*(\Omega)$, el dual de $\mathcal{H}_{J,0}(\Omega)$. Estamos interesados en el efecto regularizante, probando que u tiene una integrabilidad ligeramente mejor que f, a pesar de que el operador es de orden casi cero.

Teorema 3. *Si* $f \in L^p(\Omega)$*,* $p \ge 2$*, entonces* $u \in \mathcal{L}_{\mathcal{A},p}(\Omega)$*.*

Caracterizamos también la existencia de autovalores. La siguiente tarea es considerar problemas no lineales de la forma

$$\begin{cases} \mathfrak{L}u = f(u), & \text{en } \Omega, \\ u = 0, & \text{en } \Omega^c. \end{cases} \tag{12}$$

Buscamos soluciones no negativas en Ω.

Teorema 4. i) *Si* f *es sublineal el problema* (12) *tiene una única solución.*

ii) *Si* $f(u) = u^p$, $p > \dfrac{N+\sigma}{N-\sigma}$, *donde* σ *depende del núcleo* J, *y* Ω *es un dominio estrellado, el problema* (12) *no tiene solución.*

La prueba de no existencia para reacciones supercríticas está basada en una desigualdad de Pohozaev que obtenemos siguiendo la prueba de [61] para el caso del laplaciano fraccionario. La conjetura es que no hay solución para $p > 1$.

Finalmente estudiamos un problema tipo Neumann asociado al operador $\mathfrak{L}$

$$\begin{cases} \mathfrak{L}u = f, & \text{en } \Omega, \\ \mathcal{N}u = 0, & \text{en } \Omega^c, \end{cases}$$

donde $\mathcal{N}$ es un operador que generaliza la derivada normal, ver [27] para el caso del laplaciano fraccionario.

Parte II. Un operador no lineal del tipo p-laplaciano fraccionario

El objetivo de esta parte de la tesis es estudiar las propiedades del operador no local no lineal

$$\mathcal{L}u(x) = \mathcal{L}^{J,\psi}u(x) \equiv \int_{\mathbb{R}^N} \psi(u(x) - u(y))J(x-y)\,dy,$$

donde $\psi : \mathbb{R} \to \mathbb{R}$ es una función no decreciente, impar y no acotada, y $J : \mathbb{R}^N \to \mathbb{R}^+$ es una función medible simétrica en el rango del laplaciano fraccionario (aunque también consideraremos núcleos en el llamado caso límite de integrabilidad).

El carácter diferencial del operador viene definido por el exponente

$$q_* = \text{ínf}\left\{ q_0 > 0 \,:\, \int_{\mathbb{R}^N} \text{mín}(1, |z|^{q_0})J(z)\,dz < \infty \right\}.$$

En cuanto a la no linealidad, si ponemos $\Psi' = \psi$ consideramos funciones Ψ convexas, simétricas, satisfaciendo, para algunos $p \geq q > 1$,

$$q \leq \frac{s\Psi'(s)}{\Psi(s)} \leq p \qquad \forall s \neq 0.$$

Definimos entonces los funcionales

$$F(u) = \int_{\mathbb{R}^N} \Psi(u(x))\,dx,$$

$$E(u) = \frac{1}{2} \iint_{\mathbb{R}^{2N}} \Psi(u(x) - u(y))J(x - y)\, dxdy.$$

Las propiedades de la función ψ implican que Ψ es una función de Young estricta, por lo que podemos considerar los espacios de Orlicz y de Sobolev-Orlicz

$$L^{\Psi}(\mathbb{R}^N) = \{u : \mathbb{R}^N \to \mathbb{R},\ F(u) < \infty\}, \tag{13}$$

$$W^{J,\Psi}(\mathbb{R}^N) = \left\{u \in L^{\Psi}(\mathbb{R}^N),\ E(u) < \infty\right\}. \tag{14}$$

Nuestro principal interés radica en estudiar las propiedades de los espacios (de Banach reflexivos) (13) y (14) para no linealidades Ψ con las propiedades anteriores. En particular si $q > q_*$, entonces el funcional $E(u)$ está bien definido y es finito para funciones que satisfacen $F(\nabla u) < \infty$, por lo que $W^{1,\Psi}(\mathbb{R}^N) \subset W^{J,\Psi}(\mathbb{R}^N)$, el primero es el espacio estándar de Sobolev-Orlicz de funciones en $L^{\Psi}(\mathbb{R}^N)$ con gradiente también en $L^{\Psi}(\mathbb{R}^N)$.

Sin imponer ninguna condición de singularidad del núcleo J en el origen (no integrable, por supuesto), mostramos que existe una desigualdad de Poincaré $E(u) \geq F(u)$, que implica la inclusión

$$W_0^{J,\Psi}(\Omega) \subset L^{\Psi}(\Omega).$$

Obsérvese que si J fuera integrable entonces $W_0^{J,\Psi}(\Omega) \equiv L^{\Psi}(\Omega)$ (como en el caso $\Psi(s) = s^2$ de la primera parte). En nuestra situación, cuando J es un núcleo singular tenemos el siguiente resultado.

Teorema 5. *La inclusión* $W_0^{J,\Psi}(\Omega) \hookrightarrow L^{\Psi}(\Omega)$ *es compacta.*

Asumiendo la condición de singularidad más estricta $J(z) \geq c|z|^{-N-\alpha,}$ para $0 < |z| < 1$ y algún $\alpha > 0$, tenemos el siguiente resultado

Teorema 6. *Supongamos que* J *satisface la condición anterior con* $0 < \alpha < N$. *Entonces* $W_0^{J,\Psi}(\Omega) \subset L^{\Psi^r}(\Omega)$ *para todo* $1 \leq r \leq r^* \equiv \dfrac{N}{N-\alpha}$. *Además esta inclusión es compacta si* $r < r^*$.

Con estas propiedades estudiamos ahora el problema

$$\begin{cases} \mathcal{L}u = f(x), & \text{en } \Omega, \\ u = 0, & \text{en } \Omega^c. \end{cases} \tag{15}$$

Probamos existencia, unicidad (asumiendo condiciones técnicas sobre Ψ), así como propiedades de integrabilidad dependiendo de la singularidad del núcleo.

Teorema 7. *Dada* $f \in \left(W_0^{J,\Psi}(\Omega)\right)'$ *existe una solución al problema* (15). *Si* ψ *satisface alguna de las condiciones* (5.14) *o* (5.17) *entonces la solución es única. Si* $\Psi(s) \sim |s|^p$ *y* $J(z) \geq c|z|^{-N-\alpha}$, *entonces* $f \in L^m(\Omega)$ *implica*

i) $u \in L^{\frac{m(p-1)N}{N-m\alpha}}(\Omega)$ *si* $m < N/\alpha$*;*

ii) $u \in L^\infty(\Omega)$ *si* $m > N/\alpha$.

En cuanto al caso de reacción no lineal

$$\begin{cases} \mathcal{L}u = f(u), & \text{en } \Omega, \\ u = 0, & \text{en } \Omega^c. \end{cases} \tag{16}$$

tenemos dos situaciones para $f(u) \sim u^{m-1}$ (buscamos soluciones no negativas): el caso llamado sublineal $0 < \dfrac{m}{p} < 1$ y el caso superlineal subcrítico $1 < \dfrac{m}{p} < \dfrac{N}{N-\alpha}$. Las condiciones que nosotros consideramos son más generales y más técnicas, dependiendo de la función Ψ.

Teorema 8. *Supongamos que se satisfacen las hipótesis* (7.12) *o* (7.14) *sobre* f*, entonces el problema* (16) *tiene solución no trivial.*

Terminamos esta parte estudiando el problema generalizado de autovalores:

$$\begin{cases} \mathcal{L}u = \lambda\psi(u), & \text{en } \Omega, \\ u = 0, & \text{en } \Omega^c. \end{cases} \tag{17}$$

Construimos una familia de autovalores y autofunciones: demostramos que para cada $\mu > 0$ existe un autovalor positivo λ_μ de (17) con autofunción no negativa $u_\mu \in W_0^{J,\Psi}(\Omega)$ tal que $F(u_\mu) = \mu$.

Organización de la tesis

En la primera parte de la tesis dedicamos un capítulo preliminar Capítulo 1 para plantear las hipótesis precisas que consideramos a lo largo de esta parte. El Capítulo 2 está dedicada a estudiar el operador no local definido y la forma bilineal asociada, describiendo sus propiedades, incluyendo la acción del operador sobre diferentes funciones, dos desigualdades de Hardy y un resultado de simetrización para la forma bilineal. Las inclusiones compactas de nuestros espacios tipo Sobolev en L^2 se estudian en el Capítulo 3, así como la inclusión en algún espacio de tipo Lorentz. En el Capítulo 4 estudiamos tres problemas asociados a nuestro operador, dos problemas lineales, con condición exterior de Dirichlet o Neumann, y un problema no lineal con diferentes reacciones; mostramos existencia y unicidad para reacciones sublineales y no existencia cuando la reacción es supercrítica.

En la segunda parte comenzamos nuevamente con un capítulo preliminar donde estudiamos las propiedades de los espacios de Orlicz L^Ψ por medio de algunas desigualdades satisfechas por la no linealidad Ψ y los funcionales F y E, Capítulo 5. En el Capítulo 6 se muestran las inclusiones de Sobolev para el espacio $W^{J,\Psi}$. Finalmente el Capítulo 7 está dedicado al estudio de los problemas elípticos asociados para las diferentes reacciones comentadas anteriormente.

Introduction and summary of results

The present thesis is dedicated to the study of certain integro-differential operators with differential order close to zero, which are interesting both, from the theoretical point of view and for the applications. Integro-differential operators regularly appear in different real-world situations such as, to mention just a few:
i) Population dynamics in Biology and prey-predator relationship in Ecology; indeed, optimal search theory predicts that predators should adopt search strategies based on long jumps where prey is sparse and distributed unpredictably, instead of Brownian motion which is more efficient only for locating abundant prey, see for instance [60]. *ii*) Price fluctuation models for assets in the Economy, these processes can have sudden changes, see [52]. *iii*) Image noise processing, where nonlocal denoising algorithms are able to detect patterns and contours in a better way than the classical models, see [68]. *iv*) Fluid mechanic models, such as the surface quasi-geostrophic equation, which is used in Oceanography to model the temperature on the surface, see [20].

Also some classic theoretical results on integro-differential operators can be found in the works of S. Bochner [9], T. Kato [42], E. Lieb [49] and books like H. Landkof [46] and E. Stein [66] among many others. Finally more recent works have to do with classical applied problems, such as the obstacle problem, phase transition, stratified materials, see [1, 17, 63] and others.

All these problems share their own nonlocal nature, this means that in order to know the value of the variable that interests us at a certain point, it is necessary to know some information about their behaviour at distant points. As an illustration, we comment on the following population model.

A population model

Let us consider a single population whose density at any point $x \in \mathbb{R}^N$ and time $t \geq 0$ is represented by $u(x,t)$. We shall assume that any individual located at x may jump from x to any place in $\mathbb{R}^N$ with a certain probability distribution $\mathbb{P}(x,dy)$, which for simplicity has a density $P_0(x,y)$. Thus, the probability of jumping from x into a measurable set A is given by

$$\mathbb{P}(x,A) = \int_A \mathbb{P}(x,dy) = \int_A P_0(x,y)dy.$$

Now if we make the balance at (x,t) between the possibilities of jumping from x to some other location and those of arriving at x from other places, we get an equation which expresses the instantaneous variation of the density u as follows

$$\frac{\partial u}{\partial t}(x,t) = \int_{\mathbb{R}^N} u(y,t)P_0(x,y)dy - \int_{\mathbb{R}^N} u(x,t)P_0(y,x)\,dy.$$

A natural assumption is that $P_0(x,y) = P_0(y,x) = J(x-y)$ where J is a symmetric probability density. In this case, we may write the above equation in the form

$$\frac{\partial u}{\partial t} = J * u - u. \tag{1}$$

It is nonlocal because the diffusion of the density u at a point x and time t does not depend on $u(x,t)$ only, but also on all the values of u in a neighborhood of x (depending on the support of J) through the convolution term $J * u$.

The nonlocal character of these operators is directly related to the random properties and jump discontinuities of the phenomena they model. These types of processes are known as Lévy processes.

Lévy processes

Lévy processes can be thought of as random walks in continuous time, that is they are stochastic processes with independent and stationary increments. The most important examples are the Poisson process, the Brownian motion, the Cauchy process, and more general stable processes. Lévy processes concern many aspects of probability theory and its applications. In particular, they are prototypes of Markov processes. Good references for this processes and their applications are the books [4, 7, 16], and the papers [11, 39] among others.

We say that $X = (X_t)_{t\geq 0}$ is a Lévy process for a probability space $(\Omega, \mathfrak{A}, \mathbb{P})$ if for every $s, t \geq 0$, the increment $X_{t+s} - X_t$ is independent of the process $(X_v, 0 \leq v \leq t)$ and has the same law as X_s. Intuitively, a Lévy process represents the movement of a point whose successive displacements are random and independent, and statistically have the same distribution over different time intervals of the same length.

Given be a Lévy processe X its characteristic function ϕ_X is defined by

$$\phi_X(\xi) := \mathbb{E}\Big[e^{i\xi\cdot X_t}\Big].$$

The contributions of Kolmogorov [44], Lévy [47] and Khintchine [43] show the exact form of the characteristic function through the formula, known as the Lévy-Khintchine formula

$$-\log \mathbb{E}\Big[e^{i\xi\cdot X_t}\Big] = -ia\cdot\xi + \frac{1}{2}Q(\xi) + \int_{\mathbb{R}^N} \big(1 - e^{i\xi\cdot y} + i\xi\cdot y\mathbb{1}_{\{|y|<1\}}\big) d\mu(y),$$

where $a \in \mathbb{R}^N$, Q is a positive semi-definite quadratic form on $\mathbb{R}^N$, $\mathbb{1}$ is the indicatrix function and μ is a sigma-finite Lévy measure of X that satisfies the property $\int_{\mathbb{R}^N}(1\wedge |y|^2)\,\mu(dy) < \infty$. When $\mu(dy) = J(y)\,dy$, we say that J is a Lévy kernel. There are two extreme cases that lead to completely different models: $J \in L^1(\mathbb{R}^N)$ (or even J bounded), and the very singular power-type behaviour $J(z) = |z|^{-N-\alpha}$.

On the other hand it can be proved that the characteristic function ϕ_X is given by a continuous function $\psi : \mathbb{R}^N \to \mathbb{C}$, called a characteristic exponent of the process of Lévy X, so that

$$\mathbb{E}\Big[e^{i\xi\cdot X_t}\Big] = e^{-t\psi(\xi)}, \quad t \geq 0, \quad \xi \in \mathbb{R}^N.$$

Thus the semigroup generator corresponding to the Lévy process $\mathfrak{L}$ is characterized by the formula

$$\mathfrak{L}u(x) = -a\cdot\nabla u(x) + \frac{1}{2}\nabla\cdot Q\nabla u(x) + \text{V.P.}\int_{\mathbb{R}^N}\big(u(x-y) - u(x) - \nabla u(x)\cdot y\mathbb{1}_{\{|y|<1\}}\big)\,d\mu(y).$$

Equivalently, $\mathfrak{L}$ is a pseudo-differential operator

$$\widehat{\mathfrak{L}u}(\xi) = \widehat{u}(\xi)\psi(\xi),$$

whose symbol (the characteristic exponent $-\psi$) is given by the above Lévy-Khintchine formula.

If we focus on the non-local part of the operator, that is, we assume that there is no diffusion ($Q \equiv 0$) nor convection ($a \equiv 0$), and also the Lévy measure is given by a symmetric kernel, we have operators of the form

$$\mathfrak{L}u(x) = \text{V.P.} \int_{\mathbb{R}^N} (u(x) - u(x-y))\, J(x,y)\, dy. \tag{2}$$

The mathematical behavior of these operators depends on the properties of the Lévy measure. In the case of operators defined by integrable Lévy kernels $J \in L^1(\mathbb{R}^N)$, fix for example $\displaystyle\int_{\mathbb{R}^N} J = 1$, then $\mathfrak{L}$ is given by

$$\mathfrak{L}u = u - J * u, \tag{3}$$

and we recover the diffusion operator of the population model introduced before. In that case $\mathfrak{L}$ is a zero order operator.

On the other hand, when the Lévy kernel is a non-integrable power, $J(y) = |y|^{-N-\alpha}$ with $0 < \alpha < 2$, then the operator $\mathfrak{L}$ is a multiple of the well known fractional Laplacian

$$(-\Delta)^{\alpha/2} u(x) = C_{N,\alpha} \text{V.P.} \int_{\mathbb{R}^N} \frac{u(x) - u(x-y)}{|y|^{N+\alpha}}\, dy, \tag{4}$$

where $C_{N,\alpha}$ is a normalization constant. This is a pseudo-differential operator of order α and behaves like α derivatives. Actually, $\lim_{\alpha \to 2} (-\Delta)^{\alpha/2} u = -\Delta u$ for every $u \in C_0^2(\mathbb{R}^N)$.

These two types of operators (3) and (4) give rise to two lines of research, often disconnected. The threshold between these two types of operators is what motivates this work, characterizing the properties of the operators of the form (2) in the limit between the two ranges, what we call of *almost zero order*. We note that letting $\alpha \to 0^+$ in (4) we obtain the identity operator, precisely because of the normalizing constant, since it holds $C_{N,\alpha} \sim \alpha \to 0^+$. What we want to study here, in some very informal way, is the limit $\alpha \sim 0^+$ in the singularity of the kernel near the origin, but without the normalizing constant, see [12].

In order to study the operator (2) we first settle the appropriate functional framework, describing some properties of the associated Sobolev-type spaces in a bounded domain, such as symmetrization estimates, Hardy inequalities, compact inclusions in L^2 or the inclusion in some space of Lorentz type. We study then the effect of applying $\mathfrak{L}$ to continuous functions, explaining the denomination of operator of almost zero order. We also study when $\mathfrak{L}\mathbb{1}_\Omega$ is integrable in Ω, which leads to the concept of J-perimeter. Then we apply the described properties to study the problems of Dirichlet and Neumann type related to the equation $\mathfrak{L}u = f$ in a bounded domain Ω, together with the boundary condition $u \equiv 0$ in $\Omega^c = \mathbb{R}^N \setminus \Omega$; we consider the cases $f = f(x)$ and $f = f(u)$, including the eigenvalue problem $f(u) = \lambda u$.

Although the problems that initially motivated the study of nonlocal operators were mostly linear operators, in recent years there has been an increasing interest in the study of nonlinear models. For example, models that have to do with fractional diffusion involving nonlocal operators of the form of the fractional p-Laplacian

$$(-\Delta)_p^{s/2} u(x) = \int_{\mathbb{R}^N} \frac{\Phi(u(x) - u(x-y))}{|y|^{N+\alpha p/2}}\, dy, \qquad \Phi(z) = |z|^{p-2} z, \tag{5}$$

whit $1 < p < \infty$ and $0 < \alpha < 2$, see for example [35, 37, 45]. Note that for $p = 2$ we obtain a multiple of the standard fractional Laplacian $(-\Delta)^{\alpha/2}u$. On the other hand it is proved that in the limit $\alpha \to 2$ whit $p > 1$, after inserting a normalizing constant, we get the well known p-Laplacian operator $\Delta_p u = \nabla \cdot (|\nabla u|^{p-2}\nabla u)$.

Our interest in this part of the thesis is to study integral operators of the type of the fractional p-Laplacian for more general functions Φ than just powers, together with considering also kernels in the limit of integrability. This obligue us to study the properties of the corresponding Orlicz and Sobolev-Orlicz spaces. In particular we show a Poincaré inequality and a Sobolev inequality, depending on the singularity at the origin of the kernel J considered, which as we have said may be very weak. Both inequalities lead to compact inclusions. We then use those properties to study the associated elliptic problems of the form $\mathcal{L}u = f$ (with linear or nonlinear reaction) including the generalized eigenvalue problem $f(u) = \lambda\psi(u)$.

This work is divided into two main parts. In the first part we study linear operators defined by non integrable Lévy kernels, placing the emphasis on kernels with a very weak singularity. In the second part we study nonlinear operators of the fractional p-Laplacian type, including also weakly non-integrable kernels. We describe next the results of each part in more detail.

Part I. Lévy type operators in the limit of integrability

The aim is to study the properties of the linear operator $\mathfrak{L}$ defined by

$$\mathfrak{L}u(x) = \text{V.P.} \int_{\mathbb{R}^N} (u(x) - u(y))\mu(x, dy), \tag{6}$$

where μ is a Lévy measure, that is, it satisfies the following condition uniform in $x \in \mathbb{R}^N$

$$\int_{\mathbb{R}^N} (1 \wedge |y|^2)\mu(x, dy) < \infty. \tag{7}$$

For simplicity we assume that $\mu(x, dy) = J(x, y)\, dy$, where the kernel J belongs to the so-called non-integrable range, that is

$$J(x, y) \geq \mathcal{K}(x - y) \geq 0, \qquad \mathcal{K} \notin L^1(B_\varepsilon) \quad \forall\, \varepsilon > 0, \tag{8}$$

where $B_\varepsilon = \{z \in \mathbb{R}^N,\ |z| < \varepsilon\}$. More specifically, we write

$$\mathcal{K}(z) = |z|^{-N} \ell(|z|) \qquad \text{for } 0 < |z| < 1, \tag{9}$$

for some function $\ell : (0, 1) \to (0, \infty)$, satisfying $\displaystyle\lim_{r \to 0^+} \int_r^1 \frac{\ell(s)}{s}\, ds = \infty$.

As we are also interested primarily in the weakly singular case that separates the fractional Laplacian range from the integrable range, we impose the condition

$$\lim_{z \to 0} |z|^\alpha \ell(z) = 0 \qquad \forall\, \alpha > 0, \tag{10}$$

and we assume that the function ℓ varies slowly at the origen, that is

$$\lim_{s \to 0} \frac{\ell(\lambda s)}{\ell(s)} = 1 \qquad \forall\, \lambda > 0. \tag{11}$$

Note in that way that, in addition to the stable process case $\mathcal{K}(z) = |z|^{-N}$ for $|z| < 1$, we also include possible logarithmic perturbations of those kernels.

We first show some regularity properties that characterize the operator $\mathfrak{L}$.

Theorem 1. *Let $J(x, y) = \mathcal{K}(x - y)$ satisfy (9).*

i) *If $u \in C^\nu(\mathbb{R}^N)$ for some $\nu \in (0, 1)$ then $\mathfrak{L}u \in C(\mathbb{R}^N)$, with some modulus of continuity that depends on $\mathcal{K}$ and ν.*

ii) *If $u \in C(\mathbb{R}^N)$ with a modulus of continuity ϖ_0, then $\mathfrak{L}u \in C(\mathbb{R}^N)$ provided*

$$\lim_{R \to 0} \int_0^R \frac{\varpi_0(s)\ell(s)}{s}\, ds = 0.$$

The bilinear Dirichlet form associated to the operator $\mathfrak{L}$ (indeed associated to the kernel J) is defined by

$$\mathcal{E}(u,v) = \frac{1}{2}\iint_{Q_\Omega} (u(x)-u(y))(v(x)-v(y))J(x,y)\,dxdy,$$

where $\Omega \subset \mathbb{R}^N$ is a bounded set and $Q_\Omega = (\Omega^c \times \Omega^c)^c$. The corresponding Sobolev spaces are defined by

$$\mathcal{H}_J(\Omega) = \left\{u : \mathbb{R}^N \to \mathbb{R},\ u|_\Omega \in L^2(\Omega),\ \mathcal{E}(u,u) < \infty\right\},$$

with norm

$$\|u\|_{\mathcal{H}_J} = \left(\int_\Omega u^2 + \mathcal{E}(u,u)\right)^{1/2}.$$

When dealing with problems defined in bounded domains, since the Dirichlet conditions must be prescribed in the complement instead of just on the boundary, precisely by the nonlocal character of the operator, it is convenient to consider the space

$$\mathcal{H}_{J,0}(\Omega) = \{u \in \mathcal{H}_J(\Omega),\ u \equiv 0 \text{ in } \Omega^c\}.$$

The Lévy condition (7) implies

$$H_0^1(\Omega) \subset \mathcal{H}_{J,0}(\Omega) \subset \mathcal{H}_J(\Omega) \subset L^2(\Omega),$$

if we consider the functions in $H_0^1(\Omega)$ extended by zero outside Ω. In the fractional Laplacian case it is $\mathcal{K}(z) = |z|^{-N-\alpha}$ for some $0 < \alpha < 2$ and then, if moreover $N > \alpha$,

$$\mathcal{H}_J(\Omega) \subset H^{\alpha/2}(\Omega) \subset L^{\frac{2N}{N-\alpha}}(\Omega),$$

thanks to the Hardy-Sobolev inequality, where $H^{\alpha/2}(\Omega)$ is the usual fractional Sobolev space. We have in that case $\mathcal{H}_J(\Omega) \hookrightarrow L^2(\Omega)$ with compact inclusion.

