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Departamento de Economía
Universidad Carlos III de Madrid
Calle Madrid, 126
28903 Getafe (Spain)
Fax (341) 624-9875

STRATEGY-PROOF MECHANISMS WITH MONOTONIC PREFERENCES: THE CASE OF PURE PUBLIC GOODS ECONOMIES

Diego Moreno*

Abstract

This paper explores a typical public finance problem where there are m public goods (education, transportation, police, etc.) provided in limited amounts due to budget constraints, and where individual's preferences are not known. It is shown that all institutions (i.e., decision mechanisms) available to decide the allocation of goods have very unattractive properties: either the decision mechanisms are not compatible with individual's incentives, or they are dictatorial (i.e, they are based on a single individual's preferences).

*Diego Moreno, Departamento de Economía, Universidad Carlos III de Madrid and University of Arizona. Tel (341) 624-9653, e-mail:dmoreno@eco.uc3m.es

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1. Introduction

Consider the following problem from public finance: there are m public goods (education, transportation, police, etc.) provided in limited amounts due to budget constraints. Individuals' preferences are unknown, although they might be known to be monotonic, continuous, convex, etc.. In this paper it is shown that all the institutions (i.e., decision mechanisms) one can use to make decisions have very bad properties: either they are not compatible with individuals incentives, or they make decisions based on a single individual's preferences.

The revelation principle has established that the search for incentive compatible decision mechanisms can be restricted to those that make a decision based on individuals' reported preferences. A decision mechanism of this class is a mapping which associates a feasible outcome with each profile of reported preferences. Individuals might attempt to manipulate these mechanisms by reporting preferences different from their true ones. In order for a mechanism to be incentive compatible it must be immune to participants' manipulations; i.e., an individual must be best off reporting his true preferences, whatever preferences the other individuals report. Decision mechanisms with this property are referred to as strategy-proof.

Whether or not a decision mechanisms is strategy-proof depends on the domain of preferences on which it must decide, and on the set of outcomes it selects. Any constant decision mechanism is (trivially) strategy-proof. If constant mechanisms are ruled out, then one needs to know the set of possible (i.e., admissible) preference profiles on which a mechanism is to decide. In many cases, individuals' preferences might be known to be monotonic (i.e., such that bundles with more of each good are preferred to those with less). This will be the case when, for example, public goods are desirable or freely disposable. The purpose of this paper is to determine which strategy-proof mechanisms are available when individual preferences are known to be monotonic.

Gibbard[3] and Satterthwaite[6] independently showed that when all preferences are admissible, then only dictatorial decision mechanisms (i.e., those which always select a single individual's best outcome) are strategy-proof. Their result was obtained for the case where the set of feasible outcomes is finite, and decision mechanisms select at least three different outcomes. Barberà and Peleg[2] have established that the Gibbard-Satterthwaite Theorem remains valid when the set of feasible outcomes is infinite even if preferences must be continuous. Zhou[7] has shown that when preferences are known to be convex as well as continuous, strategy-proof decision mechanisms whose range contains a two dimensional set are dictatorial. (Strategy-proof mechanisms for this domain of preferences and with a one dimensional range were characterized by Moulin[5] as medium voter type mechanisms, among which there are nondictatorial mechanisms. Barberà and Jackson[1] have recently generalized this characterization.)

In this paper it is shown that results similar to those of Barberà-and-Peleg's and Zhou's theorems hold even when admissible preferences are further required to be monotonic. It is shown that when all the continuous and monotonic preferences are admissible, then every strategy-proof mechanism whose range contains three or more ($\geq$-maximal) outcomes is (weakly) dictatorial. When individuals' preferences are known to be convex as well as continuous and monotonic, then every strategy-proof mechanisms whose range is *sufficiently large* is also dictatorial.

In contrast with the situations accounted for in Zhou's Theorem, when only monotonic and convex preferences are admissible there are strategy-proof nondictatorial mechanisms whose range is a two dimensional set. The existence of these mechanisms is related with the fact that there are examples of two dimensional sets of outcomes on which monotonic and convex preferences are *single peak.* This possibility disappears when the range of the mechanism is sufficiently large.

2. The Model

The set of individuals is $N = \{1, ..., n\}$, where $n \geq 1$. For simplicity, individuals' consumption set is taken to be $\Re^m_+$. Individuals' preferences are represented by utility functions (i.e., real-valued functions on $\Re^m_+$). The set of feasible outcomes is denoted by X, which is assumed to be a compact subset of $\Re^m_+$.

