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LIBOR ADDITIVE MODEL CALIBRATION TO SWAPTIONS MARKETS

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Keywords: Lévy Market model, Calibration, Semidefinite Programming.

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JESÚS PÉREZ COLINO, FRANCISCO J. NOGALES, AND WINFRIED STUTE

(WORKING PAPER)

Abstract. In the current paper, we introduce a new calibration methodology for the LIBOR market model driven by LIBOR additive processes based in an inverse problem. This problem can be splitted in the calibration of the continuous and discontinuous part, linking each part of the problem with at-the-money and in/out -of -the-money swaption volatilities. The continuous part is based on a semidefinite programming (convex) problem, with constraints in terms of variability or robustnest, and the calibration of the Lévy measure is proposed to calibrate inverting the Fourier Transform.

1. Introduction

To have a robust and efficient algorithm is a central topic in the successful implementation of a model. The amount of reality that the model can collect is not only related with the sort of process that drives the model but also with the calibration methodology that you use to substract the information from the market prices. Therefore the calibration of the financial models has become an important topic in financial engineering because of the need to price increasingly complex options in a consistent way with prices of standard instruments liquidly traded in the market.

It is clear that a robust and efficient calibration algorithm is a central element in the successful implementation of a derivatives pricing model, independently if the model is driven by semimartingales or directly by a Brownian motion. Recent developments in interest rates modelling have led to a form of technological asymmetry on this topic. The theoretical performance of the usual continuous models such as the **Heath, Jarrow and Morton** (HJM) (1992) model for the instantaneous forward rates, or the LIBOR Market Model of Interest Rates by **Brace, Gaterek and Musiela** (BGM)(1997) allows a very flexible modelling and pricing of the basic interest rate options (caps and swaptions) at-the-money. However, due to the inefficiency and instability of the calibration procedures, not only a small part of the market information but also the smile-skew of volatilities, is actually exploited. On the other hand, many authors have proposed model the forward rates using semimartingales (**Björk et al.** (1997), **Jamshidian** (1999)) or directly using non-homogeneous Lévy processes (**Eberlein et al.** (2005)) but again, the calibration procedures associated to these models are not able to collect all the market complexity reflected, for example, in the smile-skew that appears in the swaption market..

On the other hand, the most common techniques to calibrate the continuous part of the model (see for example **Longstaff, Santa-Clara and Schwartz** (2000), **Rebonato** (2000) or **Brigo and Mercurio** (2002)) are limited methods. Usually it is necessary to substitute a statistical estimate to the market information on the forward LIBOR correlation matrix because the numerical complexity and instability of the calibration process makes it impossible to calibrate a full market covariance matrix. As a direct

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consequence, these calibration algorithms fail in one of their primary mission: they are very poor market risk visualization tools.

But independently of the category of the process that drives our model, the practitioner has to guarantee two properties in the calibration process:

- *first*, the calibration solution has to be **unique** and **global** and this sort of results are only possible if the calibration problem is a **convex problem**,
- and *second*, the calibration process has to provide an indication of the **sensitiveness** of our calibration against market movements (**robustness**). It is usually given as the **dual solution** of the convex problem.

Basically, the main goal in this paper is to propose a methodology to calibrate and work with the any category of market models (using piecewise stationay Lévy processes or **LIBOR additive process**) using **convex programming** methods. Notice that as a direct implication of the Lévy-Itô decomposition, every Lévy process is a combination of a Brownian motion with drift and a possibly infinite sum of independent compound Poisson process (see **Colino** (2008), Theorem 9). This also means that every **LIBOR additive process** can be approximated with arbitrary precision by a sequence of jump-diffusion processes (see **Colino** (2008), Theorems 40 and 41), that is by a sum of a sequence of Brownian motions with drifts and a sequence of compound Poisson process, a point which is useful not only in theory but also from the practitioner point of view.

Therefore this paper is structured in three different parts:

- The *first part* (Section 2) gives basically a brief introduction about what we called two-steps calibration
- The *second part* (Section 3) is devoted to the first-step calibration or calibration to the continuous part of the model. Using the at-the-money swaption volatilities, we propose a convex methodology to obtain the term structure of instantaneous volatilities and covariances from the market data.
- In the *third part* (Section 4), our aim is to propose the second-step calibration as an inverse problem to calibrate the sequence of Lévy measures according to the information given by the smile/skew in the swaption market.

1.1. **Basic assumption for the LIBOR additive model.** In this section, we mainly focus on **forward LIBOR rates** assuming, *first*, that the dynamics of **instantaneous forward rates** are specified through the **Heath, Jarrow and Morton** (1992) model, driven by **LIBOR additive processes** (piecewise homogeneous Lévy process) as in **Colino** (2008). And *second*, we additionally assume that the **LIBOR rates** can be derived from the bond prices of forward prices. In order to achieve this aim, we have to establish some assumptions that will be applied during the whole work.

1.1.1. *Assumptions related with the forward rates.*

(1) Let us define $f(t,T)$ as the **instantaneous forward rates** at time $t \in [0,T]$ for any $T < T^\star$. It corresponds to the rate that one can contract for a time t, on a loan that begins at date T and is returned an instant later. It is usually defined by
$$f(t,T) = -\frac{\partial \log B(t,T)}{\partial T} \tag{1.1}$$
where $B(t,T)$ is the value in t of a **zero-coupon bond** until maturity T, or in other words
$$B(t,T) = \exp\left\{-\int_t^T f(t,s)\,ds\right\} \tag{1.2}$$

(2) We assume that the evolution of this forward rate is driven by a d-dimensional **LIBOR additive process** (Colino (2008)) that admits the **Lévy-Itô decomposition,** such that the dynamics of the **instantaneous forward rate** $f(t,T)$ in $t \leq T \in I$, under the real-world probability $\mathbb{P}$, which we assume as follows[1]:

$$df(t,T) = \alpha_{\eta(t)}(t,T)\,dt + \sigma_{\eta(t)}(t,T)\,dW_t + \int_E \delta(t,T,x)\left(\mu_{\eta(t)} - \upsilon_{\eta(t)}\right)(dt,dx) \tag{1.3}$$

when $t \leq T \in I$ where $\eta(t) = \sup\{j \geq 0 : T_j \leq t\}$ with $j = 0,1,...,n$ (for the sake of clarity, we will denote this by the generic index j), W_t is a d-dimensional standard **Wiener process** in $\mathbb{R}^d$ and $\mu_{\eta(t)}$ is a **random measure** such that $\mu \in \mathbb{N}^+$ with the **compensator** $\upsilon_{\eta(t)}(dt,dx)$.

1.1.2. *Assumptions related with the forward LIBOR rates.*

(1) Assume that for a predetermined collection of dates $0 < T_0 < T_1 < ... < T_n$ with a fixed accrual period or **tenor** δ and any $t \leq T_i \in [0,T^\star]$, we denote by $L(t,T_i)$ the forward rate for the interval from T_i to T_{i+1} as

$$1 + \delta L(t,T_i) := F_B(t,T_i,T_{i+1}) \tag{1.4}$$

$$:= \frac{B(t,T_i)}{B(t,T_{i+1})} = \exp\left\{\int_{T_i}^{T_{i+1}} f(t,s)\,ds\right\} = B(T_i,T_{i+1})^{-1} \tag{1.5}$$

where $B(t,T_i)$ represents **value of a corporate bond.**

(2) Hence these simple forward rates should be contrasted with the instantaneous, continuously compounded defaultable forward rates (and short rates) in the framework of **Heath, Jarrow and Morton** (1992) which satisfy

$$L(t,T_i) = \frac{1}{\delta}\left(\exp\left\{\int_{T_i}^{T_{i+1}} f(t,s)\,ds\right\} - 1\right) \tag{1.6}$$

Now we can define for any $0 \leq t \leq T_i$ the **forward LIBOR spread**

$$S(t,T_i) := L(t,T_i) - L(t,T_i) \tag{1.7}$$

1.1.3. *Assumptions related with the Swap rates and Swaptions.*

(1) As usually, let us define a predetermined collection of dates $0 < T_1 < ... < T_n = T^\star$ with a fixed accrual period or **tenor** δ_j, and any $t < T_1 \in [0,T^\star]$ with $T^\star$ fixed. We first consider a fixed-for-floating forward start swap settled in arrears with notional principal N, usually equals to 1, without loss of generality. We shall frequently refer to such a contract as the **forward start payer swap**. A long position in a **forward start payer swap** corresponds to the situation when an investor , between T_α and T_β with $0 < \alpha < \beta \leq n$, will make periodic payments determined by fixed interest rates $K_{\alpha,\beta}$, and will receive in exchange payments specified by some floating rate, usually $L(t,T_{j-1})$. A short position in a forward start payer swap defines a closely related contract known as the **forward start receiver swap**.

[1] Notice that if $c = 0$ we are considering the risk-free or default-free case.

(2) Let us place ourselves within a framework of some arbitrage-free, or equivalently **risk-neutral term structure model** (under $\mathbb{P}^\star$). Then, the value at time t of the **forward start payer swap** denoted by FS_t or $FS_t(K_{\alpha,\beta})$ equals

$$
\begin{aligned}
FS_t\left(K_{\alpha,\beta}\right) &= \mathbb{E}_{\mathbb{P}^\star}\left\{\sum_{j=\alpha}^{\beta} B(t,T_j)\left(L\left(t,T_{j-1}\right)-K_{\alpha,\beta}\right)\delta_j\right\} \\
&= \sum_{j=\alpha}^{\beta} \mathbb{E}_{\mathbb{P}^\star}\left\{ B\left(t,T_j\right)\left(B(T_{j-1},T_j)^{-1}-\left(1+K_{\alpha,\beta}\delta_j\right)\right)\Big|\, \mathcal{F}_t\right\} \qquad (1.8)
\end{aligned}
$$

where writing $c_j = 1 + k\delta_j$ and rearranging we obtain the following important result:

$$
\begin{aligned}
FS_t\left(K_{\alpha,\beta}\right) &= \sum_{j=\alpha}^{\beta}\left(B\left(t,T_{j-1}\right)-c_j B(t,T_j)\right) \\
&= B(t,T_{\alpha-1}) - \sum_{j=\alpha}^{\beta} c_j B(t,T_j) \qquad (1.9)
\end{aligned}
$$

(3) Alternatively, assume that under the **forward measure** $\mathbb{P}_{T_j}$ the process $L\left(t,T_{j-1}\right)$ is a martingale. Then

$$
\begin{aligned}
FS_t\left(K_{\alpha,\beta}\right) &= \sum_{j=\alpha}^{\beta} B(t,T_j)\mathbb{E}_{\mathbb{P}_{T_j}}\left\{\left(L\left(t,T_{j-1}\right)-K_{\alpha,\beta}\right)\delta_j\right\} \\
&= \sum_{j=\alpha}^{\beta} B(t,T_j)\delta_j\left(L\left(t,T_{j-1}\right)-K_{\alpha,\beta}\right) \\
&= \sum_{j=\alpha}^{\beta}\left(B(t,T_{j-1})-B(t,T_j)-K_{\alpha,\beta}\delta_j B(t,T_j)\right) \\
&= B(t,T_{\alpha-1}) - \sum_{j=\alpha}^{\beta} c_j B(t,T_j) \qquad (1.10)
\end{aligned}
$$

where again $c_j = 1 + K\delta_j$ and we used the fact that

$$
L\left(t,T_{j-1}\right) = \frac{B(t,T_{j-1})-B(t,T_j)}{\delta_j B(t,T_j)}
$$

Therefore, as we can observe in (1.5) and (1.6) a forward swap is essentially a contract to deliver a specific coupon-bearing bond and to receive at the same time a zero-coupon bond, and this relationship provides a simple method for the replication of a swap contract.