On the very other hand, in the case of integrable kernels, $\sup_{x\in\mathbb{R}^N} \int_{\mathbb{R}^N} J(x,y)\,dy = B < \infty$, (thus not satisfying (8)), we have $\mathcal{E}(u,u) \leq B\|u\|_2^2$ and therefore $\mathcal{H}_{J,0}(\Omega) \equiv L^2(\Omega)$.

A main objective of this part is to establish the exact place were $\mathcal{H}_{J,0}(\Omega)$ and $\mathcal{H}_J(\Omega)$ lie in relation to $L^2(\Omega)$.

Theorem 2. *In the hypotheses* (8)–(11)*,*

i) *the embedding $\mathcal{H}_{J,0}(\Omega) \hookrightarrow L^2(\Omega)$ is compact;*

ii) *if moreover* $\ell(0^+) = \infty$, *then also the embedding* $\mathcal{H}_J(\Omega) \hookrightarrow L^2(\Omega)$ *is compact.*

The compactness of the inclusion $\mathcal{H}_{J,0}(\Omega) \hookrightarrow L^2(\Omega)$ can be explained by the sharper inclusion into some Lorentz space $\mathcal{H}_{J,0}(\Omega) \hookrightarrow \mathcal{L}_{\mathcal{A},2}(\Omega)$, for some function $\mathcal{A}$ depending on J. See the definition of Lorentz spaces in Section 3.2. As a tool to proving this result we establish some interesting inequalities of Hardy type plus a symmetrization result.

We then pass to study the elliptic problems associated to the operator $\mathfrak{L}$. We explain some results of the linear problem

$$\begin{cases} \mathfrak{L}u = f(x), & \text{in } \Omega, \\ u = 0, & \text{in } \Omega^c. \end{cases}$$

Existence and uniqueness are easily established for $f \in H^*(\Omega)$, the dual space of $\mathcal{H}_{J,0}(\Omega)$. We are interested in the regularizing effects, and prove that u has slightly better integrability than f, although the operator is of order almost zero.

Theorem 3. *If* $f \in L^p(\Omega)$, $p \geq 2$, *then* $u \in \mathcal{L}_{\mathcal{A},p}(\Omega)$.

We also characterize the existence of eigenvalues. The next task is to consider nonlinear problems of the form

$$\begin{cases} \mathfrak{L}u = f(u), & \text{in } \Omega, \\ u = 0, & \text{in } \Omega^c. \end{cases} \tag{12}$$

We look for solutions nonnegative in Ω.

Theorem 4. i) *If* f *is sublinear then problem* (12) *has a unique solution.*

ii) *If* $f(u) = u^p$, $p > \dfrac{N+\sigma}{N-\sigma}$, *where* σ *depends on the kernel* J, *and* Ω *is star-shaped, then* (12) *has no solution.*

The non-existence proof for supercritical reactions is based on a Pohozaev inequality that we obtain following the proof performed in [61] for the fractional Laplacian case. The conjecture is that there is no solution for any $p > 1$.

We finally study a Neumann problem associated to the operator $\mathfrak{L}$

$$\begin{cases} \mathfrak{L}u = f, & \text{in } \Omega, \\ \mathcal{N}u = 0, & \text{in } \Omega^c, \end{cases}$$

where $\mathcal{N}$ is some operator generalizing the normal derivative, see [27] for the case of the fractional Laplacian.

Part II. A nonlinear operator of fractional p-Laplacian type

The aim of this part is to study the properties of the nonlinear nonlocal operator

$$\mathcal{L}u(x) = \mathcal{L}^{J,\psi}u(x) \equiv \int_{\mathbb{R}^{\mathbb{N}}} \psi(u(x) - u(y))J(x-y)\,dy,$$

where $\psi : \mathbb{R} \to \mathbb{R}$ is a nondecreasing, continuous, unbounded odd function, and $J : \mathbb{R}^N \to \mathbb{R}^+$ is a nonnegative symmetric measurable function on the fractional Laplacian side (though we also consider kernels in the nonintegrable limit case).

The differential character of the operator is defined by the exponent

$$q_* = \inf\left\{ q_0 > 0 \,:\, \int_{\mathbb{R}^N} \min(1, |z|^{q_0})J(z)\,dz < \infty \right\}.$$

As for the non-linearity, if we put $\Psi' = \psi$ we consider functions Ψ convex and symmetric satisfying, for some $p \geq q > 1$,

$$q \leq \frac{s\Psi'(s)}{\Psi(s)} \leq p \qquad \forall s \neq 0.$$

We thus define the functionals

$$F(u) = \int_{\mathbb{R}^N} \Psi(u(x))\,dx,$$

$$E(u) = \frac{1}{2}\iint_{\mathbb{R}^{2N}} \Psi(u(x) - u(y))J(x-y)\,dxdy.$$

The properties of ψ imply that Ψ is an strict Young function, so we can consider the Orlicz and Sobolev-Orlicz spaces

$$L^{\Psi}(\mathbb{R}^N) = \{u \,:\, \mathbb{R}^N \to \mathbb{R},\ F(u) < \infty\}, \tag{13}$$

$$W^{J,\Psi}(\mathbb{R}^N) = \left\{u \in L^{\Psi}(\mathbb{R}^N),\ E(u) < \infty\right\}. \tag{14}$$

Our main interest is to study the properties of those spaces (13) and (14) which are Banach and reflexive for nonlinearities Ψ in the previous hypotheses. In particular if $q > q_*$, then the functional $E(u)$ is well defined and finite for functions that satisfy $F(\nabla u) < \infty$, so that $W^{1,\Psi}(\mathbb{R}^N) \subset W^{J,\Psi}(\mathbb{R}^N)$, the first being the standard Sobolev-Orlicz space of functions in $L^{\Psi}(\mathbb{R}^N)$ with gradient also in $L^{\Psi}(\mathbb{R}^N)$.

Without imposing any singularity condition to the kernel J at the origin, assuming just that it is not integrable, we show that there is an inequality of Poincaré type $E(u) \geq F(u)$, which implies the inclusion

$$W_0^{J,\Psi}(\Omega) \subset L^\Psi(\Omega).$$

Notice that if J were integrable then $W_0^{J,\Psi}(\Omega) \equiv L^\Psi(\Omega)$ (as in the case $\Psi(s) = s^2$ considered in the first part). In our situation, when J is a singular kernel we have the following result.

Theorem 5. *The embedding $W_0^{J,\Psi}(\Omega) \hookrightarrow L^\Psi(\Omega)$ is compact.*

Assuming the stronger singularity condition $J(z) \geq c|z|^{-N-\alpha,}$ for $0 < |z| < 1$ and $\alpha > 0$, we have the following result

Theorem 6. *Assume J satisfies the previous condition with $0 < \alpha < N$. Then $W_0^{J,\Psi}(\Omega) \subset L^{\Psi^r}(\Omega)$ for every $1 \leq r \leq r^* \equiv \dfrac{N}{N-\alpha}$. Moreover, this inclusion is compact if $r < r^*$.*

With these properties we study the problem of finding $u \in W_0^{J,\Psi}(\Omega)$

$$\begin{cases} \mathcal{L}u = f(x), & \text{in } \Omega, \\ u = 0, & \text{in } \Omega^c. \end{cases} \tag{15}$$

We prove existence, uniqueness (assuming some technical conditions on Ψ) together whit some integrability properties depending on the singularity of the kernel

Theorem 7. *For any $f \in \left(W_0^{J,\Psi}(\Omega)\right)'$ there exists a solution to problem (15). If ψ satisfies either condition (5.14) or (5.17) then the solution is unique. If $\Psi(s) \sim |s|^p$ and $J(z) \geq c|z|^{-N-\alpha,}$, then $f \in L^m(\Omega)$, implies*

i) *$u \in L^{\frac{m(p-1)N}{N-m\alpha}}(\Omega)$ if $m < N/\alpha$;*

ii) *$u \in L^\infty(\Omega)$ if $m > N/\alpha$.*

Regarding the case of non-linear reaction,

$$\begin{cases} \mathcal{L}u = f(u), & \text{in } \Omega, \\ u = 0, & \text{in } \Omega^c. \end{cases} \tag{16}$$

we have two situations for $f(u) \sim u^{m-1}$ (we look for nonnegative solutions): the so-called sublinear case $0 < \frac{m}{p} < 1$ and the superlinear subcritical case $1 < \frac{m}{p} < \frac{N}{N-\alpha}$. The conditions that we consider are more general and more technical, depending on the function.Ψ.

Theorem 8. *Suppose the hypotheses* (7.12) *or* (7.14) *on f are satisfied, then the problem* (16) *possesses at least a nontrivial solution.*

We end this part by studying the generalized eigenvalue problem

$$\begin{cases} \mathcal{L}u = \lambda\psi(u), & \text{in } \Omega, \\ u = 0, & \text{in } \Omega^c. \end{cases} \tag{17}$$

We construct a family of eigenvalues and eigenfunctions: we prove that for every $\mu > 0$ there exists a positive eigenvalue λ_μ of (17) with non-negative eigenfunction $u_\mu \in W_0^{J,\Psi}(\Omega)$ such that $F(u_\mu) = \mu$.

Organization of the thesis

In the first part of the thesis we devote a preliminary Chapter 1 to settle the precise hypotheses that we consider throughout this part. Chapter 2 is devoted to the study of the nonlocal operator and the associated bilinear form, describing their properties, including the action of the operator on different functions, two Hardy inequalities and a symmetrization result for the bilinear form. The compact embeddings of our Sobolev spaces into L^2 are studied in Chapter 3, as well as the inclusion into some Lorentz type space. In Chapter 4 three different problems are considered, two linear problems, with Dirichlet or Neumann exterir condition, and a nonlinear problem with different reactions; it is shown existence and uniqueness for sublinear reactions and nonexistence for supercriticl reactions.

In the second part we begin again with a preliminary chapter where some properties of the Orlicz spaces L^Ψ are proved by means of some interesting inequalities satisfied by the nonlinearity Ψ and the functionals F and E, Chapter 5. The Sobolev embeddings for the space $W^{J,\Psi}$ are shown in Chapter 6. Finally Chapter 7 is dedicated to the study of the associated elliptic problems for the different reactions discussed above.

LÉVY TYPE OPERATORS IN THE LIMIT OF INTEGRABILITY

Chapter 1

Preliminaries. Abstract framework

In this part we study the properties of the nonlocal operator

$$\mathfrak{L}u(x) = \text{P.V.} \int_{\mathbb{R}^N} (u(x) - u(y))J(x,y)\,dy, \tag{1.1}$$

where J is a symmetric Lévy kernel, that is, $J(x,y) = J(y,x)$ and

$$\sup_{x\in\mathbb{R}^N} \left(\int_{|x-y|<1} |x-y|^2)J(x,y)\,dy + \int_{|x-y|>1} J(x,y)\,dy \right) < \infty. \tag{H$_0$}$$

We also assume that J belongs to the non integrable range by imposing the condition $J(x,y) \geq \mathcal{K}(x-y) \geq 0$, where $\mathcal{K}$ is a Lévy kernel satisfying

$$\begin{cases} \mathcal{K}(z) = |z|^{-N}\ell(|z|) & \text{for } 0 < |z| < \rho, \\ M(r) := \displaystyle\int_r^\rho \frac{\ell(s)}{s}\,ds \to \infty & \text{as } r \to 0^+, \end{cases} \tag{H$_1$}$$

for some function $\ell : (0,\rho) \to (0,\infty)$, $\rho > 0$ satisfying $0 < c_1(\varepsilon) \leq \ell(s) \leq c_2(\varepsilon) < \infty$ for every $0 < \varepsilon < \rho$.

These hypotheses are assumed throughout the thesis without explicit mention. For some results we assume the stronger hypothesis $J(x,y) = \mathcal{K}(x-y)$ with $\mathcal{K}$ satisfying (H$_0$)–(H$_1$).

On the other hand, in order to emphasize that the kernel J neither belongs to the fractional Laplacian range, we impose the condition

$$\lim_{|x-y|\to 0} |x-y|^{N+\alpha} J(x,y) = 0 \qquad \forall\, \alpha > 0. \tag{1.2}$$

On the other hand, as a measure of the good behaviour of the kernel at the origin we assume that the function ℓ in (H_1) varies slowly at the origin, that is

$$\lim_{s\to 0} \frac{\ell(\lambda s)}{\ell(s)} = 1 \qquad \text{for every } \lambda > 0. \tag{H_2}$$

See the monograph [8] for the properties of slowly varying functions. Examples of slowly varying functions ℓ that also satisfy (H_1) are

$$\begin{aligned}
&\ell(s) = 1,\\
&\ell(s) = \log^{\beta}(2\rho/s), \quad \beta \geq -1,\\
&\ell(s) = \left(\log(2\rho/s)\log\left(\log(2\rho/s)\right)\right)^{-1} \dots
\end{aligned}$$

We do not consider highly oscillating functions like $\ell(s) = 1 - \sin(1/s)$. Observe also that using the representation formula (3.10) for slowly varying functions, see [8], it is easy to prove that (H_2) implies (1.2).

Associated to the operator $\mathfrak{L}$ we consider the bilinear form

$$\mathcal{E}(u,v) = \frac{1}{2} \iint_{Q_\Omega} (u(x)-u(y))(v(x)-v(y))J(x,y)\,dxdy, \tag{1.3}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded set and $Q_\Omega = (\Omega^c \times \Omega^c)^c$, as well as the Sobolev spaces

$$\mathcal{H}_J(\Omega) = \left\{u : \mathbb{R}^N \to \mathbb{R},\ u|_\Omega \in L^2(\Omega),\, \mathcal{E}(u,u) < \infty\right\}, \tag{1.4}$$

and

$$\mathcal{H}_{J,0}(\Omega) = \{u \in \mathcal{H}_J(\Omega),\ u \equiv 0 \text{ in } \Omega^c\}. \tag{1.5}$$

We observe that the Dirichlet form $\mathcal{E}(u,u)$ is different from the Dirichlet form related with the so-called censored processes,

$$\mathcal{E}_c(u,u) = \frac{1}{2} \int_\Omega \int_\Omega |u(x)-u(y)|^2 J(x,y)\,dxdy,$$

see for instance [10]. Actually,

$$\mathcal{E}(u,u) = \mathcal{E}_c(u,u) + \int_\Omega |u(x)|^2 \Lambda(x)\,dx, \tag{1.6}$$

where

$$\Lambda(x) = \int_{\Omega^c} J(x,y)\,dy. \tag{1.7}$$

Clearly the second integral in (1.6) is strictly positive. If for instance $J(x,y) \geq c > 0$ for every $|x-y| \leq R$, and $R > \delta = \sup_{x\in\Omega} dist(x, \partial\Omega)$ then $\Lambda(x) \geq c|\{\delta < |x-y| < R\}| = A > 0$ for every $x \in \Omega$. See Theorem 2.2.5 below for a more precise estimate of this function. This gives the Poincaré inequality

$$\mathcal{E}(u,u) \geq A\|u\|_2^2, \tag{1.8}$$

or which is the same, the property

$$\mathcal{H}_{J,0}(\Omega) \subset L^2(\Omega). \tag{1.9}$$

Even more, the Poincaré inequality holds also for the bilinear form $\mathcal{E}_c$, though the proof is not so direct. A very much weaker condition to have a Poincaré inequality is obtained in [30], where they prove that it is enough to have $|\{K(z) > 0\}| > 0$.

When J is a Lévy kernel with the only assumption of non integrability (H_1), it is not clear if there is some room between the above spaces in (1.9), like there is in the fractional Laplacian range, precisely due to the Hardy-Littlewood-Sobolev inequality. In fact, in the integrable range we do have $\mathcal{H}_{J,0}(\Omega) \equiv L^2(\Omega)$.

The next sections are devoted to prove that the inclusion (1.9) is compact and moreover some space of Lorentz type can be put in between. In order to do that we first establish some properties of the operator $\mathfrak{L}$ and the Dirichlet form $\mathcal{E}$. We finally apply all this material in order to study linear and nonlinear elliptic type problems associated to $\mathfrak{L}$.

Chapter 2

The nonlocal operator and the bilinear form

We study the nonlocal operator $\mathfrak{L}$ defined in (1.1) and the associated Dirichlet form defined in (1.3). The key point is that the nonintegrable Lévy kernels J defining $\mathfrak{L}$ posses a singularity at the origin that is weaker than that of any fractional Laplacian.

2.1 Properties of the operator $\mathfrak{L}$

2.1.1 Regularity properties

We first study the effect of applying the operator $\mathfrak{L}$ to a Hölder continuous function.

Theorem 2.1.1. *Assume $J(x,y) = \mathcal{K}(x-y)$, where $\mathcal{K}$ satisfies* (H_2). *If $u \in C^\nu(\mathbb{R}^N)$ for some $\nu \in (0,1)$, then $\mathfrak{L}u \in C(\mathbb{R}^N)$, with some modulus of continuity that depends on $\mathcal{K}$ and ν.*

Proof. Let us estimate the difference $|\mathfrak{L}u(x) - \mathfrak{L}u(y)|$ for $x, y \in \mathbb{R}^N$. Let $R < 1$ to be fixed.

$$|\mathfrak{L}u(x) - \mathfrak{L}u(y)| = \left| \int_{\mathbb{R}^N} (u(x) - u(x+z) - u(y) + u(y+z))\, \mathcal{K}(z)\, dz \right| \leq I_1 + I_2,$$

where

$$I_1 = \int_{B_R} (|u(x) - u(x+z)| + |u(y) - u(y+z)|)\, \mathcal{K}(z)\, dz,$$

$$I_2 = \int_{B_R^c} (|u(x) - u(y)| + |u(x+z) - u(y+z)|)\, \mathcal{K}(z)\, dz.$$

For I_1, using that $|u(x) - u(x+z)| \leq [u]_{C^\nu} |z|^\nu$ we get

$$I_1 \leq 2\, [u]_{C^\nu} \left| \int_{B_R} |z|^\nu \mathcal{K}(z)\, dz \right| = c \int_0^R \frac{\ell(s)}{s^{1-\nu}}\, ds = A(R).$$

For I_2, using that $|u(x+z) - u(y+z)| \leq [u]_{C^\nu} |x-y|^\nu$ we get

$$I_2 \leq 2\, [u]_{C^\nu} |x-y|^\nu \int_{B_R^c} \mathcal{K}(z)\, dz \leq c|x-y|^\nu M(R).$$

Thus picking $R = g^{-1}(|x-y|)$, where $g(R) = (A(R)/M(R))^{1/\nu}$, we obtain

$$|\mathfrak{L}u(x) - \mathfrak{L}u(y)| \leq 2c\varpi(|x-y|)$$

where $\varpi = M \circ g^{-1}$. ∎

When $\ell(0) < \infty$ we have $A(R) \sim R^\nu$ and $M(R) \sim \log(1/R)$ for $R \to 0$. In that case $g(R) \sim R \log^{-1/\nu}(1/R)$. We have then that the regularity of $\mathfrak{L}u$ is almost the same as that of u; in particular $\mathfrak{L}u \in C^{\nu-\varepsilon}(\mathbb{R}^N)$ for every $0 < \varepsilon < \nu$.

With the same technique we can obtain the following.

Corollary 2.1.2. *If u is a continuous function with a modulus of continuity ϖ_0, then $\mathfrak{L}u$ is continuous provided*

$$\lim_{R \to 0} \int_0^R \frac{\varpi_0(s)\ell(s)}{s}\, ds = 0.$$

2.1.2 Estimates of the action of $\mathfrak{L}$ on some functions

Also of interest is to obtain integrability properties of $\mathfrak{L}u$ when u is the characteristic function of some set, depending on the regularity of the boundary. Observe that for $E \subset \mathbb{R}^N$

$$\int_E \mathfrak{L}\mathbb{1}_E(x)\,dx = \int_E \int_{E^c} J(x-y)\,dydx = \mathcal{E}_1(\mathbb{1}_E, \mathbb{1}_E)$$

see the definition of $\mathcal{E}_1$ in the next section. If $E \subset \Omega$ this quantity coincides with $\mathcal{E}(\mathbb{1}_E, \mathbb{1}_E)$. This is called the J–perimeter of the set E, $P_J(E)$. See [18] for the fractional perimeter and for more general definitions when $E \not\subset \Omega$.

We say that ∂E has a modulus of continuity ϖ_0 if it is locally the graph of a function η defined on a small ball $B \subset \mathbb{R}^{N-1}$, such that

$$|\eta(z_1) - \eta(z_2)| \le \varpi_0(|z_1 - z_2|), \qquad \text{for all } z_1,\, z_2 \in B.$$

Theorem 2.1.3. *Assume $J(x,y) = \mathcal{K}(x-y)$, where $\mathcal{K}$ satisfies* (H$_2$). *If $E \subset \mathbb{R}^N$ has modulus of continuity ϖ_0, then $P_J(E) < \infty$ provided*

$$\int_0^\rho \frac{\varpi_0(s)\ell(s)}{s}\,ds < \infty.$$

Proof. We estimate $\int_{E^c} \mathcal{K}(x-y)\,dy$ for $x \in E$. To that purpose let $x \in E$, with $\delta(x) = \operatorname{dist}(x, \partial E) = r > 0$, and let $B_r = \{|y-x| < r\}$. We have

$$\int_{E^c} \mathcal{K}(x-y)\,dy \le \int_{B_r^c} \mathcal{K}(x-y)\,dy = \int_{r<|z|<\rho} \mathcal{K}(z)\,dz + \int_{|z|>\rho} \mathcal{K}(z)\,dz = c_1 M(r) + c_2,$$

if $r < \rho$; if $r \ge \rho$ we directly deduce $\int_{E^c} \mathcal{K}(x-y)\,dy < c$.

As an immediate consequence we have that if $D_0 = \{x \in E,\, \delta(x) \ge \rho\}$, then $\int_{D_0} \mathfrak{L}\mathbb{1}_E \le c$. Then it suffices to show that $\int_D M(\delta(x))\,dx < \infty$ for each set of the form

$$D = \left\{(x', x_N) \in \mathbb{R}^{N-1} \times \mathbb{R},\, |x'| < \varepsilon,\, 0 < x_N < \eta(x')\right\},$$

where $\eta < \rho$. The regularity of the function η gives that

$$\delta(x) \ge \varpi_0^{-1}(|\eta(x') - x_N|).$$

Therefore, since M is decreasing,

$$\begin{aligned}\int_D M(\delta(x))\,dx &= \int_{|x'|<\varepsilon}\int_0^{\eta(x')} M(\delta(x',x_N))\,dx_N dx' \\ &\le \int_{|x'|<\varepsilon}\int_0^{\eta(x')} M(\varpi_0^{-1}(|\eta(x')-x_N|))\,dx_N dx' \\ &\le c\int_0^\rho M(\varpi_0^{-1}(z))\,dz = c\int_0^\rho \frac{\varpi_0(s)\ell(s)}{s}\,ds.\end{aligned}$$

■

Observe that if ∂E is a domain with very weak continuity, say $\varpi_0(s) = (\log 1/s)^{-\sigma}$, $\sigma > 0$, there always exists an operator $\mathfrak{L}$ with a singularity so weak that makes $\mathfrak{L}\mathbb{1}_E$ still integrable, just choose $\ell(s) = (\log 1/s)^{-1}$ for that set. Therefore the characteristic function of a bounded domain $E \subset \Omega$ with a Lebesgue spine belongs to some Sobolev type space $\mathcal{H}_{J,0}(\Omega)$, and the set has finite J-perimeter, where $J(x,y) = |x-y|^{-N}(\log 1/|x-y|)^{-1}$.

We end this subsection estimating the action of $\mathfrak{L}$ to a specific power $|x|^{-N/2}$, precisely the one needed in the forthcoming proof of Hardy inequality.

