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NEW STOCHASTIC PROCESSES TO MODEL INTEREST RATES: LIBOR ADDITIVE PROCESSES

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Keywords: Jump processes, Processes with independent increments.

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JESÚS PÉREZ COLINO

(WORKING PAPER)

Abstract. In this paper, a new kind of additive process is proposed. Our main goal is to define, characterize and prove the existence of the LIBOR additive process as a new stochastic process. This process will be defined as a piecewise stationary process with independent increments, continuous in probability but with discontinuous trajectories, and having "*càdlàg*" sample paths. The proposed process is specifically designed to derive interest-rates modelling because it allows us to introduce a jump-term structure as an increasing sequence of Lévy measures. In this paper we characterize this process as a Markovian process with an infinitely divisible, selfsimilar, stable and self-decomposable distribution. Also, we prove that the Lévy-Khintchine characteristic function and Lévy-Itô decomposition apply to this process. Additionally we develop a basic framework for density transformations. Finally, we show some examples of LIBOR additive processes.

1. Preliminaries

1.1. **Introduction.** The story of modelling financial markets with stochastic processes began in 1900 with the study of **Louis Bachelier** (1900). He modelled stocks as a *Brownian motion* with drift. However, the model had many imperfections, including, for example, negative stock prices. It was 65 years before another, more appropriate, model was suggested by **Samuelson** (1965): *geometric Brownian motion*. Eight years later **Black and Scholes** (1973) and **Merton** (1973) demonstrated how to price *European options* based on the geometric Brownian motion. This stock-price model is now called *Black-Scholes model*, for which **Scholes** and **Merton** received the *Nobel Prize for Economics* in 1997.

However, it has become clear that this option-pricing model is inconsistent with option data. **Implied volatility models** can do better, but fundamentally, these consist of the wrong building blocks. To improve on the performance of the Black-Scholes model, **Lévy models** were proposed in the late 1980s and in the early 1990s, when there was some need for models taking into account of different stylized features of the market.

On the other hand, traditionally, **interest rates models** for Treasury bonds or Corporate bonds, in the literature, are mainly models based on Brownian motion although it is known that real-life financial markets provide a different structural and statistical behavior than that implied by these models. Some of these interest rates models have been created thinking in **Black-Scholes** framework, but they found a great number of inconveniences. Also in this field, *Lévy processes* are proposed as an appropriate tool to increase the accuracy of interest rates models. However, the nature of random sources in bond markets is different from equity markets:

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(1) Modelling a term structure is completely different from modelling a simple equity. There exists a collection of 'correlated' bonds that generate a 'multifactorial' term structure. In other words, we have a stochastic surface that is strongly linked to some no arbitrage conditions, completely different to the no-arbitrage conditions that appear in equity markets.
(2) Derivative markets for interest rates (*caps/floors* and *swaptions* markets) quote differently *'at-the-money'* volatilities according to the different maturities inside of the term structure (forward volatilities). While Brownian or Lévy processes offer nice features in terms of analytical tractability, the constraints of independence and stationarity of their increments prove to be very restrictive for this market.
(3) Also, these derivative markets quote, for each maturity, a different volatility for every strike (*'in-the-money'* or *'out-of-the-money'* options). Lévy models allow for calibration to implied volatility patterns for a single maturity but fail to reproduce option prices correctly over a range of different maturities. In addition, the existence of this term structure of 'volatility smiles' have a huge impact in order to price not only plain vanilla interest rate options but also exotic options.

In this paper, we present a stochastic process that is specifically designed to represent the random sources that appear in this market, and we develop some basic tools that any interest rate model needs in order to reproduce the risk neutral dynamics. This paper is organized as follows.

- In *Section 2* we introduce the stochastic processes of interest together with their main distributional *properties* (*infinite divisibility*, *self-similarity*, *stability* and *self-decomposability*).
- *Section 3* is mainly devoted to the *existence* of this new process, mainly under the framework for Markovian processes.
- In *Section 4* introduces the *characterization* of these processes, using or adapting the well-known *Lévy-Khintchine formula* to this framework.
- *Section 5* is dedicated to show the Libor additive *infinitesimal generator*.
- In *Section 6* we adapt the *Lévy-Itô decomposition*, and we deduce some interesting applications that appear in *Section 7*.
- Probably, *Section 8* is the most important section in order to build models using this process. Here, we expose the main tool that any financial engineering has in order to find the risk-neutral measures: *the change of measure*.
- Finally *Section 9* is devoted to expose different examples of non-homogeneous processes that can be used as a LIBOR additive process.

1.2. **Some frequently used notation and terminology.** A *probability space* $(\Omega, \mathcal{F}, \mathbb{P})$ is a triplet of a set Ω, a family $\mathcal{F}$ of subsets of Ω, and a mapping $\mathbb{P}$ from $\mathcal{F}$ into $\mathbb{R}^+$ satisfying the following conditions:

(1) $\Omega \in \mathcal{F}, \emptyset \in \mathcal{F}$ ($\emptyset$ is the empty set)

(2) If $A_n \in \mathcal{F}$ for $n = 1, 2, ...$, then $\bigcup_{n=1}^{\infty} A_n$ and $\bigcap_{n=1}^{\infty} A_n$ are in $\mathcal{F}$

(3) If $A \in \mathcal{F}$ then $A^c \in \mathcal{F}$ (A^c is $\Omega \setminus A$, the complement of A)

(4) $0 \leq \mathbb{P}[A] \leq 1$, $\mathbb{P}[\Omega] = 1$ and $\mathbb{P}[\emptyset] = 0$

(5) If $A_n \in \mathcal{F}$ for $n = 1, 2, ...$ and they are disjoint (that is $A_n \cap A_m = \emptyset$ for $n \neq m$) then

$$\mathbb{P}\left[\bigcup_{n=1}^{\infty} A_n\right] = \sum_{n=1}^{\infty} \mathbb{P}[A_n]$$

In terminology of measure theory, a probability space is a measure space with total measure 1. In general, if $\mathcal{F}$ is a family of subsets of Ω satisfying (1), (2), and (3) then $\mathcal{F}$ is called a σ-algebra on Ω.

The pair $(\Omega, \mathcal{F})$ is called a measurable space. Very often in this work, we have $\mathcal{F} = \mathcal{F}_\infty = \vee_{t\geq 0}\mathcal{F}_t$ where $(\mathcal{F}_t)_{t\geq 0}$ is a non-decreasing and right-continuous family of σ-algebras (in other words, $\mathcal{F}_s \subset \mathcal{F}_t$ for $s \leq t$, and $\mathcal{F}_t = \cap_{s>t}\mathcal{F}_s$). In such a situation a filtered probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is called a *stochastic basis* . During the whole work we denote by $\mathcal{B}\left(\mathbb{R}^d\right)$ the collection of all Borel subsets of $\mathbb{R}^d$, called the *Borel σ-algebra*. It is the σ-algebra generated by the open sets in $\mathbb{R}^d$ (that is, the smallest σ-algebra that contains all open sets in $\mathbb{R}^d$). A real-valued function $f(x)$ on $\mathbb{R}^d$ is called *measurable* if it is $\mathcal{B}\left(\mathbb{R}^d\right)$ measurable.

A mapping X from Ω into $\mathbb{R}^d$ is an $\mathbb{R}^d$-valued *random variable* (or *random variable on* $\mathbb{R}^d$) if it is $\mathcal{F}$-measurable, that is $\{\omega : X(\omega) \in B\}$ is in $\mathcal{F}$ for each $B \in \mathcal{B}\left(\mathbb{R}^d\right)$. The *distribution* of an $\mathbb{R}^d$-valued random variable X is denoted by $\mu = \mathbb{P}_X$ or $\mathcal{L}(X)$. Furthermore, $\hat{\mu}(z)$ is the *characteristic function* of a distribution μ defined as

$$\hat{\mu}(z) = \int_{\mathbb{R}^d} \exp\{i\langle z, x\rangle\}\mu(dx) \quad , \ z \in \mathbb{R}^d$$

and $\psi_\mu(z)$ is the cumulant of μ, that is, the continuous function with $\psi_\mu(0) = 0$ such that $\hat{\mu}(z) = \exp\left(\psi_\mu(z)\right)$. The *characteristic function of the distribution* $\mathbb{P}_X$ of a random variable X on $\mathbb{R}^d$ is denoted by $\hat{\mathbb{P}}_X(z)$ and given as

$$\begin{aligned}\hat{\mathbb{P}}_X(z) &= \int_{\mathbb{R}^d} \exp\{i\langle z, x\rangle\}\mathbb{P}_X(dx) \\ &= \mathbb{E}\left[\exp\{i\langle z, X\rangle\}\right], \quad \text{with } z \in \mathbb{R}^d\end{aligned}$$

A sequence of *probability measures* $\mu_n, n = 1, 2, \ldots$ *converges to a probability measure* μ, written as

$$\mu_n \to \mu \quad \text{as } n \to \infty,$$

if for every bounded continuous function f

$$\int_{\mathbb{R}^d} f(x)\mu_n(dx) \to \int_{\mathbb{R}^d} f(x)\mu(dx) \quad \text{as } n \to \infty$$

When μ and μ_n are bounded measures, the convergence $\mu_n \to \mu$ is defined in the same way. When $\{\mu_t\}$ are probability measures, we say that

$$\mu_s \to \mu_t \quad \text{as } s \to t$$

if for every bounded continuous function $f(x)$

$$\int_{\mathbb{R}^d} f(x)\mu_s(dx) \to \int_{\mathbb{R}^d} f(x)\mu_t(dx) \quad \text{as } s \to t.$$

This is equivalent to saying that $\mu_s \to \mu_t$ for every sequence s that tends to t.

We say that B is a μ-continuity set if the boundary of B has μ-measure 0. The convergence $\mu_n \to \mu$ is equivalent to the condition that $\mu_n(B) \to \mu(B)$ for every μ-continuity set $B \in \mathcal{B}\left(\mathbb{R}^d\right)$.

The *convolution* μ of two distributions μ_1 and μ_2 on $\mathbb{R}^d$ denoted by $\mu = \mu_1 * \mu_2$ is a distribution defined by

$$\mu(B) = \int\int_{\mathbb{R}^d\times\mathbb{R}^d} 1_B(x+y)\mu_1(dx)\mu_2(dy)$$

The *convolution operation* is *commutative* and *associative*. If X_1 and X_2 are independent random variables on $\mathbb{R}^d$ with distributions μ_1 and μ_2, respectively, then $X_1 + X_2$ has distribution $\mu_1 * \mu_2$. The *n-fold convolution* of μ is denoted by μ^{n*}. A probability measure μ on $\mathbb{R}^d$ is *infinitely divisible* if, for any positive integer n, there is a probability measure $\mu_{(n)}$ on $\mathbb{R}^d$ such that $\mu = \mu_{(n)}^{n*}$.

A family $\{X_t : t \geq 0\}$ of random variables on $\mathbb{R}^d$ with parameter $t \in [0, \infty)$ defined on a common probability space is called a *stochastic process*. For any fixed $0 \leq t_1 < t_2 < ... < t_n$,

$$\mathbb{P}\left[X\left(t_1\right) \in B_1, ..., X\left(t_n\right) \in B_n\right]$$

determines a probability measure on $\mathcal{B}\left(\left(\mathbb{R}^d\right)^n\right)$. The family of probability measures over all choices of n and $t_1, t_2, ..., t_n$ is called the *system of finite dimensional distributions* of $\{X_t\}$. Given two stochastic processes $\{X_t, t \geq 0\}$ and $\{Y_t, t \geq 0\}$, $\{X_t\} \stackrel{d}{=} \{Y_t\}$ means that X and Y are *identical in law* or have a common system of *finite-dimensional marginals*. A stochastic process $\{Y_t\}$ is called a *modification* of a stochastic process $\{X_t\}$ if $\mathbb{P}\left[X_t = Y_t\right] = 1$ for $t \in [0, \infty)$.

We say that X is a *semimartingale* if it is an *adapted* process ($\mathcal{F}_t$-measurable for every $t \in [0, \infty)$) such that $X_t = X_0 + M_t + V_t$ for each $t \geq 0$, where X_0 is finite-valued and $\mathcal{F}_0$-measurable, $\{M_t, t \geq 0\}$ is a *local martingale* (an adapted process, integrable for each $t \geq 0$ and $M_s = \mathbb{E}\left(M_t / \mathcal{F}_s\right)$ for $s \leq t$) and where $\{V_t, t \geq 0\}$ is an adapted process of *finite variation*.

Sometimes it will be necessary to work on the path space of *"càdlàg"* (i.e. right-continuous with left limits) semimartingales. For $I = [0, T^\star] \subseteq [0, \infty)$ with $T^\star > 0$, we denote by $\mathbb{D} = \mathbb{D}\left(\mathbb{R}^d, I\right)$ the *Skorohod space* of all *càdlàg functions* $\alpha : I \to \mathbb{R}^d$. For $I = \mathbb{R}_+ = [0, \infty)$ we denote by $\mathcal{D}_t^0\left(\mathbb{R}^d\right)$ the σ-field generated by all mappings $\alpha \mapsto \alpha(s)$ for $s \leq t$, and $\mathcal{D}_t\left(\mathbb{R}^d\right) = \cap_{s>t} \mathcal{D}_s^0\left(\mathbb{R}^d\right)$. If X is a semimartingale on $(\Omega, \mathcal{F}, \mathbb{P})$ such that $X \in \mathbb{D}\left(\mathbb{R}^d, I\right)$ we denote by $X(t)$ the value of X at time t, and by $X(t-)$ its left-hand limit at time t (with $X(0-) = X(0)$ by convention) and $\triangle X(t) = X(t) - X(t-)$.

We will use, during the whole work, the definitions as outlined in **Sato** (1999) of *infinite divisibility of processes*, *self-decomposability* (of distributions), *self-similarity and stability of processes*, *Lévy processes*, *additive processes*, *increasing or decreasing processes*. We use ':=' to mean '*is defined to be equal to*'. In particular we set $\mathbb{R}_+ := [0, \infty)$, $\mathbb{R}_+^d := [0, \infty)^d$, $\mathbb{Z}_+ := \mathbb{Z} \cap \mathbb{R}_+$, $\mathbb{Q}_+ := \mathbb{Q} \cap \mathbb{R}_+$. Elements of $\mathbb{R}^d$ are column vectors. If f and g are real numbers or real-valued functions, we define $f \vee g := \max(f, g)$, $f \wedge g := \min(f, g)$, $f^+ := f \vee 0$, $f^- := (-f) \vee 0$. In particular, we have $f = f^+ - f^-$ and $|f| = f^+ + f^-$. For $x = (x_j)_{1 \leq j \leq d}$ and $y = (y_j)_{1 \leq j \leq d}$ in $\mathbb{C}^d$ we write $\langle x, y \rangle = \sum_{j=1}^d x_j y_j$.

1.3. Basics about Lévy Processes.

The aim of this preliminary section is to give a brief introduction to the theory of Lévy processes. Inclusion of this material is justified not only because Brownian motion or Poisson processes are Lévy processes, but also because an additive process will be defined later as a nonhomogeneous Lévy process. Additionally, Lévy processes also provide one of the most important examples of Markov processes and semimartingales.

This preliminary section is an attempt to gather some basic and typical results to describe several main directions of this paper. It is not intended to give a systematic presentation of the most important results or to explain how to prove them; for these purposes one would need many more pages. A more comprehensive picture of the present knowledge can be obtained from the two books **Bertoin** (1996) and **Sato** (1999).

Definition 1. *An $\mathbb{R}^d$-valued* ***stochastic process*** *$X = \{X_t : t \geq 0\}$ is a family of $\mathbb{R}^d$-valued random variables $X_t(\omega)$ with parameter $t \in [0, \infty)$ defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$.*

Definition 2. *An $\mathbb{R}^d$-valued stochastic process $X = \{X_t : t \geq 0\}$ is called a* ***Lévy process in law*** *on $\mathbb{R}^d$ or d-dimensional Lévy process, if the following four properties are satisfied:*

- (L1) *X starts at the origin, $X_0 = 0$ a.s. (almost surely)*
- (L2) *X has* ***independent increments****, that is, for any $n \geq 1$ and $0 \leq t_0 < t_1 < ... < t_n$, the random variables $X_{t_0}, X_{t_1} - X_{t_0}, X_{t_2} - X_{t_1}, ..., X_{t_n} - X_{t_{n-1}}$ are independent.*
- (L3) *X is* ***stochastically continuous****, that is, for any $\varepsilon > 0$, $P\left[\left|X_{t+s} - X_t\right| > \varepsilon\right] \to 0$ as $s \to 0$.*

(L4) *X is time **homogeneous** (or stationary), that is, the distribution of $\{X_{t+s} - X_t : t \geq 0\}$ does not depend on s*

Notice that if X is a stochastically continuous process with independent and stationary increments (a Lévy process in law), there exists a **càdlàg** version of X with the same properties called **Lévy process** (cf. **He, Wang and Yan** (1992), Theorem 2.68). Therefore a **Lévy process** can be defined as a d-dimensional stochastic process starting in 0 with càdlàg paths and independent and stationary increments under $\mathbb{P}$ (if there is no ambiguity about the measure involved) (cf. **Bertoin** (1996)).

Proposition 1. *If X is a **Lévy process**, then $X(t)$ is **infinitely divisible** for each $t \geq 0$.*

Proof. To see this, let $t_i = it/m$ with $i = 0, 1, ..., m$ and some $t \geq 0$. Let $\mu = \mathbb{P}_{X_t}$ and $\mu_m = \mathbb{P}^{(m)}_{X_{t_i} - X_{t_{i-1}}}$ which is independent of i by temporal homogeneity. Then $\mu = \mu^{m*}_{(m)}$, since

$$X_t = \left(X^{(m)}_{t_1} - X^{(m)}_{t_0}\right) + ... + \left(X^{(m)}_{t_m} - X^{(m)}_{t_{m-1}}\right)$$

is a sum of m independent identically distributed random variables. □

Recall that the **characteristic function** of a distribution μ on $\mathbb{R}^d$ is defined by $\hat{\mu}(z) = \int_{\mathbb{R}^d} e^{i\langle z,x\rangle} \mu(dx)$, $z \in \mathbb{R}^d$. Also remember that here $\langle z, x\rangle = \sum_{j=1}^d z_j x_j$, the Euclidean inner product of $z = (z_j)$ and $x = (x_j)$ in $\mathbb{R}^d$. Thus $|x| = \langle x, x\rangle^{1/2}$.