As individuals' utility functions might be known to have certain properties, let U denote the set of a priori *admissible* utility functions. The set U with the metric d given for $u, u' \in U$ by $d(u,u') = \sup\left\{|\gamma(u(x)) - \gamma(u'(x'))|,\ x, x' \in \Re^m_+\right\}$, where $\gamma : \Re \to (-1,1)$ is defined by $\gamma(r) = \frac{r}{1+|r|}$, is a metric space.The set of admissible utility profiles is therefore U^n. Utility profiles are denoted by $\mathbf{u} = (u_1, ..., u_n)$. The set U^n with the product metric is a metric space. For $\mathbf{u} \in U^n$ and $S \subset N$, $\mathbf{u}_{-S}$ is the profile obtained from $\mathbf{u}$ by deleting the utility functions of the members of S.

A *decision mechanism* (or simply a *mechanism)* is a mapping $f : U^n \to X$. A mechanism f is *manipulable by* $i \in N$ *at* $\mathbf{u} \in U^n$ if there is $u' \in U$ such that $u_i(f(\mathbf{u}_{-i}, u')) > u_i(f_i(\mathbf{u}))$. A mechanism is *strategy-proof* if it is not manipulable by any $i \in N$ at any $\mathbf{u} \in U^n$. A mechanism f is *dictatorial on* $\Omega^n \subset U^n$ if there is an individual $i \in N$ such that for each $\mathbf{u} \in \Omega^n$, $f(\mathbf{u})$ maximizes u_i on $f(\Omega^n)$ (Individual i is then referred to as a *dictator for* f *on* Ω^n). A mechanism is *dictatorial* if it is dictatorial on U^n, and it is *nondictatorial* if it is not dictatorial. A mechanism f is *weakly dictatorial* if it is dictatorial on a dense subset Ω^n of U^n with the property that for each $u \in U$, $u(f(\Omega^n)) = u(f(U^n))$.

Strategy-proof mechanisms are those for which an individual is always best off reporting a utility function representing his true preferences. Dictatorial mechanisms always select an outcome from the set of maximizers of a single individual's (the dictator's) reported utility function. Weakly dictatorial mechanisms *almost always* select the outcome based a single individual's reported utility function. When this is the case, the condition $u(f(\Omega^n)) = u(f(U^n))$ for each $u \in U$ guarantees that the (weak) dictator obtains an outcomes that maximizes his utility function on the mechanism's range.

Given a mechanism f, write $X^f \equiv f(U^n)$, and $\bar{X}^f \equiv \left\{x \in X^f \mid \nexists x' \in X^f \text{ with } x' > x\right\}$ (i.e., $\bar{X}^f$ is the set of $\geq$-maximal points of X^f). A set $A \subset \Re^m$ is said to be *plentiful* if there are not x,$y \in A$ such that A is contained in the set $\{x' \in \Re^m \mid x' \geq \lambda x + (1-\lambda)y,\ \lambda \in [0,1]\}$. Note that if A is plentiful then it is at least a two dimensional set, although there are two dimensional sets that are not plentiful (see Example 2 below). Also denote by $\#A$ the cardinality of A.

3. The Results

For economic problems associated with allocating public goods, there are two domains that are of particular interest: the set U^C of all continuous and *increasing*[1] utility functions, and the set U^Q of continuous increasing and *strictly quasi-concave*[2] utility functions. Given an arbitrary set of utility functions U, write $\bar{U}$ for the set of utility functions in U that are strictly increasing. Theorem 1 establishes that even when utility functions are required to be in either of these sets, every strategy-proof mechanism whose range is sufficiently large is weakly dictatorial.

Theorem 1. *Let $f : U^n \to X$ be a strategy-proof mechanism. If either*

(1.1) $U^C \subset U$ and $\#\bar{X}^f \geq 3$, or

(1.2) $U^Q \subset U$ and $\bar{X}^f$ is plentiful,

then f is dictatorial on $\bar{U}^n$. Moreover, it is weakly dictatorial.