Lemma 2.1.4. *There exists $\varepsilon > 0$ small such that*

$$|x|^{N/2}\mathfrak{L}|x|^{-N/2} \ge cM(|x|) \qquad \text{for } 0<|x|<\varepsilon. \tag{2.1}$$

Proof. By the hypotheses we have $J(x,y) \ge c\ell(|x-y|)|x-y|^{-N}$ for $|x-y|<\rho$ for some $\rho > 0$. We thus have

$$\begin{aligned}|x|^{N/2}\mathfrak{L}|x|^{-N/2} &\ge c\int_{|x-y|<\rho,\,|y|>|x|}\left(1-\frac{|x|^{N/2}}{|y|^{N/2}}\right)\frac{\ell(|x-y|)}{|x-y|^N}\,dy \\ &\quad -c\int_{|y|<|x|}\left(\frac{|x|^{N/2}}{|y|^{N/2}}-1\right)\frac{\ell(|x-y|)}{|x-y|^N}\,dy \\ &\quad +\int_{|x-y|>\rho}\left(1-\frac{|x|^{N/2}}{|y|^{N/2}}\right)J(x,y)\,dy \\ &= I_1 - I_2 + I_3.\end{aligned}$$

We see that I_2 is convergent and I_3 is positive if for instance $|x|<\rho/2$. As to I_1 we get

$$\begin{aligned}I_1 &= c\int_{|z|<\rho,\,|z-x|>|x|}\frac{|z-x|^{N/2}-|x|^{N/2}}{|z-x|^{N/2}}\frac{\ell(|z|)}{|z|^N}\,dz \\ &\ge c\int_{3|x|<|z|<\rho}\frac{\ell(|z|)}{|z|^N}\,dz \ge cM(3|x|) \ge cM(|x|),\end{aligned}$$

if $|x| < \rho/3$, where we have used in the last inequality that ℓ is slowly varying. We conclude since $M(0^+) = \infty$. ■

In the fractional Laplacian case it is easy to obtain, by means of the Fourier transform,

$$(-\Delta)^{\alpha/2}|x|^{\gamma} = c_{N,\alpha,\gamma}|x|^{\gamma-\alpha},$$

whenever $\gamma > -N$. In particular

$$|x|^{\frac{N-\alpha}{2}}(-\Delta)^{\alpha/2}|x|^{-\frac{N-\alpha}{2}} = c_{N,\alpha}|x|^{-\alpha},$$

which gives the weight for the fractional Hardy inequality (2.5). Also the sharp constant can be obtained from that identity. Compared with estimate (2.1) we formally try to put $\alpha = 0$ on the left, taking care of the constant, obtaining a logarithmic type function on the right.

2.2 Properties of the bilinear form $\mathcal{E}$

We recall that the bilinear form (1.3), when applied to functions vanishing outside Ω, coincides with the global bilinear form

$$\mathcal{E}_1(u,u) = \frac{1}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} |u(x)-u(y)|^2 J(x,y)\,dxdy,$$

which is adequate also to study problems defined in the whole space. But for problems defined in a bounded domain, with a nontrivial condition in the complement of the domain, the associated bilinear form is $\mathcal{E}$ and not $\mathcal{E}_1$. See Subsection 4.2.3.

2.2.1 Symmetrization

An easy property of bilinear forms of the type (1.3) is that they decrease when taking absolute values

$$\mathcal{E}_1(|u|,|u|) \le \mathcal{E}_1(u,u). \tag{2.2}$$

This follows from the inequality $||a|-|b|| \le |a-b|$. It is also a consequence of the following more general inequality, called Stroock-Varopoulos inequality [67, 12], which will be used later on.

Proposition 2.2.1. *Let $u \in \mathcal{H}_J(\mathbb{R}^N)$ such that $F(u)$, $G(u)$, $\Phi(u) \in \mathcal{H}_J(\mathbb{R}^N)$, and assume $(\Phi')^2 \le F'G'$. Then*

$$\mathcal{E}_1(\Phi(u),\Phi(u)) \le \mathcal{E}_1(F(u),G(u)).$$

Clearly the same is true with $\mathcal{E}_1$ replaced by $\mathcal{E}$ provided $u \in \mathcal{H}_{J,0}(\mathbb{R}^N)$. It is not clear what happens in the last case for general $u \in \mathcal{H}_J(\mathbb{R}^N)$.

Another interesting property, showed in [55] for the fractional Laplacian, is the inequality

$$\mathcal{E}(m,m) + \mathcal{E}(M,M) \leq \mathcal{E}(u,u) + \mathcal{E}(v,v),$$

where

$$m(x) = min\{u(x), v(x)\}, \qquad M(x) = max\{u(x), v(x)\}.$$

The proof is based on the easy inequality

$$|m(x) - m(y)|^2 + |M(x) - M(y)|^2 \leq |u(x) - u(y)|^2 + |v(x) - v(y)|^2,$$

and obviously also works in our situation.

We prove in this section that the energy $\mathcal{E}(u,u)$ also decreases when we replace u by its symmetric rearrangement, provided the kernel is radially symmetric and decreasing. This property is well known for the norm in $H_0^{\alpha/2}(\Omega)$, $0 < \alpha \leq 2$.

For a measurable function we consider its distribution function

$$\mu(t) = |\{x \in \mathbb{R}^N : |u(x)| > t\}|.$$

We then define the decreasing rearrangement u^* of u to be the radially deacreasing function with the same distribution function as u, that is, $u^* : B_R \to \mathbb{R}^+$, with $R = \left(\dfrac{|\Omega|}{\omega_N}\right)^{1/N}$, satisfies

$$u^*(x) = \inf\{\lambda > 0 \,:\, \mu(\lambda) < \omega_N |x|^N\}.$$

Here ω_N is the measure of the unit ball in $\mathbb{R}^N$. We also have the layer cake representation

$$u^*(x) = \int_0^\infty \mathbb{1}_{\{u(x)>t\}}\, dt, \tag{2.3}$$

and

$$\int_{B_R} u^*(x)\, dx = \int_\Omega |u(x)|\, dx = \int_0^\infty \mu(t)\, dt.$$

All the L^p norms are conserved as well under symmetrization, or even the integral of any convex, nonnegative symmetric function of u. On the other hand, for Sobolev norms symmetrization decreases, that is

$$\int_{B_R} |\nabla u^*(x)|^p\, dx \leq \int_\Omega |\nabla u(x)|^p\, dx,$$

or even for fractional Sobolev norms

$$\int_{\mathbb{R}^N} |(-\Delta)^{\alpha/2} u^*(x)|^2 \, dx \leq \int_{\mathbb{R}^N} |(-\Delta)^{\alpha/2} u(x)|^2 \, dx.$$

See for instance [2]. We prove here that symmetrization also decreases the norm in the space $\mathcal{H}_J(\Omega)$.

Theorem 2.2.2. *Assume $J(x,y) = \mathcal{K}(x-y)$ where $\mathcal{K}$ is a radially nonincreasing function. If $u \in \mathcal{H}_{J,0}(\Omega)$ and u^* is its decreasing rearrangement, then*

$$\mathcal{E}(u,u) \geq \mathcal{E}(u^*,u^*).$$

We use the following result, which is the key point to prove the inequality in the fractional framework, see again [2].

Theorem 2.2.3. *Let $\Phi \in L^1(\mathbb{R}^N)$ be a positive symmetric decreasing function. Then for all non-negative measurable u one has*

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} (u(x)-u(y))^2 \Phi(x-y) \, dxdy \geq \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} (u^*(x)-u^*(y))^2 \Phi(x-y) \, dxdy.$$

Proof of Theorem 2.2.2. We write, as in [2],

$$\mathcal{E}(u,u) = \frac{1}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} |u(x)-u(y)|^2 \left(\int_0^\infty e^{\frac{-t}{\mathcal{K}(x-y)}} \, dt\right) dxdy = \int_0^\infty G(u,t) \, dt$$

where

$$G(u,t) = \frac{1}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} |u(x)-u(y)|^2 \Phi(|x-y|) \, dxdy, \qquad \Phi(z) = e^{\frac{-t}{\mathcal{K}(z)}}.$$

By adding ε times a positive integrable function to our kernel we get that $\Phi \in L^1(\mathbb{R}^N)$ satisfies the hypotheses of Theorem 2.2.3. We then conclude, by letting $\varepsilon \to 0$,

$$G(u,t) \geq G(u^*,t) \qquad \text{for every } t > 0,$$

and thus

$$\mathcal{E}(u,u) \geq \mathcal{E}(u^*,u^*).$$

∎

2.2.2 Hardy inequalities

We now establish some interesting inequalities for the bilinear form $\mathcal{E}$ which will be used afterwards in order to sharpen the inclusion (1.9). We prove two Hardy inequalities of the form

$$\mathcal{E}(u,u) \geq c \int_\Omega |u(x)|^2 \psi(x)\, dx,$$

where ψ is a weight that can be singular at the origin or at the boundary of Ω. See for instance [54] for the classical Hardy inequalities in the local case, and [28, 31, 38] for the fractional Laplacian case.

We begin with a Hardy inequality that contains a weight singular at the origin. It is to be compared with the classical Hardy inequality,

$$\int_\Omega |\nabla u(x)|^2\, dx \geq d_N \int_\Omega \frac{|u(x)|^2}{|x|^2}\, dx, \tag{2.4}$$

and the fractional Hardy inequality, corresponding to $J(x,y) = |x-y|^{-N-\alpha}$, $0 < \alpha < 2$, which is

$$\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^2}{|x-y|^{N+\alpha}}\, dxdy \geq d_{N,\alpha} \int_\Omega \frac{|u(x)|^2}{|x|^\alpha}\, dx, \tag{2.5}$$

for some explicit constants d_N and $d_{N,\alpha}$. In our situation of weakly singular kernels the weight depends on the function ℓ, and if for instance $\ell(0) = c > 0$ it is logarithmic. In the proof we use the estimate of the action of the nonlocal operator $\mathfrak{L}$ in (1.1) over some power obtained in the previous section.

Theorem 2.2.4. *For every $u \in \mathcal{H}_{J,0}(\Omega)$ it holds*

$$\mathcal{E}(u,u) \geq c \int_\Omega |u(x)|^2 M(\rho|x|/R)\, dx, \tag{2.6}$$

$R = \sup_{x\in\Omega} |x|$. *If moreover $\ell(0) > 0$ then*

$$\mathcal{E}(u,u) \geq c \int_\Omega |u(x)|^2 \left|\log(\rho|x|/R)\right|\, dx,$$

Proof. We first observe that if w is a nontrivial nonnegative function, then

$$\begin{aligned} \mathcal{E}(u,u) &= \mathcal{E}\left(\frac{u^2}{w}, w\right) + \frac{1}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} w(x)w(y)\left(\frac{u(x)}{w(x)} - \frac{u(y)}{w(y)}\right)^2 J(x,y)\, dxdy, \\ &\geq \int_\Omega \frac{u^2}{w}\,\mathfrak{L}w \end{aligned}$$

since the second integral on the right is nonnegative. Now put $w(x) = |x|^{-N/2}$ and use Lemma 2.1.4 to get

$$\mathcal{E}(u,u) \geq c \int_{|x|<\varepsilon} |u(x)|^2 M(|x|)\, dx.$$

With Poincaré inequality this gives (2.6), since M is bounded outside the origin, and M is rescaled in order to be defined at all points of Ω. ∎

Remarks. *i*) The first equality in the proof is a sort of Picone identity [58], to be compared with the classical one

$$|\nabla u|^2 = \nabla \left(\frac{u^2}{w} \right) \cdot \nabla w + u^2 \left| \frac{\nabla u}{u} - \frac{\nabla w}{w} \right|^2,$$

provided u, $w \in C^1(\Omega)$, $w \geq 0$, $w \not\equiv 0$.

ii) Observe that what we indeed have is a Hardy inequality with remainder

$$\mathcal{E}(u,u) \geq c \int_\Omega |u(x)|^2 M(|x|)\, dx + \frac{1}{2} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{\left(|x|^{N/2} u(x) - |y|^{N/2} u(y)\right)^2}{|x|^{N/2} |y|^{N/2}} J(x,y)\, dxdy.$$

iii) In general we obtain inequality (2.6) for a given weight function ψ if there exists a function w such that $\mathfrak{L} w \geq \psi w$.

iv) One of the main features of the classical Hardy inequalities (2.4) and (2.5) is that the optimal constants can be obtained in a precise way. And this depends on the fact that we can obtain an explicit function w for which $\mathfrak{L} w = \psi w$, see the comment after Lemma 2.1.4. This is not possible in our situation.

We also obtain a Hardy inequality in a ball with a weight singular at the boundary, in the spirit of [28].

Theorem 2.2.5. *Assuming hypothesis* (H$_2$)*, for every* $u \in \mathcal{H}_{J,0}(B_1)$ *it holds*

$$\mathcal{E}(u,u) \geq c \int_{B_1} |u(x)|^2 M(1 - |x|)\, dx.$$

Proof. By (1.6) we only have to estimate the function

$$\Lambda(x) = \int_{\{|x-y|>1,\, |y|<1\}} |y|^{-N} \ell(|y|)\, dy.$$

For each $|x| < 1$ given, let $R = R_x$ be the rotation that carries x to the negative first axis, that is, $w = Rx = -|x| e_1$, where $e_1 = (1, 0, \ldots, 0)$, and perform the change of variables $z = Ry$. Since

$$z_1 + |x| = z_1 + |w| > 1 \implies z_1 - w_1 > 1 \implies |z - w| > 1,$$

we get, if $N \geq 2$,

$$\begin{aligned}\Lambda(x) &\geq c \int_{1-|x|}^{1} \int_{0}^{1} (t^2+\rho^2)^{-N/2} \rho^{N-2} \ell\left(\sqrt{t^2+\rho^2}\right) d\rho dt \\ &\geq c \int_{1-|x|}^{1} \int_{0}^{1} t^{-1} w^{N-2} \ell\left(t\sqrt{1+w^2}\right) dw dt.\end{aligned}$$

Property (H_2) implies

$$\ell\left(t\sqrt{1+w^2}\right) \geq c\ell(t) \qquad \text{for every } 0 < w < 1,$$

so that

$$\Lambda(x) \geq c \int_{1-|x|}^{1} \int_{0}^{1} \frac{\ell(t)}{t} w^{N-2} \, dw dt = cM(1-|x|).$$

If $N = 1$ we get the same estimate directly. ∎

Chapter 3

The Sobolev spaces $\mathcal{H}_J$

Recall that we are considering the spaces

$$\mathcal{H}_J(\Omega) = \left\{u : \mathbb{R}^N \to \mathbb{R},\ u|_\Omega \in L^2(\Omega),\ \mathcal{E}(u,u) < \infty\right\} \tag{3.1}$$

and

$$\mathcal{H}_{J,0}(\Omega) = \{u \in \mathcal{H}_J(\Omega),\ u \equiv 0 \text{ in } \Omega^c\}\,. \tag{3.2}$$

The properties of these spaces depend on the behaviour at the origin of the kernel J defining the bilinear form $\mathcal{E}$. The condition of being of Lévy type implies

$$H_0^1(\Omega) \subset \mathcal{H}_{J,0}(\Omega) \subset \mathcal{H}_J(\Omega) \subset L^2(\Omega).$$

We want to know the exact place were $\mathcal{H}_{J,0}(\Omega)$ and $\mathcal{H}_J(\Omega)$ lie in relation to $L^2(\Omega)$. If $\ell(s) \geq s^\alpha$ for $s \sim 0$ and some $0 < \alpha < 2$, then

$$\mathcal{H}_J(\Omega) \subset H^{\alpha/2}(\Omega) \subset L^{\frac{2N}{N-\alpha}}(\Omega),$$

since the operator $\mathfrak{L}$ behaves like the fractional Laplacian. We are thus reduced to study kernels satisfying (1.2).

3.1 Compact embeddings in L^2

For operators in the limit of integrability we prove that

Theorem 3.1.1. *Under the hypothesis* (H$_2$) *the embedding* $\mathcal{H}_{J,0}(\Omega) \hookrightarrow L^2(\Omega)$ *is compact.*

We consider the bilinear form $\mathcal{E}_{\mathcal{K}}$ associated to the convolution kernel $\mathcal{K}$ of (H$_1$), that is

$$\mathcal{E}_{\mathcal{K}}(u,u) = \frac{1}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}(u(x)-u(y))^2\mathcal{K}(x-y)\,dxdy.$$

Recall that $\mathcal{E}_1(u,u) \geq \mathcal{E}_{\mathcal{K}}(u,u)$. On the other hand

$$\mathcal{E}_{\mathcal{K}}(u,u) = \int_{\mathbb{R}^N} m(\xi)|u(\xi)|^2\,d\xi,$$

where the multiplier m is given by

$$m(\xi) = \int_{\mathbb{R}^N}\big(1-\cos(z\cdot\xi)\big)\mathcal{K}(z)\,dz.$$

This multiplier has been estimated in [41] using the function $M(r)$ and hypotheses (H$_2$),

$$m(\xi) \geq cM(|\xi|^{-1}), \qquad \text{for every } |\xi| > 1.$$

We are now in a position to prove Theorem 3.1.1, which is a direct application of the following characterization of Pego, see [57].

Theorem 3.1.2. *A bounded subset* Σ *of* $L^2(\mathbb{R}^N)$ *is conditionally compact if and only if*

$$\lim_{R\to\infty}\sup_{f\in\Sigma}\int_{|x|>R}|f(x)|^2\,dx = \lim_{R\to\infty}\sup_{f\in\Sigma}\int_{|\xi|>R}|\widehat{f}(\xi)|^2\,d\xi = 0. \tag{3.3}$$

Proof of Theorem 3.1.1. For a constant $C>0$ let

$$\Sigma = \{f \in \mathcal{H}_{J,0}(\Omega) \,:\, \|f\|_{\mathcal{H}_J} \leq C\} \subset L^2(\mathbb{R}^N).$$

We first have, since Ω is bounded, that if R is large enough

$$\int_{|x|>R}|f(x)|^2\,dx = 0, \qquad \text{for every } f\in\Sigma.$$

On the other hand, from the previous calculations we have,

$$C^2 \geq \mathcal{E}(f,f) = \mathcal{E}_1(f,f) \geq \mathcal{E}_{\mathcal{K}}(f,f) \geq c\int_{|\xi|>R} M(|\xi|^{-1})|\widehat{f}(\xi)|^2\,d\xi.$$

Thus

$$\int_{|\xi|>R} |\widehat{f}(\xi)|^2 \, d\xi \leq \frac{c}{M(1/R)}, \qquad \text{for every } f \in \Sigma,$$

since M is nonincreasing. We conclude with the fact that $M(0^+) = \infty$ that (3.3) holds. ■

In order to prove the same property for the bigger space $\mathcal{H}_J(\Omega)$ we must add an extra hypothesis.

Theorem 3.1.3. *Assume* $\ell(0^+) = \infty$ *in* (H_1). *Then the embedding* $\mathcal{H}_J(\Omega) \hookrightarrow L^2(\Omega)$ *is compact.*

Proof. The proof follows the one of the classical Riesz-Fréchet-Kolmogorov Theorem, as adapted in [55] for the fractional Laplacian.

Let $\mathcal{F} \subset \mathcal{H}_J(\Omega)$ be a bounded set. We show that $\mathcal{F}$ is totally bounded in $L^2(\Omega)$, i.e., for any $\epsilon \in (0, 1)$ there exist $\beta_1, ..., \beta_M \in L^2(B_1)$ such that for any $u \in \mathcal{F}$ there exists $j \in \{1, ..., M\}$ such that

$$\|u - \beta_j\|_{L^2(\Omega)} \leq \epsilon.$$

We take a collection of disjoints cubes $Q_1, ...Q_R$ of side $\rho < 1$ such that $\Omega = \bigcup_{j=1}^{R} Q_j$. For any $x \in \Omega$ we define $j(x)$ as the unique integer in $\{1, ..., R\}$ for which $x \in Q_{j(x)}$. Also, for any $u \in \mathcal{F}$, let

$$P(u)(x) := \frac{1}{|Q_{j(x)}|} \int_{Q_{j(x)}} u(y) \, dy.$$

Notice that

$$P(u + v) = P(u) + P(v) \text{ for any } u, v \in \mathcal{F},$$

and that $P(u)$ is constant, say equal to $q_j(u)$, in any Q_j, for $j \in \{1, ..., R\}$. Therefore, we can define

$$S(u) := \rho^{N/2} \, (q_1(u), ..., q_R(u)) \in \mathbb{R}^N.$$

We observe that $S(u + v) = S(u) + S(v)$. Moreover,

$$\begin{aligned} \|P(u)\|^2_{L^2(\Omega)} &= \sum_{j=1}^{R} \int_{Q_j} |P(u)|^2 \, dx \\ &\leq \rho^N \sum_{j=1}^{R} |q_j(u)|^2 = |S(u)|^2 \leq \frac{|S(u)|^2}{\rho^N}. \end{aligned} \tag{3.4}$$

And, by Hölder inequality,

$$\begin{aligned} |S(u)|^2 &= \sum_{j=1}^R \rho^N |q_j(u)|^2 = \frac{1}{\rho^N} \sum_{j=1}^R \left| \int_{Q_j} u(y)\, dy \right|^2 \\ &\le \sum_{j=1}^R \int_{Q_j} |u(y)|^2\, dy = \int_\Omega |u(y)|^2\, dy = \|u\|^2_{L^2(\Omega)}. \end{aligned}$$

In particular,

$$\sup_{u\in\mathcal{F}} |S(u)|^2 \le C,$$

that is, the set $S(\mathcal{F})$ is bounded in $\mathbb{R}^N$ and so, since it is finite dimensional, it is totally bounded. Therefore, there exist $b_1, ..., b_K \in \mathbb{R}^N$ such that

$$S(\mathcal{F}) \subset \bigcup_{i=1}^K B_\eta(b_i). \tag{3.5}$$

For any $i \in \{1, ..., K\}$, we write the coordinates of b_i as $b_i = (b_{i,1}, ..., b_{i,N}) \in \mathbb{R}^N$. For any $x \in \Omega$, we set

$$\beta_i(x) := \rho^{-N/2} b_{i,j(x)},$$

where $j(x)$ is as above. Notice that β_i is constant on Q_j, i.e. if $x \in Q_j$ then

$$P(\beta_j)(x) = \rho^{-N/2} b_{i,j} = \beta_i(x) \tag{3.6}$$

and so $q_j(\beta_i) = \rho^{-N/2} b_{i,j}$; thus $S(\beta_i) = b_i$. Furthermore, for any $u \in \mathcal{F}$, by Hölder inequality,

$$\begin{aligned} \|u - P(u)\|^2_{L^2(\Omega)} &= \sum_{j=1}^R \int_{Q_j} |u(x) - P(u)(x)|^2\, dx \\ &= \sum_{j=1}^R \int_{Q_j} \left| u(x) - \frac{1}{|Q_j|} \int_{Q_j} u(y)\, dy \right|^2 dx \\ &= \frac{1}{\rho^{2N}} \sum_{j=1}^R \int_{Q_j} \left| \int_{Q_j} (u(x) - u(y))\, dy \right|^2 dx \\ &\le \frac{1}{\rho^N} \sum_{j=1}^R \int_{Q_j} \int_{Q_j} |u(x) - u(y)|^2\, dy\, dx \\ &\le \frac{1}{\ell(\rho)} \sum_{j=1}^R \int_{Q_j} \int_{Q_j} |u(x) - u(y)|^2 J(x-y)\, dy\, dx \le \frac{2\mathcal{E}(u,u)}{\ell(\rho)}. \end{aligned}$$

Consequently, for any $j \in \{1, ..., K\}$, recalling (3.4) and (3.6)

$$\begin{aligned} \|u - \beta_j\|_{L^2(\Omega)} &\leq \|u - P(u)\|_{L^2(\Omega)} + \|P(\beta_j) - \beta_j\|_{L^2(\Omega)} + \|P(u - \beta_j)\|_{L^2(\Omega)} \\ &\leq \frac{2\mathcal{E}(u,u)}{\ell(\rho)} + \frac{|S(u) - S(\beta)|}{\rho^{N/2}}. \end{aligned} \tag{3.7}$$

Now, given any $u \in \mathcal{F}$, we recall (3.5) and we take $j \in \{1, ..., K\}$ such that $S(u) \in B_\eta(b_j)$. Then, (3.6) and (3.7) give that

$$\|u - \beta_j\|_{L^2(\Omega)} \leq \frac{2\mathcal{E}(u,u)}{\ell(\rho)} + \frac{|S(u) - S(\beta)|}{\rho^{N/2}} \leq \frac{2\mathcal{E}(u,u)}{\ell(\rho)} + \frac{\eta}{\rho^{N/2}} < \epsilon.$$

■

We also recall the work [19], where the authors consider symmetric kernels also in the limit of integrability. But their condition

$$\sup\left\{s \geq 0 \ : \ \lim_{r\to 0^+} r^s \int_{|x-y|>r} J(x-y)\, dy = \infty\right\} > 0,$$

is not fulfilled in general by the kernels studied in this paper. In particular it requires $\limsup_{|x|\to 0^+} |x|^{N+\varepsilon} J(x) > 0$ for some $\varepsilon > 0$.