Proposition 2. *If X is a **Lévy process**, then $\hat{\mu}_{X_t}(z) = \exp\{t \cdot \Psi(z)\}$ for each $z \in \mathbb{R}^d$, $t \geq 0$ and where $\Psi(z)$ is the **Lévy exponent**.*

Proof. For the sake of clarity, define $\hat{\mu}_z(t) = \hat{\mu}_{X_t}(z)$. Then for all $s \geq 0$

$$\begin{aligned}
\hat{\mu}_z(t+s) &= \mathbb{E}\left(e^{i\langle z, X_{t+s}\rangle}\right) \\
&= \mathbb{E}\left(e^{i\langle z, X_{t+s} - X_s\rangle} e^{i\langle z, X_s\rangle}\right) \\
&= \mathbb{E}\left(e^{i\langle z, X_{t+s} - X_s\rangle}\right) \mathbb{E}\left(e^{i\langle z, X_s\rangle}\right) \\
&= \hat{\mu}_z(t)\, \hat{\mu}_z(s)
\end{aligned}$$

Notice that using $(L1)$ in Definition 2 (about Lévy processes) we have $\hat{\mu}_z(0) = 1$ and by $(L3)$ the map $t \to \hat{\mu}_z(t)$ is continuous.

However the unique solution for $\hat{\mu}_z(t+s) = \hat{\mu}_z(t)\,\hat{\mu}_z(s)$ and $\hat{\mu}_z(0) = 1$ is given by $\hat{\mu}_z(t) = \exp\{t \cdot \Psi(z)\}$ where $\Psi : \mathbb{R}^d \to \mathbb{C}$ (see e.g. **Bingham et al.** (1987) pp.4-6). Notice that $X(1)$ is infinitely divisible and Ψ is the Lévy exponent. The result follows. □

The following three theorems are fundamental. For their proofs see the monographs **Doob** (1953), **Loève** (1955), **Breiman** (1968), **Gihman and Skorohod** (1975), **Kallenberg** (1997) or **Sato** (1999).

Theorem 1. *If μ is an **infinitely divisible** distribution on $\mathbb{R}^d$, then there exists, uniquely in law, a **Lévy process in law** $\{X_t\}$ such that $\mathcal{L}(X_1) = \mu$.*

Theorem 2. *If $\{X_t\}$ is a **Lévy process in law** on $\mathbb{R}^d$, then there is a **Lévy process** $\{X'_t\}$ on $\mathbb{R}^d$ such that $\{X'_t\}$ is a modification of $\{X_t\}$, that is $X'_t = X_t$ a.s. for every $t \geq 0$.*

Theorem 3 (Lévy-Khintchine representation)**.** *If μ is **infinitely divisible**, then*

$$\hat{\mu}(z) = \exp\left[i\langle\gamma, z\rangle - \frac{1}{2}\langle z, Az\rangle + \int_{\mathbb{R}^d}\left(e^{i\langle z,x\rangle} - 1 - i\langle z, x\rangle 1_{\{|x|\le 1\}}(x)\right)\upsilon(dx)\right] \tag{1.1}$$

where A is a symmetric nonnegative-definite $d \times d$ matrix, $\gamma \in \mathbb{R}^d$, υ is a measure on $\mathbb{R}^d$ satisfying $\upsilon(\{0\}) = 0$ and $\int_{\mathbb{R}^d}\left(|x|^2 \wedge 1\right)\upsilon(dx) < \infty$.

The representation (1.1) *by A, γ and υ is unique. Conversely, for any choice of A, γ and υ satisfying the conditions above, there exists an infinitely divisible distribution μ having characteristic function* (1.1)*.*

It follows that the **Lévy process** $\{X_t\}$ corresponding to μ, by Proposition 1 and Theorem 3, has characteristic function

$$\mathbb{E}\left[e^{i\langle z,X_t\rangle}\right] = \exp\left[t\cdot\left(i\langle\gamma, z\rangle - \frac{1}{2}\langle z, Az\rangle + \int_{\mathbb{R}^d}\left(e^{i\langle z,x\rangle} - 1 - i\langle z, x\rangle 1_{\{|x|\le 1\}}(x)\right)\upsilon(dx)\right)\right]$$

Notice that we can define $\hat{\mu}_t(z) = \exp\{t\cdot\Psi(z)\}$ for each $t \ge 0$, where $\Psi(z)$ is the **Lévy exponent**. We call (γ, A, υ) the generating triplet, A the **Gaussian covariance matrix**, and υ the **Lévy measure** of μ. However, γ does not have any intrinsic meaning, since its value depends on the choice of $i\langle z, x\rangle 1_{\{|x|\le 1\}}(x)$ of the integrand in (1.1) as a term to make it υ-integrable.

If $\upsilon = 0$ then μ is **Gaussian**. If $A = 0$ then we say that μ is **purely non-Gaussian**. If $d = 1$ then A is a nonnegative real number called **Gaussian variance**. If υ satisfies $\int_{|x|\le 1}|x|\,\upsilon(dx) < \infty$ then (1.1) may be written as

$$\hat{\mu}(z) = \exp\left[i\langle\gamma_0, z\rangle - \frac{1}{2}\langle z, Az\rangle + \int_{\mathbb{R}^d}\left(e^{i\langle z,x\rangle} - 1\right)\upsilon(dx)\right]$$

with some $\gamma_0 \in \mathbb{R}^d$. This γ_0 is called the **drift**. If υ satisfies $\int_{|x|>1}|x|\,\upsilon(dx) < \infty$ then (1.1) can be written as

$$\hat{\mu}(z) = \exp\left[i\langle\gamma_1, z\rangle - \frac{1}{2}\langle z, Az\rangle + \int_{\mathbb{R}^d}\left(e^{i\langle z,x\rangle} - 1 - i\langle z, x\rangle 1_{\{|x|\le 1\}}(x)\right)\upsilon(dx)\right]$$

with some $\gamma_1 \in \mathbb{R}^d$, called the **center**.

Brownian motion is a Lévy process with $A =$ identity matrix, $\upsilon = 0$ and $\gamma = 0$, and a **Poisson process** with intensity $\lambda > 0$ is a Lévy process on $\mathbb{R}$ with $A = 0, \gamma = 0$ and $\upsilon = \lambda\delta_1$, where we denote by δ_a the distribution concentrated at a. A Lévy process on $\mathbb{R}^d$ with $A = 0, \gamma = 0$ and $\upsilon\left(\mathbb{R}^d\right) < \infty$ is called **compound Poisson process**. The **Γ-process** with parameter $q > 0$ is a Lévy process on $\mathbb{R}$ corresponding to the exponential distribution μ with mean $1/q$; that is $A = 0, \gamma = 0$ and $\upsilon(dx) = 1_{(0,\infty)}(x)\,x^{-1}e^{-qx}dx$. This μ is not a compound Poisson distribution, because υ has total mass ∞.

Any **Lévy process** $\{X_t\}$ is a **Markov process** and allows for a *càdlàg* version. Let us define two other processes: $X_- = (X_{t-})_{t\in\mathbb{R}_+}$ and $\Delta X = (X_t - X_{t-})_{t\in\mathbb{R}_+}$.

Theorem 4. *If X is a Lévy process, then for fixed $t > 0$, $\Delta X_t = 0$ (a.s).*

Proof. Let $\left(t_{(m)}, m \in \mathbb{N}_+\right)$ be a sequence in $\mathbb{R}_+$ with $t_{(m)} \uparrow t$ as $m \to \infty$. Then since X has *càdlàg* paths, $\lim_{m\to\infty} X(t_{(m)}) = X(t_-)$. However by stochastic continuity the sequence $\left(X(t_{(m)}), m \in \mathbb{N}_+\right)$ converges in probability to $X(t)$ and so has a subsequence that converges almost surely to $X(t)$. The result follows by uniqueness of limits. □

Notice that Theorem 4 is equivalent to the fact that any process with càdlàg paths and stationary and independent increments has no fixed times of discontinuity[1] (cf. **Jacod and Shiryaev** (1987), $II.4.3$).

The probabilistic meaning of the **Lévy-Khintchine representation** is explained by the following result.

Theorem 5 (Lévy-Itô decomposition of sample functions)**.** *Let $\{X_t\}$ be a Lévy process on $\mathbb{R}^d$ with the characteristic triplet (γ, A, υ). For any $G \in B_{(0,\infty)\times\mathbb{R}^d}$ let $J(G) = J(G,\omega)$ be the number of jumps at time s with height $X_s(\omega) - X_{s-}(\omega)$ such that $(s, X_s(\omega) - X_{s-}(\omega)) \in G$. Then $J(G)$ has a Poisson distribution with mean $\mu(G)$. If $G_1, ..., G_n$ are disjoint, then $J(G_1), ..., J(G_n)$ are independent. We can define a.s.,*

$$\begin{aligned} X_t^1(\omega) &= \lim_{\varepsilon \downarrow 0} \int_{(0,t]\times\{\varepsilon<|x|\leq 1\}} \{xJ(d(s,x),\omega) - x\mu(d(s,x))\} \\ &\quad + \int_{(0,t]\times\{|x|>1\}} xJ(d(s,x),\omega) \end{aligned}$$

where the convergence on the right-hand side is uniform in t in any finite time interval a.s. The process $\{X_t^1\}$ is a Lévy process with the triplet $(0,0,\upsilon)$. Let

$$X_t^2(\omega) = X_t(\omega) - X_t^1(\omega)$$

Then $\{X_t^2\}$ is a Lévy process continuous in t a.s. with the characteristic triplet $(A, \gamma, 0)$. The two processes $\{X_t^1\}$ and $\{X_t^2\}$ are independent.

In general, we may call $\{X_t^1\}$ and $\{X_t^2\}$ the **jump part** and the **continuous part** of $\{X_t\}$, respectively, but the sum of the jumps actually diverges a.s. if $\int_{|x|\leq 1} |x|\,\upsilon(dx) = \infty$ and we need the centering term $-x\upsilon(d(x))$, the so-called **compensator**, in order to achieve the convergence. Notice also that an important result from the Lévy-Itô decomposition is the relationship between Lévy processes and **semimartingales**. Using directly the Lévy-Itô decomposition we can conclude that every Lévy process is a semimartingale (cf. **Jacod and Shiryaev** (1987), Corollary II.4.19.).

2. Definition and Properties of LIBOR Additive Processes

2.1. **The LIBOR additive process.** In this section we define a stochastic process, the **LIBOR additive process**, that will drive the risk-neutral dynamics of instantaneous forward rates with independent but piecewise stationary increments (this means that it is stationary inside of each time interval, usually 6 or 12 months, defined by the tenor of the LIBOR rate). We will see that this generalization allows taking into account deterministic time inhomogeneities: the parameters describing the local behavior will now be time-dependent but non-random. Therefore, all definitions below will be applied to the *time interval* $t \in I = [0, T^\star] \subset \mathbb{R}_+$, where $T^\star$ is a fixed time horizon with $T^\star > 0$. Also, let $\mathcal{T}$ be the family of all finite subsets of $I \subset \mathbb{R}_+$. For a predetermined collection of dates $J = \{T_0, T_1, ..., T_n\} \in \mathcal{T}$, such that $0 = T_0 < T_1 < ... < T_n = T^\star$ with $J \subset I$, let $\delta_j = T_{j+1} - T_j$ denote the length or *"tenor"* of the j-th interval.

Definition 3. *A stochastic process $G = \{G_t : 0 \leq t \leq T^\star\}$ on $\mathbb{R}^d$ is a **LIBOR additive process in law** if the following conditions are satisfied:*

- (LAP1) $G_0 = 0$ *a.s.*
- (LAP2) *G is a process with the **independent increment property,** i.e., for any choice of $m \in \mathbb{N}^+$ and $0 \leq t_1 \leq ... \leq t_m \leq T^\star$, the variables $G(t_1), G(t_2) - G(t_1), ..., G(t_m) - G(t_{m-1})$ are independent.*
- (LAP3) *G is **stochastically continuous** or **continuous in probability** (but it may have discontinuous trajectories).*

[1]Recall that t is called fixed time of discontinuity of a process X if $P[\triangle X_t \neq 0] > 0$

(LAP4) *There exist* $0 = T_0 < T_1 ... < T_n = T^\star$, *such that* G *is a process with* ***piecewise stationary increments****, or homogeneous in time, inside each* $[T_j, T_{j+1}]$, *for any* $j = 0, 1, ..., n-1$.

Definition 4. *A stochastic process* $\{G_t : 0 \leq t \leq T^\star\}$ *on* $\mathbb{R}^d$ *is a* ***LIBOR additive process*** *if it satisfies* $(LAP1)$ *to* $(LAP4)$ *and if, additionally, there is* $\Omega_0 \in \mathcal{F}$ *with* $P[\Omega_0] = 1$ *such that, for every* $\omega \in \Omega_0$, $G_t(\omega)$ *is right-continuous in* $t \geq 0$ *and has left-limits in* $t > 0$ *("càdlàg" process).*

Notice that, according to Section 1.3, a **Lévy process** is defined as a stochastic process with stationary independent increments which is continuous in probability (but may have discontinuous trajectories). While **Lévy processes** offer nice features in terms of analytical tractability, the constraints of stationarity of their increments prove to be rather restrictive.

- The **first advantage** of our approach is that it allows for preserving the tractability of *Lévy processes* while enabling us to model the whole range of cap/floors or swaptions volatilities across strikes and maturities.
- A **second advantage** is that the property of *piecewise stationarity* is the usual performance that we suppose in the discretised version of the LIBOR market model where the *tenor structure* plays a relevant role, and it will be a key issue in our calibration procedure and credit risk modelling.

2.2. **Properties of LIBOR additive processes.** In this section, our aim is to briefly describe some relevant probabilistic properties[2] of this stochastic process, and specifically, some properties related with the **infinite divisibility** of its distribution, **self-similarity**, **stability** and **self-decomposability**.

2.2.1. *Infinite divisibility of LIBOR additive processes.* Recall that, for any $n \in \mathbb{N}_+$, we denoted by μ^{n*} or μ^n the n-**fold convolution** of a probability measure μ with itself, that is

$$\mu^n = \mu^{n*} = \underbrace{\mu * ... * \mu}_{n\ times} \tag{2.1}$$

μ on $\mathbb{R}^d$ is an **infinitely divisible** probability measure if, for any positive integer n, there is a probability measure $\mu_{(n)}$ on $\mathbb{R}^d$ such that $\mu = \mu_{(n)}^{n*}$, or in other words, μ can be expressed as the n-th convolution power of $\mu_{(n)}$. Equivalently, in terms of **random variables**, we say that X is **infinitely divisible** if for all $n \in \mathbb{N}_+$, there exist i.i.d. random variables $X_1^{(n)}, ..., X_n^{(n)}$ such that

$$X \stackrel{d}{=} X_1^{(n)} + ... + X_n^{(n)} \tag{2.2}$$

Proposition 3. *The following statements are equivalent:*

1. X *is infinitely divisible*
2. μ_X *has a convolution* n*-th root that is itself the law of a random variable for each* $n \in \mathbb{N}$
3. $\hat{\mu}_X(z)$ *has an* n*-th root that is itself a characteristic function of a random variable for each* $n \in \mathbb{N}$.

Proof. (1) $\Longrightarrow$ (2) The common law of the $X_j^{(n)}$ is the required convolution n-th root

(2) $\Longrightarrow$ (3) Let $X^{(n)}$ be a random variable with law $(\mu_X)^{1/n}$ then we have for each $z \in \mathbb{R}^d$:

$$\begin{aligned} \hat{\mu}_X(z) &= \int ... \int e^{i(z, x_1 + ... + x_n)} (\mu_X)^{1/n}(dx_1) ... (\mu_X)^{1/n}(dx_n) \\ &= \left(\hat{\mu}_X(z)^{(1/n)}\right)^n \end{aligned}$$

[2] Theoretical and empirical justification of the financial relevance for these properties can be found in **Carr, Geman, Madan and Yor** (2002).

where $\hat{\mu}_X(z)^{(1/n)} = \int_{\mathbb{R}^d} e^{i(z,x_j)} (\mu_X)^{1/n} (dx_j)$ and the required result follows.

(3) $\Longrightarrow$ (1) Choose $X_1^{(n)}, ..., X_n^{(n)}$ to be independent copies of the given random variable, then we have

$$\mathbb{E}\left(e^{i(z,X)}\right) = \mathbb{E}\left(e^{i\left(z,X_1^{(n)}\right)}\right) \cdot ... \cdot \mathbb{E}\left(e^{i\left(z,X_n^{(n)}\right)}\right) = \mathbb{E}\left(e^{i\left(z,X_1^{(n)}+...+X_n^{(n)}\right)}\right)$$

□

Now, let us extend the concept of infinite divisibility to **stochastic processes.**

Definition 5. *A **stochastic process** $X = \{X_t : t \geq 0\}$ on $\mathbb{R}^d$ is **infinitely divisible** if all finite-dimensional marginals of X are infinitely divisible, that is, for any choice of distinct $t_1, ..., t_m \in [0, T^\star]$ with $m \in \mathbb{N}_+$, $\left(X_{t_j}\right)_{1\leq j\leq m}$ is **infinitely divisible**. Here $\left(X_{t_j}\right)_{1\leq j\leq m}$ is an $\mathbb{R}^{md}$-valued random variable.*

Notice that it is not difficult to check that a **Lévy process** is an infinitely divisible process (see Proposition 1 in Section 1.3) due to the homogeneity property.

The **LIBOR additive process** does not preserve the homogeneity property anymore (see $(LAP4)$ in Definition 3) as in the Lévy case. Actually, it is only a piecewise stationary process. However, we can attempt to prove the infinite divisibility property using *independent increments* (see $(LAP2)$ in Definition 3) and *stochastic-continuity property* $(LAP3)$, following a similar reasoning as in **Sato** (1999). It is based on one of the fundamental limit theorems on sums of independent random variables, conjectured by **Kolmogorov** and proved by **Khintchine**.