Barberà and Peleg[2] and Zhou[7] have established similar results when nonmonotonic preferences are admissible: Barberà and Peleg Theorem establishes that if U contains the set of all continuous utility functions (increasing or not) and $\#X^f \geq 3$, then every strategy proof allocation mechanism is dictatorial. Zhou has shown that the same conclusion arises when U contains the set of continuous and strictly quasi-concave (and quadratic) utility functions, and X^f is at least

[1]A utility function u is *increasing* if for each $x, x' \in \Re^m_+$, $x > x'$ implies $u(x) \geq u(x')$, and $x \gg x'$ implies $u(x) > u(x')$; it is *strictly increasing* if $u(x) > u(x')$ whenever $x > x'$. (The convention used for vector notation is as follows: For $a, b \in Re^m$, write $a \geq b$ (resp. $a \gg b$) if $a_i \geq b_i$ (resp. $a_i > b_i$) for each $i = 1, \ldots, m$; write $a > b$ if $a \geq b$ and $a \neq b$.)

[2]A utility function u is *quasi-concave* if for each each $x, x' \in \Re^m_+$, and for each $\lambda \in (0,1)$, one has $u(\lambda x + (1-\lambda)x') \geq \min\{u(x), u(x')\}$. When this inequality is strict, u is said to be *strictly quasi-concave*.

a two dimensional set (i.e., the dimension of the smallest affine subspace that contains X^f is at least two).

The conclusion of Theorem 1 is weaker than those obtained by either Barberà-and-Peleg or Zhou. However, the following example shows that when preferences are monotonic, then there are mechanisms satisfying the assumptions of Theorem 1 that are weakly dictatorial, but not (fully) dictatorial. For $u \in U$ and $A \subset X$, denote by $\arg\max(u, A)$ the set of maximizers of u in A.

Example 1. There are two individuals and three public goods. The set of feasible outcomes is $X = \left\{(x_1, x_2, x_3) \in \Re^3_+ \mid \sum_1^3 x_i \leq 1\right\}$. Individuals' admissible utility functions are those that are continuous and increasing. Let X^f be the set $\left\{(1,0,0),(0,1,0),(\frac{1}{2},\frac{1}{2},0),(0,0,0)\right\}$, and let $\bar{u} \in U$ be given by $\bar{u}(x_1, x_2, x_3) = x_3$. Let the mechanism f be given for each $(u_1, u_2) \in U^2$ by $f(u_1, u_2) = (0,0,0)$ if $u_2 = \bar{u}$; otherwise let $f(u_1, u_2)$ be some arbitrary point in $\arg\max(u_1, X^f)$. This is a strategy-proof mechanism satisfying the assumptions of Theorem 1. It is a weakly dictatorial mechanism, but it is not a (fully) dictatorial mechanism.

Also note that conditions (1.1) and (1.2) of Theorem 1 introduce requirements on the set $\bar{X}^f$, while Barberà and Peleg's and Zhou's theorems establish conditions on X^f. However, when only increasing utility functions are admissible the set of potentially conflicting outcomes is $\bar{X}^f$. Moreover, Zhou's requirement that the set X^f be a two dimensional set is replaced here with the requirement that the set $\bar{X}^f$ be plentiful. As the following example shows, when only monotonic preferences are admissible there are strategy-proof non (weakly) dictatorial mechanisms whose range is a two dimensional set.

Example 2. There are two individuals and two public goods. The set of feasible outcomes is $X = \left\{(x_1, x_2) \in \Re^2_+ \mid x_1^2 + x_2^2 \leq 1\right\}$. Individuals' admissible utility functions are those that are continuous, increasing and quasi-concave. Define the mechanism$f$ whose range X^f is the set $\left\{(1,0),(0,1),(\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}})\right\}$ as follows. Let $\phi : 2^{X^f} \to X^f$ be a (selection) function satisfying for each $A \subset X^f$, $A \neq \emptyset$, $\phi(A) \in A$, and whenever $A \setminus \{(0,1)\} \neq \emptyset$, then $\phi(A) = \phi(A \cup \{(0,1)\})$. Let f be given for each $(u_1, u_2) \in U^2$ by $f(u_1, u_2) = \phi(\arg\max(u_1, X^f))$, if $(0,1) \in \arg\max(u_2, X^f)$; otherwise let $f(u_1, u_2) = \phi(\arg\max(u_1, X^f) \setminus \{(0,1)\})$.