3.2 Lorentz spaces

For a given function $\mathcal{A} : \mathbb{R}^+ \to \mathbb{R}^+$, increasing with $\mathcal{A}(0) = 0$, and a constant $p \geq 1$, we define the Lorentz-type space

$$\mathcal{L}_{\mathcal{A},p}(\mathbb{R}^N) = \left\{u \text{ measurable}, \ \int_0^\infty \mathcal{A}(\mu(t)) t^{p-1}\, dt < \infty\right\},$$

with seminorm

$$\|u\|_{\mathcal{A},p} = \left(p \int_0^\infty \mathcal{A}(\mu(t)) t^{p-1}\, dt\right)^{1/p},$$

where $\mu(t)$ is the distribution function of u. We may replace $\mathbb{R}^N$ by Ω in the definitions if we restrict ourselves to functions u that vanish outside Ω. In that case the above is a norm. These spaces generalize the Lebesgue spaces L^p and the standard Lorentz spaces $L_{q,p}$. In fact

$$\mathcal{L}_{\mathcal{A},p}(\Omega) = \begin{cases} L_{q,p}(\Omega) & \text{if } \mathcal{A}(s) = s^{p/q} \\ L^p(\Omega) & \text{if } \mathcal{A}(s) = s. \end{cases}$$

Theorem 3.2.1. *Assume hypothesis* (H$_2$) *and assume further that* $\mathcal{K}$ *is a radially symmetric nonincreasing function. Then there exists a function* $\mathcal{A} : [0, |\Omega|] \to \mathbb{R}^+$ *such that* $\mathcal{H}_{J,0}(\Omega) \subset \mathcal{L}_{\mathcal{A},2}(\Omega)$.

The function $\mathcal{A}$ depends on $\mathcal{K}$ through formula (3.8), and for that function the above inclusion is an improvement of the inclusion $\mathcal{H}_{J,0}(\Omega) \subset L^2(\Omega)$. In fact, as we will see, we have $\mathcal{L}_{\mathcal{A},p}(\Omega) \subsetneq L^p(\Omega)$ for every $p \geq 1$.

As a precedent we have Peetre's result [56] which asserts that $H_0^{\alpha/2}(\Omega)$ is contained in the Lorentz space $L_{\frac{2N}{N-\alpha},2}(\Omega)$, $N > \alpha$, which corresponds to taking $\mathcal{A}(s) = s^{\frac{N-\alpha}{N}}$.

We also recall that, since the norm in $\mathcal{L}_{\mathcal{A},p}(\Omega)$ depends only on the distribution function, it is invariant under rearrangement. We now show that the Lorentz space $\mathcal{L}_{\mathcal{A},p}(\Omega)$ is in fact an L^p space with weight when restricted to radially symmetric decreasing functions, $\mathcal{L}_{\mathcal{A},p,\mathrm{radial}}(\Omega) = L^p_{\mathrm{radial}}(\Omega;\psi)$, where ψ and $\mathcal{A}$ are related by the formula (3.8).

Lemma 3.2.2. *If* u *and* ψ *are non-negative, radially symmetric decreasing functions with compact support, then*

$$\int_{\mathbb{R}^N} |u(x)|^p \psi(x)\, dx = \|u\|^p_{\mathcal{A},p},$$

where

$$\mathcal{A}(s) = \int_0^s \psi\left((z/\omega_N)^{1/N}\right)\, dz. \tag{3.8}$$

Proof. Using the layer cake representation (2.3) of u^p, we have

$$\begin{aligned}
\int_{\mathbb{R}^N} |u(x)|^p \psi(x)\, dx &= \int_0^\infty \int_{\mathbb{R}^N} \mathbb{1}_{\{u(x) > t^{1/p}\}} \psi(x)\, dx dt \\
&= \int_0^\infty N\omega_N \int_0^{(\mu(t^{1/p})/\omega_N)^{1/N}} r^{N-1}\psi(r)\, dr dt = \int_0^\infty \int_0^{\mu(t^{1/p})} \psi\left((z/\omega_N)^{1/N}\right)\, dz dt \\
&= p\int_0^\infty \int_0^{\mu(\tau)} \psi\left((z/\omega_N)^{1/N}\right)\, \tau^{p-1}\, dz d\tau = p\int_0^\infty \mathcal{A}(\mu(\tau))\, \tau^{p-1}\, d\tau.
\end{aligned}$$

■

In general we have

$$\|u\|_{\mathcal{A},p} = \|u^*\|_{\mathcal{A},p} = \|u^*\|_{L^p(\Omega;\psi)} \geq \|u\|_{L^p(\Omega;\psi)}.$$

This characterization allows to see easily when the space $\mathcal{L}_{\mathcal{A},p}(\Omega)$ is strictly smaller than $L^p(\Omega)$.

Proposition 3.2.3. *If* $\mathcal{A}'(0^+) = \infty$ *and* $\lim_{s\to 0^+} \frac{\mathcal{A}'\mathcal{A}'''}{(\mathcal{A}'')^2} \geq \nu > 1$, *then* $\mathcal{L}_{\mathcal{A},p}(\Omega) \subsetneq L^p(\Omega)$ *for every* $p \geq 1$.

Proof. The inclusion is immediate since $\mathcal{A}(s) \geq cs$, and thus

$$\int_0^\infty \mathcal{A}(\mu(t))\, t^{p-1}\, dt \geq c \int_0^\infty \mu(t)\, t^{p-1}\, dt = c\|u\|_p^p.$$

To get that the inclusion is proper assume for simplicity that there exists $\rho > 0$ small such that Ω contains the ball B_ρ, and consider the function

$$u(x) = v^{1/p}(|x|)\mathbb{1}_{B_\rho}, \qquad v(s) = \frac{-\psi'(s)}{s^{N-1}\psi^\nu(s)}, \quad 1 < \nu < 2,$$

where $\psi(s) = \mathcal{A}'(\omega_N s^N)$. The condition on $\mathcal{A}$ implies that if ρ is small then v is decreasing in $(0,\rho)$. We first have $u \in L^p(\Omega)$,

$$\int_\Omega |u(x)|^p\, dx = N\omega_N \int_0^\rho \frac{-\psi'(s)}{\psi^\nu(s)}\, ds = N\omega_N \int_{\psi(\rho)}^\infty \frac{dr}{r^\nu}\, dr < \infty.$$

On the other hand, $u \notin L^p(\Omega;\psi)$, since

$$\int_\Omega |u(x)|^p \psi(x)\, dx = N\omega_N \int_0^\rho \frac{-\psi'(s)}{\psi^{\nu-1}(s)}\, ds = N\omega_N \int_{\psi(\rho)}^\infty \frac{dr}{r^{\nu-1}}\, dr = \infty.$$

■

If J satisfies hypothesis (H$_2$) then it is an exercise to check that in fact $\mathcal{A}$ satisfies the hypotheses of Proposition 3.2.3. First observe that we may assume, without loss of generality, that ℓ is differentiable near the origin and

$$\lim_{s\to 0} \frac{s\ell'(s)}{\ell(s)} = 0. \tag{3.9}$$

In fact, by the representation formula [8, Theorem 1.3.1], the function ℓ can be written as

$$\ell(x) = c(x) \exp \int_x^1 \frac{\varepsilon(t)}{t}\, dt, \tag{3.10}$$

where $c(x) \to c_0$ and $\varepsilon(x) \to 0$ when $x \to 0$, thus by L'Hôpital's rule we have

$$\lim_{x\to 0} \frac{x\ell'(x)}{\ell(x)} = \lim_{x\to 0} \frac{xc'(x)}{c(x)} = 0.$$

This implies

$$\lim_{s\to 0^+} \frac{M(s)}{\ell(s)} = \infty. \tag{3.11}$$

Then using again (3.9)

$$\begin{aligned}\lim_{s\to 0^+} \frac{\mathcal{A}'(s)\mathcal{A}'''(s)}{(\mathcal{A}''(s))^2} &= \lim_{s\to 0^+} \frac{M(s)(sM''(s)-(N-1)M'(s)}{(M'(s))^2} \\ &= \lim_{s\to 0^+} \frac{M(s)}{\ell(s)}\left(N-\frac{s\ell'(s)}{\ell(s)}\right) = \infty.\end{aligned}$$

The limit (3.11) could also be deduced by Karamata's Theorem, see for instance [8, Proposition 1.5.9 a].

Proof of Theorem 3.2.1.

The inclusion in the Lorentz space $\mathcal{L}_{\mathcal{A},2}(\Omega)$ is obtained by Theorems 2.2.4 and 5.1.1, together with Lemma 3.2.2. In fact, if $u \in \mathcal{H}_{J,0}(\Omega)$ let u^* be its decreasing rearrangement, defined in B_R. Then

$$\mathcal{E}(u,u) \geq \mathcal{E}_{\mathcal{K}}(u,u) \geq \mathcal{E}_{\mathcal{K}}(u^*,u^*) \geq c\int_{B_R} |u^*(x)|^2 M(\rho|x|/R)\,dx = c\|u^*\|^2_{\mathcal{A},2} = c\|u\|^2_{\mathcal{A},2}$$

where $\mathcal{A}$ is defined in (3.8) with $\psi(x) = M(\rho|x|/R)$. ■

Chapter 4

Elliptic problems related to $\mathfrak{L}$

4.1 The eigenvalue problem

We now consider the problem of finding the eigenvalues and eigenfunctions of $\mathfrak{L}$ in $\mathcal{H}_{J,0}(\Omega)$, that is

$$\begin{cases} \mathfrak{L}\varphi = \lambda\varphi, & \text{in } \Omega, \\ \varphi = 0, & \text{in } \Omega^c. \end{cases} \tag{4.1}$$

4.1.1 Existence of eigenvalues

Theorem 4.1.1. *Assume hypothesis* (H$_2$). *Then the first eigenvalue of the operator* $\mathfrak{L}$ *in* Ω,

$$\lambda_1 = \min_{\substack{\varphi\in\mathcal{H}_{J,0}(\Omega)\\ \|\varphi\|_2=1}} \mathcal{E}(\varphi,\varphi),$$

is positive and simple, and the first eigenfunction does not vanish in Ω.

The proof is rather standard so we only sketch the main steps.

Proof. Consider the functional $\Psi : \mathcal{H}_{J,0}(\Omega) \to \mathbb{R}^+$, defined by $\Psi(u) = \mathcal{E}(u,u)$, and let $\mathcal{M} = \{u \in \mathcal{H}_{J,0}(\Omega) : \|u\|_2 = 1\}$. Let $\{u_k\}$ be a minimizing sequence for Ψ in $\mathcal{M}$, that is

$$\lim_{k\to\infty} \Psi(u_k) = c = \inf_{u\in\mathcal{M}} \Psi(u) \geq 0.$$

Then $\{u_k\}$ is bounded in $\mathcal{H}_{J,0}(\Omega)$, so there exists a subsequence (still denoted by $\{u_k\}$), such that $u_k \rightharpoonup u^*$ in $\mathcal{H}_{J,0}(\Omega)$, and also

$$\lim_{k\to\infty} \mathcal{E}(u_k,\eta) = \mathcal{E}(u^*,\eta), \qquad \text{for every } \eta \in \mathcal{H}_{J,0}(\Omega).$$

By Theorem 3.1.1 there exists a new subsequence converging to u^* in $L^2(\Omega)$, so $\|u^*\|_2 = 1$ and $u^* \in \mathcal{M}$. This gives

$$c = \lim_{j\to\infty} \Psi(u_j) \geq \Psi(u^*) \geq c,$$

and $\Psi(u^*) = c$. The first eigenvalue is then $\lambda_1 = \Psi(u^*) > 0$, with corresponding eigenfunction $\varphi_1 = u^*$. The fact that $u^* \geq 0$ or $u^* \leq 0$ follows from (2.2). Regularity of the first eigenfunction, which is obtained in the next section, would in fact imply that u^* does not vanish in Ω. Observe that if $u^*(x) \geq 0$ and $u^*(x_0) = 0$, then

$$0 = \lambda_1 u^*(x_0) = -\int_{\mathbb{R}^N} u^*(y)J(x,y)\,dy < 0.$$

Finally suppose that there exists $v \in \mathcal{H}_{J,0}(\Omega)$ with $\|v\|_2 = 1$ such that $\mathfrak{L}v = \lambda_1 v$. Then $w = v - u^*$ also satisfies $\mathfrak{L}w = \lambda_1 w$, and thus it has a definite sign. This gives $|v| \geq |u^*|$ or the opposite $|v| \leq |u^*|$. But they have equal L^2 norm, so $|v| = |u^*|$, and thus $v = \pm u^*$, that is, λ_1 is simple. ■

We also have

Theorem 4.1.2. *In the above hypotheses there exists a sequence of eigenvalues $\{\lambda_j\}$ and eigenfunctions $\{\varphi_j\}$ to problem (4.1) with the following properties:*

i) *$\{\lambda_j\}$ is nondecreasing with limit ∞.*

ii) *If $P_j = \{\varphi \in \mathcal{H}_{J,0}(\Omega),\ \varphi \neq 0,\ \mathcal{E}(\varphi,\varphi_k) = 0\ \forall\ k = 1,\cdots,j-1\}$, then*

$$\lambda_j = \min_{\substack{\varphi\in P_j \\ \|\varphi\|_2=1}} \mathcal{E}(\varphi,\varphi) = \mathcal{E}(\varphi_j,\varphi_j).$$

iii) *The eigenfunctions form an orthonormal basis of $\mathcal{H}_{J,0}(\Omega)$, that is, for every $u \in \mathcal{H}_{J,0}(\Omega)$, it holds*

$$\lim_{n\to\infty} \mathcal{E}(u - \sum_{j=1}^{n} \mathrm{u}_j\varphi_j, \eta) = 0, \qquad \text{for every } \eta \in \mathcal{H}_{J,0}(\Omega),$$

where

$$\mathrm{u}_j = \frac{1}{\lambda_j}\mathcal{E}(u,\varphi_j).$$

Proof. The same construction as before gives the existence of the sequence of eigenvalues and eigenfunctions, with $\lambda_j = \Psi(\varphi_j)$. Two eigenfunctions φ, ψ, corresponding to two different eigenvalues λ, μ are orthogonal, in $L^2(\Omega)$ and $\mathcal{H}_{J,0}(\Omega)$ since

$$\lambda \int_\Omega \varphi\psi = \mathcal{E}(\varphi, \psi) = \mu \int_\Omega \varphi\psi,$$

so that $\displaystyle\int_\Omega \varphi\psi = 0$ and as a consequence $\mathcal{E}(\varphi, \psi) = 0$.

If the sequence $\{\lambda_j\}$ were bounded there would exist a subsequence of $\{\varphi_j\}$ converging in $L^2(\Omega)$, but orthogonality in $L^2(\Omega)$ implies $\|\varphi_j - \varphi_k\|_2 = 2$, for every j, k, which is a contradiction. ■

On the other hand, we can describe the space $\mathcal{H}_{J,0}(\Omega)$, the operator $\mathfrak{L}$ and the bilinear form $\mathcal{E}$ in terms of the eigenvalues.

Proposition 4.1.3. *In the above hypotheses,*

$$\mathcal{H}_{J,0}(\Omega) = \left\{ u \in L^2(\Omega),\ u \equiv 0 \ \textit{in}\ \Omega^c,\ \|u\|_{\mathcal{H}_{J,0}} \equiv \left(\sum_{j=1}^{\infty} \lambda_j \mathtt{u}_j^2 \right)^{1/2} < \infty \right\}.$$

and

$$\mathfrak{L}u = \sum_{j=1}^{\infty} \lambda_j \mathtt{u}_j \varphi_j, \qquad \mathfrak{L}^{1/2}u = \sum_{j=1}^{\infty} \lambda_j^{1/2} \mathtt{u}_j \varphi_j,$$
$$\mathcal{E}(u, v) = \sum_{j=1}^{\infty} \lambda_j \mathtt{u}_j \mathtt{v}_j = \int_{\mathbb{R}^N} \mathfrak{L}^{1/2}u\, \mathfrak{L}^{1/2}v\,,$$

where

$$\mathtt{u}_j = \int_\Omega u\varphi_j = \frac{1}{\lambda_j}\mathcal{E}(u, \varphi_j), \quad \mathtt{v}_j = \int_\Omega v\varphi_j = \frac{1}{\lambda_j}\mathcal{E}(v, \varphi_j).$$

Observe that $\|u\|_{\mathcal{H}_{J,0}} = \|\mathfrak{L}^{1/2}u\|_{L^2}$. With this construction we have that the operator $\mathfrak{L} : \mathcal{H}_{J,0}(\Omega) \to H^*(\Omega)$ is an isomorfism, where $H^*(\Omega)$ is the closure of the set of functions $v = \displaystyle\sum_{j=1}^{\infty} \mathtt{u}_j \varphi_j$ with the norm $\|v\|_{H^*} = \left(\displaystyle\sum_{j=1}^{\infty} \lambda_j^{-1} \mathtt{v}_j^2 \right)^{1/2}$. The duality product is

$$\langle u, v \rangle_{\mathcal{H}_{J,0} \times H^*} = \sum_{j=1}^{\infty} \lambda_j \mathtt{u}_j \lambda_j^{-1} \mathtt{v}_j = \int_\Omega uv.$$

4.1.2 Estimates of the eigenvalues

We finally estimate the first eigenvalue in terms of the size of the domain. We obtain a lower bound of the type of the one obtained in [48] in the case of the Laplacian and in [69] for the fractional Laplacian case. We consider the multiplier $m(\xi)$ associated to the kernel $\mathcal{K}$, see subsection 3.1. Observe that m is radial, $m(\xi) = \mathfrak{m}(|\xi|)$, and if ℓ is nonincreasing then $\mathfrak{m}$ is nondecreasing. To see that, for each $\xi \in \mathbb{R}^N$ given, let $R = R_\xi$ be the rotation that carries ξ to the first axis, that is, $R(\xi) = |\xi|e_1$, where $e_1 = (1, 0, \dots, 0)$, and let $S = R^{-1}$. If we put $z = S^T y|\xi|$, then

$$y \cdot \xi = y \cdot S(|\xi|e_1) = (S^T y) \cdot (|\xi|e_1) = z_1.$$

Thus

$$m(\xi) = \int_{\mathbb{R}^N} (1 - \cos(y \cdot \xi))\, \mathcal{K}(y)\, dy = \int_{\mathbb{R}^N} \frac{1 - \cos z_1}{|z|^N} \ell(|\xi|^{-1}|z|)\, dz.$$

Thus $\mathfrak{m}$ increases when ℓ decreases. Put now

$$g(t) = \int_{|\xi| \le t} m(\xi)\, d\xi. \tag{4.2}$$

We need to suppose that g satisfies

$$Ng(t) \le tg'(t). \tag{4.3}$$

In particular it implies

$$\begin{cases} J(x,y) \ge c|x-y|^{-N} & \text{if } |x-y| \le 1, \\ J(x,y) \le c|x-y|^{-N} & \text{if } |x-y| \ge 1, \end{cases}$$

which is not too restrictive in our situation of nonintegrable Lévy kernels.

Theorem 4.1.4. *Assume hypothesis* (H_2) *with* ℓ *nonincreasing and assume further that* g *in* (4.2) *satisfies* (4.3). *Then*

$$\lambda_1 \ge \frac{|\Omega|}{(2\pi)^N} g\left(\frac{2\pi}{(\omega_N |\Omega|)^{1/N}} \right). \tag{4.4}$$

Proof. The first eigenvalue satisfies

$$\lambda_1 = \mathcal{E}(\varphi_1, \varphi_1) \ge \mathcal{E}_{\mathcal{K}}(\varphi_1, \varphi_1) = \int_{\mathbb{R}^N} m(\xi) |\widehat{\varphi}_1(\xi)|^2\, d\xi,$$

where $\|\varphi_1\|_2 = \|\widehat{\varphi}_1\|_2 = 1$. Put $h(\xi) = \|\widehat{\varphi}_1\|_\infty^2 \mathbb{1}_{\{|\xi|<K\}}$, for some K to be determined. Since $\mathfrak{m}$ is increasing, we have

$$(\mathfrak{m}(|\xi|) - \mathfrak{m}(K)) \left(|\widehat{\varphi}_1(\xi)|^2 - h(\xi)\right) \geq 0.$$

Therefore

$$\mathfrak{m}(K) \left(|\widehat{\varphi}_1(\xi)|^2 - h(\xi)\right) \leq \mathfrak{m}(|\xi|) \left(|\widehat{\varphi}_1(\xi)|^2 - h(\xi)\right).$$

Integrating this inequality we get

$$\int_{\mathbb{K}^N} \mathfrak{m}(R) \left(|\widehat{\varphi}_1(\xi)|^2 - h(\xi)\right) \, d\xi \leq \int_{\mathbb{R}^N} \mathfrak{m}(|\xi|) \left(|\widehat{\varphi}_1(\xi)|^2 - h(\xi)\right) \, d\xi \leq 0,$$

provided K is chosen such that $g(K) = \dfrac{\lambda_1}{\|\widehat{\varphi}_1\|_\infty^2}$. We get

$$1 = \int_{\mathbb{R}^N} |\widehat{\varphi}_1(\xi)|^2 \, d\xi \leq \int_{\mathbb{R}^N} h(\xi) \, d\xi = \|\widehat{\varphi}_1\|_\infty^2 K^N \omega_N,$$

and therefore

$$\lambda_1 \geq \|\widehat{\varphi}_1\|_\infty^2 g\left(\left(\omega_N \|\widehat{\varphi}_1\|_\infty^2\right)^{-1/N}\right).$$

The function on the right is decreasing by (4.3), and since φ_1 hast compact support contained in Ω, by Cauchy-Schwartz inequality

$$\|\widehat{\varphi}_1\|_\infty^2 \leq \frac{|\Omega|}{(2\pi)^N},$$

so that we conclude (4.4). ■

Remarks. *i*) Since always $m(\xi) \geq c|\xi|^2$ near the origin, then (4.4) means, for large domains,

$$\lambda_1 \geq c|\Omega|^{-\frac{2}{N}}.$$

If moreover $\mathcal{K}(z) \geq c|z|^{-\alpha}$ for some $\alpha > 0$, then

$$\lambda_1 \geq c|\Omega|^{-\frac{\min\{\alpha,2\}}{N}}.$$

ii) With the same technique it can also be obtained the estimate for the sum of the eigenvalues

$$\sum_{j=1}^k \lambda_j \geq \frac{|\Omega|}{(2\pi)^N} g\left(\frac{2\pi k^{1/n}}{(\omega_N|\Omega|)^{1/N}}\right).$$

4.2 The linear problem

4.2.1 Existence

In this section we explain some results on integral regularity of solutions to elliptic problems of the form

$$\begin{cases} \mathfrak{L}u = f(x), & \text{in } \Omega, \\ u = 0, & \text{in } \Omega^c. \end{cases} \tag{4.5}$$

Here $\mathfrak{L}$ is the operator (1.1). Given a function $f \in H^*(\Omega)$, we say that $u : \mathbb{R}^N \to \mathbb{R}$ is a weak solution of (4.5), if $u \in \mathcal{H}_{J,0}(\Omega)$ is a function such that

$$\mathcal{E}(u, \phi) = \int_\Omega f\phi, \quad \text{for all } \phi \in \mathcal{H}_{J,0}(\Omega). \tag{4.6}$$

Existence and uniqueness of solution is proved in [30] in a more general framework. In fact we only need a Poincaré inequality, and then the proof is standard. We include it for completeness.

Proposition 4.2.1. *Assume Poincaré inequality* (1.8) *holds. Then problem* (4.5) *has a unique weak solution* $u \in \mathcal{H}_{J,0}(\Omega)$.

Proof. Consider the energy functional $\mathcal{G} : \mathcal{H}_{J,0}(\Omega) \to \mathbb{R}$ associated to the problem (4.5), defined by

$$\mathcal{G}(u) = \frac{1}{2}\mathcal{E}(u, u) - \int_\Omega fu.$$

This functional is well defined thanks to Poincaré inequality, it is Fréchet differentiable in $u \in \mathcal{H}_{J,0}(\Omega)$ and for any $\phi \in \mathcal{H}_{J,0}(\Omega)$

$$\langle \mathcal{G}'(u), \phi \rangle = \mathcal{E}(u, \phi) - \int_\Omega f\phi,$$

that is, critical points of $\mathcal{G}$ are weak solutions to (4.5). The result is obtained by minimizing the functional $\mathcal{G}$. Observe also that $\mathcal{H}_{J,0}(\Omega)$ is a Hilbert space so we could have used Riesz representation theorem. ■

Maximum principle and comparison principle for weak solutions (or more generally for supersolutions) to (4.5) are also easy to obtain.

Proposition 4.2.2. *If* $u \in \mathcal{H}_J(\mathbb{R}^N)$ *then* $\mathfrak{L}u \geq 0$ *in* Ω *and* $u \geq 0$ *in* Ω^c *imply* $u \geq 0$ *in* Ω.