Definition 6. *A double sequence of random variables $\left\{Z_i^{(n)} : i = 1, 2, ..., r_n; n = 1, 2, ...\right\}$ in $\mathbb{R}^d$ is called a **null array** if for each fixed n, $Z_1^{(n)}, Z_2^{(n)}, ..., Z_{r_n}^{(n)}$ are independent and if, for any $\varepsilon > 0$,*

$$\lim_{n\to\infty} \max_{1\leq i\leq r_n} \mathbb{P}\left[\left|Z_i^{(n)}\right| > \varepsilon\right] = 0 \tag{2.3}$$

*The sums $S_n = \sum_{i=1}^{r_n} Z_i^{(n)}$, $n = 1, 2, ...,$ are called the **row sums**.*

Theorem 6. *(**Khintchine (1937)**) Let $\left\{Z_i^{(n)}\right\}$ be a **null array** in $\mathbb{R}^d$ with row sums S_n. If, for some $b_n \in \mathbb{R}^d$, $n = 1, 2, ...,$ the distributions of $S_n - b_n$ converge to a distribution μ, then μ is **infinitely divisible** (id).*

Proof. cf. **Khintchine** (1937) □

Lemma 1. *The **LIBOR additive process in law** $G = \{G_t : t \geq 0\}$ is **uniformly stochastically continuous** on any finite interval $[0, T^\star]$, that is, for every $\varepsilon > 0$ and $\eta > 0$, there is $\delta > 0$ such that, if s and t are in $[0, T^\star]$ and satisfy $|s - t| < \delta$, then $\mathbb{P}[|G_s - G_t| > \varepsilon] < \eta$.*

Proof. Fix $\varepsilon > 0$ and $\eta > 0$. From property $(LAP3)$ in Definition 3, we have that for any t there is $\delta_t > 0$ such that

$$\mathbb{P}[|G_s - G_t| > \varepsilon/2] < \eta/2 \text{ for } |s - t| < \delta_t$$

Let $I_t = (t - \delta_t/2, t + \delta_t/2)$, then $\{I_t : t \in [0, T^\star]\}$ covers the interval $[0, T^\star]$.

Hence there is a finite subcovering $\left\{I_{t_j} : j = 1, ..., n\right\}$ of $[0, T^\star]$.

Let δ be the minimum of $\delta_{t_j}/2$, $j = 1, ..., n$. If $|s - t| < \delta$ and $s, t \in [0, T^\star]$ then $t \in I_{t_j}$ for some j. Hence $|s - t_j| < \delta_{t_j}$, and

$$\mathbb{P}[|G_s - G_t| > \varepsilon] \leq \mathbb{P}\left[\left|G_s - G_{t_j}\right| > \varepsilon/2\right] + \mathbb{P}\left[\left|G_t - G_{t_j}\right| > \varepsilon/2\right] < \eta$$

□

And finally, we can state the result that we look for.

Theorem 7. *If $G = \{G_t : t > 0\}$ is a **LIBOR additive process in law** on $\mathbb{R}^d$, then for every t the distribution of G_t is **infinitely divisible**.*

Proof. Fix a time interval $[0, t]$ with $t > 0$. Let $t_i^{(n)} = it/n$ for $i = 0, 1, 2, ..., n$ and $n = 1, 2, ...$ Set

$$Z_i^{(n)} = G\left(t_i^{(n)}\right) - G\left(t_{i-1}^{(n)}\right) \quad \text{for } i = 1, 2, ..., n$$

Let us recall that **Khintchine's theorem** shows that $G(t)$ is infinitely divisible if $\left\{\left\{Z_i^{(n)}\right\}_{i=1}^{n}\right\}_{n=1}^{\infty}$ is **null-array**. To prove that $Z_i^{(n)}$ is a **null array** (with $r_n = n$) we use the **uniform stochastic continuity** from Lemma 1 when $n \to \infty$

$$\max_{1 \le i \le n} \mathbb{P}\left\{\left|Z_i^{(n)}\right| > \varepsilon\right\} \le \sup_{1 \le i \le n} \mathbb{P}\left[\left|G\left(t_i^{(n)}\right) - G\left(t_{i-1}^{(n)}\right)\right| > \varepsilon\right] \underset{n \to \infty}{\to} 0$$

Therefore $\left\{\left\{Z_i^{(n)}\right\}_{i=1}^{n}\right\}_{n=1}^{\infty}$ is a **null-array**. Hence we can apply Theorem 6 or **Khintchine's Theorem** with $\mu = \mathbb{P}_{G_t}$ and $b_n = 0$. □

Remark 1. *Notice the strong relationship that exists between the concepts of **independent increments** and **stochastic continuity** with **infinitely divisible** distributions. These concepts will be helpful to work not only with the concept of **efficient financial markets**, but also with **limit law distributions class** (see Theorem 8).*

2.2.2. *Stability and self-decomposability of LIBOR additive process.* If $\{W_t : t \ge 0\}$ is the Brownian motion on $\mathbb{R}^d$ then for any $a > 0$ the process $\{W_{at} : t \ge 0\}$ is identical in law with the process $\left\{a^{1/2} W_t : t \ge 0\right\}$. This means that any change of the time scale for the Brownian motion has the same effect as some change of the spatial scale. This property is usually called **self-similarity.** There are many self-similar Lévy processes other than the Brownian motion. They constitute an important class called **strictly stable processes**.

Roughly speaking, stable processes are Lévy processes for which a change of time scale has the same effect as a change of spatial plus a linear drift. In other words, they are invariant in distribution under an appropriate scaling of time and space. They are important in probability because of their connection to **limit theorems** (see **Lamperti** (1962)) and they are of great interest in financial modelling.

In this subsection, we define both concepts and some extensions, and we determine the conditions for the self-similarity of **LIBOR additive processes**.

Definition 7. *Let μ be an infinitely divisible probability measure on $\mathbb{R}^d$. It is called **stable** if, for any $a > 0$, there are $b > 0$ and $c \in \mathbb{R}^d$ such that*

$$\hat{\mu}(z)^a = \hat{\mu}(bz)\, e^{i\langle c, z\rangle} \tag{2.4}$$

*It is **strictly stable** if, for any $a > 0$, there is $b > 0$ such that*

$$\hat{\mu}(z)^a = \hat{\mu}(bz) \tag{2.5}$$

Definition 8. *Let $\{G_t : t \ge 0\}$ be a **LIBOR additive process** on $\mathbb{R}^d$. It is called a **stable** or **strictly stable** process if the distribution of G_t at $t = 1$ is stable or strictly stable, respectively.*

Definition 9. *Let $\{X_t : t \ge 0\}$ be a stochastic process on $\mathbb{R}^d$. It is called **self-similar** if, for any $a > 0$, there is $b > 0$ such that*

$$\{X_{at} : t \ge 0\} \overset{d}{=} \{bX_t : t \ge 0\} \tag{2.6}$$

It is called ***broad-sense self-similar*** *if, for any $a>0$ there is $b>0$ and a function $c(t)$ from $[0,\infty)$ to $\mathbb{R}^d$ such that*

$$\{X_{at}:t\geq 0\}\overset{d}{=}\{bX_t+c(t):t\geq 0\} \tag{2.7}$$

Now let us consider **the Generalized Central Limit problem**. Let $Z_1, Z_2, ...$ be an independent, identically distributed sequence of random variables. Let $S^{(m)}$ be defined as the sum of m of these independent identically distributed random variables. We are interested in the case where there exists the following relationship

$$\lim_{m\to\infty} P\left(b^{(m)}S^{(m)}+c^{(m)}\leq x\right)=P\left(X\leq x\right)\ \text{for all } x\in\mathbb{R}^d \tag{2.8}$$

Theorem 8. *A probability measure μ on $\mathbb{R}^d$ is* ***stable*** *if and only if there is a random walk $\left\{S^{(m)}\right\}, b^{(m)}>0$ and $c^{(m)}\in\mathbb{R}^d$ such that $\mathbb{P}_{b^{(m)}S^{(m)}+c^{(m)}}\to\mu$ as $m\to\infty$. And in particular, it is* ***strictly stable*** *if each $c^{(m)}=0$.*

Proof. cf. **Sato** (1999) Theorem 15.7. □

It is immediate to see that it is only possible to talk about stability of the LIBOR additive process inside of the interval $[T_j, T_{j+1}]$ with $j=0,...,m$, where the increments are identically distributed. But we can generalize the definition of a stable process if we weaken the conditions on the process in the central limit theorem by requiring these to be independent but no longer necessarily identically distributed. This is the case of the **LIBOR additive process**, in which case the limiting process is called **self-decomposable**.

Definition 10. *Let μ be a probability measure on $\mathbb{R}^d$. It is called* ***self-decomposable*** *if for any $c\in(0,1)$ there is a probability measure $\rho^{(c)}$ on $\mathbb{R}^d$ such that*

$$\hat{\mu}(z)=\hat{\mu}(cz)\,\hat{\rho}^{(c)}(z) \tag{2.9}$$

It is called ***semi-selfdecomposable*** *if there are some $c\in(0,1)$ and some infinitely divisible probability measure $\rho^{(c)}$ satisfying* (2.9).

Definition 11. *A stochastic process $X=\{X_t:t\geq 0\}$ on $\mathbb{R}^d$ is* ***self-decomposable*** *if all finite-dimensional marginals of X are self-decomposable that is, for any choice of distinct $t_1,...,t_m\in[0,T^\star]$, $\left(X_{t_j}\right)_{1\leq j\leq m}$ is self-decomposable. Here $\left(X_{t_j}\right)_{1\leq j\leq m}$ is an $\mathbb{R}^{md}$-valued random variable.*

The class of self-decomposable distributions is obtained as a class of limit distributions described below.

Theorem 9. *(i) Let $\left\{X_i^{(m)}:i=1,2,...,m\right\}$ be independent random variables on $\mathbb{R}^d$ and $S^{(m)}=\sum_{i=1}^m X_i^{(m)}$. Let μ be a probability measure on $\mathbb{R}^d$. Suppose that there are $b^{(m)}>0$ and $c^{(m)}\in\mathbb{R}^d$ for $m=1,2,...$ such that*

$$\mathbb{P}_{b^{(m)}S^{(m)}+c^{(m)}}\underset{m\uparrow\infty}{\to}\mu \tag{2.10}$$

and that

$$\left\{b^{(m)}X_i^{(m)}:i=1,...,m;m=1,2,...\right\}\ \textit{is a }\boldsymbol{\textit{null array}}^{3} \tag{2.11}$$

or equivalently

$$\lim_{m\uparrow\infty}\max_{1\leq i\leq m}\mathbb{P}\left\{\left|b^{(m)}X_i^{(m)}\right|>\varepsilon\right\}=0$$

Then μ is ***self-decomposable.***

(ii) For any ***self-decomposable*** *distribution μ on $\mathbb{R}^d$ we can find $\left\{X_i^{(m)}\right\}$ independent, $b^{(m)}>0$ and $c^{(m)}\in\mathbb{R}^d$ satisfying* (2.10) *and* (2.11).

Proof. cf. **Sato** (1999) Theorem 15.3 □

Remark 2. *Limit laws are probably the best explanation for the wide-spread use of the Gaussian law in the study of financial markets. The self-decomposable laws are limit laws and this is also their appeal. Notice the relationship that appears between the concepts of independent increments and stochastic continuity with self-decomposable distributions.*

Theorem 10. *If $G = \{G_t : t > 0\}$ is a **LIBOR additive process in law** on $\mathbb{R}^d$ then for every t, the distribution of G_t is **self-decomposable**.*

Proof. Fix a time interval $[0, t]$ with $t > 0$. Let $t_i^{(n)} = it/n$ for $i = 1, 2, ..., n$ and $n = 1, 2, ...$ Let us define $Z_i^{(n)}$ as

$$Z_i^{(n)} = G\left(t_i^{(n)}\right) - G\left(t_{i-1}^{(n)}\right) \quad \text{for } i = 1, 2, ..., n$$

Let us recall that $\left\{\left\{Z_i^{(n)}\right\}_{i=1}^{n}\right\}_{n=1}^{\infty}$ is a **null-array**, by Definition 6 and uniform stochastic continuity property given in Lemma 1. The assertion follows as in the proof of Theorem 7. Therefore, by direct application of Theorem 9, we conclude that the **LIBOR additive process in law** has a **self-decomposable** distribution. □

Theorem 11. *A stochastic process $X = \{X_t : t \geq 0\}$ on $\mathbb{R}^d$ is **self-decomposable** if and only if for every $c \in (0, 1)$,*

$$X \stackrel{d}{=} cX' + U^{(c)} \tag{2.12}$$

*where $X' = \{X'_t : t \geq 0\}$ is a version of X, and $U^{(c)} = \left\{U_t^{(c)} : t \geq 0\right\}$ is a stochastic process on $\mathbb{R}^d$ and X' and $U^{(c)}$ are independent. The law of $U^{(c)}$ is uniquely determined by c and the law of X, and $U^{(c)}$ is an **infinitely divisible** process.*

Proof. cf. **Barndorff-Nielsen, Maejima and Sato** (2006). □

Theorem 12. *Let G be a **self-decomposable** LIBOR additive process on $\mathbb{R}^d$, then for every $c \in (0, 1)$ the process $U^{(c)}$ can be chosen to be a **LIBOR additive process**.*

Proof. Notice that the **LIBOR additive process** G is a self-decomposable, according with Theorem 10. Let us denote $\mu_t = \mathcal{L}(G_t)$ and $\mu_{s,t} = \mathcal{L}(G_t - G_s)$ for $0 \leq s \leq t$. According to Theorem 11, fix $c \in (0, 1)$ and denote $U_t = U_t^{(c)}$, $\rho_t = \mathcal{L}(U_t)$, and $\rho_{s,t} = \mathcal{L}(U_t - U_s)$ for $0 \leq s \leq t$.

Then

$$\hat{\mu}_t(z) = \hat{\mu}_s(z)\,\hat{\mu}_{s,t}(z) \tag{2.13}$$

where

$$\begin{aligned} \hat{\mu}_t(z) &= \hat{\mu}_t(cz)\,\hat{\rho}_t(z) \\ \hat{\mu}_{s,t}(z) &= \hat{\mu}_{s,t}(cz)\,\hat{\rho}_{s,t}(z) \end{aligned}$$

Notice that $\hat{\rho}_{s,t} \to 1$ when $s \downarrow t$ or $t \uparrow s$. It follows that $U = U^{(c)}$ is **stochastically continuous** (property ($LAP3$) in Definition 3). Obviously $U_0 = 0$ a.s ($LAP1$).

In order to prove that U is a **LIBOR additive process in law,** according to Definition 3, additionally we need the **independent increments** ($LAP2$) and **piecewise stationary property** ($LAP4$).

Let $0 = t_0 < t_1 < ... < t_n = T^\star$ and $z_1, ..., z_n \in \mathbb{R}^d$ and $z_{n+1} = 0$, then

$$\begin{aligned}
\mathbb{E}\left[\exp\left(i\sum_{j=1}^{n}\left\langle z_j, U_{t_j} - U_{t_{j-1}}\right\rangle\right)\right] &= \mathbb{E}\left[\exp\left(i\sum_{j=1}^{n}\left\langle z_j - z_{j+1}, U_{t_j}\right\rangle\right)\right] \\
&= \mathbb{E}\left[\exp\left(i\sum_{j=1}^{n}\left\langle z_j - z_{j+1}, G_{t_j}\right\rangle\right)\right] \Big/ \mathbb{E}\left[\exp\left(i\sum_{j=1}^{n}\left\langle z_j - z_{j+1}, cG'_{t_j}\right\rangle\right)\right] \\
&= \mathbb{E}\left[\exp\left(i\sum_{j=1}^{n}\left\langle z_j, G_{t_j} - G_{t_{j-1}}\right\rangle\right)\right] \Big/ \mathbb{E}\left[\exp\left(i\sum_{j=1}^{n}\left\langle z_j, cG'_{t_j} - cG'_{t_{j-1}}\right\rangle\right)\right] \\
&= \prod_{j=1}^{n}\mathbb{E}\left[\exp\left(i\left\langle z_j, G_{t_j} - G_{t_{j-1}}\right\rangle\right)\right] / \prod_{j=1}^{n}\mathbb{E}\left[\exp\left(i\left\langle z_j, cG'_{t_j} - cG'_{t_{j-1}}\right\rangle\right)\right] \\
&= \prod_{j=1}^{n}\mathbb{E}\left[\exp\left(i\left\langle z_j, U_{t_j} - U_{t_{j-1}}\right\rangle\right)\right]
\end{aligned}$$

This shows $(LAP2)$ and $(LAP4)$, therefore U is a **LIBOR additive process in law**. □

3. Existence of LIBOR additive processes

The aim of this section is to provide a simple proof of the existence of the **LIBOR additive process** according to the definition given in Section 2.1 (Definition 3). A direct construction in the Skorohod space is described in the current section, and it is based directly on the fact that the **LIBOR additive process**, in terms of trajectories, can be observed as a **piecewise stationary Lévy process**. Therefore the existence of the Lévy process guarantees the existence of LIBOR additive process in the Skorohod space.

Recall that we are given a **time interval** $[0, T^\star]$ with $T^\star$ fixed. Also, notice that, given $n \in \mathbb{N}_+$, we have a predetermined collection of fixed dates $0 = T_0 < T_1 < ... < T_n = T^\star$. Let us define η as a right-continuous function $\eta : [0, T_n] \to \{0, 1, ..., n\}$ by taking $\eta(t)$ to be $\eta(t) = \sup\{0 \le i \le n : T_i \le t\}$.