This is a nondictatorial mechanism, as Individual 2 can veto the outcome $(0,1)$. It is shown that f is a strategy-proof mechanism. Clearly, Individual 1 can never manipulate f. Suppose that Individual 2 can manipulate f at $(u_1, u_2) \in U$; i.e., there is $u' \in U$ such that $u_2(f(u_1, u')) > u_2(f(u_1, u_2))$. Note that since every $u \in U$ is quasi-convex and increasing, and since the outcome $(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$ lies above the line passing $(1,0)$ and $(0,1)$, then

(*) $u(0,1) \geq u(1,0) \Longrightarrow u(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}) > u(1,0)$, and

(**) $u(1,0) \geq u(0,1) \Longrightarrow u(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}) > u(0,1)$.

If $(0,1) \in \arg\max(u_2, X^f)$, then $f(u_1,u_2) \in \arg\max(u_1, X^f)$. Since $f(u_1,u_2) \neq (0,1)$ (because otherwise $u_2(f(u_1,u_2)) \geq u_2(f(u_1,u'))$, then by construction (see the definition of ϕ) $f(u_1,u_2) = f(u_1,u')$, which contradicts that $u_2(f(u_1,u')) > u_2(f(u_1,u_2))$.

If $(0,1) \notin \arg\max(u_2, X^f)$, then in order for $f(u_1,u') \neq f(u_1,u_2)$, it must be that case that $f(u_1,u') = (0,1)$. Therefore $\arg\max(u_1, X^f) = \{(0,1)\}$. By (*), $u_1(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}) > u_1(1,0)$; hence $f(u_1,u_2) = (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$. Because f is manipulable at (u_1,u_2) by assumption, $u_2(0,1) > u_2(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$, and as $(0,1) \notin \arg\max(u_2, X^f)$, then $u_2(1,0) > u_2(0,1) > u_2(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$, which contradicts (**).

Therefore f is a strategy-proof nondictatorial mechanism whose range X^f is a two dimensional set. It should be noticed also that this is a nonwasteful mechanism, and it would be an efficient mechanism if $X = X^f$. (The definitions of nonwasteful and efficient mechanisms are given below.)

The key feature of these example is that the underlying set of preferences that can be represented by increasing and quasi-concave utility functions in the domain of f are *single peaked* on the set X^f (i.e., there is some *natural* order $\succeq$ on X^f such that for each $u \in U^Q$ there is $x^* \in X^f$ with the property that for each $x, y \in X^f$ with $x^* \succeq x \succ y$ (or $y \succ x \succeq x^*$), then $u(x) > u(y)$)[3]. When preferences are single peak, it is well know that there are strategy-proof and non dictatorial mechanism (see Moulin[5]).

In order to prove Theorem 1 a number of preliminary results are first established. Lemma 1 states a standard *unanimity* property of strategy-proof mechanisms.

Lemma 1. *If f is a strategy-proof allocation mechanism, then for each $u \in U$ $f(u,...,u) \in \arg\max(u, X^f)$.*

Proof: Suppose not. Let $u \in U$ and $x \in X^f$ such that $u(x) > u(f(u,...,u))$. Let $\mathbf{u} \in U^n$ be such that $f(\mathbf{u}) = x$. Successive applications of strategy-proofness yield $u(f(u, \mathbf{u}_{-1})) \geq u(f(\mathbf{u})) = u(x)$, $u(f(u,u,\mathbf{u}_{-\{1,2\}})) \geq u(f(u,\mathbf{u}_{-1})) \geq u(x), ...,$ and $u(f(u,...,u)) \geq u(f(u,...,u,u_n)) \geq u(x)$, which is a contradiction.□

The following result is an immediate corollary of Lemma 1.

Corollary 1. $\bar{X}^f \subset f(\bar{U}^n)$.

Proposition 1 establishes that every strategy-proof mechanism satisfying the assumptions of Theorem 1 is dictatorial on the set $\bar{U}^n$ of profiles whose coordinate utility functions are strictly

[3]This fact was pointed out to me by Salvador Barberà.

increasing. Its proof follows the lines of the proof of the Barberà-and-Peleg's and Zhou's theorems; it is given in the Appendix.

Proposition 1. *Every strategy-proof mechanism satisfying the assumptions of Theorem 1 is dictatorial on* $\bar{U}^n$.