Proof. Property $\mathfrak{L}u \geq 0$ in Ω actually means $\mathcal{E}(u, \phi) \geq 0$ for every $\phi \in \mathcal{H}_J(\Omega)$, $\phi \geq 0$. Since $u^- \geq 0$ and $u^- \in \mathcal{H}_J(\Omega)$, by the Stroock-Varopoulos inequality we have

$$0 \geq -\mathcal{E}(u^-, u^-) \geq \mathcal{E}(u, u^-) \geq 0.$$

Hence $u^- \equiv 0$. ■

The comparison principle follows immediately as a consequence.

4.2.2 Regularity

The following result, due to Kassmann and Mimica [40], explain the weak character of the smoothing effect in problem (4.5). This result is sharp by Corollary 2.1.2

Theorem 4.2.3. *Assume hypothesis* (H_2) *and let* u *be a bounded weak solution to* (4.5) *with* $f \in L^\infty(\Omega)$. *Then there exist constants* $c > 0$ *and* $\beta \in (0, 1)$ *such that*

$$|u(x) - u(y)| \leq c\,(\|u\|_\infty + \|f\|_\infty)\,\varpi(|x - y|), \quad \textit{for every } x,\, y \in \Omega,$$

where $\varpi(s) = \dfrac{1}{M^\beta(s)}$.

We now study the smoothing effect in terms of integrability. Before that we show first that the solution is not worse that the datum. In the local case $-\Delta u = f$ there is a strong smoothing effect: u is bounded provided $f \in L^p(\Omega)$, $p > N/2$, see for instance [36, Theorem 8.15], from where some ideas are borrowed below. In fact, the same calculation allows to get easily the conclusion in the fractional Laplacian framework, $(-\Delta)^{\alpha/2} u = f$, when $p > N/\alpha$. Recall that here we are in the borderline $\alpha \sim 0$. It would be interesting to obtain $u \in L^\infty(\Omega)$ for $f \in L^p(\Omega)$ for every $p < \infty$, but $f \notin L^\infty(\Omega)$.

Theorem 4.2.4. *Assume hypothesis* (H_2) *and let* u *be a weak solution to* (4.5) *with* $f \in L^p(\Omega)$, $2 \leq p \leq \infty$. *Then,*

$$\|u\|_p \leq C\|f\|_p,$$

where the constant C *depends only on the kernel and* Ω.

Proof. Consider first the case $p = \infty$. Let B be any large ball such that $\overline{\Omega} \subset B$, and let $\eta \in C_c^\infty(B)$ be such that, $0 \leq \eta(x) \leq 1$, $x \in \mathbb{R}^N$ and $\eta \equiv 1$, in Ω. Then, for each $x \in \Omega$, we have

$$\mathfrak{L}\eta(x) = \int_{\mathbb{R}^N} (\eta(x) - \eta(y))\, J(x-y)\, dy \geq \int_{B^c} J(x-y)\, dy = c > 0.$$

Taking $\omega(x) = \dfrac{\|f\|_\infty}{c}\eta(x)$ we have $\mathfrak{L}u \leq \mathfrak{L}\omega$ in Ω, and $\omega \geq 0$ in Ω^c. Thus by the comparison principle we get $u \leq \omega$ in Ω, and hence $u \leq C\|f\|_\infty$. Similarly we have that $-u \leq C\|f\|_\infty$.

For the case $2 \leq p < \infty$ let us see first the formal calculus. Choosing as test function $\phi = |u|^{p-2}u$, and using Poincaré, Stroock-Varopoulos and Hölder inequalities, we get, modulo multiplicative constants,

$$\|u\|_p^p = \||u|^{\frac{p}{2}}\|_2 \leq \mathcal{E}(|u|^{\frac{p}{2}}, |u|^{\frac{p}{2}}) \leq \mathcal{E}(u, |u|^{p-2}u) = \int f|u|^{p-2}u \leq \|f\|_p \|u\|_p^{p-1}.$$

We would get the result if $\|u\|_p$ is finite. Also $\phi = |u|^{p-2}u$ is not an admissible test function. The justification works as usual through truncation, see for instance [36]. Let us consider for any $T > 0$ the function

$$F(s) = F_T(s) = \begin{cases} |s|^{\frac{p}{2}} & \text{if } |s| \leq T, \\ \frac{p}{2}T^{\frac{p}{2}-1}(|s| - T) + T^{\frac{p}{2}} & \text{if } |s| > T. \end{cases} \tag{4.7}$$

Since F is a Lipschitz convex function and $F(0) = 0$, we have $F(u) \in \mathcal{H}_J(\Omega)$. If we define $G = (F^2)'$ then $G' \geq 2(F')^2$, and hence Poincaré and Stroock-Varopoulos inequalities give

$$\|F(u)\|_2^2 \leq c\mathcal{E}(F(u), F(u)) \leq c\mathcal{E}(u, G(u)) = c \int_\Omega f(x)G(u(x))\, dx.$$

Now observe that $|G(u)| \leq pF(u)^{\frac{2(p-1)}{p}}$, and $|G(u)| \leq c|u|$ for $|u| > T$, so that $G(u) \in L^{\frac{p}{p-1}}(\Omega)$. Applying then Hölder inequality to the last integral we get

$$\int_\Omega fG(u) \leq c\|f\|_p \|F(u)\|_2^{\frac{2(p-1)}{p}},$$

and hence

$$\|F(u)\|_2^{\frac{2}{p}} \leq c\|f\|_p\,,$$

with c independent of T. We conclude by taking the limit as $T \to \infty$, since we have $\|F_T(u)\|_2^{\frac{2}{p}} \to \|u\|_p$. ■

Theorem 4.2.5. *If $f \in L^p(\Omega)$, $p \geq 2$, then $u \in \mathcal{L}_{\mathcal{A},p}(\Omega)$.*

Proof. By the above proof we know that $|u|^{\frac{p}{2}} \in \mathcal{H}_J(\Omega)$, and also that

$$\mathcal{E}(|u|^{\frac{p}{2}}, |u|^{\frac{p}{2}}) \le \|f\|_p \|u\|_p^{p-1} \le c\|f\|_p^p.$$

Therefore, using Theorem 3.2.1 and estimate (4.2.4), we get

$$\|u\|_{\mathcal{A},p}^p = \||u|^{\frac{p}{2}}\|_{\mathcal{A},2}^2 \le c\mathcal{E}(|u|^{\frac{p}{2}}, |u|^{\frac{p}{2}}) \le c\|f\|_p^p.$$

■

4.2.3 Nonhomogeneous exterior datum

Now we want to study the problem

$$\begin{cases} \mathfrak{L}u = f, & \text{in } \Omega, \\ u = g, & \text{in } \Omega^c, \end{cases} \tag{4.8}$$

where $f \in H^*(\Omega)$, and $g \in \mathcal{H}_J(\mathbb{R}^N)$.

We observe that when multiplying the equation by a test function, we get

$$\int_\Omega f\varphi = \int_\Omega \int_{\mathbb{R}^N} (u(x) - u(y))\varphi(x)J(x,y)\,dydx.$$

Since u does not necessarily vanish outside Ω, the right-hand side is different from $\mathcal{E}_1(u,\varphi)$, and this is the reason of the introduction of the bilinear form $\mathcal{E}$ in (1.3).

The solution to problem (4.8) is a function $u \in \mathcal{H}_J(\Omega)$ such that $u - g \in \mathcal{H}_{J,0}(\Omega)$ and (4.6) holds. We can solve (4.8) by considering the problem satisfied by $w = u - g$, and noting that $\mathfrak{L}g \in H^*(\Omega)$. We remark the recent work [29], where conditions on the data g defined only on Ω^c are imposed to guarantee that the problem is well posed, i.e., g can be extended properly into Ω.

4.2.4 Neumann problem

We consider in this section Neumann type problems associated to the operator (1.1), following the construction made in [27] for the fractional Laplacian. We therefore study the problem

$$\begin{cases} \mathfrak{L}u = f, & \text{in } \Omega, \\ \mathcal{N}u = 0, & \text{in } \Omega^c, \end{cases} \tag{4.9}$$

where
$$\mathfrak{L}u(x) = \int_{\mathbb{R}^N} (u(x)-u(y))J(x,y)\,dy, \qquad x\in\Omega,$$
and
$$\mathcal{N}u(x) = \int_{\Omega} (u(x)-u(y))J(x,y)\,dy, \qquad x\in\Omega^c.$$
The introduction of the exterior operator $\mathcal{N}$, which plays the role of a Neumann operator, is motivated by the following property, which can be interpreted as an integration by parts formula: For every u, $v\in\mathcal{H}_J(\Omega)$ it holds
$$\int_\Omega v\mathfrak{L}u + \int_{\Omega^c} v\mathcal{N}u = \mathcal{E}(u,v).$$
The weak formulation of problem (4.9) is to find $u\in\mathcal{H}_J(\Omega)$ such that
$$\mathcal{E}(u,\varphi) = \int_\Omega f\varphi,$$
for every $\varphi\in\mathcal{H}_J(\Omega)$. Using the constant function $\varphi = 1\in\mathcal{H}_J(\Omega)$ we get that a necessary condition to have a solution to problem (4.9) is
$$\int_\Omega f = 0. \tag{4.10}$$
Observe that for a constant it holds $\mathfrak{L}c = \mathcal{N}c = 0$. The following maximum principle is also immediate. Compare with Proposition 4.2.2.

Proposition 4.2.6. *If $u\in\mathcal{H}_J(\Omega)$ satisfies $\mathfrak{L}u\geq 0$ in Ω and $\mathcal{N}u\geq 0$ in Ω^c then u is constant.*

Existence of solution is now proved using the compactness result obtained in Section 3.1.

Theorem 4.2.7. *Assume $\ell(0^+)=\infty$ in hypothesis* (H$_1$). *Then given any $f\in L^2(\Omega)$ satisfying* (4.10) *there exists a solution $u\in\mathcal{H}_J(\Omega)$ to problem* (4.9), *unique up to additive constants.*

Proof. Let $T_0 : L^2(\Omega)\to\mathcal{H}_J(\Omega)$ be the operator defined by $T_0h = v$, where v is the unique solution to the problem
$$\begin{cases} v+\mathfrak{L}v = h, & \text{in } \Omega,\\ \mathcal{N}v = 0, & \text{in } \Omega^c.\end{cases}$$

The existence of such a solution follows from Riesz representation Theorem. Let $T : L^2(\Omega) \to L^2(\Omega)$ be defined by $Th = T_0 h|_{\Omega}$. Thanks to Theorem 3.1.3 this operator T is compact, and it is also easily seen to be self-adjoint. Proposition 4.2.6 implies that $ker(I-T)$ consists only on constant functions. Therefore, for every $f \in (ker(I-T))^{\perp}$, that is, for every $f \in L^2(\Omega)$ with $\int_{\Omega} f = 0$, there exists a function $w \in L^2(\Omega)$ satisfying $(I-T)w = f$. The function $u = T_0 w$ satisfies

$$\begin{cases} u + \mathfrak{L}u = w, & \text{in } \Omega, \\ \mathcal{N}u = 0, & \text{in } \Omega^c, \end{cases}$$

but in Ω it holds $w = f + Tw = f + T_0 w = f + u$, so that u solves problem (4.9). ■

If the kernel does not decay too fast at infinity then any solution stabilizes to a certain average. Let

$$W(y) = \lim_{|x|\to\infty} \frac{J(x,y)}{J(x,0)} \qquad \text{for } y \in \Omega,$$

and assume $0 < c_1 \le W(y) \le c_2 < \infty$ for every $y \in \Omega$.

Proposition 4.2.8. *If u is a solution to problem* (4.9) *then*

$$\lim_{|x|\to\infty} u(x) = \frac{\displaystyle\int_{\Omega} u(y)W(y)\,dy}{\displaystyle\int_{\Omega} W(y)\,dy}.$$

Proof. For every $\varepsilon > 0$ there exists some $R > 0$ such that

$$(1-\varepsilon)J(x,0)W(y) < J(x,y) < (1+\varepsilon)J(x,0)W(y)$$

for every $y \in \Omega$, $|x| > R$. Now the condition

$$0 = \mathcal{N}u(x) = \int_{\Omega} (u(x) - u(y))J(x,y)\,dy$$

outside Ω implies

$$u(x) = \frac{\displaystyle\int_{\Omega} u(y)J(x,y)\,dy}{\displaystyle\int_{\Omega} J(x,y)\,dy},$$

and therefore, for $|x| > R$,

$$\frac{1-\varepsilon}{1+\varepsilon}\,\frac{\displaystyle\int_\Omega u(y)W(y)\,dy}{\displaystyle\int_\Omega W(y)\,dy} < u(x) < \frac{1+\varepsilon}{1-\varepsilon}\,\frac{\displaystyle\int_\Omega u(y)W(y)\,dy}{\displaystyle\int_\Omega W(y)\,dy}.$$

■

If J decays at infinity like a power of $|x-y|$ then $W(y) = 1$ and any solution stabilizes to its standard mean in Ω,

$$\lim_{|x|\to\infty} u(x) = \frac{1}{|\Omega|}\int_\Omega u(y)\,dy.$$

This is not true if J decays exponentially or even has compact support.

We finally may consider also the problem with nontrivial Neumann data

$$\begin{cases} \mathfrak{L}u = f, & \text{in } \Omega,\\ \mathcal{N}u = g, & \text{in } \Omega^c, \end{cases}$$

In that case we must assume that there exists some regular function ψ such that $\mathcal{N}\psi = g$ in Ω^c, something that is not clear. We then would obtain that the function $z = u - \psi$ satisfies the homogenous problem and we are reduced to the previous situation.

4.3 Nonlinear problems

We study in this section the nonlinear elliptic type problem

$$\begin{cases} \mathfrak{L}u = f(u), & \text{in } \Omega,\\ u \geq 0,\ u \not\equiv 0, & \text{in } \Omega,\\ u = 0, & \text{in } \Omega^c. \end{cases} \tag{4.11}$$

4.3.1 Sublinear reaction

We first show existence in the sublinear case, i.e., when $f : [0,\infty) \to \mathbb{R}$ is a continuous function satisfying

$$\frac{f(t)}{t} \text{ is nonincreasing on } (0,\infty) \quad \text{and } \lim_{t\to\infty}\frac{f(t)}{t} = 0. \tag{4.12}$$

See [15] for the classical case when $\mathfrak{L} = -\Delta$.

Theorem 4.3.1. *Under the assumption* (4.12) *problem* (4.11) *admits a unique solution.*

Proof. Consider the energy functional $\Phi : \mathcal{H}_{J,0}(\Omega) \to \mathbb{R}$ defined by

$$\Phi(u) = \frac{1}{2}\mathcal{E}(u,u) - \int_\Omega F(u),$$

where $F(u) = \displaystyle\int_0^u f$. From (4.12) it follows that there exist $\sigma \in (0,1)$ and $a, b > 0$ such that

$$|f(t)| \leq a + b\,t^\sigma, \quad \forall t \geq 0.$$

This functional is well defined since $|F(u)| \leq a_1 + b_1 u^{\sigma+1}$, and then

$$\left|\int_\Omega F(u)\,dx\right| \leq C_1 + C_2\|u\|_{\sigma+1}^{\sigma+1} < \infty.$$

On the other hand, this estimate also gives coercivity since $\sigma + 1 < 2$,

$$\Phi(u) \geq \frac{1}{2}\|u\|_{\mathcal{H}_J}^2 - C_2\|u\|_{\sigma+1}^{\sigma+1} - C_1 \geq \frac{1}{2}\|u\|_{\mathcal{H}_J}^2 - C_3\|u\|_{\mathcal{H}_J}^{\sigma+1} - C_1.$$

Let now $\{u_k\} \subset \mathcal{H}_{J,0}(\Omega)$ be a minimizing sequence for Φ; this sequence is bounded in $\mathcal{H}_{J,0}(\Omega)$, and therefore there is a subsequence, still denoted $\{u_k\}$, such that $u_k \rightharpoonup u$ in $\mathcal{H}_{J,0}(\Omega)$, and therefore $u_k \to u$ in $L^2(\Omega)$. We thus deduce

$$\int_\Omega F(u_k)\,dx \to \int_\Omega F(u)\,dx,$$

so that

$$\Phi(u) \leq \liminf_{k\to\infty}\left(\frac{1}{2}\mathcal{E}(u_k,u_k) - \int_\Omega F(u_k)\right) = \liminf_{k\to\infty}\Phi(u_k) = \inf_{u\in\mathcal{H}_{J,0}(\Omega)}\Phi(u).$$

This shows that u is a global minimum for Φ, and hence it is a critical point, namely a solution to (4.11). It is non trivial since the energy is negative close to the origin. In fact, by hypotheses we have $f(t) \geq c > 0$ for $0 < t < \varepsilon$. Thus, if $w \in \mathcal{H}_{J,0}(\Omega)$ is any given function, we have

$$\Phi(tw) \leq At^2 - Bt < 0$$

for any $t > 0$ small, where $A = \dfrac{1}{2}\|w\|_{\mathcal{H}_J}$, $B = c\|w\|_1$.

It has a sign since $\Phi(|u|) \leq \Phi(u)$ by (2.2), so we may take $u \geq 0$.

Uniqueness follows a standard argument. Suppose u_1 and u_2 are two solutions of (4.11), and use $\varphi_1 = \dfrac{u_1^2 - u_2^2}{u_1}$ and $\varphi_2 = \dfrac{u_1^2 - u_2^2}{u_2}$ as test functions, respectively. Then

$$\mathcal{E}(u_1, \varphi_1) - \mathcal{E}(u_2, \varphi_2) = \int_\Omega \left(\frac{f(u_1)}{u_1} - \frac{f(u_2)}{u_2} \right) \left(u_1^2 - u_2^2 \right) \leq 0,$$

since $f(t)/t$ is nonincreasing. On the other hand, as in the proof of Theorem 2.2.4, we have

$$\mathcal{E}(u_1, \varphi_1) - \mathcal{E}(u_2, \varphi_2) = \mathcal{E}(u_1, u_1) - \mathcal{E}\left(u_2, \frac{u_1^2}{u_2}\right) + \mathcal{E}(u_2, u_2) - \mathcal{E}\left(u_1, \frac{u_2^2}{u_1}\right) \geq 0.$$

We conclude $u_1 = u_2$. ■

4.3.2 Supercritical reactions. Pohozaev inequality

We now show nonexistence for supercritical reactions $f(u)$ when Ω is star-shaped, where supercritical means above some exponent depending on the kernel. In the fractional Laplacian case the critical exponent is $p_* = \dfrac{N+\alpha}{N-\alpha}$, and is proved in [61] by means of a Pohozaev inequality. We follow their proof and establish an inequality adapted to our bilinear form $\mathcal{E}$. Let, for $\lambda > 1$,

$$\gamma(\lambda) = \lambda^{-N} \sup_{\substack{x,y \in \mathbb{R}^N \\ x \neq y}} \frac{J(x/\lambda, y/\lambda)}{J(x,y)},$$

and assume $\gamma(\lambda) < \infty$ for λ close to 1.

Theorem 4.3.2. *If u is a solution to problem* (4.11) *and Ω is star-shaped, then*

$$\int_\Omega u f(u) \leq \frac{2N}{N-\sigma} \int_\Omega F(u), \tag{4.13}$$

where $\sigma = \gamma'(1^+)$ and $F' = f$.

Corollary 4.3.3. *Problem* (4.11) *with $f(u) = u^p$ and Ω star-shaped has no solution for any exponent $p > p_* = \dfrac{N+\sigma}{N-\sigma}$.*

In the power case (fractional Laplacian type)

$$J(x,y) = \begin{cases} |x-y|^{-N-\alpha_1} & \text{if } |x-y| < 1, \\ |x-y|^{-N-\alpha_2} & \text{if } |x-y| > 1, \end{cases}$$

$\alpha_1 < 2$, $\alpha_2 > 0$, we get $\sigma = \max\{\alpha_1, \alpha_2\}$.

Proof of Theorem 4.3.2. We put $\phi = u_\lambda$ as test in (4.6), where $u_\lambda(x) = u(\lambda x)$. Since Ω is star-shaped, when $\lambda > 1$ we have that u_λ vanishes outside Ω, and then $u_\lambda \in \mathcal{H}_{J,0}(\Omega)$. We have then

$$\mathcal{E}(u, u_\lambda) = \int_\Omega f(u)u_\lambda, \quad \text{for all } \lambda > 1. \tag{4.14}$$

We observe that, with the above definition of $\gamma(\lambda)$, we have

$$\begin{aligned}
\mathcal{E}(u_\lambda, u_\lambda) &= \frac{1}{2}\iint_{\mathbb{R}^{2N}} |u(\lambda x) - u(\lambda y)|^2 J(x,y)\,dxdy \\
&= \frac{1}{2}\lambda^{-2N}\iint_{\mathbb{R}^{2N}} |u(x) - u(y)|^2 J(x/\lambda, y/\lambda)\,dxdy \\
&\le \frac{1}{2}\lambda^{-N}\gamma(\lambda)\iint_{\mathbb{R}^{2N}} |u(x) - u(y)|^2 J(x,y)\,dxdy \\
&= \lambda^{-N}\gamma(\lambda)\mathcal{E}(u,u),
\end{aligned}$$

so that

$$\mathcal{E}(u, u_\lambda) \le (\mathcal{E}(u_\lambda, u_\lambda))^{1/2}(\mathcal{E}(u,u))^{1/2} \le \lambda^{-N/2}\sqrt{\gamma(\lambda)}\mathcal{E}(u,u).$$

Therefore, if $I(\lambda) = \dfrac{\lambda^{N/2}}{\sqrt{\gamma(\lambda)}}\mathcal{E}(u, u_\lambda)$, we deduce that $I(\lambda) \le I(1)$ for $\lambda > 1$, and thus $I'(1^+) \le 0$.

With this information we differentiate both sides of equality (4.14) with respect to λ at $\lambda = 1$. On one hand

$$\begin{aligned}
\frac{d}{d\lambda}\bigg|_{\lambda=1^+}\mathcal{E}(u, u_\lambda) &= \frac{d}{d\lambda}\bigg|_{\lambda=1^+}\left(\lambda^{-N/2}\sqrt{\gamma(\lambda)}I(\lambda)\right) \\
&= \left(-\frac{N}{2} + \frac{\gamma'(1^+)}{2}\right) I(1^+) + I'(1^+) \\
&\le -\frac{1}{2}(N - \gamma'(1^+))\mathcal{E}(u,u) \\
&= -\frac{1}{2}(N - \gamma'(1^+))\int_\Omega f(u)u.
\end{aligned}$$

On the other hand,

$$\frac{d}{d\lambda}\bigg|_{\lambda=1^+}\int_\Omega f(u)u_\lambda = \int_\Omega x\cdot\nabla u\, f(u) = -N\int_\Omega F(u).$$

Putting together this two estimates we get (4.13). ∎

A NONLINEAR OPERATOR OF FRACTIONAL p-LAPLACIAN TYPE

Chapter 5

Preliminaries. Abstract framework

The aim of this part is to study the properties of the nonlinear nonlocal operator

$$\mathcal{L}u(x) = \mathcal{L}^{J,\psi}u(x) \equiv \int_{\mathbb{R}^N} \psi(u(x)-u(y))J(x-y)\,dy, \tag{5.1}$$

where $\psi : \mathbb{R} \to \mathbb{R}$ is a nondecreasing, continuous, unbounded odd function, and $J : \mathbb{R}^N \to \mathbb{R}^+$ is a nonnegative symmetric measurable function satisfying

$$\begin{cases} \lim_{|z|\to 0^+} |z|^N J(z) = \infty, \\ \int_{\mathbb{R}^N} \min(1, |z|^{q_0}) J(z)\,dz < \infty, \quad \text{for some } q_0 > 0. \end{cases} \tag{Q$_0$}$$

We also denote

$$q_* = \inf\{q_0 > 0 \,:\, (\mathrm{Q}_0) \text{ holds }\}, \tag{5.2}$$

which measures in some sense the differential character of the operator. The power case $\psi(s) = |s|^{p-2}s$ for some $p > 1$, $J(z) = |z|^{-N-\alpha p/2}$ for some $0 < \alpha < 2$, is known as the fractional p–Laplacian operator.

$$(-\Delta)_p^{\alpha/2}u(x) = \int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))}{|x-y|^{N+\frac{\alpha p}{2}}}\,dy.$$

But we are also interested in the limit case of integrability, which in our context means $q_* = 0$, that is, the singularity of the kernel can be weaker than that of any fractional Laplacian or fractional p–Laplacian. Also we consider general functions ψ more than just powers. Some of the results also hold for more general kernels, $J = J(x, y)$, satisfying only a lower estimate $J(x, y) \geq J_0(x - y)$, with J_0 in the above hypotheses, but we prefer to keep the proofs in a simpler way.