Let $G^{(j)}$, with $0 \le j \le n$, be a family of $n+1$ independent Lévy processes such that $G^{(j)}$ has the triplet $(\gamma_j, A_j, \upsilon_j)$. Set for $0 \le t \le T^\star$

$$G(t) = \sum_{j=0}^{\eta(t)}\left(G^{(j)}(T_{j+1} \wedge t) - G^{(j)}(T_j)\right) \tag{3.1}$$

Then $G(t)$ has characteristic function, due to the independence of the $G^{(j)}$'s:

$$e^{T_1\psi_1(z) + (T_2 - T_1)\psi_2(z) + ... + \left(t - T_{\eta(t)}\right)\psi_{\eta(t)}(z)} = e^{\sum_{j \le \eta(t)}((T_{j+1}\wedge t) - T_j)\psi_j(z)}$$

where $\psi_j(z)$ is the Lévy exponent of $G^{(j)}$ (see Theorem 14 for further details). Since $G^{(j)}$ has sample paths in the Skorohod space, it immediately follows from the above construction that the same is true for G. Let us prove that G is a **LIBOR additive process.**

Theorem 13. *Let $\{G_t, t \ge 0\}$ be a stochastic process on $\mathbb{R}^d$, defined by (1.15). Then G_t is a* ***LIBOR additive process****.*

Proof. Notice that to prove that G is a **LIBOR additive process**, we just need to prove properties $(LAP1)$ to $(LAP4)$ in Definition 3 as properties of G. $(LAP1)$ is obvious taking into account property $(L1)$ in Definition 2 of **Lévy process in law**. To prove $(LAP2)$ in Definition 3, notice that using the property $(L2)$ in Definition 2, we have that for any $s, t \in [0, T^\star]$ belonging to the same sub-interval $T_j \leq s \leq t < T_{j+1}$:

$$G_t - G_s = G_t^{(j)} - G_s^{(j)}$$

is identical with the increment of $G^{(j)}$ that has independent increments by $(L2)$ in Definition 2 of **Lévy process in law**. If $s < T_j \leq t < T_{j+1}$ are in adjacent intervals, then

$$\begin{aligned} G_t - G_s &= G_t - G_{T_j}^{(j)} + G_{T_j}^{(j)} - G_s \\ &= G_t^{(j+1)} - G_{T_j}^{(j+1)} + \left(G_{T_j}^{(j)} - G_{T_{j-1}}^{(j)}\right) - \left(G_s^{(j)} - G_{T_{j-1}}^{(j)}\right) \\ &= \left(G_t^{(j+1)} - G_{T_j}^{(j+1)}\right) + \left(G_{T_j}^{(j)} - G_s^{(j)}\right) \end{aligned}$$

Hence this increment may be decomposed into two independent increments over adjacent intervals. These two facts finally lead to the independence property of the adjacent increments and hence to increments in general.

Since each G has no fixed discontinuities (Theorem 4), the stochastic continuity $(LAP3)$ is guaranteed, and finally $(LAP4)$ is obvious using (1.15). □

4. Characterization of LIBOR additive processes

The following results recall the representation of characteristic functions of infinitely divisible distributions, briefly shown in Section 1.3. The **Lévy-Khintchine formula** was obtained on $\mathbb{R}$ around 1930 by **De Finetti** and **Kolmogorov** in special cases, and then mentioned by **Lévy** (1934) in the general case. It was immediately extended to $\mathbb{R}^d$. This theorem is essential to the whole theory, and a simpler proof was given by **Khintchine** (1937) and **Gnedenko and Kolmogorov** (1954). Here we show a detailed version, however, the proof is omitted.

Theorem 14 (Lévy-Khintchine)**.** *(i) If μ is an **infinitely divisible** distribution on $\mathbb{R}^d$, then its **characteristic function** $\hat{\mu}(z)$ has the form*

$$\hat{\mu}(z) = \exp\left[\psi\left(z\right)\right] \tag{4.1}$$

*where the **Lévy exponent** $\psi(z)$ with $z \in \mathbb{R}^d$ equals*

$$\psi\left(z\right) = i\left\langle \gamma, z\right\rangle - \frac{1}{2}\left\langle z, Az\right\rangle + \int_{\mathbb{R}^d}\left(e^{i\langle z,g\rangle} - 1 - i\left\langle z, g\right\rangle 1_{\{|g|\leq 1\}}\right)\upsilon\left(dg\right) \tag{4.2}$$

*and where A is a **symmetric nonnegative-definite** $d \times d$ **matrix**, $\gamma \in \mathbb{R}^d$, υ is a **Radon measure** on $\mathbb{R}^d \backslash \{0\}$ and $g \in \mathbb{R}^d$ satisfying*

$$\upsilon\left(\{0\}\right) = 0 \text{ and } \int_{\mathbb{R}^d}\left(|g|^2 \wedge 1\right)\upsilon\left(dg\right) < \infty \tag{4.3}$$

*(ii) The representation of $\hat{\mu}(z)$ in (i) by A, υ, and γ is **unique**.*

*(iii) Conversely, if A is a **symmetric nonnegative-definite** $d\times d$ **matrix**, υ is a **measure** satisfying* (1.22) *and $\gamma \in \mathbb{R}^d$, then there exists an **infinitely divisible** distribution μ whose characteristic function is given by* (1.20).

Proof. cf. **Sato** (1999) Theorem 8.1. □

Definition 12. *We call (A, υ, γ) in Theorem* 31 *the **generating triplet** of μ. The A and the υ are called, respectively, the **Gaussian covariance matrix** and the **Lévy measure** of μ. When $A = 0$, μ is called purely non-gaussian.*

Corollary 1. *If* μ *has the generating triplet* (A, υ, γ) *, then* μ^t *has the generating triplet* $(tA, t\upsilon, t\gamma)$ *.*

Now, let $G = \{G_t; t \geq 0\}$ be the **LIBOR additive process** with a given tenor structure $0 = T_0 < T_1 < ... < T_n = T^\star$ with $T^\star$ fixed. Let us recall that, given a set of n-Lévy processes, the **LIBOR additive process** can be constructed as $G_t = \sum_{j=0}^{\eta(t)} \left(G^{(j)}(T_{j+1} \wedge t) - G^{(j)}(T_j) \right)$. Let us call $\mu^{(j)}$ as the distribution (or law) associated to the **Lévy process** $G^{(j)}$, and additionally, let us define $\mu^{(j)}_{T_j, T_{j+1}}$ and $\hat{\mu}^{(j)}_{T_j, T_{j+1}}$ as the distribution and characteristic function respectively associated to the increment $\left(G^{(j)}(T_{j+1}) - G^{(j)}(T_j) \right)$. The next theorem shows how the distribution function of G_t, is characterized by the **sequence of triplets** $\left\{ \left(A_j, \upsilon_j, \gamma_j \right) \right\}_{j \leq \eta(t)}$.

Theorem 15. *A d-dimensional process $G = \{G_t; t \geq 0\}$ is a **LIBOR additive process** if and only if it is a semimartingale admitting a **sequence of triplets** $\left\{ \left(A_j, \upsilon_j, \gamma_j \right) \right\}_{j \leq \eta(t)}$ such that for all $t \in \mathbb{R}^+$ and $z \in \mathbb{R}^d$ we have*

$$\hat{\mu}_t(z) = \prod_{j \leq \eta(t)} \hat{\mu}^{(j)}_{T_j, t \wedge T_{j+1}}(z) \tag{4.4}$$

where $\left(A_j, \nu_j, \gamma_j \right)$ is the triplet associated with the characteristic function of $\mu_{T_j, T_{j+1}}$, for any $j = 0, 1, ..., \eta(t)$.

Proof. For any $t \in [0, T^\star]$ and given a tenor structure $0 = T_0 < T_1 ... < T_n = T^\star$, then with $G_t = \sum_{j \leq \eta(t)} \left(G^{(j)}(T_{j+1} \wedge t) - G^{(j)}(T_j) \right)$ and using the **independent increments property** we have

$$\begin{aligned} & \mathbb{E}\left[f\left(G_{T_0}, ..., G_{T_{\eta(t)}}, G_t \right) \right] \\ = & \int ... \int f\left(g_0, g_0 + g_1, ..., g_0 + ... + g_{\eta(t)} \right) \mu^{(0)}(dg_0) \times \mu^{(1)}(dg_1) \times ... \times \mu^{(\eta(t))}\left(dg_{\eta(t)} \right) \end{aligned}$$

for any bounded measurable function f. Let $z_1, ..., z_n \in \mathbb{R}^d$ and

$$f\left(g_0, g_1, ..., g_{\eta(t)} \right) = \exp\left(i \sum_{j=0}^{\eta(t)} \langle z_j, g_{j+1} - g_j \rangle \right)$$

Therefore

$$\begin{aligned} \hat{\mu}_t(z) &= \mathbb{E}\left[\exp\left(i \sum_{j=0}^{\eta(t)} \langle z_j, g_{j+1} - g_j \rangle \right) \right] \\ &= \int ... \int \left[\exp\left(i \sum_{j=0}^{\eta(t)} \langle z_j, g_{j+1} - g_j \rangle \right) \right] \mu^{(0)}_0(dg_0) \times \mu^{(1)}_{T_0, T_1}(dg_1) \times ... \times \mu^{(\eta(t))}_{T_{\eta(t)}, t}\left(dg_{\eta(t)} \right) \\ &= \prod_{j \leq \eta(t)} \int ... \int \exp\left(i \langle z_j, g_{j+1} - g_j \rangle \right) \mu^{(j)}_{T_j, T_{j+1} \wedge t}(dg_j) \\ &= \prod_{j \leq \eta(t)} \mathbb{E}\left(\exp\left[iz\left(G^{(j)}_{t \wedge T_{j+1}} - G^{(j)}_{T_j} \right) \right] \right) \end{aligned}$$

and using **Lévy-Khintchine** Theorem 14.i) we have

$$\begin{aligned} \hat{\mu}_t(z) &= \exp\left[\sum_{j \leq \eta(t)} \left(t \wedge T_{j+1} - T_j \right) \Psi_j(z) \right] \\ &= \prod_{j \leq \eta(t)} \hat{\mu}^{(j)}_{T_j, t \wedge T_{j+1}}(z) \end{aligned} \tag{4.5}$$

where

$$\begin{aligned}\hat{\mu}^{(j)}_{T_j,T_{j+1}}(z) &= \mathbb{E}\left[\exp\left(iz\left(G^{(j)}_{T_{j+1}} - G^{(j)}_{T_j}\right)\right)\right] \\ &= \exp\left[\left(T_{j+1} - T_j\right)\psi_j(z)\right]\end{aligned} \tag{4.6}$$

and

$$\psi_j(z) = i\left\langle \gamma_j, z\right\rangle - \frac{1}{2}\left\langle z, A_j z\right\rangle + \int_{\mathbb{R}^d}\left(e^{i\langle z,g\rangle} - 1 - i\left\langle z, g\right\rangle 1_{\{|g|\leq 1\}}\right)\upsilon_j(dg)$$

The sufficient condition came as a direct consequence of (iii) in the Lévy-Khintchine theorem. □

Remark 3. *Let $G = \{G_t : t > 0\}$ be a **LIBOR additive process.** Notice that, given a tenor structure $0 = T_0 < T_1 < ... < T_n = T^\star$ with $T^\star$ fixed, and a set of n infinitely divisible measures $\{\mu_0, \mu_1, ..., \mu_n\}$ associated to this tenor structure, then, for any $s, t \in [0, T^\star]$ with $s < t$, $\mu_{s,t}$, the distribution of $G_t - G_s$, is **uniquely determined in law** by its **sequence of triplets** $\left\{\left(A_j, \upsilon_j, \gamma_j\right)\right\}_{\eta(s)\leq j\leq \eta(t)}$ (as a direct consequence from Theorem 14, ii) and Theorem 15).*

5. Infinitesimal generators of LIBOR additive processes

Here we turn our attention to the **infinitesimal generator** of the **LIBOR additive process** which will play an essential role to show the markovianity and martinagle characteristic of this process. This section is an attempt to gather some basic and typical results that will necesary later. It is not intended to give a systematic presentation of the most important results or to explain how to prove them; for these purposes the reader can find a more comprehensive picture in **Ethier and Kurtz** (1986) or **Sato** (1999).

This section has been divided in two subsections:

- In the *first*, we review a number of basic definitions and theorems related with the infinitesimal generator of its transition semigroup. Basically, the aim of this preliminary section is to give a brief introduction to the theory of semigroups of linear operators.
- In the *second* subsection, we apply directly these definitions to the LIBOR additive process. Further developments can be found in **Colino** (2008).

5.1. **Strongly continuous contraction semigroup and infinitesimal generator.** Let $\mathbb{B}$ be a real (or complex) **Banach space**. That is, $\mathbb{B}$ is a vector space over the real (or complex) scalar field equipped with a mapping $\|f\|$ from $\mathbb{B}$ into $\mathbb{R}$, called the norm, satisfying

$$\begin{array}{lll}(1) & \|af\| = |a|\,\|f\| & \text{for } f \in \mathbb{B},\ a \in \mathbb{R} \text{ (or } a \in \mathbb{C})\\ (2) & \|f+g\| \leq \|f\| + \|g\| & \text{for } f, g \in \mathbb{B}\\ (3) & \|f\| = 0 & \text{if and only if } f = 0\end{array}$$

such that if a sequence $\{f_n\}$ in $\mathbb{B}$ satisfies $\lim_{n,m\to\infty}\|f_n - f_m\| = 0$, then there is $f \in \mathbb{B}$ with $\lim_{n\to\infty}\|f_n - f\| = 0$.

A **linear operator** P in $\mathbb{B}$ is a mapping from a linear subspace $\mathcal{D}(P)$ of $\mathbb{B}$ into $\mathbb{B}$ such that

$$P(af + bg) = aPf + bPg \quad \text{for } f, g \in \mathcal{D}(P) \quad a, b \in \mathbb{R} \text{ (or } a, b \in \mathbb{C})$$

The set $\mathcal{D}(P)$ is called the **domain** of P.

A linear operator P is called **bounded** if $\mathcal{D}(P) = \mathbb{B}$ and $\sup_{\|f\|\leq 1}\|Pf\|$ called the norm of P and denoted by $\|P\|$, is finite. A linear operator P with $\mathcal{D}(P) = \mathbb{B}$ is bounded if and only if P is continuous in the sense that $f_n \to f$ implies $Pf_n \to Pf$.

A linear operator P is said to be **closed** if $f_n \in \mathcal{D}(P)$, $f_n \to f$ and $Pf_n \to g$ imply $f \in \mathcal{D}(P)$ and $Pf = g$, in other words, if the graph of P, $\{(f, Pf) : f \in \mathcal{D}(P)\}$ is a closed set in $\mathbb{B} \times \mathbb{B}$. The set $\{Pf : f \in \mathcal{D}(P)\}$ called the **range** of P, is denoted by $\mathcal{R}(P)$. The **identity operator** on $\mathbb{B}$ is denoted by I. A subset $\mathcal{D}_1$ of $\mathbb{B}$ is said to be **dense** in $\mathbb{B}$ if, for any $f \in \mathbb{B}$, there is a sequence $\{f_n\}$ in $\mathcal{D}_1$ such that $f_n \to f$.

Definition 13. *A family $\{P_t : t \geq 0\}$ of bounded linear operator on $\mathbb{B}$ is called a* ***strongly continuous semigroup*** *if*

(1) $P_t P_s = P_{t+s}$ *for* $t, s \in [0, \infty)$
(2) $P_0 = I$
(3) $\lim_{t \downarrow 0} P_t f = f$ *for any* $f \in \mathbb{B}$

It is called a **strongly continuous contraction semigroup** if, moreover,

$$\|P_t\| \leq 1$$

Definition 14. *The* ***infinitesimal generator*** *L of a* ***strongly continuous contraction semigroup*** *$\{P_t\}$ is defined by*

$$Lf = \lim_{t \downarrow 0} \frac{1}{t} (P_t f - f) \tag{5.1}$$

with $\mathcal{D}(L)$ being the set of f such that the right-hand side of (5.1) *exists.*

One of the major theorem of the theory of semigroups of operators is as follows. It was independently proved by **Hille** (1948) and **Yosida** (1948), and the proof can be found in **Ethier and Kurtz** (1986).

Theorem 16. (*i*) *If L is the infinitesimal generator of a strongly continuous semigroup $\{P_t\}$, then L is closed, $\mathcal{D}(L)$ is dense and, for any $q > 0$, $\mathcal{R}(qI - L) = \mathbb{B}$, $qI - L$ is one-to-one, $\left\|(qI - L)^{-1}\right\| \leq 1/q$, and*

$$(qI - L)^{-1} f = \int_0^\infty e^{-qt} P_t f dt \quad \text{for } f \in \mathbb{B}$$

(*ii*) *The infinitesimal generator determines the semigroup. That is, two strongly continuous contraction semigroups coincide if their infinitesimal generators coincide.*

(*iii*) *If a linear operator L in $\mathbb{B}$ has a dense domain $\mathcal{D}(L)$ and, for any $q > 0$, $\mathcal{R}(qI - L) = \mathbb{B}$, $qI - L$ is one-to-one, $\left\|(qI - L)^{-1}\right\| \leq 1/q$, then L is the infinitesimal generator of a strongly continuous semigroup on $\mathbb{B}$.*

Proof. cf. **Ethier and Kurtz** (1986) □

5.2. **Infinitesimal generators of LIBOR additive processes.** Let $C_0 = C_0(\mathbb{R}^d)$ be the real **Banach space** of continuous functions f from $\mathbb{R}^d$ into $\mathbb{R}$ satisfying $\lim_{|x| \to \infty} f(x) = 0$ with norm $\|f\| = \sup_x |f(x)|$. Let C_0^n be the set of $f \in C_0$ such that f is n times differentiable and the partial derivatives of f with order $\leq n$ belong to C_0.