(4) The **forward swap rate** $K_{\alpha,\beta}$ at time t for the date T_α is that value of the fixed rate K that makes the value of the n-period forward swap zero, i.e. that value of K for which $FS_t(K) = 0$

(5) Let us define the **payer** (respectively, **receiver**) **swaption** with **strike rate** $K_{\alpha,\beta}$ as the financial derivative that gives to the owner the right to enter at time T_α the underlying **forward payer** (respectively, **receiver**) **swap** settled in arrears with maturity T_β, with $0 < \alpha < \beta \leq n$. Because $FS_{T_\alpha}\left(K_{\alpha,\beta}\right)$ is the value at time t of the forward payer swap with the fixed interest rate $K_{\alpha,\beta}$, it is clear that the price of the **payer swaption** at time t equals

$$
PS\left(t,T_\alpha,T_\beta,\sigma^\star_{\alpha,\beta},K_{\alpha,\beta}\right) = \mathbb{E}_{\mathbb{P}^\star}\left\{ B(t,T_\alpha)\left(FS_{T_\alpha}\left(K_{\alpha,\beta}\right)\right)^+\Big|\,\mathcal{F}_t\right\} \qquad (1.11)
$$

where t is the moment of valuation, T_α is the moment where the forward start swap begins and T_β is the swap maturity, $\sigma^\star_{\alpha,\beta}$ is the implied volatility quoted in the swaption market for the strike $K_{\alpha,\beta}$. Therefore it is apparent that the option, in the payer swaption, is exercised at time T_α if and only if the value of the underlying forward swap with maturity T_β, is positive.

And for the **receiver swaption** we have

$$RS\left(t,T_\alpha,T_\beta,\sigma^\star_{\alpha,\beta},K_{\alpha,\beta}\right)=\mathbb{E}_{\mathbb{P}^\star}\left\{B(t,T_\alpha)\left(-FS_{T_\alpha}\left(K_{\alpha,\beta}\right)\right)^+\Big|\,\mathcal{F}_t\right\}\tag{1.12}$$

2. The two-steps calibration for the LIBOR additive process as an inverse problem

From the beginning of the present paper, the reader would wonder to know the advantages of this new stochastic process, the LIBOR additive process, in the interest rates modelling. This subsection will show briefly the main reason: the double-calibration of this process against a non-homogeneous swaption market with volatility smiles. Basically the idea is simple: according to the **Lévy-Itô** decomposition of the LIBOR additive process (Theorems 40 and 41 in **Colino** (2008)), we can define the calibration problem as an **inverse problem** of a sequence of triplets that completely characterize the entire process.

Let us consider the following discretization in $[0,T^\star]$, $0<T_1<...<T_n=T^\star$. Let us define the price of a payer swaption at time t as $PS(t,T_\alpha,T_\beta,\sigma^\star_{i,j},K^h_{\alpha,\beta})$ as a call-option to get into a swap that begins in T_α and finish in T_β with the swap rate $S\left(t,T_\alpha,T_\beta\right)$ (underlying of the option) with strike $K_{\alpha,\beta}$ and $0<\alpha\leq\beta\leq n$, and where $\sigma^\star_{\alpha,\beta}$ is the **Black** (1976) cumulative variance of swaption on $S\left(t,T_\alpha,T_\beta\right)$ for the mentioned strike $K_{\alpha,\beta}$ quoted in the swaption market.

Proposition 1. *The general calibration problem at the moment t can be written as an **inverse problem** defined as*

$$\left(\gamma_i,A_i,\upsilon_i\right)_{i\in\{1,\ldots,n\}}=\arg\inf\sum_{h=-m}^{m}\sum_{j=i}^{n}\left[\omega^h_{ij}\left\|PS^\sigma(t,T_\alpha,T_\beta,A_i,\upsilon_i,K^h_{\alpha,\beta})-PS^M(t,T_\alpha,T_\beta,\sigma^\star_{i,n},K^h_{\alpha,\beta})\right\|\right]\tag{2.1}$$

*where, on the right part of the equality, we have denoted as $PS^\sigma(t,T_\alpha,T_\beta,A_i,\upsilon_i,K^h_{\alpha,\beta})$ the **theoretical payer swaption price** in t, given by our model, and defined as the value of an option with maturity T_α that gives to the holder the right to get into a forward payer swap between T_α and T_β settled in arrears, with $0<\alpha\leq\beta\leq n$, and $PS^M(t,T_\alpha,T_\beta,\sigma^\star_{i,n},K^h_{\alpha,\beta})$ as the **market value of a payer swaption** in t, priced using **Black-76** model, with strike $K^h_{\alpha,\beta}$, where $h\in\mathbb{N}$ such that if $h=0$ then $FS_t\left(K^0_{\alpha,\beta}\right)=0$ (at-the-money case), and market volatility $\sigma^\star_{\alpha,\beta}$ such that*

$$\begin{cases} PS^\sigma(t,T_\alpha,T_\beta,A_i,\upsilon_i,K^h_{\alpha,\beta})=\mathbb{E}_{\mathbb{P}^\star}\left\{B(t,T_\alpha)\left(FS_{T_\alpha}\left(K^h_{\alpha,\beta}\right)\right)^+\Big|\,\mathcal{F}_t\right\}\\ PS^M(t,T_\alpha,T_\beta,\sigma^\star_{\alpha,\beta},K^h_{\alpha,\beta})=Black(t,T_\alpha,T_\beta,\sigma^\star_{\alpha,\beta},K^h_{\alpha,\beta})\sum_{j=1}^n B\left(t,T_j\right)\delta_j\end{cases}$$

and $\left(\gamma_i,A_i,\upsilon_i\right)_{i=1,\ldots,n}$ is the sequence of triplets consisting of

$$\begin{cases}\gamma_i\in\mathbb{R}^d\\ A_i=\left(\sigma_{ij}\right),\ \text{a } d\times d \text{ symmetric non-negative matrix, with } d=n-i+1\\ \upsilon_i \text{ a positive measure on } \mathbb{R}^d\setminus\{0\} \text{ with } \int_{\mathbb{R}^d}\left(|g|^2\wedge1\right)\upsilon\left(dg\right)<\infty\end{cases}$$

Proof. Notice that the basic underlying that is moved by the **LIBOR additive process** is the **forward LIBOR rates**, according to the model specified in this paper. Let us consider the usual payer swaption definition that we have to price it using our model is

$$\begin{aligned}PS^\sigma(t,T_\alpha,T_\beta,\sigma^\star_{\alpha,\beta},K^h_{\alpha,\beta}) &= \mathbb{E}_{\mathbb{P}^\star}\left\{B(t,T_\alpha)\left(FS_{T_\alpha}\left(K^h_{\alpha,\beta}\right)\right)^+\Big|\,\mathcal{F}_t\right\}\\ &= \mathbb{E}_{\mathbb{P}^\star}\left\{B(t,T_\alpha)\left(\mathbb{E}_{\mathbb{P}^\star}\left(\sum_{j=\alpha}^{\beta}B\left(T_\alpha,T_j\right)\left(L\left(t,T_{j-1}\right)-K^h_{\alpha,\beta}\right)\delta_j\Bigg|\,\mathcal{F}_{T_\alpha}\right)\right)^+\Bigg|\,\mathcal{F}_t\right\}\end{aligned}$$

that after simple manipulations and using assumptions in 1.1.2 related with swaps and swaptions, then this yields, as expected

$$PS^{\sigma}(t,T_\alpha,T_\beta,\sigma^{\star}_{\alpha,\beta},K^h_{\alpha,\beta}) = \mathbb{E}_{\mathbb{P}^{\star}}\left\{ B\left(t,T_\alpha\right)\left(1-\sum_{j=\alpha+1}^{\beta} c_j B(T_\alpha,T_j)\right)^{+} \middle| \mathcal{F}_t \right\}$$

or in other words, the payer swaption may also be seen as a put option on a coupon-bearing bond, where $c_j = K^h_{\alpha,\beta}\delta_j$ when $j=\alpha,...,\beta-1$ and $c_j = 1+K^h_{\alpha,\beta}\delta_j$ with $j=\beta$.

Additionally, notice that using the same approximation that has already been employed by **Brace, Gatarek and Musiela** (1997) we can write the payer swaption value as a function of the LIBOR rates as

$$\begin{aligned} PS^{\sigma}(t,T_\alpha,T_\beta,\sigma^{\star}_{\alpha,\beta},K_{\alpha,\beta}) &= B\left(t,T_\alpha\right)\mathbb{E}_{\mathbb{P}_{T^\star}}\left\{\left(1-\sum_{j=\alpha+1}^{\beta} c_j B(T_\alpha,T_j)\right)^{+}\middle|\mathcal{F}_t\right\} \\ &= B\left(t,T_\alpha\right)\mathbb{E}_{\mathbb{P}_{T^\star}}\left\{\left(1-\sum_{j=\alpha+1}^{\beta} c_j \prod_{l=\alpha+1}^{j-1}\left(1+\delta_j L\left(T_\alpha,T_l\right)\right)^{-1}\right)^{+}\middle|\mathcal{F}_t\right\} \\ &= B\left(t,T^\star\right)\mathbb{E}_{\mathbb{P}_{T^\star}}\left\{\prod_{l=\alpha+1}^{n-1}\left(1+\delta_j L\left(T_\alpha,T_l\right)\right)\left(1-\sum_{j=\alpha+1}^{\beta} c_j \prod_{l=\alpha}^{j-1}\left(1+\delta_j L\left(T_\alpha,T_l\right)\right)^{-1}\right)^{+}\middle|\mathcal{F}_t\right\} \\ &= B\left(t,T^\star\right)\mathbb{E}_{\mathbb{P}_{T^\star}}\left\{\left(-\sum_{j=\alpha}^{\beta}\left(c_j\prod_{l=j}^{n-1}\left(1+\delta_j L\left(T_\alpha,T_l\right)\right)\right)\right)^{+}\middle|\mathcal{F}_t\right\} \end{aligned}$$

where the dynamics of the forward LIBOR rates driven by a **LIBOR additive process**, is specified as

$$L\left(t,T_k\right) = L\left(0,T_k\right)\exp\left(\int_0^t \lambda_j\left(s,T_k\right)dG_s^{T^\star}\right)$$

We have proved in **Colino** (2008) (Section 1.3) that this process is uniquely determined in law by its sequence of triplets $(\gamma_i, A_i, \upsilon_i)_{i\in\{1,\ldots,n\}}$ consisting of

$$\begin{cases} \gamma_i \in \mathbb{R}^d \\ A_i = (\sigma_{ij})\,, \text{ a } d\times d \text{ symmetric non-negative matrix, with } d=n-i+1 \\ \nu_i \text{ a positive measure on } \mathbb{R}^d\backslash\{0\} \text{ with } \int_{\mathbb{R}^d}\left(|g|^2\wedge 1\right)\upsilon\left(dg\right)<\infty \end{cases}$$

□

Notice that using directly the independence property between the continuous and the jump part, implicit in the **Lévy-Itô** theorem (theorems 40 and 41 in **Colino** (2008)), we can prove directly the following proposition,

Proposition 2. *The inverse problem* (2.1) *can be split in two different and independent inverse problems*

(1) $A_i = \arg\inf \sum_{j=i}^{n}\left[\omega^0_{ij}\left\|PS^{\sigma}(t,T_\alpha,T_\beta,A_i,\upsilon_i,K^0_{\alpha,\beta}) - PS^{M}(t,T_\alpha,T_\beta,\sigma^{\star}_{i,n},K^0_{\alpha,\beta})\right\|\right]$

(2) $\upsilon_i = \arg\inf \sum_{h=-m}^{m}\sum_{j=i+1}^{n}\left[\omega^h_{ij}\left\|PS^{\sigma}(t,T_\alpha,T_\beta,A_i,\upsilon_i,K^h_{\alpha,\beta}) - PS^{M}(t,T_\alpha,T_\beta,\sigma^{\star}_{i,n},K^h_{\alpha,\beta})\right\|\right]$

Therefore we can calibrate this sequence of triplets against the market separately: on one hand, the continuous part calibration will be achieved using at-the-money swaption volatilities, and on the other hand, the sequence of Lévy measures will be estimated using in/out -of -the-money swaption smiles for different maturities.