5.1 The associated Orlicz spaces

Formula (5.1) makes sense pointwise for regular functions with some extra restriction on the nonlinearity ψ and the kernel J, see Section 5.3. In order to define the operator $\mathcal{L}$ in weak sense we consider the *nonlocal nonlinear interaction energy* (linear in the second variable)

$$\mathcal{E}(u;\varphi) = \frac{1}{2}\iint_{\mathbb{R}^{2N}} \psi(u(x)-u(y))(\varphi(x)-\varphi(y))J(x-y)\,dxdy, \tag{5.3}$$

and we put

$$\langle \mathcal{L}u, \varphi\rangle = \mathcal{E}(u;\varphi).$$

Clearly, by the symmetry properties of ψ and J we have $\mathcal{E}(u;\varphi) = \displaystyle\int_{\mathbb{R}^N} \mathcal{L}u\varphi$ for regular functions. But the above allows to define $\mathcal{L}$ also for functions in a Sobolev type space. To this end we define the functionals

$$F(u) = \int_{\mathbb{R}^N} \Psi(u(x))\,dx, \tag{5.4}$$

$$E(u) = \frac{1}{2}\iint_{\mathbb{R}^{2N}} \Psi(u(x)-u(y))J(x-y)\,dxdy, \tag{5.5}$$

with $\Psi' = \psi$. The properties of ψ imply that Ψ is an strict Young function, so we can consider the Orlicz spaces

$$L^{\Psi}(\mathbb{R}^N) = \{u : \mathbb{R}^N \to \mathbb{R},\ F(u) < \infty\}, \tag{5.6}$$

$$W^{J,\Psi}(\mathbb{R}^N) = \left\{u \in L^{\Psi}(\mathbb{R}^N),\ E(u) < \infty\right\}. \tag{5.7}$$

Observe that in general $\mathcal{E}(u;u) \neq cE(u)$ for any constant $c > 0$, the equality being true only in the power case $\psi(u) = k|u|^{p-2}u$, and then $c = p$. What we have is that $\mathcal{E}$ is the Euler-Lagrange operator associated to the functional E, that is,

$$\langle E'(u), \varphi\rangle = \mathcal{E}(u;\varphi)$$

for every $u, \varphi \in W^{J,\Psi}(\mathbb{R}^N)$.

The above spaces do not have good properties unless we impose some conditions on the nonlinearity Ψ. The simplest case is when

$$c_1 s^{p-1} \leq \Psi'(s) \leq c_2 s^{p-1}, \qquad s > 0, \quad p > 1, \tag{5.8}$$

so that the space $L^\Psi(\mathbb{R}^N)$ coincides with $L^p(\mathbb{R}^N)$, and the Sobolev space $W^{J,\Psi}(\mathbb{R}^N)$ is denoted by $W^{J,p}(\mathbb{R}^N)$. But we are interested in more general functions. Thus we consider the set, for some $p \geq q > 1$,

$$\begin{aligned} \Gamma_{p,q} = \Big\{ \Psi : \mathbb{R} \to \mathbb{R}^+, \ \ &\text{convex, symmetric, satisfying } \Psi(0) = 0,\ \Psi(1) = 1, \\ &q \leq \frac{s\Psi'(s)}{\Psi(s)} \leq p \quad \forall s \neq 0 \Big\}. \end{aligned} \tag{5.9}$$

The condition $\Psi(1) = 1$ is for normalization purposes and simplifies some expression. We thus deal with functions that lie between two powers, for instance a sum of powers, but we also allow for perturbation of powers like $\Psi(s) = c|s|^p|\log(1+s)|^r$, $\min\{p, p+r\} > 1$. The first property deduced from (5.9) is the relation between the interaction energy $\mathcal{E}$ and the functional E,

$$qE(u) \leq \mathcal{E}(u; u) \leq pE(u). \tag{5.10}$$

Our main interest lies in studying the properties of the spaces (5.6) and (5.7) for nonlinearities Ψ in the class $\Gamma_{p,q}$. In particular we have that $L^\Psi(\mathbb{R}^N)$ and $W^{J,\Psi}(\mathbb{R}^N)$ are reflexive Banach spaces, with norms defined, for instance, in (5.19) and (6.1). On the other hand, if $q > q_*$, see (5.2), then the functional $E(u)$ is well defined and finite for functions satisfying $F(\nabla u) < \infty$, see Proposition 6.1.1. This means the inclusion $W^{1,\Psi}(\mathbb{R}^N) \subset W^{J,\Psi}(\mathbb{R}^N)$, the former being the standard Orlicz-Sobolev space of functions in $L^\Psi(\mathbb{R}^N)$ with gradient also in $L^\Psi(\mathbb{R}^N)$.

To end this subsection we point out that a result on symmetrization analogous to what was proved in Section 2.2.1 also holds in the present situation: the energy $E(u)$ decreases when we replace u by its symmetric rearrangement (the radially decreasing function with the same distribution function as u). The same proof performed there can be used to get the result, so we omit the details.

Theorem 5.1.1. *If $u \in W^{J,\Psi}(\mathbb{R}^N)$ and u^* is its decreasing rearrangement, then*

$$E(u) \geq E(u^*).$$

We remark that this property is well known for the norm in $W_0^{\sigma/2,p}(\Omega)$, $0 < \sigma \leq 2$, $p > 1$, see [2] and [5].

5.2 Some useful inequalities

We now turn our attention to the operator $\mathcal{L}$. The pointwise expression (5.1) does not always have a meaning. Let us look at some easy situations where $\mathcal{L}u$ is well defined.

We may take, for instance, Ψ'' nondecreasing and $u \in C_0^2(\mathbb{R}^N)$. Another less trivial example is $q > q_* + 1$ and $u \in C^\alpha(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ with $\frac{q_*}{q-1} < \alpha < 1$, so that

$$|\mathcal{L}u(x)| \leq \|u\|_\infty \int_{|x-y|>1} J(x-y)\,dy + \int_{|x-y|<1} |x-y|^{(q-1)\alpha} J(x-y)\,dy < \infty.$$

We now show some useful inequalities. The first one is a Kato type inequality, that is, the result of applying the operator $\mathcal{L}$ to a convex function of u. We refer to [42] and [21], respectively, for the well-known inequalities

$$-\Delta|u| \leq \operatorname{sign}(u)(-\Delta)u, \qquad (-\Delta)^{\sigma/2}(u^2) \leq 2u(-\Delta)^{\sigma/2}u.$$

Proposition 5.2.1. *If A is a positive convex function and $\mathcal{L}u$ is well defined, then $\mathcal{L}(A(u))$ is also well defined and*

$$\mathcal{L}(A(u)) \leq \gamma_\psi^+(A'(u))\mathcal{L}u.$$

Proof. We just observe that since A is convex and ψ is nondecreasing, we have

$$\psi(A(u(x)) - A(u(y))) \leq \psi(A'(u(x))(u(x)-u(y))) \leq \gamma_\psi^+(A'(u(x)))\psi(u(x)-u(y)).$$

Now integrate with respect to $J(x-y)\,dy$ to get the result. ■

As a Corollary we obtain an integral version of the Kato inequality, useful in the applications.

Corollary 5.2.2. *Assume $G \geq \gamma_\psi^+(A')A$. Then*

$$\mathcal{E}(u, G(u)) \geq qE(A(u)).$$

Of later use are also the following two inequalities

$$\mathcal{E}(u, u^+) \geq \mathcal{E}(u^+, u^+), \qquad E(u) \geq E(|u|), \tag{5.11}$$

whose proof is immediate just looking at the signs of the corresponding functions.

Related to those inequalities is the well known Stroock-Varopoulos inequality, see [67] for the linear case $\Psi(s) = |s|^2$ and $J(z) = |z|^{-N-\sigma}$ for some $0 < \sigma < 2$, and [12] for general Lévy kernels J. It is of the type of the integral Kato inequality, but the functions for which it holds is different. In the case of powers they coincide but for the coefficient, which is always better in the Stroock-Varopoulos inequality. We show here a generalized Stroock-Varopoulos inequality.

Proposition 5.2.3. *Assume* $\delta = \inf_{s>0} \dfrac{\psi(s)}{\gamma_\psi^+(s)} > 0$ *and let* $u \in W^{J,\Psi}(\mathbb{R}^N)$ *such that* $G(u)$, $A(u) \in W^{J,\Psi}(\mathbb{R}^N)$, *where* A *and* G *satisfy* $G' \geq |\Psi(A')|$. *Then*

$$\mathcal{E}(u; G(u)) \geq \frac{\delta q}{p} E(A(u)). \tag{5.12}$$

Proof. The proof follows from a calculus estimate. For any $d > c$ we have that

$$\begin{aligned} \Psi\left(|A(d) - A(c)|\right) &\leq \Psi\left(\int_c^d |A'(s)|\,ds\right) \leq \gamma_\Psi^+(d-c)\Psi\left(\frac{1}{d-c}\int_c^d |A'(s)|\,ds\right) \\ &\leq \frac{\gamma_\Psi^+(d-c)}{d-c}\int_c^d \Psi\left(|A'(s)|\,\right)ds \leq \frac{p}{q}\gamma_\psi^+(d-c)|G(d) - G(c)| \\ &\leq \frac{p}{\delta q}\psi(d-c)|G(d) - G(c)|. \end{aligned}$$

The same inequality is obtained for $d \leq c$. We now deduce (5.12) by choosing $d = u(x)$, $c = u(y)$ and integrate with respect to $J(x-y)\,dxdy$. ∎

For instance in the case of a sum of powers, $\Psi(s) = \sum_{i=1}^{M} k_i s^{p_i}$, $p_1 < p_2 < \cdots < p_M$, we have $\gamma_\psi^+(s) = \max\{s^{p_1-1},\ s^{p_M-1}\}$ and $\delta = \min\{k_1 p_1,\ k_M p_M\}$.

All the above inequalities hold also, with different constants, for nonlinearities that behave like a power, i.e., when they satisfy (5.8) instead of (5.9). In particular in that case the integral Kato inequality and the Stroock-Varopoulos inequality coincide, but for the coefficient, both giving

$$\mathcal{E}(u; |u|^{r-1}u) \geq cE(|u|^{\frac{r+p-1}{p}}). \tag{5.13}$$

We also obtain some calculus inequalities needed in proving uniqueness results in the last sections. We borrow ideas from [33] and [51] that deal with the exact power case.

Lemma 5.2.4. *Let* ψ *be a nonnegative, nondecreasing, continuous odd function and let* $\psi = \Psi'$.

i) If ψ satisfies

$$\frac{s\psi'(s)}{\psi(s)} \geq 1 \qquad \text{for every } s \neq 0, \tag{5.14}$$

then

$$\left(\psi(a) - \psi(b)\right)(a-b) \geq 4\Psi\left(\frac{a-b}{2}\right). \tag{5.15}$$

ii) If ψ is concave in $(0, \infty)$ then

$$\left(\psi(a) - \psi(b)\right)(a-b) \geq \psi'(|a|+|b|)(a-b)^2. \tag{5.16}$$

iii) If ψ satisfies

$$c_1|s|^{p-2} \leq \psi'(s) \leq c_2|s|^{p-2} \qquad \text{for some } 1 < p < 2 \text{ and every } s \neq 0, \tag{5.17}$$

then

$$\left(\psi(a) - \psi(b)\right)(a-b) \geq \frac{c(\Psi(a-b))^{\frac{2}{p}}}{(\Psi(a)+\Psi(b))^{\frac{2-p}{p}}}. \tag{5.18}$$

Proof. i) We begin by proving a Clarkson inequality. Condition (5.14) implies that the function $g(s) = \Psi(\sqrt{|s|})$ is convex. Therefore

$$\begin{aligned}
\Psi\left(\frac{a+b}{2}\right) + \Psi\left(\frac{a-b}{2}\right) &= g\left(\left(\frac{a+b}{2}\right)^2\right) + g\left(\left(\frac{a-b}{2}\right)^2\right) \\
&\leq g\left(\left(\frac{a+b}{2}\right)^2 + \left(\frac{a-b}{2}\right)^2\right) = g\left(\frac{a^2+b^2}{2}\right) \\
&\leq \frac{1}{2}\left(g(a^2) + g(b^2)\right) = \frac{1}{2}\left(\Psi(a) + \Psi(b)\right).
\end{aligned}$$

Now the convexity of Ψ implies

$$\Psi(a) \geq \Psi(b) + \Psi'(b)(a-b),$$

and also

$$\Psi\left(\frac{a+b}{2}\right) \geq \Psi(b) + \frac{1}{2}\Psi'(b)(a-b),$$

so that

$$\Psi(a) + \Psi(b) \geq 2\Psi\left(\frac{a+b}{2}\right) + 2\Psi\left(\frac{a-b}{2}\right) \geq 2\Psi(b) + \Psi'(b)(a-b) + 2\Psi\left(\frac{a-b}{2}\right).$$

This gives

$$\Psi(a) \geq \Psi(b) + \Psi'(b)(a-b) + 2\Psi\left(\frac{a-b}{2}\right),$$

and reversing the roles of a and b,

$$\Psi(b) \geq \Psi(a) + \Psi'(a)(b-a) + 2\Psi\left(\frac{a-b}{2}\right).$$

Adding these two inequalities we get (5.15).

ii) Developing the function Ψ around the point $s = a$ we get

$$\begin{aligned}\Psi(b) &= \Psi(a) + \Psi'(a)(b-a) + (b-a)^2 \int_0^1 (1-s)\Psi''(a+s(b-a))\,ds \\ &\geq \Psi(a) + \Psi'(a)(b-a) + (b-a)^2\Psi''(a+b)\int_0^1 (1-s)\,ds.\end{aligned}$$

We have used that $|a+s(b-a)| \leq |a|+|b|$ and Ψ'' is nonincreasing in $(0,\infty)$. Observe that though Ψ'' is singular at zero, the integral is convergent. We conclude as before.

iii) As *ii*), using (5.17) in the last step.

■

5.3 Properties of the Orlicz space L^Ψ

In this part we study in detail the properties of the Orlicz space $L^\Psi(\mathbb{R}^N)$ defined in (5.6), and the corresponding space in a bounded domain Ω. We refer to [59] for instance for the general theory of Orlicz spaces.

We begin by studying the Young functions in the set $\Gamma_{p,q}$. First observe that $\Psi \in \Gamma_{p,q}$, $p \geq q \geq 0$, implies

$$\min\{|s|^p,\,|s|^q\} \leq \Psi(s) \leq \max\{|s|^p,\,|s|^q\}.$$

Associated to any given positive function $g : \mathbb{R}^+ \to \mathbb{R}^+$ we consider its characteristic functions, for $s > 0$,

$$\gamma_g^-(s) = \inf_{x>0}\frac{g(sx)}{g(x)}, \qquad \gamma_g^+(s) = \sup_{x>0}\frac{g(sx)}{g(x)}.$$

These are nondecreasing functions that satisfy

Lemma 5.3.1. *For any $\Psi \in \Gamma_{p,q}$, $p \geq q > 0$,*

$$\min\{s^p,\,s^q\} \leq \gamma_\Psi^-(s) \leq \gamma_\Psi^+(s) \leq \max\{s^p,\,s^q\}$$

$$\frac{q}{p}\,\frac{\gamma_\Psi^-(s)}{s} \leq \gamma_{\Psi'}^-(s) \leq \gamma_{\Psi'}^+(s) \leq \frac{p}{q}\,\frac{\gamma_\Psi^+(s)}{s}.$$

Proof. If $s > 1$ we have that

$$\log\left(\frac{\Psi(sx)}{\Psi(x)}\right) = \int_x^{sx} \frac{\Psi'(t)}{\Psi(t)}\,dt \le p\int_x^{sx} \frac{1}{t}\,dt = p\log s,$$

and thus $\Psi(sx) \le s^p \Psi(x)$. The other estimates for Ψ are analogous. The inequalities for Ψ' are deduced from the definition of $\Gamma_{p,q}$. ∎

The complementary function Φ of a Young function Ψ is defined such that $(\Phi')^{-1} = \Psi'$. If we normalize it to satisfy $\Phi(1) = 1$ we have, for every $p \ge q > 1$ ([59, Corollary 1.1.3])

$$\Psi \in \Gamma_{p,q} \;\Leftrightarrow\; \Phi \in \Gamma_{q',p'}, \qquad p' = \frac{p}{p-1},\; q' = \frac{q}{q-1}.$$

These two functions satisfy the Young inequality

$$ab \le \Psi(a) + \Phi(b), \qquad a,\, b \in \mathbb{R},$$

and equality holds only if $b = \Psi'(|a|)\operatorname{sign} a$. From this point on we always assume $q > 1$.

Let then $\Psi \in \Gamma_{p,q}$ be fixed and consider the corresponding Orlicz space $L^\Psi(\mathbb{R}^N)$. It is a linear space that satisfies

$$L^p(\mathbb{R}^N) \cap L^q(\mathbb{R}^N) \subset L^\Psi(\mathbb{R}^N) \subset L^p(\mathbb{R}^N) + L^q(\mathbb{R}^N),$$

and in the case of bounded domains

$$L^p(\Omega) \subset L^\Psi(\Omega) \subset L^q(\Omega).$$

Also it is a Banach space with norm, called *Luxemburg norm*,

$$\|u\|_{L^\Psi} = \inf\{k > 0 \,:\, F(u/k) \le 1\}. \tag{5.19}$$

We recall that other equivalent norms are also used in the literature. The following result allows us to use $F(u)$ instead of $\|u\|_{L^\Psi}$ in most calculations.

Lemma 5.3.2.

$$\gamma_\Psi^-(\|u\|_{L^\Psi}) \le F(u) \le \gamma_\Psi^+(\|u\|_{L^\Psi}). \tag{5.20}$$

Proof. Let $a = \|u\|_{L^\Psi}$. We clearly have $F\left(u/a\right) \le 1$. Then

$$F(u) = \int_{\mathbb{R}^N} \Psi(u(x))\,dx \le \gamma_\Psi^+(a)\int_{\mathbb{R}^N} \Psi\left(\frac{u(x)}{a}\right)dx \le \gamma_\Psi^+(a).$$

On the other hand, for every $\varepsilon > 0$ we have $F\left(u/(a+\varepsilon)\right) > 1$, so that

$$F(u) \geq \gamma_\Psi^-(a+\varepsilon) \int_{\mathbb{R}^N} \Psi\left(\frac{u(x)}{a+\varepsilon}\right) dx \geq \gamma_\Psi^-(a+\varepsilon).$$

■

The dual space of $L^\Psi(\mathbb{R}^N)$ is $L^\Phi(\mathbb{R}^N)$, where Φ is the complementary function, and thus they are both reflexive Banach spaces.

Chapter 6

The Sobolev-Orlicz space $W^{J,\Psi}$

6.1 Basic properties

The Sobolev type space $W^{J,\Psi}(\mathbb{R}^N)$ defined in (5.7), in the same way as in the previous section, is a Banach space with norm

$$\|u\|_{W^{J,\Psi}} = \|u\|_{L^\Psi} + [u]_{W^{J,\Psi}} \equiv \|u\|_{L^\Psi} + \inf\{k > 0 \,:\, E(u/k) \leq 1\}. \tag{6.1}$$

The second term is a kind of Gagliardo seminorm in the context of Young functions. For this seminorm an analogous property as that of Lemma 5.3.2 also holds,

$$\gamma_\Psi^-([u]_{W^{J,\Psi}}) \leq E(u) \leq \gamma_\Psi^+([u]_{W^{J,\Psi}}). \tag{6.2}$$

In order to show that this space is reflexive as well we consider the weighted space

$$L^\Psi(\mathbb{R}^{2N}, J) = \left\{ w : \mathbb{R}^{2N} \to \mathbb{R},\ \iint_{\mathbb{R}^{2N}} \Psi(w(x,y))J(x-y)\,dxdy < \infty \right\}$$

and put $M = L^\Psi(\mathbb{R}^N) \times L^\Psi(\mathbb{R}^{2N}, J)$. Clearly the product space M is reflexive. The operator $T : W^{J,\Psi}(\mathbb{R}^N) \to M$ defined by $Tu = [u, w]$, where $w(x,y) = u(x) - u(y)$, is an isometry. Since $W^{J,\Psi}(\mathbb{R}^N)$ is a Banach space, $T(W^{J,\Psi}(\mathbb{R}^N))$ is a closed subspace of M. It follows that $T(W^{J,\Psi}(\mathbb{R}^N))$ is reflexive (see [14, Proposition 3.20]), and consequently $W^{J,\Psi}(\mathbb{R}^N)$ is also reflexive.

We now take a look at the properties of the space $W^{J,\Psi}(\mathbb{R}^N)$ in terms of the properties of the kernel J, in particular its singularity at the origin, which is reflected in the exponent q_* of (5.2).

Proposition 6.1.1. *If $\Psi \in \Gamma_{p,q}$ with $p \geq q > q_*$ then*

$$W^{1,\Psi}(\mathbb{R}^N) \subset W^{J,\Psi}(\mathbb{R}^N)$$

and moreover

$$E(u) \leq c(F(u) + F(\nabla u)). \tag{6.3}$$

Proof. We decompose the integral

$$\begin{aligned} E(u) &= \frac{1}{2}\int_{\mathbb{R}^N}\left(\int_{|z|<1}\Psi(u(x)-u(x+z))J(z)\,dz\right.\\ &\qquad\left.+\int_{|z|>1}\Psi(u(x)-u(x+z))J(z)\,dz\right)dx = \frac{1}{2}(I_1+I_2). \end{aligned}$$

The far away integral is easy to estimate

$$I_2 \leq 2\int_{\mathbb{R}^N}\Psi(u(x))dx\int_{|z|>1}J(z)\,dz = cF(u).$$

As to the inner integral, we have

$$\begin{aligned} I_1 &\leq \int_{\mathbb{R}^N}\int_{|z|<1}\Psi\left(\frac{u(x)-u(x+z)}{|z|}\right)\gamma^+(|z|)J(z)\,dzdx\\ &\leq \int_{\mathbb{R}^N}\int_{|z|<1}\Psi\left(\int_0^1|\nabla u(x+tz)|\,dt\right)\gamma^+(|z|)J(z)\,dzdx\\ &\leq \int_{\mathbb{R}^N}\int_{|z|<1}\int_0^1\Psi\left(|\nabla u(x+tz)|\right)\,dt\gamma^+(|z|)J(z)\,dzdx\\ &\leq \int_{|z|<1}\int_0^1\int_{\mathbb{R}^N}\Psi\left(|\nabla u(x+tz)|\right)\,dx\,dt\gamma^+(|z|)J(z)\,dz\\ &= \int_{\mathbb{R}^N}\Psi\left(|\nabla u(x)|\right)\,dx\int_{|z|<1}\gamma^+(|z|)J(z)\,dz = cF\left(|\nabla u|\right), \end{aligned}$$

since $\gamma^+(|z|) = \gamma^+_\Psi(|z|) \leq |z|^q$ in the set $\{|z|<1\}$, and using hypothesis (Q_0). ■

If the kernel J behaves like that of the fractional Laplacian

$$c_1|z|^{-N-\alpha} \leq J(z) \leq c_2|z|^{-N-\alpha}, \tag{6.4}$$

then we also have the following interpolation estimate

Proposition 6.1.2. *If J satisfies* (6.4) *for some $\alpha > 0$, and $\Psi \in \Gamma_{p,q}$ with $p \geq q > \alpha$ then*

$$E(u) \leq cF(u) \max\left\{ \left(\frac{F(\nabla u)}{F(u)}\right)^{\alpha/p}, \left(\frac{F(\nabla u)}{F(u)}\right)^{\alpha/q} \right\}. \tag{6.5}$$

Proof. We apply inequality (6.3) to the rescaled function $u_\lambda(x) = u(\lambda x)$. We first observe that by (6.4),

$$\begin{aligned} E(u_\lambda) &= \frac{1}{2} \iint_{\mathbb{R}^{2N}} \Psi(u(\lambda x) - u(\lambda y)) J(x-y)\, dxdy \\ &= \frac{1}{2} \iint_{\mathbb{R}^{2N}} \lambda^{-2N} \Psi(u(x) - u(y)) J(\lambda^{-1}(x-y))\, dxdy \\ &\geq c\frac{1}{2} \iint_{\mathbb{R}^{2N}} \lambda^{-N+\alpha} \Psi(u(x) - u(y)) J(x-y)\, dxdy = c\lambda^{-N+\alpha} E(u). \end{aligned}$$

Thus

$$\begin{aligned} E(u) &\leq c\lambda^{N-\alpha} E(u_\lambda) \leq \lambda^{N-\alpha} c(F(u_\lambda) + F(\nabla u_\lambda)) \\ &\leq c\lambda^{N-\alpha}(\lambda^{-N} F(u) + \gamma_\Psi^+(\lambda)\lambda^{-N} F(\nabla u)) \\ &\leq c\lambda^{-\alpha}(a + b\max\{\lambda^p,\, \lambda^q\}) \equiv cg(\lambda), \end{aligned}$$

$a = F(u)$, $b = F(\nabla u)$. Minimizing the right-hand side in λ we get

$$\min\{g(\lambda)\} = \begin{cases} a(b/a)^{\alpha/q} & \text{if } a/b < q/\alpha - 1, \\ a(b/a)^{\alpha/p} & \text{if } a/b > p/\alpha - 1, \\ a(1 + b/a) & \text{if } q/\alpha - 1 < a/b < p/\alpha - 1. \end{cases}$$

From this we easily deduce (6.5). ■

In the power-like case we obtain from the above the well-known interpolation result.