Suppose now that $\{G_t\}$ is a **LIBOR additive process** on $\mathbb{R}^d$ and the **transition function** $\mathbb{P}_{0,t}(g_0, B)$ is defined by

$$\mathbb{P}_{0,t}(g_0, B) := \mu_t(B - g_0)$$

for $t \geq 0, g_0 \in \mathbb{R}^d$ and $B \in \mathcal{B}(\mathbb{R}^d)$

Define, for $f \in C_0$,

$$\begin{aligned} (\mathbb{P}_t f)(g_0) &= \int_{\mathbb{R}^d} f(g)\, \mathbb{P}_t(g_0, dg) \\ &= \int_{\mathbb{R}^d} f(g_0 + g)\, \mu_t(dg) \\ &= \mathbb{E}\left[f(g_0 + G_t)\right] \end{aligned}$$

Theorem 17. *The family of operators $\{\mathbb{P}_t : t \geq 0\}$ defined above from a* ***LIBOR additive process*** *$\{G_t\}$ on $\mathbb{R}^d$ is a* ***strongly continuous semigroup*** *on $C_0\left(\mathbb{R}^d\right)$ with norm $\|\mathbb{P}_t\| = 1$. Let L be its* ***infinitesimal generator****.*

Then

$$\begin{aligned} Lf(g) &= \frac{1}{2}\sum_{n,m=1}^{d}\left(a_{\eta(t)}(n,m)\right)\frac{\partial^2 f}{\partial g_n \partial g_m}(g) + \sum_{n=1}^{d}\left(\gamma_{\eta(t)}(n)\right)\frac{\partial f}{\partial g_n}(g) \\ &\quad + \int_{\mathbb{R}^d}\left(f(g+x) - f(g) - \sum_{n=1}^{d} x_n \frac{\partial f}{\partial g_n}(g)\, 1_{|x|\leq 1}(x)\right)\upsilon_{\eta(t)}(dx) \end{aligned} \tag{5.2}$$

for $f \in C_0^2$, where $\left\{\left(\gamma_{\eta(t)}, A_{\eta(t)}, \upsilon_{\eta(t)}\right)_{t\geq 0}\right\}$ is the generating triplet of $\{G_t\}$ with $t \geq 0$ and $A_{\eta(t)} = \left(a_{\eta(t)}(n,m)\right)_{n,m\leq d}$, $\gamma_{\eta(t)} = \left(\gamma_{\eta(t)}(n)\right)_{n\leq d}$

Proof. Notice that, according with Definition 14, the infinitesimal generator of the LIBOR additive process is the Lévy infinitesimal generator and the proof in. **Sato** (1999) Theorem 31.5 applies here (see also in **Barles, Buckdahn and Pardoux** (1997) Theorem 3.4. or **Nualart and Schoutens** (2001) for Lévy processes, or **Pardoux, Pradeilles and Rao** (1997) in the no-homogenous case). □

6. The Lévy-Itô decomposition of LIBOR additive processes

In the present subsection we exhibit the **canonical representation** for multidimensional semimartingales, and for practical purposes, we introduce here the **Lévy-Itô decomposition of sample functions**. This decomposition expresses *sample functions of a LIBOR additive process as a sum of two independent parts*; **a continuous part** and a part expressible as **a compensated sum of independent sums**. This decomposition was conceived by **Paul Lévy** (1934) using a direct analysis of the paths of Lévy processes, and formulated and proved by **Kiyosi Itô** (1942) using many pages. However there are many proofs available in the literature.

Let us recall that we assume a **stochastic basis** $(\Omega, \mathbb{G}, \mathbb{P})$ equipped with the "*usual*" filtration $\mathbb{G} = (\mathcal{G}_t)_{t\in[0,T^\star]}$ which satisfies the "*usual conditions*". Additionally, let us consider an **auxiliary measurable space** $(E, \mathcal{E})$ which we assume to be a **Blackwell**[4] space. Further

$$E = [0,\infty) \times \left(\mathbb{R}^d \backslash \{0\}\right) = [0,\infty) \times D_{0,\infty} \tag{6.1}$$

where $D_{0,\infty} = \mathbb{R}^d \backslash \{0\}$. Recall that we defined a **time interval** $[0, T^\star]$ with $T^\star$ fixed. Also, notice that we have a predetermined collection of fixed dates $0 = T_0 < T_1 < ... < T_n = T^\star$. Recall also that η is a right-continuous function $\eta : [0, T_n] \to \{0, 1, ..., n\}$ by taking $\eta(t)$ to be $\eta(t) = \sup\{i \geq 0 : T_i \leq t\}$.

A **marked point** ε in E (usually $E = \mathbb{R}^d$) is denoted by $\varepsilon = (s, x)$ with $s \in (0,\infty)$ and $x \in D_{0,\infty}$. The Borel σ-algebra of E is denoted by $\mathcal{E} = \mathcal{B}(E)$. Let us define a **random measure** on $\mathbb{R}_+ \times E$ as a family

[4] In all the sequel, E will actually be $\mathbb{R}_+^d$ or $\mathbb{R}^d$, or at most a Polish spaces with its Borel $\sigma-$fields

$\upsilon = (\upsilon(\omega; dt, dg) : \omega \in \Omega)$ of nonnegative measures on $(\mathbb{R}_+ \times E, \mathcal{R}_+ \times \mathcal{E})$ satisfying $\upsilon(\omega; \{0\} \times E) = 0$ identically. Hence, the integral of $\delta(\varepsilon)$ with respect to a measure υ on E is written as

$$\int_E \delta(\varepsilon)\, \upsilon(d\varepsilon) = \int_{(0,\infty)\times D_{(0,\infty)}} \delta(s,x)\, \upsilon(ds,dx) \tag{6.2}$$

Now we formulate the **Lévy-Itô decomposition for the LIBOR additive process** as the main theorem of this subsection[5].

Theorem 18 (Lévy-Itô 1)**.** *Let $G = \{G_t : t \geq 0\}$ be a* ***LIBOR additive process*** *on $\mathbb{R}^d$ defined on a* ***stochastic basis*** *$(\Omega, \mathcal{G}, \mathbb{P})$ with the system of generating triplets $\left\{\left(\gamma_{\eta(t)}(t), A_{\eta(t)}, \upsilon_{\eta(t)}\right)_{t\geq 0}\right\}$ and define the measure $\mu_{\eta(t)}$ on E by $\mu((0,t] \times B) = \mu_{\eta(t)}(B)$ for $B \in \mathcal{B}(\mathbb{R}^d)$. Using Ω_0 from Definition 11 of* ***LIBOR additive process****, for $B \in \mathcal{B}(E)$*

$$\mu_{\eta(t)}(B,\omega) = \begin{cases} \sharp\{t : (t, G_t(\omega) - G_{t-}(\omega)) \in B\} \text{ for } \omega \in \Omega_0 \\ 0 \text{ for } \omega \notin \Omega_0 \end{cases} \tag{6.3}$$

Then the following holds:

(i) $\left\{\mu_{\eta(t)}(B) : B \in \mathcal{B}(E)\right\}$ is an ***integer-valued random measure*** *(Poisson) on E and $\upsilon_{\eta(t)}$ is a predictable random measure namely the* ***compensator of the random measure*** *$\mu_{\eta(t)}(B)$ associated to the jumps of G.*

(ii) There is $\hat{\Omega} \in \mathcal{F}$ with $\mathbb{P}\left[\hat{\Omega}\right] = 1$ such that, for any $\omega \in \hat{\Omega}$

$$\begin{aligned} \hat{G}_t(\omega) &= \lim_{\varepsilon \downarrow 0} \int_{s\in(0,t],x\in D(\varepsilon,1]} \left\{x\ \mu_{\eta(t)}(d(s,x),\omega) - x\ \upsilon_{\eta(t)}(d(s,x))\right\} \\ &\quad + \int_{s\in(0,t],x\in D(1,\infty)} x\ \mu_{\eta(t)}(d(s,x),\omega) \end{aligned} \tag{6.4}$$

is defined for all $t \in [0,\infty)$ and the convergence is uniform in t on any bounded interval. The process $\left\{G_t^1\right\}$ is a ***LIBOR additive process*** *on $\mathbb{R}^d$ with $\left\{\left(0, \upsilon_{\eta(t)}, 0\right)\right\}$ as the system of generating triplets.*

(iii) Define

$$\tilde{G}_t(\omega) = G_t(\omega) - \hat{G}_t(\omega) \text{ for } \omega \in \Omega_1 \tag{6.5}$$

There is $\tilde{\Omega} \in \mathcal{F}$ with $\mathbb{P}\left[\tilde{\Omega}\right] = 1$ such that, for any $\omega \in \tilde{\Omega}$, $\tilde{G}_t^2(\omega)$ is continuous in t. The process $\left\{\tilde{G}_t^2\right\}$ is a ***LIBOR additive process*** *on $\mathbb{R}^d$ with $\left\{\left(\gamma_{\eta(t)}(t), A_{\eta(t)}, 0\right)_{t\geq 0}\right\}$ as a system of generating triplets*

*(iv) The two processes $\left\{\hat{G}_t\right\}$ (**jump part**) and $\left\{\tilde{G}_t\right\}$ (**continuous part**) are **independent.***

Proof. cf. **Sato** (1999) Section 20 ☐

Theorem 19 (Lévy-Itô 2)**.** *Suppose that the* ***LIBOR additive process*** *$G = \{G_t : t \geq 0\}$ in the last Theorem satisfies $\int_{|x|\leq 1} |x|\, \upsilon_{\eta(t)}(dx) < \infty$ for all $t > 0$. Let $\gamma_0(t)$ be the drift of G_t. Then, there is $\hat{\Omega} \in \mathcal{F}$ with $\mathbb{P}\left[\hat{\Omega}\right] = 1$ such that, for any $\omega \in \hat{\Omega}$*

$$\hat{G}_t(\omega) = \int_{(0,t]\times D(0,\infty)} x\mu_{\eta(t)}(d(s,x),\omega) \tag{6.6}$$

[5]For the sake of simplicity, let us define $D_{a,b} = D(a,b] = \left\{x \in \mathbb{R}^d : a < |x| \leq b\right\}$ and $D_{a,\infty} = D(a,\infty] = \left\{x \in \mathbb{R}^d : a < |x| < \infty\right\}$

is defined for all $t \geq 0$. *The process* $\left\{\hat{G}_t\right\}$ *is a* ***LIBOR additive process*** *on* $\mathbb{R}^d$ *such that*

$$\mathbb{E}\left[e^{i\left\langle z,\hat{G}_t\right\rangle}\right] = \exp\left[\int_{\mathbb{R}^d}\left(e^{i\langle z,x\rangle}-1\right)\upsilon_{\eta(t)}\left(dx\right)\right] \tag{6.7}$$

Define

$$\tilde{G}_t\left(\omega\right) = G_t\left(\omega\right) - \hat{G}_t\left(\omega\right) \;\; \textit{for } \omega \in \Omega_3 \tag{6.8}$$

Then, for any $\omega \in \tilde{\Omega} \cap \hat{\Omega}$, $\tilde{G}_t\left(\omega\right)$ *is continuous in* t *and* $\left\{\tilde{G}_t\right\}$ *is an* ***LIBOR additive process*** *on* $\mathbb{R}^d$ *such that*

$$\mathbb{E}\left[e^{i\left\langle z,\tilde{G}_t\right\rangle}\right] = \exp\left[-\frac{1}{2}\left\langle z, A_{\eta(t)}z\right\rangle + i\left\langle \gamma_{\eta(t)}(t), z\right\rangle\right] \tag{6.9}$$

The two processes $\left\{\hat{G}_t\right\}$ *(**jump part**) and* $\left\{\tilde{G}_t\right\}$ *(**continuous part**) are **independent**.*

Proof. cf. **Sato** (1999) Section 20 □

Theorems 18 and 19 are called the **Lévy-Itô decomposition for the LIBOR additive process**.

As we have already mentioned, several proofs of the **Lévy-Itô theorem** exist and they are very well known in the literature, even for additive processes (**Sato** (1999)). The simplest proof begins, *first*, with the construction of the Poisson random measures. *Second*, given any **LIBOR additive process** $\{G_t\}_{t\geq 0}$ we use its system of generating triplets $\left\{\left(\gamma_{\eta(t)}(t), A_{\eta(t)}, \upsilon_{\eta(t)}\right)_{t\geq 0}\right\}$ in order to construct an additive process $\{Y_t\}_{t\geq 0}$ such that $\{Y_t\}_{t\geq 0} \overset{d}{=} \{G_t\}_{t\geq 0}$ and $\{Y_t\}_{t\geq 0}$ has the Lévy-Itô decomposition. *Third*, using the facts that $\{G_t\}_{t\geq 0}$ and $\{Y_t\}_{t\geq 0}$ induce an identical probability measure on the **Skorohod space** $\mathbb{D} = \mathbb{D}\left([0,\infty), \mathbb{R}^d\right)$ of right continuous paths with left limits with the σ-algebra $\mathcal{F}_D$ generated by the Borel cylinder sets and that all relevant quantities are $\mathcal{F}_D-$ measurable, we can prove that $\{G_t\}_{t\geq 0}$ also has the Lévy-Itô decomposition.

7. Applications to sample-function properties: continuity, jumping times and increasingness

From the **Lévy-Itô decomposition** we can deduce many **sample function properties** of **LIBOR additive processes**. Following **Sato** (1999), we devote this subsection to studying fundamental properties of sample functions of processes with independent increments and piecewise stationarity, such as continuity, jumping times, and increasingness.

Theorem 20 (Continuity)**.** *Sample functions of* $\{G_t\}_{t\geq 0}$ *are **continuous** a.s. if and only if* $\upsilon_{\eta(t)} = 0$ *for every* $t \in [0, T^\star]$

Proof. By Theorem 18 (Lévy-Itô 1) the number of jumping times satisfying $\left|G_t - G_{t_-}\right| \in D_{\varepsilon,\infty}$ has mean $t\int_{|g|>\varepsilon}\upsilon_{\eta_{(t)}}\left(dg\right)$. Hence the number of jumps is 0 a.s. if and only if $\upsilon_{\eta_{(s)}} = 0$ for every $s \in [0,t]$. □

Theorem 21 (Jumping times)**.** *If* $\upsilon_{\eta(t)}\left(\mathbb{R}^d\right) = \infty$ *for every* $t \in [0, T^\star]$, *then, almost surely, jumping times are countable and dense in* $[0,\infty)$. *If* $0 < \upsilon_{\eta(t)}\left(\mathbb{R}^d\right) < \infty$ *for every* $t \in [0, T^\star]$ *then, almost surely, jumping times are infinitely many and countable in increasing order, and the first jumping time* $T(\omega)$ *has exponential distribution with mean* $1/\,\upsilon_{\eta(t)}\left(\mathbb{R}^d\right)$.

Proof. Countability of jumps is a consequence of right-continuity with left-limits. For $\varepsilon > 0$ and $\omega \in \Omega_0$ let $T_\varepsilon(\omega)$ be the first time that $G_t(\omega)$ jumps with size $> \varepsilon$. Let $T_\varepsilon(\omega) = \infty$ if $G_t(\omega)$ does not have any jump with size $> \varepsilon$. Since $T_\varepsilon(\omega) \leq t$ is equivalent to if $\int_{(0,t]\times(\varepsilon,\infty)} \mu_{\eta(t)}(d(s,z),\omega) \geq 1$,

$$\mathbb{P}[T_\varepsilon \leq t] = 1 - \exp\left[-t\int_{D_{\varepsilon,\infty}} \upsilon_{\eta(t)}(dg)\right]$$

by Theorem 18 (Lévy-Itô 1). Hence, if $\int_{D_{\varepsilon,\infty}} \upsilon_{\eta(t)}(dg) = c > 0$, then T_ε has exponential distribution with mean $1/c$.

Suppose that $\upsilon_{\eta(t)}(\mathbb{R}^d) = \infty$ for every $t \in [0, T^\star]$, then $\lim_{\varepsilon\downarrow 0} P[T_\varepsilon \leq t] = 1$ for any $t > 0$, and hence $\lim_{\varepsilon\downarrow 0} T_\varepsilon = 0$ a.s. Hence there is $E_0 \in \mathcal{F}$ with $P[E_0] = 1$ such that for any $\omega \in E_0$ the time 0 is a limiting point of jumping times of $G_t(\omega)$, and for any $s > 0$ there is $E_s \in \mathcal{F}$ with $P[E_s] = 1$ such that for any $\omega \in E_s$ the set of jumping times s as a limiting point from the right. Consider $E = \cap_{s\in\mathbb{Q}^+} E_s$. Jumping times are dense in $[0,\infty)$ for any $\omega \in E$.

Now, suppose that $0 < \upsilon_{\eta(t)}(\mathbb{R}^d) < \infty$ for every $t \in [0, T^\star]$. By Theorem 18 (Lévy-Itô 1), $\mu_{\eta(t)}$ has a Poisson distribution with mean $t\upsilon_{\eta(t)}(\mathbb{R}^d)$ and $\mu_{\eta(t)} < \infty$ a.s. Hence the jumping times are enumerable in increasing order. The first jumping time T has exponential distribution with mean $1/\upsilon_{\eta(t)}(\mathbb{R}^d)$ because $P[T \leq t] = P[\mu(t) \geq 1] = 1 - e^{-t\upsilon_{\eta(t)}(\mathbb{R}^d)}$. It follows that $T(\omega) < \infty$ a.s. Let $T^{(s)}$ be the first jumping time after s, hence $T^{(s)} < \infty$ a.s. Hence there are infinitely many jumps, a.s. $\square$

Notice that in the case of $0 < \upsilon_{\eta(t)}(\mathbb{R}^d) < \infty$ we can actually say more: if we denote the nth jumping time by $U_n(\omega)$ and $U_0(\omega) = 0$, then $\{U_n - U_{n-1} : n \in \mathbb{N}\}$ constitutes independent identically distributed random variables, each distributed with mean $1/\upsilon_{\eta(t)}(\mathbb{R}^d)$ and $\lim_{n\longrightarrow\infty} U_n(\omega) = \infty$ a.s. To see this note that $\left\{\mu_{\eta(t)}\right\}$ is a Poisson process with parameter $\upsilon_{\eta(t)}(\mathbb{R}^d)$.

Definition 15. *A **LIBOR additive process** $\{Z_t\}_{t\geq 0}$ is said to be **increasing** if $Z_t(\omega)$ is increasing as a function of t, a.s.*

Theorem 22 (Increasingness)**.** *Let $d = 1$. A **LIBOR additive process** $\{Z_t\}_{t\geq 0}$ on $\mathbb{R}$ is **increasing** if and only if $A_{\eta(t)} = 0$, $\int_{(-\infty,0)} \upsilon_{\eta(t)}(dz) = 0$, $\int_{(0,\infty)} z\cdot\upsilon_{\eta(t)}(dz) < \infty$ and $\gamma_{\eta(t)} \geq 0$, for every $t \in [0, T^\star]$.*

Proof. The *'only if'* part follows from the fact that $\int_{(-\infty,0)} \upsilon_{\eta(t)}(dz) = 0$ implies $\mu_{\eta(t)}((0,t]\times(-\infty,0)) = 0$ and therefore $\{Z_t\}_{t\geq 0}$ does not have negative jumps. Hence, by the last theorem we have that

$$Z_t = t\cdot\gamma_{\eta(t)} + \int_{(0,t]\times(0,\infty)} z\cdot\mu_{\eta(t)}(d(s,z)) \text{ a.s. for every } t \in [0, T^\star]$$

because the continuous part $\tilde{Z}_t = t\cdot\gamma_{\eta(t)}$ and this shows that $\{Z_t\}_{t\geq 0}$ is increasing.