The main goal of this two-step calibration is to collect as much information as we can from the market prices, in order to simulate the most realistic possible scenarios, but also provide robust and global solutions to the calibration problem. The methodology behind this calibration procedure is not new, several authors have introduced these ideas in the continuous process framework or in stock market. However, our approach has two improvements.

- In the *first-step,* we introduce some relevant changes in the SDP problem that guarantee not only the convexity but also the stability of the solution, something essential to achieve the correct simulation in the full-rank LIBOR additive model.
- And with the *second-step* of the calibration, we introduce the information given by the smile in the swaption market directly in the sequence of the Lévy measures, creating a direct link between jumps and smile.

3. First-step: semidefinite-programming to calibrate the continuous Market model

The forward rates covariance matrix plays an increasingly important role in exotic interest rate derivatives modelling and there is a need for a calibration algorithm that allows the retrieval of a maximum amount of covariance information from the market. As far as we know, **Brace and Womersley** (2000) and **d'Aspremont** (2003) are the only that propose a methodology to calibrate a multivariate LIBOR market model without assuming any 'a priory' structure to the covariance matrix, based on semidefinite programming, allowing at the same time robust and global solutions. They showed how semidefinite programming based calibration methods provide integrated calibration and risk-management results with guaranteed numerical performance, the dual program having a very natural interpretation in terms of hedging instruments and sensitivity.

Our main goal in this subsection, is to provide an extension of **Brace and Womersley** (2000) or **d'Aspremont** (2003) methodologies that has the goal of solving the calibration problem for the continuous part of our model, improving the stability and robustness of the **Brace and Womersley** (2000) or **d'Aspremont** (2003) solutions. Basically, our proposal is based on a relevant change in the objective function that allows us to skip the dependence of the solutions with respect to how to formulate the objective function.

3.1. **Introduction to the swaption calibration problem.** Let us study the swaption market under the **Brace, Gaterek and Musiela** model. Let us define the **swap rate** as the fixed rate that zeroes the present value of a set of periodical exchanges of fixed against floating coupons on a LIBOR rate of given maturity at futures dates. Denoting by $S(t, T_1, T_n)$ a forward swap rate at time t for an interest rate swap with first reset at T_1 and exchanging payments at $T_1, ..., T_n$. It is clear that it is stochastic and under the appropriate measure swap measure $\mathbb{Q}^{1,n}$ we can assume a lognormal dynamics for the continuous part of the swap dynamics

$$d\tilde{S}(t, T_1, T_n) = S(t, T_1, T_n)\sigma_{1,n}(t)\, dW_t^{\mathbb{Q}^{1,n}}$$

However, analytical approximations are available for swaptions in the LIBOR Market model framework. Indeed, **Brace, Dun and Barton** (1999) suggest to adopt the LIBOR forward market as the central model for the two markets, mainly for its mathematical tractability. We will stick to their suggestion, also because of the fact that forward rates are somehow more natural and more representative coordinates of the yield-curve than swap rates.

To introduce the formula we will use in the following, note that a crucial role in the swap market model is played by the Black swap volatility $\sigma^{\star}_{1,n}(T_1)$ entering Black's formula for swaptions, expressed by

$$\begin{aligned}\sigma^{\star}_{1,n}(T_1) \quad : \quad &= \frac{1}{T_1}\int_0^{T_1} (\sigma_{1,n}(t))^2\, dt \\ &= \frac{1}{T_1}\int_0^{T_1} \left(d\ln \tilde{S}(t,T_1,T_n)\right)\left(d\ln \tilde{S}(t,T_1,T_n)\right)\end{aligned}$$

Notice that if we choose the LIBOR Market Model as a central model, we must resort to different pricing techniques. It is possible to price swaptions with a Monte Carlo simulation, by simulating the forward rates involved in the payoff through a discretization of the dynamics presented above, so as to obtain the relevant zero coupon bonds and the forward swap rate. In fact, recall that we can write the forward-swap as a basket of forwards (see **Rebonato** (1998))

$$S(t,T_1,T_n) = \sum_{i=1}^{n} \omega_i(t)\, F(t,T_i) \tag{3.1}$$

where ω_i are the weights (with an explicit expression) such that $0 \leq \omega_i \leq 1$ (in fact, they are always positive, monotone and sum to one) such that

$$\begin{aligned}\omega_i(t) \quad &= \quad \frac{\tau_i B(t,T_1,T_i)}{\sum_{i=2}^{n}\tau_i B(t,T_1,T_i)} \\ &= \quad \frac{\tau_i \prod_{j=2}^{i} \frac{1}{1+\tau_j F(t,T_j)}}{\sum_{i=2}^{n}\tau_i \prod_{j=2}^{i} \frac{1}{1+\tau_j F(t,T_j)}}\end{aligned}$$

One can compute, under a number of approximations, based on "partially freezing the drift" and on "collapsing all measures" in the original dynamics, an analogous quantity $\sigma^{\star}_{1,n}(T_1)$ in the LMM.

We present here one of the simplest ways to deduce this formula based on a similar setting, appeared earlier for example in **Rebonato** (1998), and tested against Monte Carlo simulations for instance in **Brigo and Mercurio** (2001). Such approximated formulae, are easily obtained, first, freezing the weight's at time 0, so as to obtain

$$S(t,T_1,T_n) = \sum_{i=1}^{n} \omega_i(0)\, F(t,T_i).$$

Notice that this approximation is justified by the fact that the variability of the ω's is much smaller than the variability of the forward rates. This can be tested both historically and through simulations of the forward rates via Monte Carlo methods (see **Brigo and Mercurio** (2001)).

Then, let us differentiate both sides and we obtain

$$\begin{aligned}d\tilde{S}(t,T_1,T_n) \quad &\approx \quad \sum_{i=1}^{n} \omega_i(0)\, dF(t,T_i) \\ &= \quad (...)\, dt + \sum_{i=1}^{n} \omega_i(0)\, \sigma_i(t)\, F(t,T_i)\, dW_i(t)\end{aligned}$$

under any of the forward-adjusted measures, and compute the quadratic variation

$$d\tilde{S}(t,T_1,T_n)d\tilde{S}(t,T_1,T_n) \approx \sum_{i,j=1}^{n} \omega_i(0)\,\omega_j(0)\, F(t,T_i)\, F(t,T_j)\, \rho_{i,j}\sigma_i(t)\,\sigma_j(t)\, dt$$

and the percentage quadratic variation is

$$\left(\frac{d\tilde{S}(t,T_1,T_n)}{S(t,T_1,T_n)}\right)\left(\frac{d\tilde{S}(t,T_1,T_n)}{S(t,T_1,T_n)}\right) = \left(d\ln\tilde{S}(t,T_1,T_n)\right)\left(d\ln\tilde{S}(t,T_1,T_n)\right)$$
$$\approx \frac{1}{T_1}\frac{\sum_{i,j=1}^{n}\omega_i(0)\,\omega_j(0)\,F(t,T_i)\,F(t,T_j)\,\rho_{i,j}\sigma_i(t)\,\sigma_j(t)}{S(t,T_1,T_n)^2}dt$$

Now we can assume that freezing all forward rates in the above formula to their time-zero value

$$\left(d\ln\tilde{S}(t,T_1,T_n)\right)\left(d\ln\tilde{S}(t,T_1,T_n)\right) \approx \frac{1}{T_1}\sum_{i,j=1}^{n}\frac{\omega_i(0)\,\omega_j(0)\,F_B(0,T_i)\,F_B(0,T_j)}{S(t,T_1,T_n)^2}\rho_{i,j}\sigma_i(t)\,\sigma_j(t)\,dt$$

and finally we can obtain the expression for $\left(\sigma^{\star}_{1,n}(T_1)\right)^2$ such that

$$\int_0^{T_1}\left(d\ln\tilde{S}(t,T_1,T_n)\right)\left(d\ln\tilde{S}(t,T_1,T_n)\right)$$
$$\approx \frac{1}{T_1}\sum_{i,j=1}^{n}\frac{\omega_i(0)\,\omega_j(0)\,F(0,T_i)\,F(0,T_j)}{S(t,T_1,T_n)^2}\rho_{i,j}\int_0^{T_1}\sigma_i(t)\,\sigma_j(t)\,dt := \left(\sigma^{\star}_{1,n}(T_1)\right)^2$$

and this result proves the following proposition, given by **Rebonato** (1998)

Proposition 3. *The LIBOR market model (squared) swaption volatility can be approximated by*

$$\left(\sigma^{\star}_{1,n}(T_1)\right)^2 T_1 = \sum_{i,j=1}^{n}\frac{\omega_i(0)\,\omega_j(0)\,F(0,T_i)\,F(0,T_j)}{S(t,T_1,T_n)^2}\rho_{i,j}\int_0^{T_1}\sigma_i(t)\,\sigma_j(t)\,dt \tag{3.2}$$

The quantity $\sigma^{\star}_{1,n}(T_1)$ can be used as a proxy for the **Black** volatility of the swap rate $S(t,T_1,T_n)$. Putting this quantity in Black's formula for swaption allows one to compute approximated swaptions prices with the LIBOR market model (continuous part). Notice this result is obtained under a number of assumptions, and at first one would imagine its quality to be rather poor. However, it turns out that the approximation is very accurate as also pointed out by **Brace, Dun and Barton** (1998) and **Brigo and Mercurio** (2001).

3.2. **The calibration problem.** In this subsection we introduce the practical implementation of the calibration program using the swaption pricing approximation detailed above. Now, for the sake of reality, let us introduce a change of the notation.