Corollary 6.1.3. *If J satisfies* (6.4) *for some $\alpha > 0$, and ψ satisfies* (5.8) *with $p > \alpha$ then*

$$E(u) \leq cF^{1-\alpha/p}(u) F^{\alpha/p}(\nabla u),$$

or which is the same

$$\|u\|_{W^{\alpha/2,p}} \leq c\|u\|_p^{1-\alpha/p}\, \|\nabla u\|_p^{\alpha/p}.$$

In this section we consider a nonlinearity $\Psi \in \Gamma_{p,q}$, $p \geq q > \max\{q_*, 1\}$ fixed, and in order to get the announced Sobolev embeddings, we assume from this point on that the kernel J, besides condition (Q_0), also satisfies the singularity condition at the origin

$$J(z) \geq c|z|^{-N-\alpha} \quad \text{for } 0 < |z| < 1, \quad \alpha > 0. \tag{6.6}$$

Clearly it must be $\alpha \leq q_*$. In fact in the fractional p–Laplacian case it is $\alpha = \sigma p/2$. Other kernels could also be considered, for instance $J(z) = |z|^{-N-\mu} \left|\log(|z|/2)\right|^{\beta}$, for $0 < |z| < 1$, with $\mu \geq 0$, (and $\beta \geq -1$ if $\mu = 0$). In that case it is $q_* = \mu$. If $\mu > 0$ then J satisfies (6.6) with $\alpha = \mu$ if $\beta \geq 0$, but if $\beta < 0$ it satisfies (6.6) only with $0 < \alpha < \mu$. A more intricate example can be constructed by the following piecewise definition of J,

$$J(z) = \begin{cases} |z|^{-N} & \text{if} \quad 2^{-2k-1} < |z| \leq 2^{-2k}, \\ |z|^{-N-\mu} & \text{if} \quad 2^{-2k} < |z| \leq 2^{-2k+1}, \end{cases}$$

$k \geq 1$, $\mu > 0$. Here we have $q_* = \mu$ while condition (6.6) does not hold for any $\alpha > 0$.

Assume now that u has support contained in $\overline{\Omega}$. Then

$$\begin{aligned} E(u) &= \frac{1}{2} \iint_{\mathbb{R}^{2N}} \Psi(u(x) - u(y)) J(x-y)\, dxdy \\ &= \frac{1}{2} \int_{\Omega}\int_{\Omega} \Psi(u(x) - u(y)) J(x-y)\, dxdy + \int_{\Omega}\int_{\Omega^c} \Psi(u(x)) J(x-y)\, dxdy \\ &\geq \int_{\Omega} \Psi(u(x)) \Lambda(\Omega; x)\, dx, \end{aligned}$$

where

$$\Lambda(\Omega; x) = \int_{\Omega^c} J(x-y)\, dy.$$

If

$$\mu = \min\{J(z) \,:\, |z| \leq R\} > 0, \qquad R > \delta = \sup_{x\in\Omega} dist(x, \Omega^c),$$

then

$$\Lambda(\Omega; x) \geq \mu|\{\delta < |z| < R\}| = A > 0 \qquad \text{for every } x \in \Omega.$$

This gives the Poincaré inequality

$$E(u) \geq AF(u), \tag{6.7}$$

and the inclusion

$$W_0^{J,\Psi}(\Omega) \subset L^{\Psi}(\Omega). \tag{6.8}$$

We remark that in the case of integrable kernel J we immediately would get $E(u) \leq c\|J\|_1 F(u)$, and thus $W_0^{J,\Psi}(\Omega) \equiv L^{\Psi}(\Omega)$.

In order to obtain better energy estimates in the case $q_* > 0$, which would yield better space embeddings, we need a better estimate of the function $\Lambda(\Omega;\cdot)$ in terms of the kernel J. The following result is essentially contained in [65, Lemma A.1].

Proposition 6.1.4.

$$\Lambda(\Omega; x) \geq P\left(\left(\frac{|\Omega|}{\omega_N}\right)^{1/N}\right) \qquad \text{for every } x \in \Omega,$$

where $P(s) = \displaystyle\int_{|z|>s} J(z)\,dz$. *In particular, if* J *satisfies* (6.6) *then*

$$\Lambda(\Omega'; x) \geq c(\Omega)|\Omega'|^{-\alpha/N} \qquad \text{for every } \Omega' \subset \Omega.$$

6.2 The Sobolev embedding

This estimate allows us to prove, assuming condition (6.6), the Sobolev embedding $W_0^{J,\Psi}(\Omega) \subset L^{\Psi^r}(\Omega)$ for every $1 \leq r \leq r^* \equiv \dfrac{N}{N-\alpha}$, if $\alpha < N$, for every $1 \leq r < \infty$ if $\alpha \geq N$. The proof uses ideas of [26] and [64]. If $\alpha \geq N$ we obtain the result substituting α by any number below N and close to N, since (6.6) still holds for that exponent.

Theorem 6.2.1. *Assume* J *satisfies condition* (6.6) *with* $0 < \alpha < N$. *Then there exists a positive constant* $C = C(N, p, q, \alpha, \Omega)$ *such that, for any function* $u \in W_0^{J,\Psi}(\Omega)$ *we have* $u \in L^{\Psi^r}(\Omega)$ *for every* $1 \leq r \leq r^* \equiv \dfrac{N}{N-\alpha}$ *and*

$$\|\Psi(u)\|_r \leq CE(u). \tag{6.9}$$

Proof. We prove the inequality for $r = r^*$, and then the result for $r < r^*$ follows by Hölder inequality. We can assume, without loss of generality, that u is radially deacreasing and $\Omega = B_{R^*}$, since substituting u by its symmetric decreasing rearrangement u^*, we have by Theorem 5.1.1,

$$\|\Psi(u)\|_r = \|\Psi(u^*)\|_r \leq CE(u^*) \leq CE(u).$$

We may also consider the case of u bounded, since if not, taking the sequence $u_T = \min\{u, T\}$, and thanks to the Dominated Convergence Theorem, we would get the result in the limit $T \to \infty$. We now define

$$A_k := \{x \in \mathbb{R}^N \,:\, u(x) > 2^k\}, \qquad a_k = |A_k|,$$

$$D_k := A_k \setminus A_{k+1}, \qquad d_k = |D_k|.$$

We have $A_k = B_{R_k}$, with $R_{k+1} \le R_k \le R^*$. Also $a_k = d_k = 0$ for all large k, say for $k > M$. Now we compute,

$$\|\Psi(u)\|_r = \left(\sum_{k=-\infty}^{M} \int_{D_k} \Psi^r(u(x))\,dx\right)^{1/r} \le \sum_{k=-\infty}^{M} \Psi(2^{k+1}) d_k^{1/r} \le c \sum_{k=-\infty}^{M} \Psi(2^k) a_k^{1/r},$$

since $r > 1$. On the other hand, if $x \in D_i$ and $y \in D_j$, with $j \le i-2$, then

$$|u(x) - u(y)| \ge 2^i - 2^{j+1} \ge 2^{i-1}.$$

Thus

$$\begin{aligned}
&\sum_{i=-\infty}^{M} \sum_{j=-\infty}^{i-2} \int_{D_i} \int_{D_j} \Psi(u(x) - u(y)) J(x-y)\,dydx \\
&\qquad \ge \sum_{i=-\infty}^{M} \Psi(2^{i-1}) \int_{D_i} \sum_{j \le i-2} \int_{D_j} J(x-y)\,dydx \\
&\qquad \ge \sum_{i=-\infty}^{M} \Psi(2^{i-1}) \int_{D_i} \int_{A_{i-1}^c} J(x-y)\,dydx \ge c \sum_{i=-\infty}^{M} \Psi(2^i) a_{i-1}^{-\alpha/N} d_i \\
&\qquad = c \sum_{i=-\infty}^{M} \Psi(2^i) a_{i-1}^{-\alpha/N} \left(a_i - \sum_{k=i+1}^{M} d_k\right) = c(A - B).
\end{aligned}$$

The second term can be estimated as

$$\begin{aligned}
B &= \sum_{i=-\infty}^{M} \sum_{k=i+1}^{M} \Psi(2^i) a_{i-1}^{-\alpha/N} d_k = \sum_{k=-\infty}^{M} \sum_{i=-\infty}^{k-1} \Psi(2^i) a_{i-1}^{-\alpha/N} d_k \\
&\le \sum_{k=-\infty}^{M} \sum_{i=-\infty}^{k-1} \Psi(2^i) a_{k-1}^{-\alpha/N} d_k \le \sum_{k=-\infty}^{M} \Psi(2^k) a_{k-1}^{-\alpha/N} d_k \sum_{i=-\infty}^{k-1} \gamma^+(2^{i-k}) \\
&= \sum_{k=-\infty}^{M} \Psi(2^k) a_{k-1}^{-\alpha/N} d_k \sum_{m=1}^{\infty} \gamma^+(2^{-m}) = c \sum_{k=-\infty}^{M} \Psi(2^k) a_{k-1}^{-\alpha/N} d_k = c(A - B).
\end{aligned}$$

We deduce the estimate

$$\begin{aligned}
E(u) &\ge \sum_{i=-\infty}^{M} \sum_{j=-\infty}^{i-2} \int_{D_i} \int_{D_j} \Psi(u(x) - u(y)) J(x-y)\,dydx \\
&\ge CA = C \sum_{i=-\infty}^{M} \Psi(2^i) a_{i-1}^{-\alpha/N} a_i.
\end{aligned}$$

We conclude, using [64, Lemma 5], since $\dfrac{1}{r} = 1 - \dfrac{\alpha}{N}$,

$$E(u) \geq C \sum_{i=-\infty}^{M} \Psi(2^i) a_i^{1-\alpha/N} \geq C \|\Psi(u)\|_r.$$

■

6.3 Compactness

We also prove that the above embedding is compact provided $r < r^*$. To this end we first show the compactness of the inclusion for $r = 1$ and then interpolate with the continuity for $r = r^*$. It is important to remark that the inclusion $W_0^{J,\Psi}(\Omega) \hookrightarrow L^\Psi(\Omega)$ is compact even when $q_* = 0$, which implies $r^* = 1$, provided the following conditions on the kernel at the origin hold

$$\lim_{|z| \to 0^+} |z|^N J(z) = \infty, \tag{6.10}$$

$$J(z_1) \geq cJ(z_2) \qquad \text{for every } 0 < |z_1| \leq |z_2| \leq 1, \text{ and some } c > 0. \tag{6.11}$$

This implies some kind of minimal singularity and some monotonicity near the origin. In particular this allows to consider for instance a kernel of the form $J(z) = |z|^{-N} |\log |z||^\beta$, $\beta > 0$, for $|z| \sim 0$, as in the fisrt part where $\Psi(s) = |s|^2$.

Theorem 6.3.1. *Assume J satisfies* (6.10) *and* (6.11)*. Then the embedding*

$$W_0^{J,\Psi}(\Omega) \hookrightarrow L^\Psi(\Omega) \tag{6.12}$$

is compact.

Proof. As in Section 3.1 the idea of the proof goes back to the Riesz-Fréchet-Kolmogorov work. We follow here the adaptation to the nonlineal fractional Laplacian framework performed in [26].

Let $\mathcal{A} \subset W_0^{J,\Psi}(\Omega)$ be a bounded set. We show that $\mathcal{A}$ is totally bounded in $L^\Psi(\Omega)$, i.e., for any $\epsilon \in (0,1)$ there exist $\beta_1, ..., \beta_M \in L^\Psi(B_1)$ such that for any $u \in \mathcal{A}$ there exists $j \in \{1, ..., M\}$ such that

$$F(u - \beta_j) \leq \epsilon. \tag{6.13}$$

We take a collection of disjoints cubes $Q_1, ...Q_{M'}$ of side $\rho < 1$ such that $\Omega \subset \bigcup_{j=1}^{M'} Q_j$. For any $x \in \Omega$ we define $j(x)$ as the unique integer in $\{1, ..., M'\}$ for which $x \in Q_{j(x)}$. Also, for any $u \in \mathcal{A}$, let

$$Q(u)(x) := \frac{1}{|Q_{j(x)}|} \int_{Q_{j(x)}} u(y)\, dy.$$

Notice that

$$Q(u+v) = Q(u) + Q(v) \text{ for any } u, v \in \mathcal{A},$$

and that $Q(u)$ is constant, say equal to $q_j(u)$, in any Q_j, for $j \in \{1, ..., M'\}$. Therefore, we can define

$$S(u) := \rho^N \left(\Psi(q_1(u)), ..., \Psi(q_{M'}(u))\right) \in \mathbb{R}^{M'},$$

and consider the spatial 1-norm in $\mathbb{R}^{M'}$ as

$$\|v\|_1 := \sum_{j=1}^{M'} |y_j|, \qquad \text{for any } v = (y_1, \dots, y_{M'}) \in \mathbb{R}^{M'}.$$

We observe that

$$F(Q(u)) = \sum_{j=1}^{M'} \int_{Q_j} \Psi\left(Q(u)(x)\right)\, dx \leq \rho^N \sum_{j=1}^{M'} \Psi\left(q_j(u)\right) = \|S(u)\|_1, \tag{6.14}$$

and also, by Jensen inequality and (6.7),

$$\begin{aligned} \|S(u)\|_1 &= \sum_{j=1}^{M'} \rho^N \left|\Psi(q_j(u))\right| = \rho^N \sum_{j=1}^{M'} \Psi\left(\frac{1}{\rho^N} \int_{Q_j} u(y)\, dy\right) \\ &\leq \sum_{j=1}^{M'} \int_{Q_j} \Psi(u(y))\, dy = \int_\Omega \Psi(u(y))\, dy \leq c. \end{aligned} \tag{6.15}$$

In the same way,

$$F(Q(u) - a) = F(Q(u) - Q(a)) \leq \|S(u) - S(a))\|_1$$

for every constant a. In particular from (6.15) we obtain that the set $S(\mathcal{A})$ is bounded in $\mathbb{R}^{M'}$ and so, since it is finite dimensional, it is totally bounded. Therefore, there exist $b_1, ..., b_K \in \mathbb{R}^{M'}$ such that

$$S(\mathcal{A}) \subset \bigcup_{i=1}^{K} B_\eta(b_i), \tag{6.16}$$

where $B_\eta(b_i)$ are the 1–balls of radius η centered at b_i. For any $i \in \{1, ..., K\}$, we write the coordinates of b_i as $b_i = (b_{i,1}, ..., b_{i,M'}) \in \mathbb{R}^{M'}$. For any $x \in \Omega$ we set

$$\beta_i(x) = \Psi^{-1}(\rho^{-N} b_{i,j(x)}),$$

where $j(x)$ is as above. Notice that β_i is constant on Q_j, i.e. if $x \in Q_j$ then

$$Q(\beta_i)(x) = \Psi^{-1}(\rho^{-N} b_{i,j(x)}) = \Psi^{-1}(\rho^{-N} b_{i,j}) = \beta_i(x) \tag{6.17}$$

and so $q_j(\beta_i) = \Psi^{-1}(\rho^{-N} b_{i,j})$; thus $S(\beta_i) = b_i$. Furthermore, again by Jensen inequality,

$$\begin{aligned} F(u - Q(u)) &= \sum_{j=1}^{M'} \int_{Q_j} \Psi\left(u(x) - Q(u)(x)\right) dx \\ &= \sum_{j=1}^{M'} \int_{Q_j} \Psi\left(\frac{1}{\rho^N} \int_{Q_j} (u(x) - u(y))\, dy\right) dx \\ &\leq \frac{1}{\rho^N} \sum_{j=1}^{M'} \int_{Q_j} \int_{Q_j} \Psi\left(u(x) - u(y)\right) dy\, dx \\ &\leq \frac{1}{\ell(\rho)} \sum_{j=1}^{M'} \int_{Q_j} \int_{Q_j} \Psi\left(u(x) - u(y)\right) J(x-y)\, dy\, dx \leq \frac{c}{\ell(\rho)}, \end{aligned}$$

where ℓ is some function satisfying $\ell(z) \leq |z|^N J(z)$. Using (6.10), (6.11) we can take ℓ radial satisfying $\ell(0^+) = \infty$. Consequently, for any $j \in \{1, ..., K\}$, recalling (6.14) and (6.17)

$$\begin{aligned} F(u - \beta_j) &\leq F(u - Q(u)) + F(Q(u) - Q(\beta_j)) + F(Q(\beta_j) - \beta_j) \\ &\leq c\left(\frac{1}{\ell(\rho)} + \|S(u) - S(\beta_j))\|_1\right). \end{aligned}$$

Now recalling (6.16) we take $j \in \{1, ..., K\}$ such that $S(u) \in B_\eta(b_j)$, that is

$$\|S(u) - S(\beta_j))\|_1 = \|S(u) - b_j\|_1 < \eta.$$

We conclude by choosing ρ and η small so as to have $c\left(\dfrac{1}{\ell(\rho)} + \eta\right) < \epsilon$. ■

As a corollary we obtain the full compactness result in the fractional case.

Theorem 6.3.2. *Assume J satisfies* (6.6) *and* (6.11)*. Then the embedding*

$$W_0^{J,\Psi}(\Omega) \hookrightarrow L^{\Psi^r}(\Omega) \tag{6.18}$$

is compact for every $1 \leq r < r^*$ if $\alpha < N$, for every $1 \leq r < \infty$ if $\alpha \geq N$.

Proof. As before if $\alpha \geq N$ we obtain the result substituting α by any number below N. By classical interpolation

$$\|\Psi(u)\|_r \leq \|\Psi(u)\|_1^\lambda \|\Psi(u)\|_{r^*}^{1-\lambda} \leq cF(u)^\lambda,$$

where $\dfrac{1}{r} = \lambda + \dfrac{1-\lambda}{r^*}$. Therefore we can obtain, instead of (6.13), the estimate

$$\int_\Omega \Psi^r(u(x) - \beta_j)\, dx \leq c\epsilon^{\lambda r},$$

and we are done. ∎

Chapter 7

Elliptic problems

With the machinery developped in the previous two chapters, we next study the problem

$$\begin{cases} \mathcal{L}u = f(x,u), & \text{in } \Omega, \\ u = 0, & \text{in } \Omega^c. \end{cases} \tag{7.1}$$

This problem must be considered in weak sense with the aid of the interaction energy $\mathcal{E}$, that is, any solution u satisfies

$$\mathcal{E}(u;\varphi) = \int_\Omega f\varphi, \qquad \forall\, \varphi \in W_0^{J,\Psi}(\Omega). \tag{7.2}$$

We study first the case $f = f(x)$ in an appropriate space. We obtain existence and uniqueness of a solution and show some integrability properties in terms of the data f when ψ is restricted to the power-like case (5.8). We do not address regularity issues in this work. For Hölder regularity results in the case of the fractional p–Laplacian we refer to [24, 25, 50, 53].

7.1 The problem with reaction $f = f(x)$

We start with this section the study of some elliptic type problems associated to our nonlinear nonlocal operator $\mathcal{L}$.

Here we consider the problem

$$\begin{cases} \mathcal{L}u = f(x), & \text{in } \Omega, \\ u = 0, & \text{in } \Omega^c. \end{cases} \tag{7.3}$$

Given any $f \in \left(W_0^{J,\Psi}(\Omega)\right)'$, the dual space, we say that $u \in W_0^{J,\Psi}(\Omega)$ is a weak solution to (7.3) if (7.2) holds.

7.1.1 Existence

By Poincaré inequality (6.7) we have that $f \in \left(W_0^{J,\Psi}(\Omega)\right)'$ for instance provided $f \in L^\Phi(\Omega)$, where Φ is the complementary function of Ψ.

We next show that problem (7.3) has a weak solution. We do not know if this solution is a strong solution, that is if $\mathcal{L}u$ is defined pointwise and the equality in (7.3) holds almost everywhere. On the other hand, we are able to show uniqueness assuming some extra conditions on the function Ψ. In the exact power case $\Psi(s) = |s|^p$ these extra conditions cover the full range $p > 1$.

Theorem 7.1.1. *For any $f \in \left(W_0^{J,\Psi}(\Omega)\right)'$ there exists a solution $u \in W_0^{J,\Psi}(\Omega)$ to problem* (7.3)*. If ψ satisfies either condition* (5.14) *or* (5.17) *then the solution is unique.*

Proof. Existence follows by minimizing in $W_0^{J,\Psi}(\Omega)$ the functional

$$I(v) = E(v) - \int_\Omega fv.$$

Clearly it is well defined, lower semicontinuous and Fréchet differentiable with

$$\langle I'(v), \varphi \rangle = \mathcal{E}(v; \varphi) - \int_\Omega f\varphi$$

for every v, $\varphi \in W_0^{J,\Psi}(\Omega)$. To see that it is coercive we first observe that

$$\|v\|_{W^{J,\Psi}} \to \infty \quad \Longrightarrow \quad E(v) \to \infty.$$

Actually, by (5.20), (6.2) and Poincaré inequality,

$$\begin{aligned} \|v\|_{W^{J,\Psi}} &= \|v\|_{L^\Psi} + [v]_{W^{J,\Psi}} \le c\left((\gamma_\Psi^-)^{-1}(F(v)) + (\gamma_\Psi^-)^{-1}(E(v))\right) \\ &\le c \max\left\{(E(v))^{1/p}, (E(v))^{1/q}\right\}. \end{aligned}$$

Now use Hölder inequality in Orlicz spaces,

$$\left|\int_\Omega fv\right| \le c\|v\|_{W_0^{J,\Psi}}, \qquad c = \sup_{\|w\|_{W_0^{J,\Psi}}=1} \left|\int_\Omega fw\right|.$$

The last quantity is known as the Orlicz norm of f in $\left(W_0^{J,\Psi}(\Omega)\right)'$, an is equivalent to the Luxemburg norm, see [59]. We thus get

$$I(v) \ge E(v) - c\,(E(v))^{1/q} \to \infty$$

as $\|v\|_{W^{J,\Psi}} \to \infty$. Therefore there exists a minimum of I, attained by compactness for some function $u \in W_0^{J,\Psi}(\Omega)$, which is a weak solution to our problem.

We now show uniqueness. Suppose by contradiction that there exist two functions u_1, $u_2 \in W_0^{J,\Psi}(\Omega)$ such that

$$\mathcal{E}(u_1;\varphi) = \mathcal{E}(u_2;\varphi) \qquad \forall\,\varphi \in W_0^{J,\Psi}(\Omega). \tag{7.4}$$

Assume first that (5.14) holds. We have, denoting $a = u_1(x) - u_1(y)$, $b = u_2(x) - u_2(y)$, and using (5.15),

$$\begin{aligned} E(u_1 - u_2) &= \frac{1}{2}\iint_{\mathbb{R}^{2N}} \Psi(a-b)J(x-y)\,dxdy \\ &\le c\iint_{\mathbb{R}^{2N}} (\psi(a) - \psi(b))\,(a-b)J(x-y)\,dxdy \\ &= c\,(\mathcal{E}(u_1;u_1-u_2) - \mathcal{E}(u_2;u_1-u_2)) = 0 \end{aligned}$$

by (7.4). This implies $u_1 \equiv u_2$.

Assume now condition (5.17). We calculate, using Hölder inequality and (5.18),

$$\begin{aligned} &E(u_1 - u_2) = \frac{1}{2}\iint_{\mathbb{R}^{2N}} \Psi(a-b)J(x-y)\,dxdy \\ &\le c\left(\iint_{\mathbb{R}^{2N}} \frac{(\Psi(a-b))^{2/p}}{(\Psi(a)+\Psi(b))^{\frac{2-p}{p}}} J(x-y)\,dxdy\right)^{\frac{p}{2}} \\ &\qquad\qquad \cdot\left(\iint_{\mathbb{R}^{2N}} (\Psi(a)+\Psi(b))\,J(x-y)\,dxdy\right)^{1-\frac{p}{2}} \\ &\le c\left(\iint_{\mathbb{R}^{2N}} (\psi(a)-\psi(b))\,(a-b)J(x-y)\,dxdy\right)^{\frac{p}{2}} \\ &\qquad\qquad \cdot\left(\iint_{\mathbb{R}^{2N}} (\Psi(a)+\Psi(b))\,J(x-y)\,dxdy\right)^{1-\frac{p}{2}} \\ &= c\,(\mathcal{E}(u_1;u_1-u_2) - \mathcal{E}(u_2;u_1-u_2))^{\frac{p}{2}}\,(E(u_1)+E(u_2))^{1-\frac{p}{2}} = 0. \end{aligned}$$

■

A maximum principle is easy to obtain.