And the *'if'* part follows from the fact that since $\{Z_t\}_{t\geq 0}$ has no negative jumps, we have that $\upsilon_{\eta(t)}((-\infty,0)) = 0$ for every $t \in [0, T^\star]$. Since an increasing function remains increasing after a finite number of its jumps are deleted, we have $Z(t) - Z_\varepsilon(t) \geq 0$, hence

$$\begin{aligned} Z'(t) &= \lim_{\varepsilon\downarrow 0} Z_\varepsilon(t) \\ &= \int_{(0,t]\times(0,\infty)} z\cdot\mu_{\eta(t)}(d(s,z)) \end{aligned}$$

exists and is bounded above by $Z(t)$.

Hence, we have that the *generating function* or *Laplace transform* of its distribution is

$$\begin{aligned}\mathbb{E}\left[e^{-uZ_{\varepsilon}(t)}\right] &= \exp\left[t\int_{(\varepsilon,\infty)}\left(e^{-uz}-1\right)\cdot \upsilon_{\eta(t)}(dz)\right]\\ &= \exp\left[t\int_{(\varepsilon,\infty)}\left(e^{-uz}-1+uz1_{(0,1]}(z)\right)\cdot \upsilon_{\eta(t)}(dz)-tu\int_{(\varepsilon,1]}z\cdot \upsilon_{\eta(t)}(dz)\right]\end{aligned}$$

for $u>0$.

As $\varepsilon\downarrow 0$, $\mathbb{E}\left[e^{-uZ_{\varepsilon}(t)}\right]$ tends to $\mathbb{E}\left[e^{-uZ'(t)}\right]$ which is positive, and

$$\int_{(\varepsilon,\infty)}\left(e^{-uz}-1+uz1_{(0,1]}(z)\right)\cdot \upsilon_{\eta(t)}(dz)$$

tends to the integral over $(0,\infty)$ which is finite.

Hence, we have that $\int_{(0,\infty)} z\cdot \upsilon_{\eta(t)}(dz)<\infty$, and directly by application of Theorem 19 (Lévy-Itô 2) we have that $Z(t)=\hat{Z}(t)+\tilde{Z}(t)$, where the jump part $\hat{Z}(t)=Z'(t)$ and the continuous part $\tilde{Z}(t)$ has the generating system of triplets $\left(A_{\eta(t)},0,\gamma_{\eta(t)}\right)$. But $\tilde{Z}(t)=Z(t)-Z'(t)\geq 0$ and therefore $A_{\eta(t)}=0$ and $\gamma_{\eta(t)}\geq 0$ for every $t\in[0,T^{\star}]$. □

According to the last theorem, notice that a **LIBOR additive process** on $\mathbb{R}$ generated by $\left\{\left(\gamma_{\eta(t)}(t),A_{\eta(t)},\upsilon_{\eta(t)}\right)\right\}$ with $A_{\eta(t)}=0,\ \upsilon_{\eta(t)}((-\infty,0))=0$ and $\int_{(0,1]} z\cdot \upsilon_{\eta(t)}(dz)=\infty$ for every $t\in[0,T^{\star}]$, has **positive jumps** only, does not have Brownian-like part, but it is fluctuating, **not increasing**, no matter how large γ is. An explanation is that such a process can exist only with infinitely strong drift in the negative direction, which cancels the divergence of the sum of jumps; but it causes a random continuous motion in the negative direction. It is clear that an **increasing LIBOR additive process** will not have negative jumps, but also **the drift has to be positive or zero**, for every interval in I.

8. Density Transformations of LIBOR additive processes

In this subsection we cite the most important results from **Jacod and Shiryaev** (1987) Chapter III, **Bjork, Kabanov and Runggaldier** (1997) and **Sato** (1999) Chapter 6, concerning **Girsanov's theorem** and the explicit computation of density processes of absolutely continuous probability measures. This subsection mainly serves the purpose of preparation for the usual change of measure for financial models driven by **LIBOR additive processes**.

Roughly speaking, the basic idea is the following. Let G be a semimartingale on some stochastic basis $(\Omega,\mathcal{F},\mathbb{P})$. Then it is well known that the class of semimartingales is invariant with respect to equivalent transformation of measure, or in other words, G remains a semimartingale on $(\Omega,\mathcal{F},\mathbb{Q})$ (see **Rogers and Williams** (1987) IV.38) where $\mathbb{Q}$ is locally absolutely continuous to $\mathbb{P}$. This change of measure can be described by two sequences β_i and Y_i (we will give an explicit expression later) called **Girsanov quantities**, in the sense that the density process Z of $\mathbb{Q}$ with respect to $\mathbb{P}$ can be expressed via β_i and Y_i.

As usual, we assume that two measures $\mathbb{P}$ and $\mathbb{Q}$ on a common measurable space $(\Omega,\mathbb{G})$ are called *mutually absolutely continuous* or **equivalent measures**, written $\mathbb{P}\approx\mathbb{Q}$ if the collection $\{B\in\mathcal{G}_t:\mathbb{P}(B)=0\}$ is identical with $\{B\in\mathcal{G}_t:\mathbb{Q}(B)=0\}$. The **Radon-Nikodym derivative** of $\mathbb{Q}$ with respect to $\mathbb{P}$ is denoted by $\frac{d\mathbb{Q}}{d\mathbb{P}}$. If $\mathbb{P}\approx\mathbb{Q}$ then $\frac{d\mathbb{Q}}{d\mathbb{P}}$ is positive and finite $\mathbb{P}$-almost everywhere.

Let us start with the following useful theorems about **stochastic exponentials**, and **nice-versions of triplets characteristics** for the **LIBOR additive process**.

Theorem 23. *Let $G = \{G_t : t \geq 0\}$ be a* ***LIBOR additive process*** *and consider the stochastic differential equation*

$$dZ = Z_- dG, \quad Z_0 = 1$$

This equation has a unique (up to indistinguishability) "càdlàg" adapted solution, called the ***stochastic exponential*** *of G, which is a semimartingale and is denoted by $\mathcal{E}(G)$. Explicitly*

$$\mathcal{E}(G)_t = \exp\left(G_t - \frac{1}{2}\left\langle \tilde{G}\right\rangle_t\right) \prod_{s \leq t} \left(1 + \hat{G}_s\right) e^{-\hat{G}_s}$$

where $\tilde{G}_t$ is the continuous part in t, and $\hat{G}_t$ is the jump part of G_t. If we define $\tau := \inf\left\{t \geq 0 : \hat{G}_t = -1\right\}$ then $\mathcal{E}(G) \neq 0$ on $[0, \tau)$, and $\mathcal{E}(G) = 0$ on $[\tau, \infty)$.

Proof. cf. **Jacod and Shiryaev** (1987) Theorem I.4.61 □

Theorem 24. *Let $G = \{G_t : t \geq 0\}$ be a* ***LIBOR additive process*** *on $\mathbb{R}^d$. Then there exist a* ***"nice-version"*** *of the triplet characteristics for G which is of the form*

$$\begin{cases} A_{\eta_{(t)}} = \left(A_{\eta_{(t)}}, (i,j)\right)_{i,j \leq d} = \sum_{j \leq \eta(t)} \int_{T_{j-1}}^{T_j \wedge t} \hat{a}_{\eta_{(t)}}(i,j) \cdot dH_{\eta_{(t)}}(\omega, t) \\ \upsilon_{\eta_{(t)}}(\omega; dt, dg) = K_{\omega,t}(dg) \cdot dH_{\eta_{(t)}}(\omega) \\ \gamma_{\eta_{(t)},(i)} = \sum_{j \leq \eta(t)} \int_{T_{j-1}}^{T_j \wedge t} b_{\eta_{(t)}},(i) \cdot dH_{\eta_{(t)}}(\omega, t) \end{cases}$$

where $H_{\eta_{(t)}}$ is a real-valued predictable, increasing and locally integrable process, $\hat{\gamma}_{\eta_{(t)}}(t) = \left(b_{\eta_{(t)}}(i)\right)_{1 \leq i \leq d}$ is an $\mathbb{R}^d$-valued predictable process, $\hat{A}_{\eta_{(t)}} = \left(\hat{a}_{\eta_{(t)}}(i,j)\right)_{1 \leq i,j \leq d}$ is a predictable process with values in the set of all symmetric nonnegative definite $d \times d$ matrices, and $K_{\omega,t}(dg)$ is a transition kernel from $(\Omega \times \mathbb{R}_+, \mathbb{P})$ into $\left(\mathbb{R}^d, \mathcal{B}^d\right)$ which satisfies for any $t \in [0, T^\star]$

$$K_{\omega,t}(\{0\}) = 0 \text{ and } \int_{\mathbb{R}^d} \left(|g|^2 \wedge 1\right) K_{\omega,t}(dg) \leq 1$$

Proof. cf. **Jacod and Shiryaev** (1987) Proposition $II.2.9$. □

Theorem 25 (**Girsanov's theorem for semimartingales**)**.** *Assume that $\mathbb{P}$ and $\mathbb{Q}$ are (locally) equivalent measures, and let $G = \{G_t : t \geq 0\}$ be a* ***LIBOR additive process*** *(piecewise homogeneous semimartingale) with $\mathbb{P}$-characteristics $\left(A_{\eta_{(t)}}, \nu_{\eta_{(t)}}, \gamma_{\eta_{(t)}}(t)\right)\Big|_{\mathbb{P}}$. Let $\hat{a}_{\eta_{(t)}}$ and $H_{\eta_{(t)}}$ be the processes of the "nice-version" from Theorem 47 for every $\eta_{(t)} = 0, 1, ..., n$ and $t \in [0, T^\star]$, and let $\mathbb{P} \approx \mathbb{Q}$. Then there exists a sequence $\left(Y_{\eta_{(t)}}, \beta_{\eta_{(t)}}\right)$ where $Y_{\eta_{(t)}j}$ is a $\mathbb{P}$-measurable nonnegative function and $\beta_{\eta_{(t)}} = \left(\beta_{\eta_{(t)}}(i)\right)_{i \leq d}$ is a sequence of predictable processes, for any $t \in [0, T^\star]$ satisfying*

$$\begin{array}{l} \int \left(Y_{\eta_{(t)}} - 1\right) c(g)\, dv^{\mathbb{P}}_{\eta_{(t)}} < \infty \\ \sum_{j \leq \eta(t)} \int_{T_{j-1}}^{T_j \wedge t} \left|\hat{a}_{\eta_{(t)}} \beta_{\eta_{(t)}}\right| dH_{\eta_{(t)}} < \infty \\ \sum_{j \leq \eta(t)} \int_{T_{j-1}}^{T_j \wedge t} \left(\beta'_{\eta_{(t)}} \hat{a}_{\eta_{(t)}} \beta_{\eta_{(t)}}\right) dH_{\eta_{(t)}} < \infty \end{array}$$

$\mathbb{Q}$-a.s. for any $t \in [0, T^\star]$ and such that a version of the characteristics of G relative to $\mathbb{Q}$ are $\left(A^{\mathbb{Q}}_{\eta_{(t)}}, \nu^{\mathbb{Q}}_{\eta_{(t)}}, \gamma^{\mathbb{Q}}_{\eta_{(t)}}(t)\right)$, such that

$$\begin{cases} A^{\mathbb{Q}}_{\eta_{(t)}} = A^{\mathbb{P}}_{\eta_{(t)}} \\ \upsilon^{\mathbb{Q}}_{\eta_{(t)}} = Y_{\eta_{(t)}} \cdot \upsilon^{\mathbb{P}}_{\eta_{(t)}} \\ \gamma^{\mathbb{Q}}_{\eta_{(t)}}(t) = \gamma^{\mathbb{P}}_{\eta_{(t)}} + \sum_{j \leq \eta(t)} \int_{T_{j-1}}^{T_j \wedge t} \left|\hat{a}_j \beta_j\right| dH_j + \int_{\mathbb{R}} \left(Y_{\eta_{(t)}} - 1\right) c(g)\, \upsilon^{\mathbb{P}}_{\eta_{(t)}}(dg) \end{cases}$$

Proof. cf. **Jacod and Shiryaev** (1987) Theorem $III.3.24$ □

Definition 16. *The quantities β_j and Y_j for any $j = 0, ..., \eta(t)$ from Theorem* 25 *are called **Girsanov quantities** of $\mathbb{Q}$ with respect to $\mathbb{P}$ relative to G, or simply Girsanov quantities of $\mathbb{Q}$.*

Remark 4. *Notice that the $Y_{\eta_{(t)}}$ describe how the jump distribution of G change when we pass from $\mathbb{P}$ to $\mathbb{Q}$, and $\beta_{\eta_{(t)}}$ together with $Y_{\eta_{(t)}}$ determines the changes in drift. On the other hand, notice that the **Girsanov quantities** are not unique: from the uniqueness of $\upsilon^{\mathbb{P}}_{\eta_{(t)}}$ and $\upsilon^{\mathbb{Q}}_{\eta_{(t)}}$ we only get uniqueness of $Y_{\eta_{(t)}}$ on supp $\upsilon^{\mathbb{P}}_{\eta_{(t)}}$. And with the uniqueness of $\gamma^{\mathbb{Q}}_{\eta_{(t)}}(t)$ and $\gamma^{\mathbb{P}}_{\eta_{(t)}}$ we only get the uniqueness of $\hat{a}_{\eta_{(t)}}\beta_{\eta_{(t)}}$ for fixed $\hat{a}_{\eta_{(t)}}$, for any $t \in [0, T^\star]$.*

Example 1. *Let W be a standard $\mathbb{R}^d$-valued $\mathbb{P}$-Brownian motion and let A be a $d \times d$ matrix. Set $G := A \cdot W$ and take $\mathbb{P} \approx \mathbb{Q}$ with Girsanov quantities β and Y relative to G. Then $\tilde{G} = G$, so $\langle G \rangle_t = AA't$, and the $\mathbb{P}$-characteristic triplet of G are given by*

$$\begin{aligned} A^{\mathbb{P}}_t &= at \\ \nu^{\mathbb{P}}_t &= 0 \\ \gamma^{\mathbb{P}}(t) &= 0 \end{aligned}$$

where $a := AA'$ is the covariance matrix of G. By Theorem 25 *we get the $\mathbb{Q}$-characteristic triplet of G by*

$$\begin{aligned} A^{\mathbb{Q}}_t &= at \\ \nu^{\mathbb{Q}}_t &= 0 \\ \gamma^{\mathbb{Q}}(t) &= \int_0^t (a\beta_s)\, ds \end{aligned}$$

Note that G remains a Lévy process under $\mathbb{Q}$ if and only if β is deterministic and independent of time. In this case, G is a linear transformation of a standard Brownian motion with constant drift, i.e. $G_t = AW^{\mathbb{Q}}_t + a\beta t$.

9. Examples of LIBOR additive processes

In the following sections we briefly list a number of popular processes that can be studied as special cases of the **LIBOR additive processes** or non-homogeneous Lévy processes. We pay special attention to their density function, their characteristic function, their characteristic triplet together with some other properties.

9.1. **The non-homogeneous Poisson Process.**

Given a tenor structure $0 = T_0 < T_1 ... < T_n = T^\star$, the non-homogeneous **Poisson process** is the simplest LIBOR additive process we can think of. It is based on the Poisson $(\lambda_j)_{j=1...n}$ distribution which has

$$\hat{\mu}_j(z) = \exp\left[\lambda_j\left(\exp[iz] - 1\right)\right]$$

as a characteristic function for any $j = 0, 1, ..., n$.

Like an ordinary Poisson process it has independent increments and these increments are Poisson distributed, but increments over different intervals of equal length can have different means. In particular, the number of jumps in an interval $(T_j, T_{j+1}]$ has a Poisson distribution with mean $\Lambda(T_{j+1}) - \Lambda(T_j)$ where

$$\Lambda(t) = \sum_{j \le \eta(t)} \int_{T_j}^{t \wedge T_{j+1}} \lambda_j(s)\, ds$$

Therefore, the Poisson distribution lives on the nonnegative integers $\{0,1,2,...\}$, such that

$$\mathbb{P}(G_t = g) = \frac{e^{-\Lambda(t)}\Lambda(t)^g}{g!}$$

Since the in-homogeneous Poisson $(\mu_j)_{j=1...n}$ distribution is infinitely divisible we can define an in-homogeneous Poisson process as the process that starts at zero, has independent increments property, is stochastically continuous and has piecewise stationary increments. The in-homogeneous Poisson process turns out to be an increasing pure jump process, with jump sizes always equal to 1. This means that the additive process triplet is given by

$$\left[0, 0, \lambda_{\eta(t)}\delta(1)\right]$$

where $\delta(1)$ denotes the Dirac measure at point 1.

9.2. **The non-homogeneous Compound Poisson Process.** Let $N = \{N_t, t \geq 0\}$ be a non-homogeneous Poisson process with intensity parameters $(\lambda_j)_{j=1...n}$ and let J_k be independent and identically distributed (i.i.d.) random variables independent of N and following a law, μ_J say, with characteristic function $\hat{\mu}_J(z)$. Then we say that $G = \{G_t : t \geq 0\}$ is a **non-homogeneous Compound Poisson process** if

$$G_t = \sum_{k=1}^{N_t} J_k, t \geq 0$$

The value of the process at time t, G_t, is a sum of N_t random numbers with law μ_J. Notice that the ordinary non-homogeneous Poisson process corresponds to the case where $J_k, k = 1, 2, ...$ i.e. have law μ_J degenerate at the point 1.