PROOF OF THEOREM 1: Let f be a strategy-proof allocation mechanism satisfying the assumptions of the theorem. By Proposition 1 f is dictatorial on $\bar{U}^n$. Moreover, $\bar{U}$ is dense in U, and therefore Corollary 1 implies that f is weakly dictatorial.□

Propositions 2 to 4 establish that under the assumptions of Theorem 1, strategy-proof mechanisms that are weakly dictatorial but not (fully) dictatorial have other bad properties: they are discontinuous (Proposition 2), or they produce outcomes that are not Pareto optimal (Proposition 3), or they are wasteful (Proposition 4). For each $u \in U$ and each integer k, let $u^k \in \bar{U}$ denote the utility function defined for $x \in \Re^m_+$ by $u^k(x) = u(x) + \frac{\|x\|}{k(1+\|x\|)}$.

Proposition 2. *Let f be a strategy-proof mechanism satisfying the assumptions of Theorem 1. If f is continuous on U, then it is dictatorial.*

Proof: By Proposition 1 f is dictatorial on $\bar{U}^n$. Without loss of generality, assume that Individual 1 is the dictator for f on $\bar{U}^n$. Suppose by way of contradiction that f is not dictatorial; i.e., there is $(u_1, ..., u_n) \in U^n$ and $\bar{x} \in X^f$ such that $u_1(\bar{x}) > u_1(f(u_1, ..., u_n))$. We can assume (see Lemma A.3) that $\bar{x} \in \bar{X}^f$. Thus, let $u_1' \in \bar{U}$ be such that $\arg\max(u_1', X^f) = \{\bar{x}\}$. Since for each k, $u_i^k \in \bar{U}$, one has $f(u_1', u_2^k, ..., u_n^k) = \bar{x}$, and since f is continuous $f(u_1', u_2, ..., u_n) = \lim_{k\to\infty} f(u_1', u_2^k, ..., u_n^k) = \bar{x}$. Hence

$$u_1(f(u_1', u_2, ..., u_n)) = u_1(\bar{x}) > u_1(f(u_1, u_2, ..., u_n)),$$

and therefore Individual 1 can manipulate f at $(u_1, ..., u_n)$. This contradicts the fact that f is strategy-proof and establishes Proposition 2.□

An outcome $x \in X$ is said to be *Pareto optimal with respect to* $\mathbf{u} = (u_1, ..., u_n) \in U^n$ if no $x' \in X$ exists such that for each $i \in N$, $u_i(x') \geq u_i(x)$, and the inequality is strict for some $j \in N$. A mechanism f is *efficient* if for each $\mathbf{u} \in U^n$, $f(\mathbf{u})$ is Pareto optimal with respect to $\mathbf{u}$.

Proposition 3. *Let f be a strategy-proof mechanism satisfying the assumptions of Theorem 1. If f is efficient , then it is dictatorial.*

Write U^* for the set of utility function in U with a unique maximizer on X^f. Note that for each $u \in U^*$, $\arg\max(u, X^f)$ is a member of $\bar{X}^f$. Before proving Proposition 3, the following lemma is established.

Lemma 2. *If f is a strategy-proof mechanism satisfying the assumptions of Theorem 1, then there is $i \in N$ such that for each $(u_i, \mathbf{u}_{-i}) \in U^* \times U^{n-1}$, $f(u_i, \mathbf{u}_{-i}) \leq \arg\max(u_i, X^f)$*

Proof: W.l.o.g. assume that Individual 1 is the dictator of Proposition 1. It is shown that Lemma 2 holds for $i = 1$. Suppose not; let $(u_1, u_2, ..., u_n) \in U^* \times U^{n-1}$ and suppose that it is not the case that $x = f(u_1, u_2, ..., u_n) \leq \arg\max(u_1, X^f)$. Note this implies in particular that $x \neq \arg\max(u_1, X^f)$. Let $u \in \bar{U}$ be such that $u(x) > u(\arg\max(u_1, X^f))$. Since $x \in f(u_1 \times U^{n-1})$, Lemma 1 yields

$$u(f(u_1, u, ..., u)) \geq u(x).$$

Hence $f(u_1, u, ..., u) \neq \arg\max(u_1, X^f)$.