We suppose that the calibration data set is made of m swaptions with option maturity T_α written on swaps of maturity $T_\beta - T_\alpha$ for $\alpha < \beta$, where $\alpha,\beta \in \mathbb{N}^+$, and $T_\alpha, T_\beta \in \{T_1,...,T_n\}$ with $n \in \mathbb{N}^+$, with market volatility given by $\sigma^{\star}_{\alpha,\beta}(T_\alpha)$. Notice that in our non-homogeneous case where $\sigma_i(t)$ is of the form $\sigma(t,T_i,T_{i+1})$ for any $i = 1,...,n$, with $t < T_i$ and piecewise constant on intervals of size $\tau = T_{i+1} - T_i$.

Therefore, the expression of the market cumulative variance, according to **Rebonato**'s formula (3.14), can be expressed as

$$\begin{aligned}\left(\sigma^{\star}_{\alpha,\beta}(T_\alpha)\right)^2 T_\alpha &= \int_0^{T_\alpha} \sum_{i,j=\alpha}^{\beta} \frac{\omega_i(0) F(0,T_i)}{S(0,T_\alpha,T_\beta)} \frac{\omega_j(0) F(0,T_j)}{S(0,T_\alpha,T_\beta)} \rho_{i,j}\sigma_i(s)\sigma_j(s)\,ds \\ &= \int_0^{T_\alpha} \sum_{i,j=\alpha}^{\beta} \hat{\omega}_i(0)\hat{\omega}_j(0)\cdot\sigma_{i,j}(s)\cdot ds \\ &= \int_0^{T_\alpha} Tr\left(\Omega_{\alpha,\beta}A_s\right) ds \\ &= Tr\left(\Omega_{\alpha,\beta}\int_0^{T_\alpha} A_s ds\right) \qquad (3.3)\end{aligned}$$

where $A_s = (\sigma_{i,j}(s))_{i,j\in[\alpha,\beta]} = \left(\rho_{i,j}\sigma_i(s)\sigma_j(s)\right)_{i,j\in[\alpha,\beta]}$ and $\Omega_{\alpha,\beta} = \hat{\omega}(0)\hat{\omega}(0)' = (\hat{\omega}_i(0)\hat{\omega}_j(0))_{i,j\in[\alpha,\beta]}$.

These conditions show that the cumulative market variance of a particular swaption can be written as the linear function of the forward covariance matrix, or equivalently we can say that here swaptions are priced as basket options with constant coefficients. As detailed in **Brace and Womersley** (2000) or **d'Aspremont** (2003), this simple approximation creates a relative error on swaption prices of $1-2\%$, which is well within bid-ask spreads.

On the other hand, if we want to formulate the calibration problem, this conditions has to be extended for every $\alpha = 1, ..., n$ in order to capture all the swaptions volatilities that the market quotes. Therefore the calibration problem becomes, using the approximate swaption variance formula in (3.16):

$$\begin{array}{ll}\text{Find} & A \\ \text{subject to} & Tr\left(\Omega_{i,n}A\right) = \left(\sigma^{\star}_{i,n}(T_i)\right)^2 T_i \quad \text{with } i = 1,...,n \\ & A \succeq 0\end{array}$$

which is a semidefinite feasible problem in the covariance matrix $A \in \mathbb{S}^N$ and $\Omega_{i,n} \in \mathbb{S}^N$, and $\left(\sigma^{\star}_{i,n}(T_i)\right)^2 T_i \in R^+$ is given by the swaption market as the **Black** (1976) cumulative variance of swaption on $S(0, T_i, T_n)$. Notice that A and Ω are block diagonal matrices that represent how the different factors disappear with the time (see **Brigo and Mercurio** (2001) section 7.1) such that $N = n(n-1)/2$ and $A_i \in \mathbb{S}^{n-(i-1)}$ in the following sense

$$A = \begin{bmatrix} A_1 & 0 & \cdots & 0 \\ 0 & A_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A_n \end{bmatrix}$$

where $(A_i)_{i=1,...,n}$ represents the sequence of semidefinite covariance matrices that characterize the continuous part of the LIBOR additive process.

The general form of the problem proposed by **Brace and Womersley** (2000) or **d'Aspremont** (2003) is the following:

$$\begin{array}{ll}\text{minimize} & Tr\left(CA\right) \\ \text{subject to} & Tr\left(\Omega_i A\right) = \left(\sigma^{\star}_{i,n}(T_i)\right)^2 T_i \quad \text{with } i = 1,...,n \\ & A \succeq 0\end{array}$$

however the calibration problem gives an entire set of solutions, extremely sensible to the matrix in the objective function $C \in \mathbb{S}^N$. That is clearly the biggest drawback in this framework. The general form of the problem that we propose to solve is the following:

Proposition 4. *The general calibration problem can be written as an infinite-dimensional linear matrix inequality with the following objective function:*

$$\begin{array}{lc} \textit{find} & k, A \\ \textit{subject to} & Tr\left(\Omega_i A\right) = \left(\sigma^{\star}_{i,n}(T_i)\right)^2 T_i \quad \textit{with } i = 1, ..., n \\ & \left\|A - A_{hist}\right\|_{Fr} \leq k I_N \\ & A \succeq 0 \end{array} \tag{3.4}$$

or equivalently,

$$\begin{array}{lc} \textit{minimize} & \left\|A - A_{hist}\right\|_{Fr} \\ \textit{subject to} & Tr\left(\Omega_i A\right) = \left(\sigma^{\star}_{i,n}(T_i)\right)^2 T_i \quad \textit{with } i = 1, ..., n \\ & A \succeq 0 \end{array} \tag{3.5}$$

which is a semidefinite feasibility problem in the covariance matrix $A \in \mathbb{S}^N$ *where* $A_{hist} \in \mathbb{S}^N$ *is the historical covariance matrix,* $k \in \mathbb{R}^+, \Omega_{i,n} \in \mathbb{S}^N$*, and* $\left(\sigma^{\star}_{i,n}(T_i)\right)^2 T_i \in \mathbb{R}^+$ *is given by the swaption market as the* ***Black*** *(1976) cumulative variance of swaption on* $S\left(0, T_i, T_n\right)$ *where* T_i *is the maturity of the option over a swap rate at time* 0 *for an interest rate swap with first reset at* T_i *and exchanging payments at* $T_i, ..., T_n$.

3.3. **Primal-Dual SDP problem and Algorithm Implementation.** According to the last proposition the general form of the problem to be solved is given by:

$$\begin{array}{lc} \text{minimize} & \left\|A - A_{hist}\right\|_{Fr} \\ \text{subject to} & Tr\left(\Omega_i A\right) = \left(\sigma^{\star}_{i,n}(T_i)\right)^2 T_i \quad \text{with } i = 1, ..., n \\ & A \succeq 0 \end{array}$$

Because their feasible set is the intersection of an affine subspace with the convex cone of nonnegative vectors, the objective being linear, these programs are convex. Or in other words, their solution set is convex as the intersection of an affine subspace with the (convex) cone of positive semidefinite matrices and a particular solution can be found by choosing A_{hist} and solving the corresponding semidefinite program. If the program is feasible, convexity guarantees the existence of a unique (up to degeneracy or unboundedness) optimal solution.

The first method used to solve these programs in practice was the simplex method. This algorithm works well in most cases but is known to have an exponential worst case complexity. In practice, this means that convergence of the simplex method cannot be guaranteed. Since the work of **Nemirovskii and Yudin** (1979) and **Karmarkar** (1984) however, we know that these programs can be solved in polynomial time by interior point methods and most modern solver implement both techniques. More importantly for our purposes here, the interior point methods used to prove polynomial time solvability of linear programs have been generalized to a larger class of convex problems. One of these extensions is called semidefinite programming. **Nesterov and Nemirovskii** (1994) showed that these programs can be solved in polynomial time. A number of efficient solvers are available to solve them, the one used in this work is called **SEDUMI** by **Sturm** (1999). In practice, a program with $n = 50$ will be solved in less than a second.

Now, let us show how the dual solution to the calibration program provides a complete description of the sensitivity to changes in market condition. In fact, because the mentioned algorithms used to solve the calibration problem jointly solve the problem and its dual, the sensitivity of the calibrated covariance matrix is readily available from the dual solution to the calibration program. Notice that according to the standard form of the primal semidefinite program, we can write the following Lagrangian

$$L(A, \lambda) = -\left\|A - A_{hist}\right\|_{Fr} + \sum_{i=1}^{n} \lambda_i \left(\left(\sigma^{\star}_{i,n}(T_i)\right)^2 T_i - Tr\left(\Omega_i A\right)\right)$$

and because the semidefinite cone is self-dual, we find that $L(A,\lambda)$ is bounded below in $A \succeq 0$, hence the dual semidefinite problem becomes:

$$\begin{array}{cc} \text{maximize} & -\sum_{i=1}^{n} \lambda_i \left(\sigma^{\star}_{i,n}(T_i)\right)^2 T_i \\ \text{s.t.} & -\left\|A - A_{hist}\right\|_{Fr} - \sum_{i=1}^{n} \lambda_i Tr\left(\Omega_i A\right) \succeq 0 \end{array}$$

For a general overview of semi-definite programming algorithms see **Vandenberghe and Boyd** (1996), **Nesterov and Nemirovskii** (1994) or **Alizadeh, Haeberly and Overton** (1998). We followed the implementation structure given in **Toh, Todd and Tütüncü** (1996), having adapted in C the Mathematica algorithm by **Brixius, Potra and Sheng** (1996). Some more recent libraries including a more efficient formulation of the SOCP (quadratic, smoothness, euclidean distance ...) and L.P. constraints are available. These include the SEDUMI 1.1 library package by **Sturm** (1999) for symmetric cone programming, which we have extensively used here. One of the most efficient ways to use this library of function in Matlab is using CVX programming. CVX is a Matlab-based modeling system for convex optimization developed by **Grant, Boyd and Ye** (2005).

All modern solvers as SEDUMI 1.1 in **Sturm** (1999) or SDPT3 in **Toh, Todd and Tütüncü** (1998) can produce both primal and dual solutions to this problem. It is clear that this dual solution can be used for risk-management purposes, and it is shown here as a indicator of sensibility of our calibration problem.