Proposition 7.1.2. *If $u \in W^{J,\Psi}(\mathbb{R}^N)$ then*

$$\left.\begin{array}{l}\mathcal{E}(u,\varphi) \geq 0 \quad \forall\, \varphi \in W^{J,\Psi}(\mathbb{R}^N),\ \varphi \geq 0 \\ u \geq 0 \ \ in\ \Omega^c\end{array}\right\} \Rightarrow\ u \geq 0\ in\ \Omega.$$

Proof. Since $u^- \geq 0$ and $u^- \in W^{J,\Psi}(\mathbb{R}^N)$, we have, by (5.11),

$$0 \geq -\mathcal{E}(u^-,u^-) \geq \mathcal{E}(u,u^-) \geq 0.$$

Hence $u^- \equiv 0$. ■

7.1.2 Regularity

We now study the integrability properties of the solution in terms of the integrability of the datum in the power-like case (5.8). In the exact power case of the fractional p–Laplacian these integrability properties have been obtained in [6]. Our proofs in the more general case treated here differ from theirs in that we are using Stroock-Varopoulos inequality instead of Kato inequality, and that we allow for the limit case $q_* = 0$, which does not make sense in the fractional p–Laplacian. All the proofs are based on the well known Moser iteration technique for the standard Laplacian case, see for example the book [36].

The first result uses no singularity condition on the kernel J, besides being nonintegrable.

Theorem 7.1.3. *Assume condition* (5.8)*. If u is a weak solution to problem* (7.3) *with $f \in L^m(\Omega)$ then $u \in L^{m(p-1)}(\Omega)$.*

Of course this result is not trivial only if $m > \dfrac{p}{p-1}$, since u being a weak solution it belongs to $W_0^{J,\Psi}(\Omega) \subset L^p(\Omega)$.

Proof. Without loss of generality we may assume $u \geq 0$, and this simplifies notation; the general case is obtained in a similar way. We define for $\beta \geq 1$ and $K > 0$ the function

$$H(s) = \begin{cases} s^\beta, & s \leq K, \\ \text{linear}, & s > K. \end{cases}$$

We choose as test function $\varphi = G(u) = \int_0^u \Psi(H'(s))\,ds$. It is easy to check that $\varphi \in W_0^{J,\Psi}(\Omega)$. In fact

$$E(\varphi) \leq \gamma_\Psi^+(\Psi(\beta K^{\beta-1}))E(u) < \infty.$$

We obtain on one hand, using the Stroock-Varopoulos inequality (5.12) and the Poincaré inequality (6.7),

$$\mathcal{E}(u; G(u)) \geq cE(H(u)) \geq cF(H(u)), \tag{7.5}$$

and on the other hand, using Hölder inequality,

$$\int_\Omega fG(u) \leq \|f\|_m \|G(u)\|_{m'}. \tag{7.6}$$

Letting $K \to \infty$ in the definition of H, the inequalities (7.5) and (7.6) give

$$\|u\|_{p\beta}^{p\beta} \leq c\|f\|_m \|u\|_{((\beta-1)p+1)m'}^{(\beta-1)p+1}. \tag{7.7}$$

Choosing now $\beta = \dfrac{m(p-1)}{p}$, we get

$$\|u\|_{m(p-1)} \leq c\|f\|_m^{\frac{1}{p-1}}.$$

■

The same proof allows to gain more integrability when condition (6.6) holds.

Theorem 7.1.4. *Assume conditions* (5.8) *and* (6.6) *and let u be a weak solution to problem* (7.3)*, where $f \in L^m(\Omega)$, $m < N/\alpha$. Then $u \in L^{\frac{m(p-1)N}{N-m\alpha}}(\Omega)$.*

Again this result is not trivial only if $m > \dfrac{Np}{Np - N + \alpha}$, since then

$$\frac{m(p-1)N}{N-m\alpha} > \frac{Np}{N-\alpha}.$$

Proof. In the previous proof, using Sobolev inequality (6.9) instead of Poincaré inequality, we obtain in (7.7)

$$\|u\|_{p\beta r^*}^{p\beta} \leq \beta\|f\|_m \|u\|_{((\beta-1)p+1)m'}^{(\beta-1)p+1}$$

where $r^* = \dfrac{N}{N-\alpha}$. Choosing now $\beta = \dfrac{m'(p-1)}{p(r^*-m')}$, we get

$$\|u\|_{\frac{m(p-1)N}{N-m\alpha}} \leq c\|f\|_m^{\frac{1}{p-1}}.$$

■

Even more, assuming a better integrability condition on f we get that the solution is bounded. This is a well known result for the standard Laplacian or the fractional Laplacian.

Theorem 7.1.5. *Assume conditions* (5.8) *and* (6.6). *If u is a weak solution to problem* (7.3)*, where $f \in L^m(\Omega)$ with $m > N/\alpha$, then $u \in L^\infty(\Omega)$.*

Proof. We change here slightly the test function used in the previous two proofs. We define for $\beta \geq 1$ and $K \geq k$ (k to be chosen later) a $\mathcal{C}^1([k,\infty))$ function H, as follows:

$$H(s) = \begin{cases} s^\beta - k^\beta, & s \in [k, K], \\ \text{linear}, & s > K. \end{cases}$$

Let us also define $v = u + k$, and choose as test function $\varphi = G(v) = \displaystyle\int_k^v \Psi(H'(s))\,ds$. We obtain on one hand, using the Stroock-Varopoulos inequality (5.12) and the Sobolev inequality (6.9),

$$\mathcal{E}(u; G(v)) \geq cE(H(v)) \geq c\|\Psi(H(v))\|_{r^*}\,, \tag{7.8}$$

and on the other hand, using Hölder inequality,

$$\int_\Omega fG(v) \leq \int_\Omega fv\Psi(H'(v)) \leq \frac{1}{k^{p-1}} \int_\Omega fv^p\Psi(H'(v)) \leq \frac{c}{k^{p-1}}\|f\|_m \|vH'(v)\|_{pm'}^p\,, \tag{7.9}$$

since $v \geq k$. Inequality (7.8) together with (7.9), and the properties of Ψ, lead to

$$\|H(v)\|_{r^*p} \leq \left(\frac{c\|f\|_m}{k^{p-1}}\right)^{1/p} \|vH'(v)\|_{pm'}\,. \tag{7.10}$$

We choose $k = (c\|f\|_m)^{\frac{1}{p-1}}$ and let $K \to \infty$ in the definition of H, so that the inequality (7.10) becomes

$$\|u\|_{r^*p\beta} \leq \beta\|u\|_{pm'\beta}.$$

Hence for all $\beta \geq 1$ the inclusion $u \in L^{pm'\beta}(\Omega)$ implies the stronger inclusion $u \in L^{r^*p\beta}(\Omega)$, since $r^* = \dfrac{N}{N-\alpha} > m' = \dfrac{m}{m-1}$ provided $m > \dfrac{N}{\alpha}$. Observe that u being a weak solution it belongs to $W_0^{J,\Psi}(\Omega)$, and thus $u \in L^{\frac{Np}{N-\alpha}}(\Omega)$. The result follows now iterating the estimate starting with $\beta = \dfrac{N(m-1)}{(N-\alpha)m} > 1$, see for example [36, Theorem 8.15] for the details in the standard Laplacian case. This gives $u \in L^\infty(\Omega)$. In fact we get the estimate

$$\|u\|_\infty \leq c(E(u)^{\frac{1}{p}} + \|f\|_m^{\frac{1}{p-1}}).$$

∎

7.2 The problem with reaction $f = f(u)$

We study in this section the nonlinear elliptic type problem

$$\begin{cases} \mathcal{L}u = f(u), & \text{in } \Omega, \\ u \geq 0,\ u \not\equiv 0, & \text{in } \Omega, \\ u = 0, & \text{in } \Omega^c. \end{cases} \tag{7.11}$$

7.2.1 The lower range

We first show existence in the *lower* case, i.e., when $f : [0,\infty) \to \mathbb{R}$ is a continuous function satisfying

$$\exists\, 0 < \mu < \frac{q-1}{p} \quad : \qquad |f(t)| \leq c_1 + c_2 \Psi^\mu(t), \qquad \liminf_{t\to 0^+} \frac{f(t)}{\Psi^\mu(t)} \geq c_3 > 0. \tag{7.12}$$

In the power-like case (5.8) with $f(t) = t^{m-1}$ this means $0 < m < p$. See [15] for the classical sublinear problem for $\mathcal{L} = -\Delta$ and Section 4.3.1 for general linear $\mathcal{L}$ with $q_* \geq 0$, both in the case $\Psi(s) = |s|^2$.

Theorem 7.2.1. *Under the assumption* (7.12) *problem* (7.11) *has a solution* $u \in W_0^{J,\Psi}(\Omega)$.

Proof. We define the energy functional $I : W_0^{J,\Psi}(\Omega) \to \mathbb{R}$ defined by

$$I(v) = E(v) - \int_\Omega G(v),$$

where $G(u) = \displaystyle\int_0^u f(s)\,ds$. This functional is easily seen to be weakly lower semicontinuous, and is well defined since

$$\left| \int_\Omega G(v) \right| \leq c_1|\Omega| + c_2|\Omega|^{1-\mu}(F(v))^\mu < \infty. \tag{7.13}$$

On the other hand, this same estimate also gives coercivity since $\mu < 1$, and then

$$I(v) \geq E(v) - c\,(E(v))^\mu \to \infty \qquad \text{as } \|v\|_{W^{J,\Psi}} \to \infty.$$

Let now $\{v_n\} \subset W_0^{J,\Psi}(\Omega)$ be a minimizing sequence for I, that is

$$\liminf_{n\to\infty} I(v_n) = \nu = \inf_{u \in W_0^{J,\Psi}(\Omega)} I(u).$$

This sequence is bounded in $W_0^{J,\Psi}(\Omega)$, and therefore we can assume that there is a subsequence, still denoted $\{v_n\}$, such that $v_n \rightharpoonup u$ in $W_0^{J,\Psi}(\Omega)$. Therefore $v_n \to u$ in $L^{\Psi}(\Omega)$. We thus deduce by (7.13)

$$\int_\Omega G(v_n) \to \int_\Omega G(u),$$

so that

$$\nu \le I(u) \le \liminf_{n\to\infty} \left(E(v_n) - \int_\Omega G(v_n) \right) = \liminf_{n\to\infty} I(v_n) = \nu.$$

This shows that $I(u) = \nu$ and u is a global minimum for I, hence a solution to (7.11). It is easy to see that we can replace u by $|u|$ since $I(|u|) \le I(u)$. In order to show that u is nontrivial let us check that $I(u) < 0$. In fact, given any $v \in W_0^{J,\Psi}(\Omega)$ we have

$$I(\varepsilon v) \le \gamma_\Psi^+(\varepsilon)E(v) - \varepsilon \left(\gamma_\Psi^-(\varepsilon)\right)^\mu \int_\Omega G(u) \le \varepsilon^q E(v) - \varepsilon^{1+p\mu} \int_\Omega G(u) < 0,$$

for small $\varepsilon > 0$, since $q > 1 + p\mu$. We deduce that $\nu < 0$ and $u \not\equiv 0$. ■

Unfortunately we are only able to prove uniqueness in the exact power case $\Psi(s) = |s|^p$. In fact uniqueness follows in that case using, as in Section 4.3.1, a standard argument by means of a Picone inequality proved in [31], see [15]. Though a Picone inequality could be obtained also assuming that (5.8) is satisfied, it is not sharp enough to prove uniqueness. In the more general case of $\Psi \in \Gamma_{p,q}$ such Picone type inequality is not even known to hold.

7.2.2 The intermediate range: subcritical reaction

We now assume condition (6.6) and consider nonlinear functions f in the intermediate range, that is above the power $p-1$ but subcritical in the sense of Sobolev, see Theorem 6.2.1. The precise conditions on f are

$$\begin{array}{lll} \exists\, \rho > p : & tf(t) \ge \rho G(t) & \forall\, t > 0; \\ \exists\, 1 < r < r^*,\, t_0 > 0 : & tf(t) \le c\psi^r(t) & \forall\, t > 0; \\ \exists\, \lambda_0 > 0 : & f(\lambda t) \ge \lambda^\rho f(t) & \forall\, t > 0,\, \lambda > \lambda_0, \end{array} \tag{7.14}$$

where $G' = f$. When $f(t) = t^{m-1}$ these conditions hold with $\rho = m$ provided $p < m < \dfrac{Nq}{N-\alpha}$.

Theorem 7.2.2. *Assume J satisfies* (6.6)*, ψ satisfies either* (5.14) *or* (5.17)*, and f is a nondecreasing function satisfying* (7.14)*. Then problem* (7.11) *has a solution $u \in W_0^{J,\Psi}(\Omega)$.*

Proof. As before we consider the functional

$$I(v) = E(v) - \int_\Omega G(v),$$

whose critical points are the solutions to our problem. This functional is well defined in $W_0^{J,\Psi}(\Omega)$ thanks to the Sobolev embedding and the second condition in (7.14). We therefore apply the standard variational technique based on the Mountain Pass Theorem [3]. We only have to prove that the functional satisfies the Palais-Smale condition and has the appropriate geometry.

We first prove that any Palais-Smale sequence has a convergent subsequence. Let $\{v_n\}$ be a sequence satisfying

$$I(v_n) \to \nu, \qquad \langle I'(v_n), \varphi\rangle \to 0 \quad \forall\, \varphi \in \left(W_0^{J,\Psi}(\Omega)\right)'.$$

By the first condition in (7.14), and using (5.10), we have

$$\langle I'(v_n), v_n\rangle = \mathcal{E}(v_n; v_n) - \int_\Omega v_n f(v_n) \le pE(v_n) - \rho \int_\Omega G(v_n).$$

On the other hand, for all large n such that $\|I'(v_n)\| \le 1$ we have

$$|\langle I'(v_n), v_n\rangle| \le \|v_n\|_{W^{J,\Psi}}.$$

Therefore

$$\begin{aligned} \nu + 1 \ &\ge I(v_n) = I(v_n) - \frac{1}{\rho}\langle I'(v_n), v_n\rangle + \frac{1}{\rho}\langle I'(v_n), v_n\rangle \\ &\ge \left(1 - \frac{p}{\rho}\right) E(v_n) - \frac{1}{\rho}\|v_n\|_{W^{J,\Psi}} \\ &\ge \left(1 - \frac{p}{\rho}\right) \min\left\{\|v_n\|^p_{W^{J,\Psi}},\, \|v_n\|^q_{W^{J,\Psi}}\right\} - \frac{1}{\rho}\|v_n\|_{W^{J,\Psi}}. \end{aligned}$$

This implies $\|v_n\|_{W^{J,\Psi}} \le k$ for every n, so that there exists a subsequence, still denoted $\{v_n\}$, converging weakly to some $u \in W_0^{J,\Psi}(\Omega)$, and by Theorem 6.3.2 it is $v_n \to v_\infty$ strongly in $L^{\Psi^r}(\Omega)$ for every $1 \le r < r^*$. The second condition in (7.14) implies $v_n f(v_n) \to v_\infty f(v_\infty)$ in $L^1(\Omega)$. Now write,

$$\begin{aligned} \mathcal{E}(v_n; v_n - v_\infty) - \mathcal{E}(v_\infty; v_n - v_\infty) \ &= \langle I'(v_n), v_n - v_\infty\rangle - \langle I'(v_\infty), v_n - v_\infty\rangle \\ &\quad + \int_\Omega (f(v_n) - f(v_\infty))(v_n - v_\infty) \to 0. \end{aligned}$$

Using inequalities (5.15) or (5.18) as in the proof of uniqueness in Theorem 7.1.1, we obtain

$$E(v_n - v_\infty) \to 0,$$

that is $v_n \to v_\infty$ in $W_0^{J,\Psi}(\Omega)$, and Palais-Smale condition holds.

Let us now look at the behaviour of I close to the origin and far from it. First $I(0) = 0$. Also, given any $v \in W_0^{J,\Psi}(\Omega)$, we have by Sobolev inequality and the second condition in (7.14)

$$I(v) = E(v) - \int_\Omega G(v) \geq c_1 \|\Psi(v)\|_r - c_2 \|\Psi(v)\|_r^r > 0$$

for every $\|\Psi(v)\|_r$ small. But $\|v\|_{W^{J,\Psi}}$ small implies $\|\Psi(v)\|_r$ small. We have obtained

$$\exists\, \varepsilon, \delta > 0 \;:\; I(v) > I(0) + \delta \quad \forall\, v \in W_0^{J,\Psi}(\Omega),\ \|v\|_{W^{J,\Psi}} = \varepsilon.$$

On the other hand, if $\lambda > 0$ is large, using the third condition in (7.14), we get

$$I(\lambda v) \leq \lambda^p E(v) - \lambda^\rho \int_\Omega G(v) < 0,$$

since $p < \rho$. Thus

$$\exists\, \overline{v} \in W_0^{J,\Psi}(\Omega),\ \|\overline{v}\|_{W^{J,\Psi}} > \varepsilon \;:\; I(\overline{v}) < I(0).$$

This ends the proof by an application of the Mountain Pass Theorem. Actually, if we define

$$\Theta = \{h \in C([0,1]; W_0^{J,\Psi}(\Omega)) \;:\; h(0) = 0,\, h(1) = \overline{v}\},$$

then

$$\eta = \inf_{h \in \Theta} \max_{t \in [0,1]} I(h(t))$$

is a critical value with $I(u) = \eta$ for some $u \in W_0^{J,\Psi}(\Omega)$, which is a solution to our problem.

■

7.2.3 Supercritical reaction

The exponent r^* in (7.14) is sharp in the fractional p–Laplacian case. In fact, in the fractional Laplacian case $p = 2$ this has been proved in [61] by means of a Pohozaev identity when Ω is star-shaped. We adapted their proof in Section 4.3.2 for more general kernels again with $\Psi(s) = |s|^2$, obtaining an exponent which depends on the kernel and is presumed not to be optimal. The proof of this last result works verbatim for general powers $\Psi(s) = |s|^p$, but not for other functions, since homogeneity is crucial in the argument.

Let, for $\lambda > 1$,

$$\mu(\lambda) = \lambda^{-N} \sup_{\substack{z\in\mathbb{R}^N \\ z\neq 0}} \frac{J(z/\lambda)}{J(z)}, \tag{7.15}$$

and assume $\mu(\lambda) < \infty$ for λ close to 1.

Theorem 7.2.3. *If u is a bounded solution to problem (7.11) with $\Psi(s) = |s|^p$ and Ω is star-shaped, then*

$$\int_\Omega uf(u) \leq \frac{Np}{N-\delta} \int_\Omega G(u),$$

where $\delta = \mu'(1^+)$ and $G' = f$.

Corollary 7.2.4. *Problem (7.11) with $f(u) = u^{m-1}$, $\Psi(s) = |s|^p$ and Ω star-shaped has no bounded solutions for any exponent $m > m_* = \dfrac{Np}{N-\delta}$.*

We observe that this nonexistence result depends not only on the behaviour of the kernel at the origin, but on its global behaviour, see (7.15). In fact when the kernel is

$$J(z) = \begin{cases} |z|^{-N-\alpha_1} & \text{if } |z| < 1, \\ |z|^{-N-\alpha_2} & \text{if } |z| > 1, \end{cases}$$

$\alpha_1 < p$, $\alpha_2 > 0$, we get $\sigma = \max\{\alpha_1, \alpha_2\}$, see again Section 4.3.2. It will be interesting to know if only the singularity of J at the origin determines by its own the existence or nonexistence of solution. If this is the case we would get, in the critical singularity exponent $q_* = 0$ that there is no solution for any $m > p$. This, together with the existence result for $m < p$ of Theorem 7.2.1, leaves only the case $m = p$ to be studied. We dedicate next section to this task.

7.3 The generalized eigenvalue problem

In this last section we study the parametric problem

$$\begin{cases} \mathcal{L}u = \lambda\psi(u), & \text{in } \Omega, \\ u = 0, & \text{in } \Omega^c. \end{cases} \tag{7.16}$$

Though the problem is not homogeneous due to the presence of the non homogeneous function ψ, since it is precisely this same function ψ that defines both, the operator and the reaction, it can also be called *generalized eigenvalue* problem, as is usual for the p–Laplacian or the fractional p–Laplacian, see [50, 51]. The first (generalized) eigenvalue and eigenfunction would be obtained minimizing

$$I(v) = \frac{E(v)}{F(v)}, \qquad v \in W_0^{J,\Psi}(\Omega) \setminus \{0\}.$$

In fact, if u is a minimum, the function $g(t) = I(u + t\varphi)$, for any admissible function φ satisfies $g(0) = \dfrac{E(u)}{F(u)} = \lambda_1$, $g'(0) = 0$, that is,

$$\langle E'(u), \varphi\rangle = \lambda_1 \langle F'(u), \varphi\rangle,$$

which is the associated Euler-Lagrange equation, the weak formulation for problem (7.16).

On the other hand, since neither the operator nor the reaction function are in general suitable for rescaling, the equation is not invariant under the usual change of variables in that kind of problems. We therefore minimize $E(u)$ for a fixed value of $F(u)$ and obtain in that way a family of eigenvalues. See also [62] for the analysis for a closely related nonlinear nonlocal operator. In the fractional Laplacian case this family reduces to a single number, the first eigenvalue, see [32] and [50].

Theorem 7.3.1. *For every $\mu > 0$ there exists a positive eigenvalue λ_μ of* (7.16) *with non-negative eigenfunction $u_\mu \in W_0^{J,\Psi}(\Omega)$ such that $F(u_\mu) = \mu$. The family $\Lambda = \{\lambda_\mu : \mu > 0\}$ is bounded from below by a positive constant. Each eigenfunction is moreover bounded if* (5.8) *and* (6.6) *holds for some $\alpha > 0$.*

Proof. Define

$$\nu_\mu = \inf\{E(v) : v \in W_0^{J,\Psi}(\Omega), F(v) = \mu\}.$$

The inequality (6.7) immediately gives $\nu_\mu \geq A\mu > 0$ for every $\mu > 0$. Let $\{v_n\}$ be a minimizing sequence, that is

$$\lim_{n\to\infty} E(v_n) = \nu_\mu, \qquad F(v_n) = \mu.$$

Then $\{v_n\}$ is bounded in $W_0^{J,\Psi}(\Omega)$, so there exists a subsequence, still denoted by $\{v_n\}$, such that $v_n \rightharpoonup u_\mu$ in $W_0^{J,\Psi}(\Omega)$. As usual, by Theorem 6.3.2 there exists a subsequence converging to u_μ in $L^\Psi(\Omega)$, so $F(u_\mu) = \mu$. This gives

$$\nu_\mu \leq E(u_\mu) \leq \lim_{n\to\infty} E(v_n) = \nu_\mu,$$

and then $E(u_\mu) = \nu_\mu$. The functionals E and T are differentiable, and so by the Lagrange multiplier rule there exists a number λ_μ such that $E'(u_\mu) = \lambda_\mu F'(u_\mu)$, that is

$$\mathcal{E}(u_\mu;\varphi) = \lambda_\mu \int_\Omega \psi(u_\mu)\varphi,$$

for every $\varphi \in W_0^{J,\Psi}(\Omega)$. Putting $\varphi = u_\mu$ we get $\lambda_\mu > 0$. The fact that the eigenfunction is nonnegative or nonpositive follows by (5.11) which implies $E(\pm|u|) \le E(u)$. The relation between λ_μ and ν_μ is easy:

$$q\nu_\mu = qE(u_\mu) \le \mathcal{E}(u_\mu;u_\mu) = \lambda_\mu \int_\Omega \psi(u_\mu)u_\mu \le p\lambda_\mu F(u_\mu) = p\mu\lambda_\mu,$$

and also $q\mu\lambda_\mu \le p\nu_\mu$. In particular Λ is bounded from below by Aq/p.

Finally the boundedness of u_μ assuming (6.6) is easily proved again by the Moser iterative scheme as performed in [13]. The key point is the use of the Stroock-Varopoulos inequality (5.13) and condition (5.8), and finally apply Theorem 7.1.5. See also [32]. ■

In the fractional Laplacian case, that is, when $p = q$, we have $\lambda_\mu = \nu_\mu/\mu$, and in fact by homogeneity $\lambda_\mu = \lambda_1$ for every $\mu > 0$.

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