Let $u_1' \in \bar{U}$ be such that $\arg\max(u_1', X^f) = \left\{\arg\max(u_1, X^f)\right\}$. As $(u_1', u, ..., u) \in \bar{U}^n$ and Individual 1 is a dictator for f on $\bar{U}^n$, one has $f(u_1', u, ..., u) = \arg\max(u_1, X^f)$. Thus f is manipulable by Individual 1 at $(u_1', u, ..., u)$, which contradicts the fact that f is strategy-proof and proves the lemma.□

With Lemma 2 in hand, one can easily proof Proposition 3.

PROOF OF PROPOSITION 3: W.l.o.g. assume that Lemma 2 is satisfied for $i = 1$. It is shown that Individual 1 is a dictator for f. Suppose not; let $\mathbf{u} \in U^n$ and $x \in \bar{X}^f$ be such that $u_1(x) > u_1(f(\mathbf{u}))$. Let $u_1' \in U^*$ be such that $\arg\max(u_1', X^f) = \{x\}$. Then $f(u_1', \mathbf{u}_{-1}) \leq x$ by Lemma 2, and since f is efficient, then $f(u_1', \mathbf{u}_{-1}) = x$. Thus Individual 1 can manipulate f at $\mathbf{u}$, contradicting that f is strategy-proof. Hence Individual 1 is a dictator for f on U^n, and therefore f is dictatorial.□

The conclusion of Proposition 3 holds even when mechanisms are less than efficient, as long as they do not waste resources. A mechanism f is *nonwasteful* if $X^f \subset \bar{X}$. Proposition 4 establishes that nonwasteful mechanisms satisfying the assumptions of Theorem 1 are dictatorial. The proof of Proposition 4 is immediate from Lemma 3; thus it is omitted.

Proposition 4. *Let f be a strategy-proof mechanism satisfying the assumptions of Theorem 1. If f is non wasteful, then it is dictatorial.*

Finally, it is of interest to determine whether one can design mechanisms that perform better when individuals' preferences are strictly monotonic (i.e., when individuals can claim only strictly increasing utility functions). Theorem 2 shows that for continuous mechanisms the answer to this question is negative: In this case, only dictatorial mechanisms are strategy-proof. It is unknown whether there are discontinuous mechanisms that are non dictatorial and strategy-proof.

Theorem 2. *Let $f : \bar{U}^n \to X$ be a strategy-proof mechanism. If f is continuous and either*

(2.1) $\bar{U} = \bar{U}^C$ and $\#\bar{X}^f \geq 3$, or

(2.2) $\bar{U} = \bar{U}^Q$ and $\bar{X}^f$ is plentiful,

then f is dictatorial.

Proof: Let f be a strategy-proof mechanism satisfying the assumptions of Theorem 2, and denote by U the set U^C if assumption (2.1) is satisfied and U^Q if assumption (2.2) is satisfied. Theorem 2 is proved by showing that f can be extended to a strategy-proof mechanism F whose domain is U. The mechanism F will therefore be dictatorial on $\bar{U}^n$ by Theorem 1. As F coincides with f on $\bar{U}^n$ (equal to either $(\bar{U}^C)^n$ or $(\bar{U}^Q)^n$), then f itself is dictatorial.

For each $\mathbf{u} \in U^n$, let $L(\mathbf{u})$ denote the set of limit points of the sequence $\{f(\mathbf{u}^k)\}$, where $\mathbf{u}^k = (u_1^k, ..., u_n^k)$. Note that $L(\mathbf{u})$ is never empty as X is compact. Let $\phi : 2^X \to X$ be an arbitrary (selection) function that for each $A \subset X$, $A \neq \emptyset$, assigns $\phi(A) \in A$. Finally, let the mechanism $F : U^n \to X$ be given for each $\mathbf{u} \in U^n$ by $F(\mathbf{u}) = f(\mathbf{u})$, if $\mathbf{u} \in \bar{U}^n$; otherwise let $F(\mathbf{u}) = \phi(L(\mathbf{u}))$. It is shown that F is strategy-proof.

Suppose by way of contradiction that F is manipulable by $i \in N$ at $\mathbf{u} \in U^n$; i.e., there is $u' \in U$ such that $u_i(F(u', \mathbf{u}_{-i})) > u_i(F(\mathbf{u}))$. Since u_i is continuous and X is compact, there are $\delta > 0$ sufficiently small and an integer K sufficiently large that for each $k, k' > K$ and each $x, x' \in X$ such that $\|x - F(\mathbf{u})\| < 2\delta$, $\|x' - F(u', \mathbf{u}_{-i})\| < \delta$, one has $u_i^{k'}(x') > u_i^k(x)$.