3.4. **Numerical Results.** As a first attempt of calibration, let us use the well-known data from **Brigo and Morini** (2005). It will allow the reader to compare the results. Therefore, we have introduced the following inputs to the problem:

(1) Initial curve of annual forward rates, as a vector with the following components (February 1, 2002 from **Brigo and Morini** (2005))

$F(0;0,1)$	0.036712
$F(0;1,2)$	0.04632
$F(0;2,3)$	0.050171
$F(0;3,4)$	0.05222
$F(0;4,5)$	0.054595
$F(0;5,6)$	0.056231
$F(0;6,7)$	0.057006
$F(0;7,8)$	0.057699
$F(0;8,9)$	0.05691
$F(0;9,10)$	0.057746

(2) Swaption **Black** volatilities (February 1, 2002 from **Brigo and Morini** (2005)), where we substract just the following annual data, from 1 to 10 years,

	$1y$	$2y$	$3y$	$4y$	$5y$	$6y$	$7y$	$8y$	$9y$	$10y$
$1y$	0.179	0.165	0.153	0.144	0.137	0.132	0.128	0.125	0.123	0.12
$2y$	0.154	0.142	0.136	0.13	0.126	0.122	0.12	0.117	0.115	0.113
$3y$	0.143	0.133	0.127	0.122	0.119	0.117	0.115	0.113	0.111	0.109
$4y$	0.136	0.127	0.121	0.117	0.114	0.113	0.111	0.109	0.108	0.107
$5y$	0.129	0.121	0.117	0.113	0.111	0.109	0.108	0.106	0.105	0.104
$6y$	0.125	0.118	0.114	0.1095	0.1075	0.106	0.105	0.104	0.1035	0.1025
$7y$	0.121	0.115	0.111	0.106	0.104	0.103	0.102	0.102	0.102	0.101
$8y$	0.118	0.112	0.1083	0.104	0.1023	0.1017	0.101	0.101	0.1007	0.1
$9y$	0.115	0.109	0.1057	0.102	0.1007	0.1003	0.1	0.1	0.0993	0.099
$10y$	0.112	0.106	0.103	0.1	0.099	0.099	0.099	0.099	0.098	0.098

(3) Historical forward rate correlations (February 1, 2002 from **Brigo and Morini** (2005)),

	$1y$	$2y$	$3y$	$4y$	$5y$	$6y$	$7y$	$8y$	$9y$	$10y$
$1y$	1	0.82	0.69	0.65	0.58	0.47	0.29	0.23	0.43	0.47
$2y$	0.82	1	0.8	0.73	0.68	0.55	0.45	0.4	0.53	0.57
$3y$	0.69	0.8	1	0.76	0.72	0.63	0.47	0.56	0.67	0.61
$4y$	0.65	0.73	0.76	1	0.78	0.67	0.58	0.56	0.68	0.7
$5y$	0.58	0.68	0.72	0.78	1	0.84	0.66	0.67	0.71	0.73
$6y$	0.47	0.55	0.63	0.67	0.84	1	0.77	0.68	0.73	0.69
$7y$	0.29	0.45	0.47	0.58	0.66	0.77	1	0.72	0.71	0.65
$8y$	0.23	0.4	0.56	0.56	0.67	0.68	0.72	1	0.73	0.66
$9y$	0.43	0.53	0.67	0.68	0.71	0.73	0.71	0.73	1	0.75
$10y$	0.47	0.57	0.61	0.7	0.73	0.69	0.65	0.66	0.75	1

The following figures compare some of the most relevant results, in terms of variance-covariance and correlation matrices:

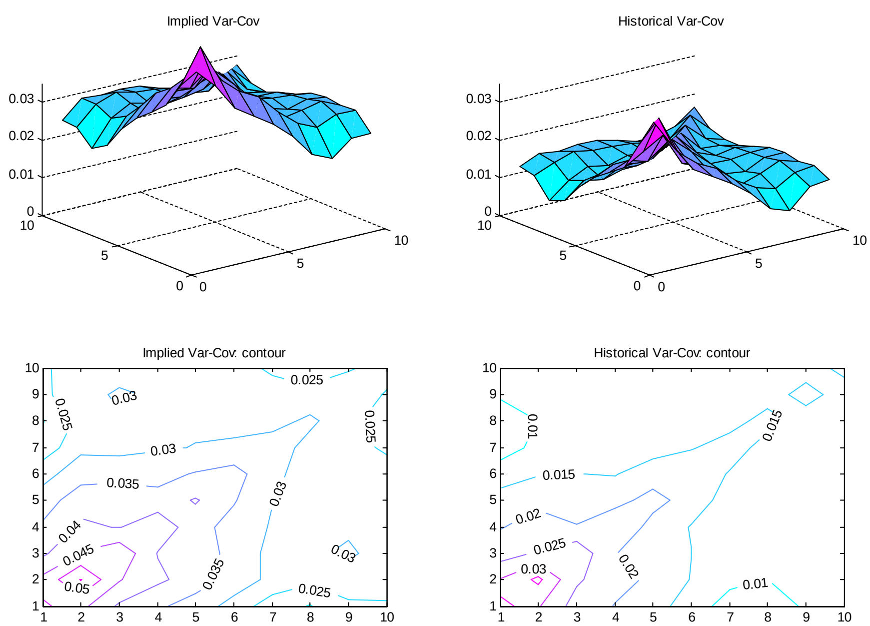


Figure (1) Covariance Matrix: Historical estimation vs. risk-neutral calibration

The solution of the primal SDP problem proposed here, is the following calibrated instantaneous forward volatility structure:

0.2264	0.2369	0.2346	0.2351	0.2398	0.2445	0.2495	0.2625	0.2788	0.3099
0.2350	0.2394	0.2487	0.2565	0.2655	0.2734	0.2831	0.3011	0.3206	0
0.2208	0.233	0.2469	0.2563	0.2662	0.2766	0.2878	0.3055	0	0
0.2091	0.2251	0.2398	0.2492	0.2611	0.2724	0.2826	0	0	0
0.2011	0.218	0.2327	0.2441	0.2570	0.2672	0	0	0	0
0.1927	0.2091	0.2256	0.2379	0.2492	0	0	0	0	0
0.1834	0.2014	0.2183	0.2288	0	0	0	0	0	0
0.1770	0.1954	0.2100	0	0	0	0	0	0	0
0.1717	0.1871	0	0	0	0	0	0	0	0
0.1663	0	0	0	0	0	0	0	0	0

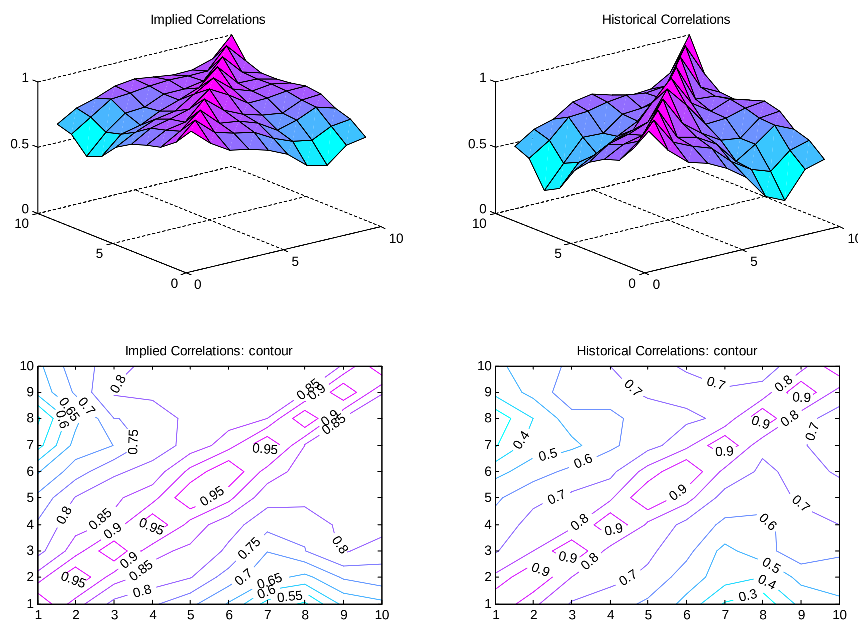


Figure (2) Correlation Matrices: Historical estimation vs. Risk-neutral calibration

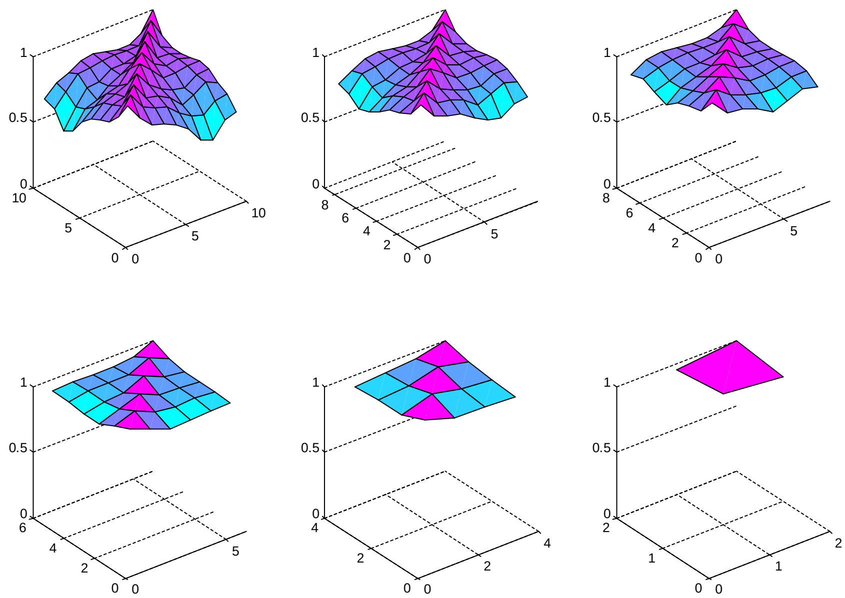


Figure (3) Implied Term Structure in the Correlation Matrix (first row:1y-2y-3y and second row: 4y-6y-8y)

Figures (1) and **(2)** show two interesting properties: First, how the shape of the implied covariance matrix and historical covariance matrix look quite similar. That is an expected consequence of the objective function. Second, the implied covariance matrix has higher value in absolute terms than the historical covariance matrix. This characteristic is usual in many different derivatives markets. Additionally, **Figure (3)** shows us the future dynamics of the variance-covariance matrix where the number of factors decrease with time. It is easy to see the effect of interpolations in the long term part of the volatility structure, basically because of the low liquidity of the 6, 8 and 9 years swaptions.

The reader can compare that the numerical results from the original **Brigo and Morini** (2005) (after rank reduction to enforce positive forward volatilities). Let us briefly summarize here some differences:

- It is clear that a *first* relevant improvement is that the procedure here exposed guarantee a semidefinite positive covariance matrix, something essential to achieve a correct simulation in the full-rank LIBOR additive model.
- The *second* relevant property is that our procedure always guarantees a unique global solution. In some of the other cited methodologies, usually one has multiplicity of possible solutions, some of them with negative volatilities.
- And *finally*, this procedure does not require any rank reduction, and allows us the retrieval of a maximum amount of covariance information from the market.