Let us write, for a Borel set A, the distribution function of the law μ_J as follows

$$\mathbb{P}(J_k \in A)_t = \frac{\upsilon_{\eta(t)}(A)}{\lambda_{\eta(t)}}$$

where $\upsilon_{\eta(t)}(\mathbb{R}) = \lambda_{\eta(t)} < \infty$ with $\upsilon_{\eta(t)}(\{0\}) = 0$. Then the characteristic function of G_t is given by

$$\begin{aligned}
\mathbb{E}[\exp(izG_t)] &= \prod_{j \leq \eta(t)} \mathbb{E}\left(\exp\left[iz\left(G_{t \wedge T_{j+1}} - G_{T_j}\right)\right]\right) \\
&= \prod_{j \leq \eta(t)} \exp\left(\left(t \wedge T_{j+1} - T_j\right) \int_{\mathbb{R}} \left(e^{izg} - 1\right) \upsilon_j(dg)\right) \\
&= \prod_{j \leq \eta(t)} \exp\left(\left(t \wedge T_{j+1} - T_j\right) \lambda_j \left(\mu_J(z) - 1\right)\right)
\end{aligned}$$

From this we can easily obtain the characteristic triplet

$$\left[\int_{-1}^{+1} g\upsilon_{\eta(t)}(dg), 0, \upsilon_{\eta(t)}(dg)\right]$$

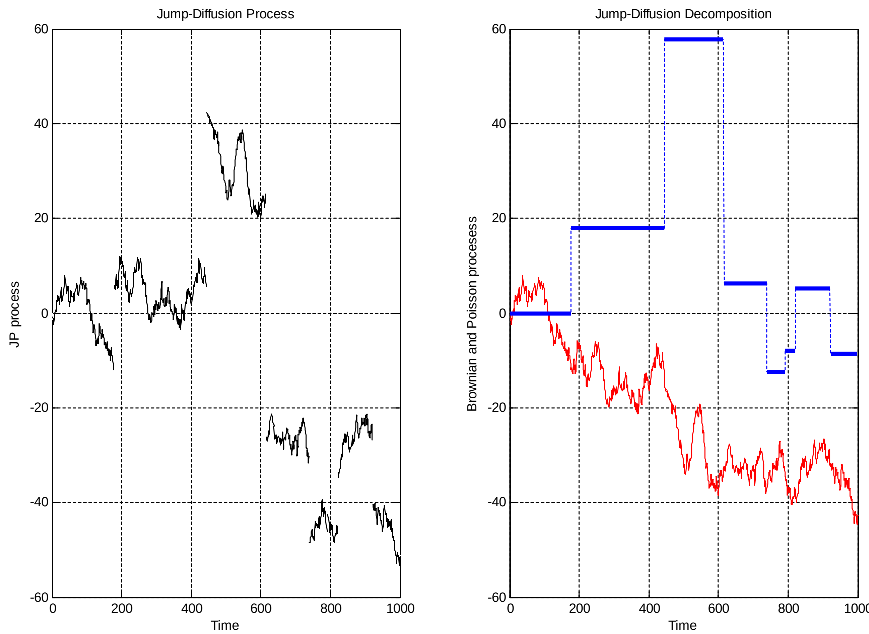


Figure (1): Sample-path of a jump-diffusion process, with the Lévy-Itò decomposition

The non-homogeneous compound Poisson process is usually introduced in the simulation of jump-diffusion sample paths on a fixed grid is shown in **Figure** (1). On the left side, the figure shows us two independent sample paths, the usual continuous-part given by a Brownian motion, and the jump-part simulated using a compound Poisson process. Both together, by application of Lévy-Itô Theorem 18, form the jump-diffusion sample-path that appear in the right part of Figure (1). Efficient algorithms to simulate Poisson, exponential and Gaussian processes can be found in **Press et al.** (1992). Different methods of approximation of the small jumps are discussed in **Schoutens** (2003).

9.3. **The non-homogeneous Gamma Process.** Here the **non-homogeneous Gamma process** $G = \{G_t : t \geq 0\}$ with parameters $a_j, b_j > 0$ for every $j = 0, 1, ..., n$, is defined as the stochastic process which starts at zero and has independent increments, is stochastically continuous and has independent Gamma distributed increments inside of each interval for every $j = 0, 1, ..., n$.

The density function of the **non-homogeneous Gamma distribution** (a_j, b_j) with $a_j > 0$ and $b_j > 0$ for every $j = 0, 1, ..., n$ is given by

$$\mu_j (g; a_j, b_j) = \frac{b_j^{a_j}}{\Gamma(a_j)} g^{a_j - 1} \exp(-g b_j), \text{ with } g > 0$$

Notice that this function has a semi-heavy (right) tail.

The characteristic function is given by

$$\hat{\mu}_j (z; a_j, b_j) = \left(1 - \frac{iz}{b_j}\right)^{-a_j}$$

and obviously, it is infinitely divisible. The characteristic triplet of a non-homogeneous Gamma process is given by

$$\left[\frac{a_j (1 - \exp(b_j))}{b_j}, 0, a_j \frac{\exp(-b_j g) 1_{\{g>0\}} dg}{g}\right]$$

The most common method of simulation for Gamma processes can be found in **Marsaglia and Tsang** (2000).

9.4. **The non-homogeneous Inverse Gaussian Process.** Let $T^{(a,b)}$ be the first time a standard Brownian motion with drift $b>0$ reaches the positive level $a>0$. It is well known that this random time follows the so-called **Inverse Gaussian** $IG(a,b)$ law. The IG distribution is infinitely divisible. Hence we can define the IG process $G=\{G_t : t\geq 0\}$ with parameters $a_j, b_j>0$ for any $j=0,1,...,n$ as the process which starts at zero, is stochastically continuous and has independent IG distributed increments, homogeneous only inside of each interval for every $j=0,1,...,n$ such that for any $t\in[0,T^\star]$

$$\begin{aligned}\mathbb{E}\left(\exp\left[izG_t\right]\right) &= \prod_{j\leq\eta(t)}\mathbb{E}\left(\exp\left[iz\left(G_{t\wedge T_{j+1}}-G_{T_j}\right)\right]\right)\\ &= \prod_{j\leq\eta(t)}\exp\left[-a_j\left(t\wedge T_{j+1}-T_j\right)\left(\sqrt{-2iz+b_j^2}-b_j\right)\right]\end{aligned}$$

The density function of the $IG(a_j,b_j)$, for any $j=0,1,...,n$ is explicitly known:

$$\mu_j\left(g;a_j,b_j\right)=\frac{a_j}{\sqrt{2\pi}}\exp\left(a_jb_j\right)g^{-3/2}\exp\left(-\frac{1}{2}\left(\frac{a_j^2}{g}+b_j^2g\right)\right)$$

and the Lévy measure associated to the $IG(a_j,b_j)$ law is given by

$$\upsilon_j\left(dg\right)=\frac{a_j}{\sqrt{2\pi}}g^{-3/2}\exp\left(-\frac{1}{2}b_j^2g\right)1_{\{g>0\}}dg$$

The first component of the characteristic triplet equals

$$\gamma_j=\frac{a_j}{b_j}\left(2\Phi\left(b_j\right)-1\right)$$

where $\Phi(x)$ is the Normal distribution function.

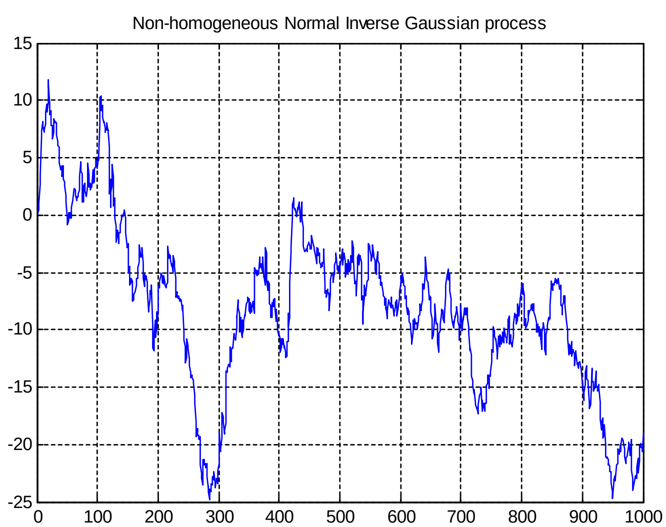


Figure (2): Non-homogeneous IG process with parameters $(a_j,0)$.

Figure (2) represents a non-homogeneous inverse Gaussian process with parameters $(a_j,0)$ where a_j goes from $a_0=1$ to $a_{1000}=2$. Simulation algorithm can be found in **Prause** (1997).

9.5. **The non-homogeneous Generalized Inverse Gaussian Process.** The **Inverse Gaussian** $IG(a,b)$ law can be generalized to what is called the **Generalized Inverse Gaussian** distribution $GIG(a,b)$. This distribution on the positive half-line is given in terms of its density function

$$\mu(g;\lambda,a,b)=\frac{(b/a)^{\lambda}}{2K_{\lambda}(ab)}g^{\lambda-1}\exp\left(-\frac{1}{2}\left(\frac{a^2}{g}+b^2g\right)\right)$$

The parameters λ, a and b are such that $\lambda\in\mathbb{R}$ while a and b are both nonnegative and not simultaneously 0.

The characteristic function is given by

$$\hat{\mu}(k;\lambda,a,b)=\frac{1}{K_{\lambda}(ab)}\left(1-2ik/b^2\right)^{\lambda/2}K_{\lambda}\left(ab\sqrt{1-2ik/b^{-2}}\right)$$

where $K_{\lambda}(g)$ denotes the modified Bessel function with the index λ.

Hence we can define the non-homogeneous GIG process as a stochastic process that starts at zero, is continuous in probability with independent increments, and is piecewise stationary or stationary inside each interval $[T_j,T_{j+1}]$ for $j=0,1,...,n$. According to **Barndorff-Nielsen and Shephard** (2001), it has an infinitely divisible distribution with the following Lévy measure

$$\upsilon_j(dg)=\frac{\exp\left(-\frac{1}{2}b_jg\right)}{g}\left(a_j^2\int_0^{\infty}\exp(-gz)\,h_j(z)dz+\max\{0,\lambda_j\}\right)dg$$

where

$$h_j(z)=\left(\pi^2a_jz\left(J_{|\lambda|}^2\left(a_j\sqrt{2z}\right)+N_{|\lambda|}^2\left(a_j\sqrt{2z}\right)\right)\right)^{-1}$$

and where J and N are Bessel functions. Simulation algorithm can be found in **Prause** (1997).

9.6. **The non-homogeneous α-Stable Process.** Since **Mandelbrot** (1963) introduced the α-stable distribution to model the empirical distribution of asset prices, the α-stable distribution became a popular alternative to the normal distribution which has been rejected by numerous empirical studies that have found financial return series to be heavy-tailed and possibly skewed.

More explicitly, we can define the non-homogeneous α-stable process as a stochastic process that starts at zero, is continuous in probability with independent increments, and is piecewise stationary or stationary inside each interval $[T_j,T_{j+1}]$ for $j=0,1,...,n$ with a Lévy measure of the form

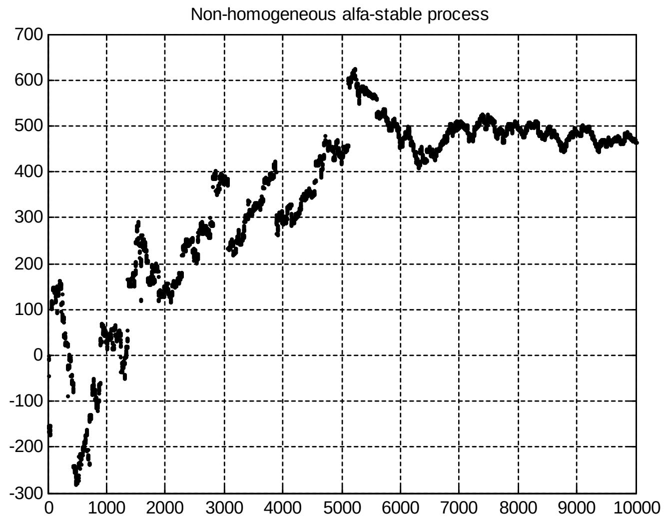
$$\upsilon_j(dg)=\frac{A}{g^{\alpha_j+1}}1_{\{g>0\}}dg+\frac{B}{g^{\alpha_j+1}}1_{\{g<0\}}dg$$

for some positive constants A and B[6]. The characteristic function of a real-valued non-homogeneous stable random variable G has the form

$$\hat{\mu}_j(z)=\begin{cases}\exp\left\{-\sigma_j^{\alpha_j}|z|^{\alpha_j}\left(1-i\beta_j\ sgn\ z\ \tan\frac{\pi\alpha_j}{2}\right)+i\theta z\right\}, & \text{if } \alpha\neq 1\\ \exp\left\{-\sigma_j|z|\left(1+i\beta_j\frac{2}{\pi}\ sgn\ z\ \log|z|\right)+i\theta z\right\}, & \text{if } \alpha=1\end{cases}$$

where $\alpha_j\in(0,2]$, $\sigma_j\geq 0$, $\beta_j\in[-1,1]$ and $\theta\in\mathbb{R}$,

[6]Note the link between between α-stable processes and TS processes. A tempered stable process (TS) is usually obtained by taking a one-dimensional stable process and multiplying the Lévy measure with a decreasing exponential on each half of the real axis.



Figure(3): Non-homogeneous α-stable process, where α goes from 1 to 2 (Brownian case)

Chambers, Mallows and Stuck (1976) describe a simulation method for generating α-stable processes with a set of admissible parameters, and also provide a list of Fortran programs to simulate this process. **Figure (3)** shows us a sample path of a non-homogeneous α-stable process where α moves uniformly with the time from 1 to 2 (Brownian case).

9.7. **The non-homogeneous Tempered Stable Process.** The class of the **Tempered Stable** (TS) **distributions** was proposed by **Tweedie** (1984) and **Koponen** (1995), but this class of distributions may be generalized to the so called class of **Modified Stable distributions** due to **Barndorff-Nielsen and Shephard** (2003) and **Rosinski** (2006). The distribution function is not available in closed form but the characteristic function of the **Tempered Stable** (TS) distribution law $TS(\lambda, a, b)$ with $a > 0$, $b \geq 0$ and $0 > \lambda > 1$, is given by

$$\hat{\mu}\left(z;\lambda,a,b\right) = \exp\left(ab - a\left(b^{1/\lambda} - 2iz\right)^{\lambda}\right)$$

We can define the **non-homogeneous Tempered Stable** (TS) process $G = \{G_t : t \geq 0\}$ as the process which starts at zero, has independent increments and is stochastically continuous with piecewise stationary increments. It has an infinitely divisible distribution and, from the characteristic function, we can derive the Lévy measure of the non-homogeneous TS process

$$\upsilon_j\left(dg\right) = a_j 2^{\lambda}\frac{\lambda}{\Gamma\left(1-\lambda\right)} g^{-\lambda-1}\exp\left(-\frac{1}{2}b_j^{1/\lambda}g\right)1_{\{g>0\}}dg$$

The process is a subordinator and has infinite activity. The first term of the characteristic triplet is given by

$$\gamma_j = a_j 2^{\lambda}\frac{\lambda}{\Gamma\left(1-\lambda\right)}\int_0^1 g^{-\lambda}\exp\left(-\frac{1}{2}b_j^{1/\lambda}g\right)dg$$

As we have mentioned in the case of the α-stable case, neither the density function nor specific random number generators are available. In order to simulate other techniques are available in the literature. The most common method is based on the so-called rejection-method by **Rosinski** (2002).

9.8. **The non-homogeneous Variance Gamma Process.** The class of **Variance Gamma** (VG) distributions was introduced by **Madan and Seneta** (1990) and **Madan and Milne** (1991) as a model for stock returns, and it generates a finite variation process with infinite but relatively low activity of small jumps. The VG process proposed in **Madan et al.** (1998) is obtained by evaluating arithmetic Brownian motion with drift θ and volatility σ at a random time given by a gamma process having a mean rate per unit time of 1 and a variance rate of v. Specifically, we have

$$G_t^{VG}(\sigma, v, \theta) = \theta G_t^v + \sigma W_{G_t^v}$$

where G_t^v is the gamma process with mean rate 1 and variance rate v, independent of W.

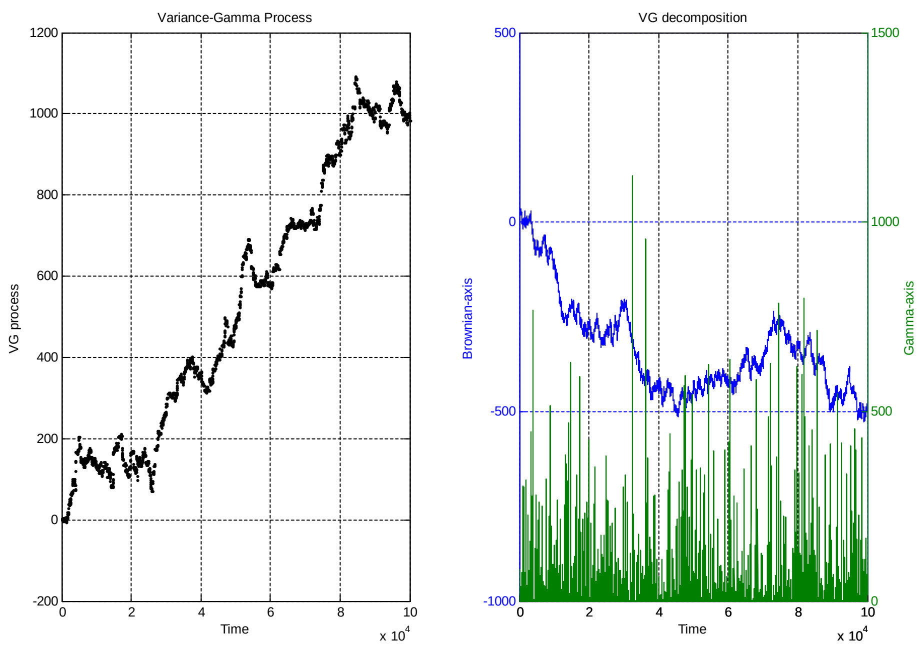


Figure (4): Simulation of a VG Process as a Time-Changed Brownian Motion

Figure (4) shows this composition of processes on the right side, and the resulting process $G_t(\sigma, v, \theta)$ (on the left side) is a pure jump process with infinite activity that has two additional parameters, providing control over skewness and kurtosis, respectively.