Let $k_1, k_2 > K$ be such that both $\left\|F(\mathbf{u}^{k_1}) - F(\mathbf{u})\right\| < \delta$ and $\left\|F(\mathbf{u}_{-i}^{k_2}, u'^{k_2}) - F(\mathbf{u}_{-i}, u')\right\| < \delta$. Since for each k, $F(\mathbf{u}^k) = f(\mathbf{u}^k)$ and f is continuous, k_1, k_2 can be chosen sufficiently large that $\left\|F(\mathbf{u}^{k_1}) - F(\mathbf{u}^{k_2})\right\| < \delta$. The triangle inequality implies that $\left\|F(\mathbf{u}^{k_2}) - F(\mathbf{u})\right\| < 2\delta$, and therefore

$$u_i^{k_2}(F(\mathbf{u}_{-i}^{k_2}, u'^{k_2})) > u_i^{k_2}(F(\mathbf{u}^{k_2})).$$

Note, however, that $F(\mathbf{u}_{-i}^{k_2}, u'^{k_2}) = f(\mathbf{u}_{-i}^{k_2}, u'^{k_2})$, and $F(\mathbf{u}^{k_2}) = f(\mathbf{u}^{k_2})$, as both $(\mathbf{u}_{-i}^{k_2}, u'^{k_2})$ and $\mathbf{u}^{k_2}$ are members of $\bar{U}^n$. Hence f is manipulable by Individual i at $\mathbf{u}^{k_2}$. This contradicts the fact that f is strategy-proof, thereby establishing Theorem 2.□

The results presented here show that the Gibbard-Satterthwaite result remains valid even when it is known that individuals' preferences are monotonic. The conclusions here are not perhaps as *clean* as the ones obtained when satiated preferences are admissible: strategy-proof mechanisms are characterized as weakly dictatorial rather than fully dictatorial, and conditions under which this result arises require one to measure appropriately the *size* of the range of a mechanism. However, their implications are virtually the same: all strategy-proof mechanisms have very bad properties (i.e., they are (weakly) dictatorial).

When individuals' preferences are convex continuous and monotonic, it has been shown that strategy-proof and nondictatorial mechanisms do exists, although their range must be a relatively small set (i.e., it must not be plentiful). It will be of interest to characterize this class of mechanisms, and to study in which cases there are mechanisms in this class with interesting properties.

Appendix: Proof of Proposition 1

First, a number of preliminary results are established as lemmas. For lemmas A.1 to A.4 let $f : U \to X$ be a strategy-proof allocation mechanism such that $U^Q \subset U$. Lemma A.1 establishes a *modified* version of the *strong positive association property.* Its proof is omitted (see Lemma 4.8 in Barberà and Peleg[2]).

Lemma A.1 *(MSPAP). For each* $\mathbf{u} = (u_1, ..., u_n) \in U^n$, $u \in U$ *and* $i \in N$ *such that for every* $x \in X^f \setminus \{f(\mathbf{u})\}$, $u(x) \geq u(f(\mathbf{u}))$ *implies* $u_i(x) > u_i(f(\mathbf{u}))$, *one has* $f(\mathbf{u}_{-i}, u) = f(\mathbf{u})$.

Lemma A.2 establishes that the image of any profile whose coordinate utility functions are strictly increasing is a member of $\bar{X}^f$.

Lemma A.2. *If* $\mathbf{u} \in \bar{U}^n$, *then* $f(\mathbf{u}) \in \bar{X}^f$.

Proof: Suppose by way of contradiction that there is $\mathbf{u} = (u_1, ..., u_n) \in \bar{U}^n$ such that $f(\mathbf{u}) \notin \bar{X}^f$. Write $f(\mathbf{u}) = \bar{x}$, and choose $\alpha \in \Re^n$ such that $u_1(\bar{x}) + \alpha_1 = u_2(\bar{x}) + \alpha_2 = ... = u_n(\bar{x}) + \alpha_n$. Consider the utility function $u \in \bar{U}$ given by $u(x) = \min\{u_1(x) + \alpha_1, ..., u_n(x) + \alpha_n\}$, and note that for each $x \in X \setminus \{\bar{x}\}$, whenever $u(x) \geq u(\bar{x})$, then $u_i(x) \geq u_i(\bar{x})$ for each $i \in N$. Thus, by slightly bending the indifference curve of u through $\bar{x}$ in the direction of the main diagonal, one can obtain a utility function $\bar{u} \in \bar{U}$ satisfying for each $x \in X \setminus \{\bar{x}\}$, that $\bar{u}(x) \geq \bar{u}(\bar{x})$ implies $u_i(x) > u_i(\bar{x})$, for each $i \in N$. MSPAP (Lemma A.1) yields