3.5. **Computing sensitivity and risk-model management.** In this subsection, following **d'Aspremont** (2005), we investigate how the dual optimal solution can be exploited to manage the sensibility of the primal solution to movements in the market. Let us suppose that we have solved both the primal and the dual calibration problems above with market constraints $\sigma^{\star}_{i,n}(T_i)$ and let us denote with X and Y the optimal primal and dual solutions, respectively. Suppose also that the market price constraints in the original calibration problem are modified by a small amount $\Delta \in \mathbb{R}^n$. The new calibration problem becomes the following semidefinite program:

$$\begin{array}{ll} \text{minimize} & \|A - A_{hist}\|_{Fr} \\ \text{subject to} & Tr\left(\Omega_i A\right) = \left(\sigma^{\star}_{i,n}(T_i)\right)^2 T_i + \Delta_i \quad \text{with } i = 1, ..., n \\ & A \succeq 0 \end{array} \tag{3.6}$$

where $A \in \mathbb{S}^N$ is the covariance matrix that we look for, and $A_{hist} \in \mathbb{S}^N$ is the historical covariance matrix, $k \in \mathbb{R}^+, \Omega_{i,n} \in \mathbb{S}^N$, and $\left(\sigma^{\star}_{i,n}(T_i)\right)^2 T_i \in \mathbb{R}^+$ is given by the swaption market as the **Black** (1976) cumulative variance of swaption on $S\left(0, T_i, T_n\right)$. If we note $A(\Delta)$ the primal optimal solution to the revised problem, we get the sensitivity of the solution to a change in market condition as:

$$\frac{\partial A(\Delta)}{\partial \Delta_i} = -\lambda_i \tag{3.7}$$

where λ is the optimal solution to the dual problem (see **Boyd and Vandenberghe** (2004) for details). More specifically, the dual solution for the calibration problem proposed in 3.1.4 is (x 1.0e-007)

0.15590	0.01058	0.06334	0.03071	0.17700	0.11090	0.55298	0.01178	0.07040	0
0.23461	0.01677	0.09225	0.06832	0.16764	0.12237	0.55440	0.04306	0.05752	0
0.26118	0.02001	0.16164	0.07118	0.21080	0.11018	0.24672	0.02131	0	0
0.25988	0.03141	0.16433	0.04552	0.24894	0.14307	0.37939	0	0	0
0.43288	0.03126	0.12174	0.04007	0.32598	0.11148	0	0	0	0
0.44196	0.02360	0.10298	0.05126	0.26754	0	0	0	0	0
0.35816	0.02253	0.14567	0.04492	0	0	0	0	0	0
0.32410	0.02378	0.11929	0	0	0	0	0	0	0
0.36771	0.02268	0	0	0	0	0	0	0	0
0.30537	0	0	0	0	0	0	0	0	0

This result, represented in **Figure (4)**, shows the degree of stability of our primal solution, providing a direct indicator of robustness, but also it illustrates how a semidefinite programming based calibration allows to test various realistic scenarios at a minimum numerical cost and improves on the classical non-convex methods that either had to "bump the market data and recalibrate" the model for every scenario with the risk of jumping from one local optimum to the next, or simulate unrealistic market movements by directly adjusting the covariance matrix.

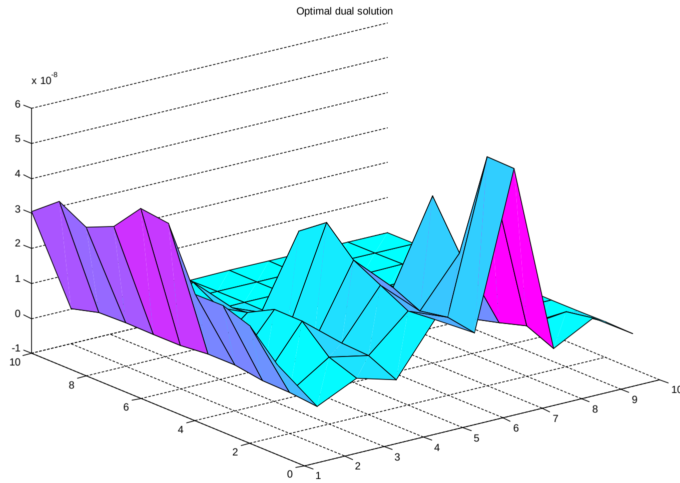


Figure (4) Sensibility of the primal solution (optimal dual solution)

3.5.1. *Robust dynamic calibration.* The previous sections were focused on how to compute the impact of a change in market conditions. Here we propose two different dynamic calibration solutions to dynamically provide a robust matrix for a certain period of market sessions (10 days). Let us assume that the initial problem (3.6) for $t = 0$, therefore, in order to improve dynamically the robustness of our calibration, we solve the following sequence of problems

$$\begin{array}{ll} \text{minimize} & \left\|A^{(t)} - A^{(t-1)}\right\|_{Fr} \\ \text{subject to} & Tr\left(\Omega_i A^{(t)}\right) = \left(\sigma_{i,n}^{\star(t)}(T_i)\right)^2 T_i \quad \text{with } i = 1, ..., n \\ & A^{(t)} \succeq 0 \end{array} \tag{3.8}$$

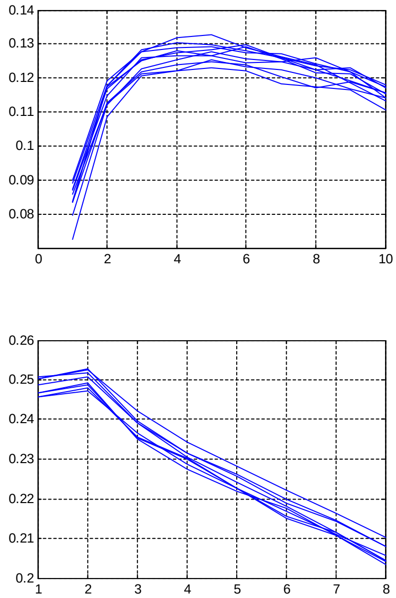
or, alternatively

$$\begin{array}{ll} \text{minimize} & \left\|A^{(t)} - A^{(t-1)}\right\|_{Fr} + \varphi \left\|A^{(t)} - A_{hist}\right\|_{Fr} \\ \text{subject to} & Tr\left(\Omega_i A^{(t)}\right) = \left(\sigma_{i,n}^{\star(t)}(T_i)\right)^2 T_i \qquad \text{with } i = 1, ..., n \\ & A^{(t)} \succeq 0 \end{array} \tag{3.9}$$

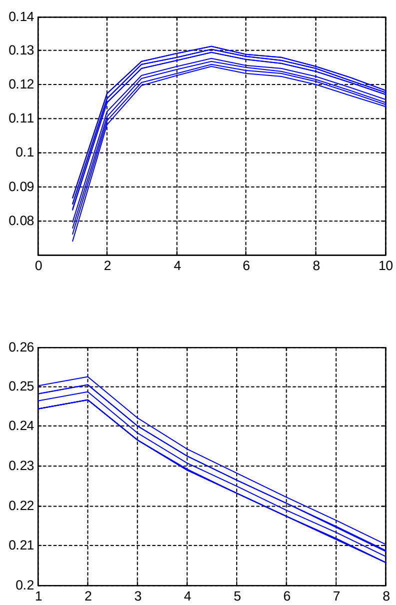
for every $t = 1, ..., n$, where $A^{(t)} \in \mathbb{S}^N$ is the covariance matrix that we look for, and $A^{(t-1)} \in \mathbb{S}^N$ is the previous optimal covariance matrix solved in $t-1$; $i, n \in \mathbb{R}^+$ and $1 \leq i \leq n$, $\Omega_{i,n} \in \mathbb{S}^N$, and $\left(\sigma_{i,n}^{\star(t)}(T_i)\right)^2 T_i \in \mathbb{R}^+$ is given by the swaption market as the **Black** (1976) cumulative variance of swaption with underlying $S(t, T_i, T_n)$, and φ, in problem (3.9), in a regularization constant.

Here, we have three examples of calibration of LIBOR market model proposed. In all cases, we have used the same forward term structure and swaptions volatilities, that the market quoted from March $12th$ to April, 2007 $23th$[2]. In the first example (*Case 1*), we solve the calibration related with the SDP problem (3.6), and the second example (*Case 2*) is related with the proposed robust dynamic calibration as the SDP problem (3.8) for the 10 consecutive mentioned market sessions. Additionally, we solve the third calibration problem, regularized for a $\varphi = 0.1$, proposed as SDP problem in (3.9) as (*Case 3*).

[2]Data courtesy of **Caja Madrid**, Fixed Income Derivatives desk.



Figure(5) (*Case 1*) Dynamic solution from (3.6). Forward volatility structure calibrated using correlation (from initial period (10 factors) to fourth period (7 factors))



Figure(6) (*Case 2*) Dynamic solution from (3.8). Forward volatility structure calibrated using robust dynamic calibration (from initial period (10 factors) to fourth period (7 factors))

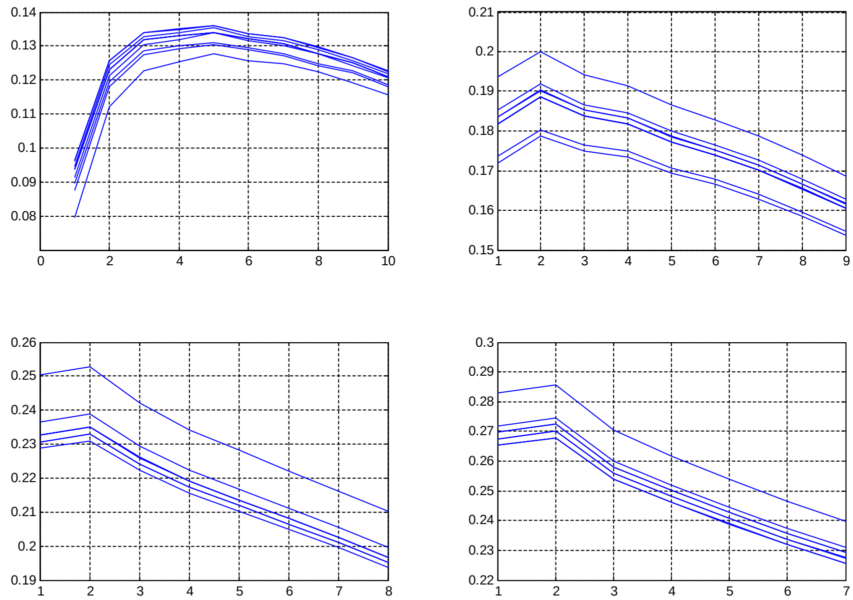


Figure (7) (*Case 3* with $\varphi = 0.1$) Dynamic solution from (3.9). Forward volatility structure calibrated using robust dynamic calibration (from initial period (10 factors) to fourth period (7 factors))

It is not surprising to see in **Figure (5)** and **Figure (6)** how the second case produces more stable or robust results than the first case. Because of that, the usual daily variation in the market prices will not affect the derivatives valuation and it will minimize the variability in the hedging and greeks computation. **Figure (7)** shows us an intermediate situation basically because we have introduced a regularization term in the objective function. This third case is useful if the trader wants to introduce any personal or historical view related with the covariance matrix but without loosing robustness.

4. Second-step: the jump-part calibration

In the previous section, we proposed a new method to obtain the calibration of the diffusion or continuous part, against the at-the-money swaption prices. However, during the whole thesis, we have introduced the jump framework in order to fit the implied forward volatility structure for in-the-money and out-of-the-money swaptions implied volatility. In order to obtain a practical solution to the calibration problem and fit the smile, many authors have resorted to minimizing the in-sample quadratic pricing error (see, for example, **Andersen and Andreasen** (2000), **Bates**(1996)) or **Cont and Tankov** (2004) but always in the equity framework. Here, we extend some of these ideas to the Lévy-calibration problem in the swaptions market.

Basically the idea that we expose here is related with the calibration of a sequence of Lévy-measures under the jump-diffusion framework. As we have shown previously, the calibration of the diffusion part can be made just for the at-the-money swaption prices. It guarantees the compatibility with **Black** at-the-money prices. However, if the trader or practitioner wants to introduce the effect of the smile in the pricing of exotic derivatives, then we need to introduce a sequence of jump measures that adjust every of the maturities of the option, and for the different strikes.

For a predetermined collection of dates $0 < T_0 < T_1 < ... < T_n$ with a fixed accrual period or **tenor** δ, and for any $t \leq T_i \in [0, T^\star]$, let us denote by $L(t, T_i)$ the **forward rate** for the interval from T_i to T_{i+1}, and $PS^\sigma(t, T_\alpha, T_\beta, (x_{i,j}), \upsilon_i, K_{\alpha,\beta})$ and $PS^M(t, T_\alpha, T_\beta, \sigma^\star_{i,n}, K_{\alpha,\beta})$ are respectively the **payer-swaption price** given by our model and payer-swaption price given by the swaption market (according to **Black** (1976) model).