The **characteristic function** of the $VG(\sigma, v, \theta)$ law is easily evaluated as

$$\hat{\mu}(z; \sigma, v, \theta) = \left(1 - iz\theta v + \frac{1}{2}\sigma^2 v z^2\right)^{-1/v}$$

- Hence, we can define the **non-homogeneous Variance Gamma** (VG) **process** $\{G_t : t \geq 0\}$ as the process that starts at zero, has independent and piecewise stationary increments, with the following characteristic function

$$\begin{aligned}\mathbb{E}(\exp[izG_t]) &= \prod_{j \leq \eta(t)} \mathbb{E}\left(\exp\left[iz\left(G_{t\wedge T_{j+1}} - G_{T_j}\right)\right]\right) \\ &= \prod_{j \leq \eta(t)} \left(1 - iz\theta_j v_j + \frac{1}{2}\sigma_j^2 v_j z^2\right)^{-(t\wedge T_{j+1} - T_j)/v_j}\end{aligned}$$

The Lévy density for the variance gamma process may be derived directly from the **Lévy-Khintchine** theorem. Alternatively, one may exploit the representation of the variance gamma process as the difference of two independent gamma processes. It is shown in **Carr et al.** (2002) that this characterization leads to the following **Lévy measure**

$$\upsilon_j\left(dg\right) = \begin{cases} C_j \exp\left(G_j g\right) \left|g\right|^{-1} dg & g < 0 \\ C_j \exp\left(G_j g\right) \left|g\right|^{-1} dg & g > 0 \end{cases}$$

for any $j = 0, 1, ..., n$ where

$$\begin{aligned} C_j &= 1/\upsilon_j > 0 \\ G_j &= \left(\sqrt{\tfrac{1}{4}\theta_j^2\upsilon_j^2 + \tfrac{1}{2}\sigma_j^2\upsilon_j} - \tfrac{1}{2}\theta_j\upsilon_j\right)^{-1} > 0 \\ M_j &= \left(\sqrt{\tfrac{1}{4}\theta_j^2\upsilon_j^2 + \tfrac{1}{2}\sigma_j^2\upsilon_j} + \tfrac{1}{2}\theta_j\upsilon_j\right)^{-1} > 0 \end{aligned}$$

With this parametrization, we are implicitly assuming that the non-homogeneous VG process is expressed as the difference of two independent non-homogeneous Gamma processes, where $G^{(1)}$ is a Gamma processes with parameters $a_j = C_j$ and $b_j = M_j$, whereas $G^{(2)}$ is an independent Gamma process with $a'_j = C'_j$ and $b'_j = G'_j$. **Figure (5)** shows how these two mentioned gamma processes (right side) can generated the sample-path for the variance gamma process (left side).

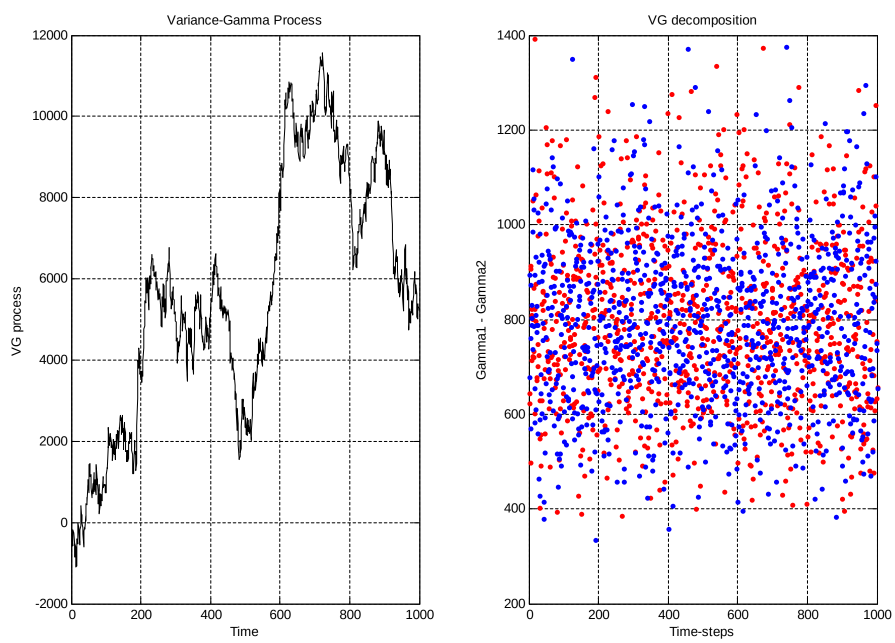


Figure (5): Simulation of a VG Process as the Difference of Two Gamma Processes

On the other hand, the Lévy measure has infinite mass, and hence a VG process has infinitely many jumps in any finite interval. Since

$$\int_{-1}^{1} |g|\, \upsilon_j\left(dg\right) < \infty$$

a VG process has paths of finite variation. A VG process has no Brownian motion component and its characteristic triplet is given by $\left[\gamma_j, 0, \upsilon_j\left(dg\right)\right]$ where

$$\gamma_j = \frac{-C_j\left(G_j\left(\exp\left(-M_j\right) - 1\right) - M_j\left(\exp\left(-G_j\right) - 1\right)\right)}{M_j G_j}$$

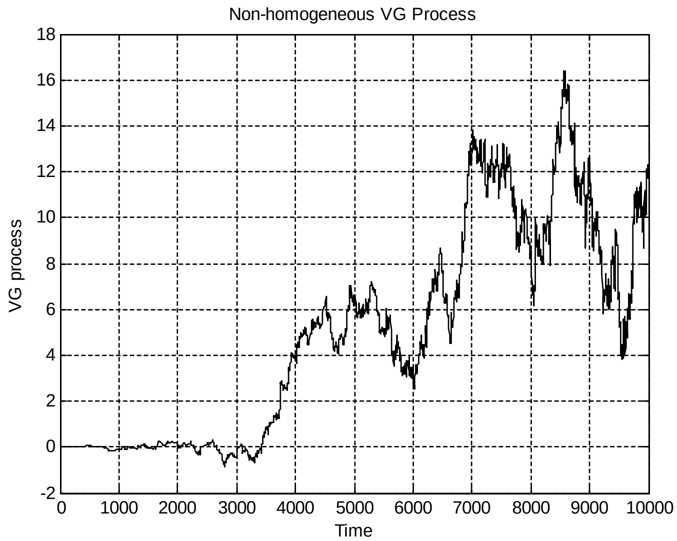


Figure (6): Non-homogeneous Variance Gamma Process.

9.9. **The non-homogeneous Normal Inverse Gaussian Process.** Following **Barndorff-Nielsen** (1995), the **Normal Inverse Gaussian** (NIG) distribution with parameters $\alpha > 0, -\alpha < \beta < \alpha$ and $\delta > 0$, $NIG(\alpha, \beta, \delta)$ has characteristic function

$$\hat{\mu}(z; \alpha, \beta, \delta) = \exp\left(-\delta\left(\sqrt{\alpha^2 - (\beta + iz)^2} - \sqrt{\alpha^2 - \beta^2}\right)\right)$$

and we can define the non-homogeneous NIG process $\{G_t : t \geq 0\}$ as a process with $G_{\{0\}} = 0$ a.s. with independent NIG distributed increments, continuous in probability and piecewise stationary.

The Lévy measure for the NIG process is given by

$$\upsilon_j(dg) = \frac{\delta_j \alpha_j}{\pi} \frac{\exp(\beta_j g) K_1(\alpha_j |g|)}{|g|} dg$$

where $K_\lambda(g)$ denotes the modified Bessel function of the third kind with index λ.

A NIG process has no Brownian component and its Lévy triplet is given by $[0, \gamma_j, \upsilon_j(dg)]$ where

$$\gamma_j = \frac{2\delta_j \alpha_j}{\pi} \int_0^1 \sinh(\beta_j g) K_1(\alpha_j g)\, dg$$

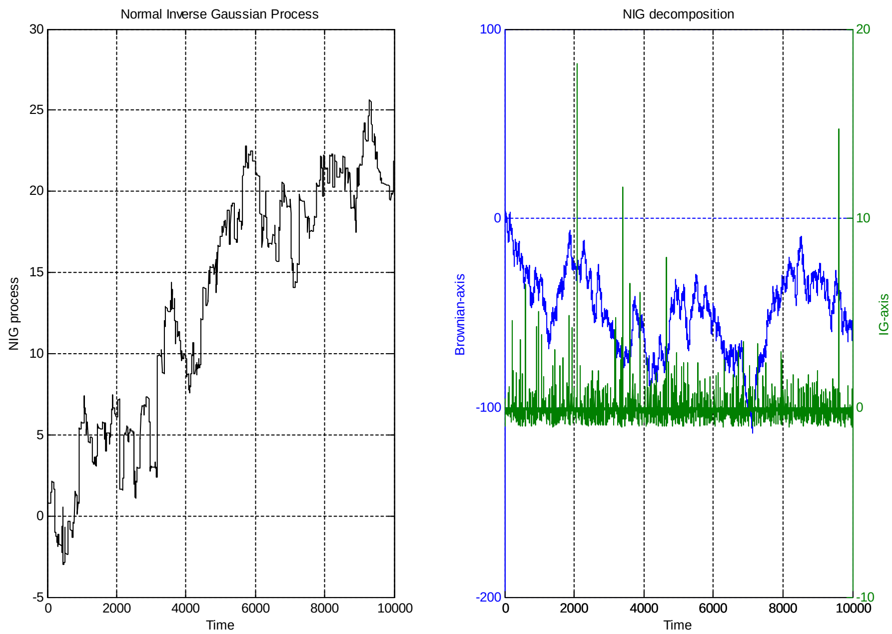


Figure (7): Simulation of a NIG process as a Time-Changed Brownian Motion

As in the VG case, we can also simulate an NIG process as a time-changed Brownian motion. **Figure (7)** shows a path of an NIG process (left side) obtained by sampling a standard Brownian motion and an IG process (right graph).

9.10. **The non-homogeneous CGMY Process.** In order to obtain a more flexible process than the Variance Gamma process, that has finite or infinite activity and infinite variation, the additional parameter Y was introduced by **Carr, Madan, Geman and Yor** (2002). Later, in **Carr et al.** (2003), this four-parameter distribution was generalized to a six-parameter case, however we present here the first case, the $CGMY(C,G,M,Y)$ distribution, with characteristic function

$$\hat{\mu}(z; C, G, M, Y) = \exp\left(C\Gamma(-Y)\left((M-iz)^Y - M^Y + (G+iz)^Y - G^Y\right)\right)$$

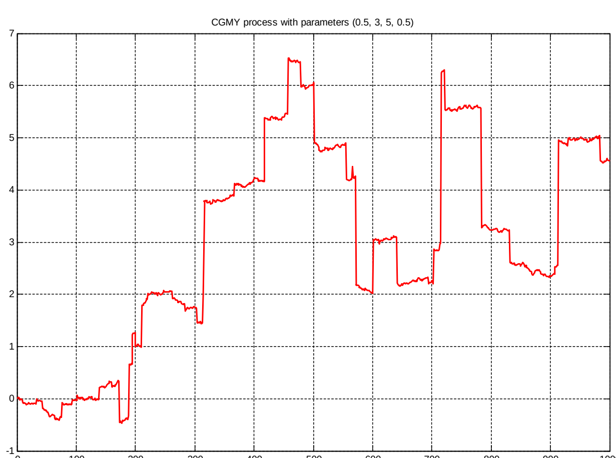


Figure (8): Simulation of a CGMY process

Based on this distribution, we can define a **non-homogeneous CGMY process** $\{G_t : t \geq 0\}$ as the process that starts at zero, has independent and piecewise stationary increments, with the following characteristic function

$$\begin{aligned}\mathbb{E}\left(\exp\left[izG_t\right]\right) &= \prod_{j\leq\eta(t)} \mathbb{E}\left(\exp\left[iz\left(G_{t\wedge T_{j+1}} - G_{T_j}\right)\right]\right) \\ &= \prod_{j\leq\eta(t)} \hat{\mu}\left(k; C_j, G_j, M_j, Y_j\right)^{(t\wedge T_{j+1}-T_j)} \\ &= \prod_{j\leq\eta(t)} \hat{\mu}\left(k; \left(t\wedge T_{j+1} - T_j\right) C_j, G_j, M_j, Y_j\right) \\ &= \prod_{j\leq\eta(t)} \exp\left(C_j\left(t\wedge T_{j+1} - T_j\right)\Gamma\left(-Y_j\right)\left(\left(M_j - iz\right)^{Y_j} - M_j^{Y_j} + \left(G_j + iz\right)^{Y_j} - G_j^{Y_j}\right)\right)\end{aligned}$$

The CGMY process is a pure jump process with triplet $\left[\gamma_j, 0, \upsilon_j\left(dg\right)\right]$, that is, it contains no Brownian part. The path behavior is determined by the Y_j parameter, which has a value restricted to $Y_j < 2$ for every $j = 0, 1, ..., n$. If $Y_j < 0$ the paths have finite jumps in any finite interval; if not, the paths have infinitely many jumps in any finite time interval, i.e. the process has infinite activity. Moreover, if the $Y_j \in [1, 2)$, the process is of infinite variation.

The Lévy measure for the nonhomogeneous CGMY process is given by

$$\upsilon_j\left(dg\right) = \begin{cases} C_j \exp\left(G_j g\right)\left|-g\right|^{-1-Y} dg & g < 0 \\ C_j \exp\left(-G_j g\right)\left|g\right|^{-1-Y} dg & g > 0 \end{cases}$$

and the first parameter of the characteristic triplet equals

$$\gamma_j = C_j\left(\int_0^1 \exp\left(-M_j g\right) g^{-Y_j} dg - \int_{-1}^0 \exp\left(G_j g\right) g^{-Y_j} dg\right)$$

9.11. **The non-homogeneous Meixner Process.** The Meixner process was introduced in **Schoutens and Teugels** (1998), **Schoutens** (2000) and **Grigelionis** (1999) later suggested that it may serve for fitting stock returns. This application to finance was worked out in **Schoutens** (2001, 2002). The density of the **Meixner distribution** (Meixner(α, β, δ)) is given by

$$\mu\left(g; \alpha, \beta, \delta\right) = \frac{\left(2\cos\left(\beta/2\right)\right)^{2\delta}}{2\alpha\pi\Gamma\left(2d\right)} \exp\left(\frac{\beta g}{\alpha}\right)\left|\Gamma\left(\delta + \frac{ig}{\alpha}\right)\right|^2$$

where $\alpha > 0, -\pi < \beta < \pi, \delta > 0$. The characteristic function of the Meixner (α, β, δ) distribution is given by

$$\hat{\mu}(z; \alpha, \beta, \delta) = \left(\frac{\cos\left(\beta/2\right)}{\cosh\left(\left(\alpha z - i\beta\right)/2\right)}\right)^{2\delta}$$

Hence we can define the **Meixner process** $\{G_t : t \geq 0\}$ as the process that starts at zero, has independent and piecewise stationary increments, with a distribution given by the Meixner distribution function Meixner$(\alpha_j, \beta_j, \delta_j t)$.

According to **Grigelionis** (1999), this non-homogeneous Meixner process has no Brownian part while the pure jump part is governed by the Lévy measure

$$\upsilon_j\left(dg\right) = \delta_j \frac{\exp\left(\beta_j g/\alpha_j\right)}{g\sinh\left(\pi g/\alpha_j\right)} dg$$

The first parameter in the characteristic triplet equals

$$\gamma_j = \alpha_j \delta_j \tan\left(\beta_j/2\right) - 2\delta_j \int_1^{\infty} \frac{\sinh\left(\beta_j g/\alpha_j\right)}{\sinh\left(\pi g/\alpha_j\right)} dg$$

This process has infinite variation due to $\int_{-1}^{+1} |g|\, v_j\,(dg) = \infty$, for any $j = 0, 1, ..., n$.

9.12. **The non-homogeneous Generalized Hyperbolic Process.** The **Generalized Hyperbolic** (GH) distributions were introduced by **Barndorff-Nielsen** (1977) as a model for the grain-size distribution of wind-blown sand. In order to use this distribution in financial modelling, two subclasses of the GH distribution appear in 1995. **Eberlein and Keller** (1995) used the Hyperbolic distribution and in the same year **Barndorff-Nielsen** (1995) proposed the NIG as a special case of a GH distribution.

Following **Barndorff-Nielsen** (1977) the **Generalized Hyperbolic** (GH) distribution $GH\left(\alpha, \beta, \delta, v\right)$ is defined through its characteristic function

$$\hat{\mu}\left(k; \alpha, \beta, \delta, v\right) = \left(\frac{\alpha^2 - \beta^2}{\alpha^2 - \left(\beta + iz\right)^2}\right)^{v/2} \frac{K_v\left(\delta\sqrt{\alpha^2 - \left(\beta + iz\right)^2}\right)}{K_v\left(\delta\sqrt{\alpha^2 - \beta^2}\right)}$$

where K_v is the modified Bessel function, α and β determine the shape of the distribution and δ is the scale of the parameter.

The density of the $GH\left(\alpha, \beta, \delta, v\right)$ distribution is given by

$$\mu\left(g; \alpha, \beta, \delta, v\right) = a\left(\alpha, \beta, \delta, v\right)\left(\delta^2 + g^2\right)^{\frac{v}{2} - \frac{1}{4}} K_{v - \frac{1}{2}}\left(\alpha\sqrt{\delta^2 + g^2}\right)\exp\left(\beta g\right)$$

with

$$a\left(\alpha, \beta, \delta, v\right) = \frac{\left(\alpha^2 - \beta^2\right)^{v/2}}{\sqrt{2\pi}\alpha^{v-1/2}\delta^v K_v\left(\delta\sqrt{\alpha^2 - \beta^2}\right)}$$

where

$$\begin{array}{lll} \delta \geq 0, & |\beta| < \alpha & \text{if} \quad v > 0 \\ \delta > 0, & |\beta| < \alpha & \text{if} \quad v = 0 \\ \delta > 0, & |\beta| \leq \alpha & \text{if} \quad v < 0 \end{array}$$

and using this distribution and characteristic function, we can define a non-homogeneous GH process $\{G_t : t \geq 0\}$ as the process that starts at zero, has independent and piecewise stationary increments, and where the distribution of G_t has characteristic function

$$\begin{aligned} \mathbb{E}\left(\exp\left[izG_t\right]\right) &= \prod_{j \leq \eta(t)} \mathbb{E}\left(\exp\left[iz\left(G_{t \wedge T_{j+1}} - G_{T_j}\right)\right]\right) \\ &= \prod_{j \leq \eta(t)} \hat{\mu}_j\left(g; \alpha_j, \beta_j, \delta_j, v_j\right)^{\left(t \wedge T_{j+1} - T_{j+1}\right)} \end{aligned}$$

It is an infinite variation process without Gaussian part (in the general case). The Lévy measure $v_j\,(dg)$ is known, but the expression is rather complicate as it involves integrals of special functions.

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