Following the similar idea in **Andersen and Andreasen** (2000) or **Bates**(1996) for equity markets, notice that we can formulate the following sequence of calibration problems, for every $i = 1, ..., n$, we calibrate the Lévy measures minimizing the in-sample quadratic pricing error as

$$(\upsilon_i) = \arg\inf \sum_{h=-m}^{m} \sum_{j=\alpha+1}^{\beta} \left[\omega_{ij}^h \left| PS^\sigma(t, T_\alpha, T_\beta, A_i, \upsilon_i, K^h_{\alpha,\beta}) - PS^M(t, T_\alpha, T_\beta, \sigma^\star_{i,n}, K^h_{\alpha,\beta}) \right|^2 \right] \tag{4.1}$$

where $\alpha, \beta \in \{1, ..., n\}$, and $K^h_{\alpha,\beta} \in \left\{ K^{-m}_{\alpha,\beta}, ..., K^m_{\alpha,\beta} \right\}$ are the different strikes that the swaption market quotes, with $h \in \mathbb{N}$ such that if $h = 0$ then $FS_t\left(K^0_{\alpha,\beta}\right) = 0$ (at-the-money case), or in other words if $K^0_{\alpha,\beta} = FS_{\alpha,\beta}$ then we have introduced at-the-money swaption, and no relevant information is added to the calibration problem because $PS^\sigma(t, T_\alpha, T_\beta, (x_{i,j}), \upsilon_i, FS_{\alpha,\beta}) = PS^M(t, T_\alpha, T_\beta, \sigma^\star_{i,n}, FS_{\alpha,\beta})$.

Let us recall that an **European payer swaption** is an option giving the right (and no obligation) to enter in a IRS at a given future time, the swaption maturity T_α. Usually the swaption maturity coincides with the first reset date of the underlying interest rates swap (IRS). The underlying IRS length $(T_\beta - T_\alpha)$ is called the tenor of the swaption. As we have previously mentioned, it is the market practice to value swaptions with a **Black** (1976) formula.

Precisely let us define the price of a payer swaption at time t as $PS^M(t, T_\alpha, T_\beta, \sigma^\star_{i,j}, K^h_{\alpha,\beta})$ given by the swaption market, as a call-option to get into a swap that begins in T_α and finish in T_β with the swap rate $S(t, T_\alpha, T_\beta)$ (underlying of the option) with strike $K_{\alpha,\beta}$, and where $\sigma^\star_{\alpha,\beta}$ is the **Black** (1976) cumulative variance of swaption on $S(t, T_\alpha, T_\beta)$ for the mentioned strike $K_{\alpha,\beta}$ quoted in the swaption market, such that

$$PS^M(t, T_\alpha, T_\beta, \sigma^\star_{\alpha,\beta}, K_{\alpha,\beta}) = Black(t, T_\alpha, T_\beta, \sigma^\star_{\alpha,\beta}, K_{\alpha,\beta}) \sum_{j=\alpha+1}^{\beta} B(t, T_j)\, \delta_j$$

On the other hand, the usual swaption definition that we have to price it using our model is

$$\begin{aligned}
PS^\sigma(t, T_\alpha, T_\beta, \sigma^\star_{\alpha,\beta}, K_{\alpha,\beta}) &= \mathbb{E}_{\mathbb{P}^\star} \left\{ B(t, T_\alpha) \left(FS_t(K_{\alpha,\beta}) \right)^+ \sum_{j=\alpha+1}^{\beta} B(t, T_j)\, \delta_j \right\} \\
&= \mathbb{E}_{\mathbb{P}_{\alpha,\beta}} \left\{ \sum_{j=\alpha+1}^{\beta} B(t, T_j)\, \delta_j \left((S(t, T_\alpha, T_\beta) - K_{\alpha,\beta}) \right)^+ \right\} \\
&= \mathbb{E}_{\mathbb{P}_{\alpha,\beta}} \left\{ \left((S(t, T_\alpha, T_\beta) - K_{\alpha,\beta}) \right)^+ \right\} \sum_{j=\alpha+1}^{\beta} B(t, T_j)\, \delta_j \\
&= \mathbb{E}_{\mathbb{P}_{\alpha,\beta}} \left\{ \left(\left(\sum_{i=\alpha}^{\beta-1} \omega_i(t)\, L(t, T_i) - K_{\alpha,\beta} \right) \right)^+ \right\} \sum_{j=\alpha+1}^{\beta} B(t, T_j)\, \delta_j
\end{aligned}$$

where

$$\begin{aligned} L(t,T_i) &= \sum_{j=1}^{\eta(t)} \left(L(t\wedge T_j, T_i) - L(T_{j-1}, T_i) \right) \\ &= L(0,T_i) \sum_{j=1}^{\eta(t)} \exp\left\{ \left(\alpha_j(T_j,T_i) - \frac{1}{2}\sigma_j(T_j,T_i)^2 \right)(t\wedge T_j - T_{j-1}) \right. \\ &\quad \left. + \sigma_j(T_j,T_i)\left(W_{t\wedge T_j} - W_{T_{j-1}} \right) \right\} \prod_{l=N_{j-1}(T_{j-1})}^{N_j(t\wedge T_j)} e^{Y_l} \end{aligned}$$

4.1. **An example of calibration under the double exponential jump-diffusion model.** Let us assume that our LIBOR additive process follows a jump-diffusion scheme or more specifically, a **double exponential jump-diffusion** or **Kou** (2003) model, which has two components, a continuous part modeled as **Brownian motion**, and a jump-part with jumps having a **double exponential distribution** and jump times driven by a **Poisson process**, assuming that

$$G_t^{T^\star} := \sum_{j\le\eta(t)} \int_{T_j}^{t\wedge T_{j+1}} \alpha_j(s,T_l)\,ds + \sum_{j\le\eta(t)} \int_{T_j}^{t\wedge T_{j+1}} \sigma_j(s,T_l)\,dW_s + \sum_{j\le\eta(t)} \sum_{l=1}^{N_j(t)} Y_l$$

where α_j under risk neutral measure has an specific form, W_t is the standard Brownian motion, N_j is a Poisson process with rate λ_j and Y_i is a sequence of independent and identically distributed of jumps with double exponential distribution i.e. the common density of Y is given by

$$f_Y(dy) = p\cdot\eta_1 e^{-\eta_1 y} 1_{\{y\ge 0\}} + q\cdot\eta_2 e^{\eta_2 y} 1_{\{y<0\}}$$

where $p,q \ge 0, p+q=1, \lambda_j \ge 0$ for every $j=0,1,...,n$, and $\eta_1 > 1, \eta_2 > 0$. Note that the means of the two exponential distribution are $1/\eta_1$ and $1/\eta_2$ respectively.

It is clear that the **Fourier transform** (or characteristic function) of $G_t^{T^\star}$ admits the (unique) representation given by the **Lévy-Khintchine** Theorem (Theorem 31 in **Colino** (2008)), in this case

$$\begin{aligned} \hat{\mu}_t(z) &= \mathbb{E}\left[e^{\langle iz, G_t\rangle} \right] \\ &= \prod_{j\le\eta(t)} \mathbb{E}\left(\exp\left[iz\left(G_{t\wedge T_{j+1}} - G_{T_j} \right) \right] \right) \\ &= \exp\left[\sum_{j\le\eta(t)} (t\wedge T_{j+1} - T_j)\cdot \Psi_j(z) \right] \end{aligned}$$

where the characteristic exponent $\Psi_j(z)$ equals

$$\Psi_j(z) = i\langle \alpha_j, z\rangle - \tfrac{1}{2}\langle z, A_j z\rangle + \int_{\mathbb{R}^d} \left(e^{i\langle z,y\rangle} - 1 - i\langle z,y\rangle 1_{\{|y|\le 1\}} \right) \upsilon_j(dy) \quad \text{with } z\in\mathbb{R}^d$$

where A_j is a symmetric nonnegative-definite $(n-j+1)\times(n-j+1)$ matrix given as a solution by the first calibration problem, with $j=0,1,...,n$, and α_j under risk-neutral measure follows (see **Glasserman and Kou** (2003) and **Kou and Wang** (2004))

$$\begin{aligned} \alpha_j(t,T_i) &= \sum_{k=\eta(t)}^{i} \frac{\delta\sigma_j(t,T_k)\,\sigma_j(t,T_i)\,L(t-,T_k)}{1+\delta L(t-,T_k)} \\ &\quad + \int_{\mathbb{R}^r} y \left(1 - \prod_{k=\eta(t)}^{i} \frac{1+\delta L(t-,T_k)}{1+\delta L(t-,T_k)(1+y)} \right) \lambda_j \left(p\cdot\eta_1 e^{-\eta_1 y} 1_{\{y\ge0\}} + q\cdot\eta_2 e^{\eta_2 y} 1_{\{y<0\}} \right) dy \end{aligned}$$

and the Lévy measure can be defined as

$$\upsilon_j(dy) = \lambda_j\left(p\cdot\eta_1 e^{-\eta_1 y} 1_{\{y\ge0\}} + q\cdot\eta_2 e^{\eta_2 y} 1_{\{y<0\}} \right) dy$$

where $p,q \geq 0, p+q=1$, $\eta_2 > 0$ and additionally the condition that $\eta_1 > 1$ is imposed to ensured that $L(t-,T_i)$ has finite expectations.

Therefore, the forward LIBOR rate can be described with the following SDE

$$\frac{dL(t,T_i)}{L(t-,T_i)} = \alpha_j(t,T_i)\,dt + \sigma_j(t,T_i)\,dW_t + d\left(\sum_{l=1}^{N_j(t)} e^{Y_l} - 1\right)$$

Let us recall that we interpret the swap rate as a linear combination of forward rates, and in our case forward LIBOR rates, such that the payer swaption $PS^{\sigma}(t,T_\alpha,T_\beta,\sigma^\star_{\alpha,\beta},K_{\alpha,\beta})$ may also be seen as

$$\begin{aligned} PS^{\sigma}(t,T_\alpha,T_\beta,\sigma^\star_{\alpha,\beta},K_{\alpha,\beta}) &= \mathbb{E}_{\mathbb{P}_{\alpha,\beta}}\left\{\sum_{j=\alpha+1}^{\beta} B(t,T_j)\,\delta_j\,(S(t,T_\alpha,T_\beta)-K_{\alpha,\beta})^+\right\} \\ &= \mathbb{E}_{\mathbb{P}_{\alpha,\beta}}\left\{\left(\sum_{j=\alpha}^{\beta-1}\omega_j(t)\,L(t,T_j)-K_{\alpha,\beta}\right)^+\right\}\sum_{j=\alpha+1}^{\beta} B(t,T_j)\,\delta_j \end{aligned}$$

Let us define the following two auxiliary variables

$$\xi_j := \ln \omega_j(0)\,L(0,T_j)$$

and

$$X_t := \ln \sum_{j=1}^{\eta(t)} \exp\left\{\left(\alpha_j(s,T_i)+\frac{1}{2}\sigma_j(s,T_i)^2\right)(t\wedge T_j - T_j) + \sigma_j(s,T_i)\left(W_{t\wedge T_j}-W_{T_j}\right)\right\}\prod_{l=N_j(T_{j-1})}^{N_j(t\wedge T_j)} e^{Y_l}$$

then we have

$$S(t,T_\alpha,T_\beta) = \sum_{j=\alpha}^{\beta-1} e^{\xi_j+X_j}$$

will allows us to redefine the value of a payer swaption in t as

$$PS^{\sigma}(t,T_\alpha,T_\beta,\xi_i,K_{\alpha,\beta}) = \mathbb{E}_{\mathbb{P}_{\alpha,\beta}}\left\{\left(\sum_{j=\alpha}^{\beta-1} e^{\xi_j+X_j}-K_{\alpha,\beta}\right)^+\right\}\sum_{j=\alpha+1}^{\beta} B(t,T_j)\,\delta_j$$

We consider the modified payoff $\Phi(x,K_{\alpha,\beta}) = \left(\sum_{j=\alpha}^{\beta-1} e^{x_j} - K_{\alpha,\beta}\right)^+$ and, for the sake of clarity, setting $\xi = (\xi_j)_{\alpha\leq j\leq\beta}$, $X = (X_j)_{\alpha\leq j\leq\beta}$ and $K_{\alpha,\beta}$ for T_α, T_β given inside the tenor structure $\{T_1,...,T_n\} \in [0,T^\star]$,

$$\begin{aligned} PS^{\sigma}(\xi,K_{\alpha,\beta}) &= \mathbb{E}_{\mathbb{P}_{\alpha,\beta}}\{\Phi(\xi+X,K_{\alpha,\beta})\}\sum_{j=\alpha+1}^{\beta} B(t,T_j)\,\delta_j \\ &= \sum_{j=\alpha+1}^{\beta} B(t,T_j)\,\delta_j \cdot \int_{\mathbb{R}} \Phi(\xi+X,K_{\alpha,\beta})\,\phi(x)\,dx \end{aligned}$$

where ϕ is the 'unknown' density of X (however, we know the **Fourier transform** or **Lévy-Khitchine** characteristic function $\hat{\mu}_X(z) = \prod_{\alpha+1\leq j\leq\beta} \hat{\mu}_{X_j}(z)$).

The expectation under $\mathbb{P}_{\alpha,\beta}$ can be computed inverting the Fourier transform, according with **Raible** (2000), in the following way

$$\begin{aligned} PS^{\sigma}(\xi, K_{\alpha,\beta}) &= \frac{1}{2\pi i}\int_{R-i\infty}^{R+i\infty} e^{\xi z}\hat{\mu}_X(z)\,dz \\ &= \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{\xi_i(R+iu)}\hat{\mu}_X(R+iu)\,du \\ &= \frac{e^{\xi_i R}}{2\pi}\lim_{N,M\to\infty}\int_{-M}^{N} e^{\xi_i iu}\hat{\mu}_X(R+iu)\,du \\ &= \frac{e^{\xi_i R}}{2\pi}\lim_{N,M\to\infty}\int_{-M}^{N} e^{\xi_i iu}\prod_{\alpha+1\le j\le\beta}\hat{\mu}_{X_j}(R+iu)\,du \end{aligned}$$

that can be solved numerically, after some additional transformations, using **FFT**[3].

Fourier transforms have been widely used in valuing financial derivatives. For example, **Carr and Madan** (1998) propose Fourier transforms with respect to the log-strikes prices; **Geman and Yor** (1993), **Fu, Madan and Wang** (1999) use Fourier transform to price Asian options in the Black Scholes setting; Fourier transforms for the double-barrier and lookback options under the CEV model are given in **Davydov and Linetsky** (2001); **Petrella and Kou** (2004) use a recursion and Fourier transforms to price discretely monitored barrier and lookback options. **Raible** (2000) proposed a method for the evaluation of European stock options in a Lévy setting by using bilateral (or, two-sided) Fourier transforms. This approach is based on the observation that the pricing formula for European options can be represented as a convolution. Then one can use the fact that the bilateral Laplace transform of a convolution is the product of the bilateral Laplace transforms of the factors (the latter transforms are usually known explicitly).

Therefore, the second step calibration problem can be formulated in the following way

$$\begin{aligned} (\lambda_i, \eta_1, \eta_2, p) = \quad & \arg\inf \textstyle\sum_{h=-m}^{m}\sum_{j=i+1}^{n}\left[\omega_{ij}^{h}\left\| PS^{\sigma}(t, T_\alpha, T_\beta, A_i, \upsilon_i, K_{\alpha,\beta}^{h}) - PS^{M}(t, T_\alpha, T_\beta, \sigma_{i,n}^{\star}, K_{\alpha,\beta}^{h})\right\|\right] \\ s.t. \quad & p + q = 1 \\ & p, q \ge 0 \\ & \eta_1 > 1, \eta_2 > 0 \end{aligned}$$

where the weights ω_{ij}^{h} are positive and sum to one and they reflect the pricing error tolerance for the swaption with maturity T_α and swap ends at T_β with strike $K_{\alpha,\beta}^{h}$, $PS^{M}(t, T_\alpha, T_\beta, \sigma_{i,n}^{\star}, K_{\alpha,\beta}^{h})$ is directly given by the market price of a payer swaption, $K_{\alpha,\beta}^{h} \in \left\{K_{\alpha,\beta}^{-m}, ..., K_{\alpha,\beta}^{m}\right\}$ are the different strikes that the swaption market quotes, with $h \in \mathbb{N}$ such that if $h = 0$ then $FS_t\left(K_{\alpha,\beta}^{0}\right) = 0$ (at-the-money case).

5. Numerical Performance

The reliability of the two steps swaptions calibration depends mainly on the accuracy of the underlying approximation (3.1) and (3.2) in the first calibration step. This formula has already been tested, for instance by **Brigo and Mercurio** (2001) and **Jackel and Rebonato** (2000). Here we extend similar tests, based on **Monte Carlo simulation** of the LMM dynamics calibrated using semidefinite programming, and we compare, for payer and receivers European swaptions at-the-money, the estimated prices using the Monte Carlo versus the theoretical **Black**-76 swaption value.

As we have mentioned before, let us define a predetermined collection of dates $0 < T_0 < T_1 < ... < T_n$ with a fixed accrual period or **tenor** δ, and for any $t \le T_i \in [0, T^{\star}]$, and by $L(t, T_i)$ we define the **forward**

[3]See **Carr and Madan** (1998)

rate for the interval from T_i to T_{i+1}, and $PS^{\sigma}(t, T_\alpha, T_\beta, (x_{i,j}), \upsilon_i, K_{\alpha,\beta})$ and $PS^{M}(t, T_\alpha, T_\beta, \sigma^{\star}_{i,n}, K_{\alpha,\beta})$ are respectively the **payer-swaption price** given by our model and payer-swaption price given by the swaption market (according to **Black** (1976) model). Let us, first of all, discretize the continuous dynamics (see Section 3.1.3 in **Colino** (2008)). Taking logs in order to get the stronger convergence of Milstein scheme, one obtains for every $\alpha \leq j \leq \beta \leq n$ with

$$\begin{aligned} \log L(t+\Delta t, T_j) &= \log L(t, T_j) + \sigma_j(t) \sum_{i=\alpha+1}^{j} \frac{\rho_{i,j}\sigma_i(t)\delta_i L(t, T_i)}{1+\delta_i L(t, T_i)} \Delta t \\ &\quad - \frac{\sigma_j(t)^2}{2} \Delta t + \sigma_j(t)\left(W(t+\Delta t) - W(t)\right) \end{aligned} \tag{5.1}$$

where the instantaneous volatility $\sigma_i(t)$ and correlations $\rho_{i,j}(t)$ between the n-factors has been estimated solving the SDP problem (see Proposition 4) and they are piecewise stationary. The base scenario we use for most of our results sets $\delta = 1$ year, $n = 10$ years corresponding to a ten-years term structure of annual rates, and generating 10 points of data per year. Therefore α will take values between 1 and 5, and β between 1 and 10 (with $\alpha < \beta \leq 10$) where T_α indicates the expiry date of the option and T_β is the maturity of the swaption. We generate 10.000 simulations under the terminal measure in order to reach a two-side 98% window, according to the standard error of Monte Carlo method[4]. All the swaptions priced here using this methodology, are at-the-money swaptions, and the data set used in this simulation are real market data (quoted March 12th, 2007)[5].

Figure (8) compares the prices between both methodologies and estimated relative errors (in percentage) between the prices generated using Monte Carlo with a LMM and **Black**-76. In the x-axis, the swap maturity in years is represented as the market used to quoted swaptions, without the option term $(\beta - a)$, and in the z-axis, the option expiry (payer in the left and receiver in the right side) in years.

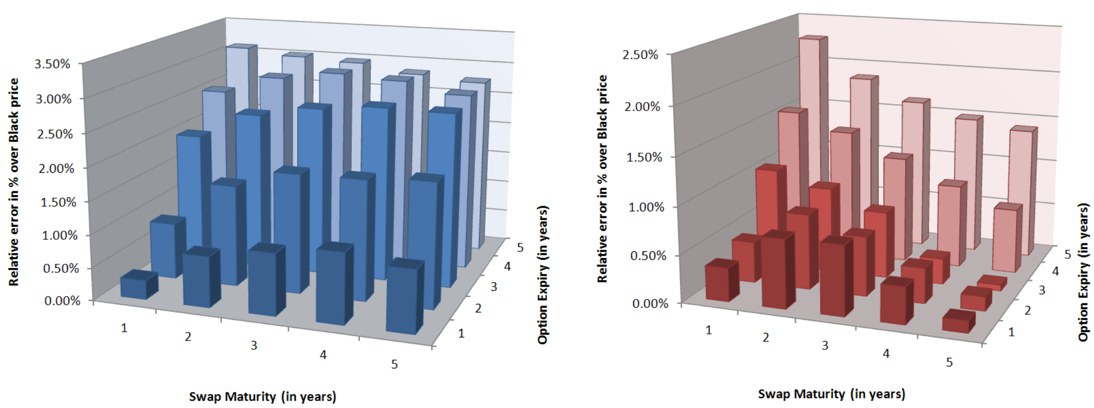


Figure (**8**): Relative error (in percentage) in the **payer-swaption** (left) and **receiver-swaption** (right) price using LMM-valuation (Monte Carlo) calibrated using SDP versus **Black**-76.

Not surprisingly, considering the structure of the SDP problem, **Figure (8)** shows that all swaptions seem to fit reasonably well, except for the longest underlying, in line with the **d'Aspremont** (2003) or **Longstaff et al.** (2000). With the exception of the payer-swaption with longest option expiry, in the rest of the cases, the price approximation appears good enough, and it confirms the accuracy of a full-rank calibration using SDP for at-the-money swaptions market (with 10 factors). It helps us to confirms the

[4]See **Glasserman and Zhao** (2000) for a complete description and implementation methodology.

[5]Market volatilities kindly provided by Caja Madrid Capital Markets (Fixed Income Derivatives Desk) using reliable market sources. All errors are my own.

advantages of this methodology to price and hedge more sophisticated exotic swaptions, that usually have a great price dependence of the covariance structure.

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