$$\bar{x} = f(\mathbf{u}) = f(\bar{u}, u_1, ..., u_n) = ... = f(\bar{u}, \bar{u}, ..., \bar{u}, u_n) = f(\bar{u}, \bar{u}, ..., \bar{u}).$$

Note, however, that $\bar{x} \notin \bar{X}^f$, and since $\bar{u}$ is a strictly increasing function, this implies that $f(\bar{u}, \bar{u}, ..., \bar{u}) \notin \arg\max(\bar{u}, X^f)$, which contradicts Lemma 1 and establishes Lemma A.2.□

Lemma A.3 establishes that the set $\bar{X}^f$ is closed, and that moreover, the limit of any increasing sequence in X^f (i.e., any sequence $\{x_t\}$ in X^f such that $x_{t+1} \geq x_t$) is also a member of $\bar{X}^f$. For each $x \in X$, let $I(x)$ be the set of indexes $k \in \{1, ..., m\}$ such that $x_k > 0$.

Lemma A.3. *The set* $\bar{X}^f$ *is closed. Moreover, if* $\bar{x}$ *is the limit of any increasing sequence in* X^f, *then* $\bar{x} \in \bar{X}^f$.

Proof: Let $\bar{x}$ be a limit point of $\bar{X}^f$ or the limit of any increasing sequence in X^f. It is shown that $\bar{x}$ is a member of $\bar{X}^f$. Let $\alpha \in \Re^m$, $\alpha \gg 0$, be such that for each $k, k' \in I(\bar{x})$, $\alpha_k \bar{x}_k = \alpha_{k'} \bar{x}_{k'}$, and let $u \in U$ be given by $u(x) = \min\{\alpha_k x_k,\ k \in I(\bar{x})\}$. If $\bar{x} \notin \bar{X}^f$, then $\arg\max(u, X^f) = \emptyset$, and therefore $f(u, ..., u) \notin \arg\max(u, X^f)$, contradicting Lemma 1. Hence $\bar{x} \in \bar{X}^f$.□

For each $u_n \in U$, let $O(u_n) = \left\{x \in \bar{X}^f \mid \exists \mathbf{u}_{-n} \in U^{n-1} \text{ with } f(\mathbf{u}_{-n}, u_n) = x\right\}$. Lemma A.3 implies that $O(u_n)$ is closed ($O(u_n) = \bar{X}^f \cap \bar{X}^{f^{u_n}}$, where the mechanism $f^{u_n} : U^{n-1} \to X$ is given by $f^{u_n}(\mathbf{u}_{-n}) = f(\mathbf{u}_{-n}, u_n)$). Lemmas A.4-A.7 establish other properties of these sets. Write U^* for the set of utility functions in U with a unique maximizer on X^f. Note that for each $u \in U^*$, $\arg\max(u, X^f)$ is a member of $\bar{X}^f$.

Lemma A.4. *If* $u_n \in U^*$, *then* $\arg\max(u_n, X^f) \in O(u_n)$.

Proof: Let $u_n \in U^*$ arbitrary. Choose $u \in U^*$ such that $\arg\max(u, X^f) = \arg\max(u_n, X^f)$. Lemma 1 implies that $f(u, ..., u) = \arg\max(u_n, X^f)$. As f is strategy-proof, then $f(u, ..., u, u_n) = \arg\max(u_n, X^f)$. Hence $\arg\max(u_n, X^f) \in X^{f^{u_n}}$, and therefore $\arg\max(u_n, X^f) \in O(u_n)$.□

A *generalized Leontieff utility function* is a function of the form $u(x) = \min\{u^1(x), ..., u^p(x)\}$, where p is an arbitrary integer, and each u^j is a linear utility function. Note the a generalized Leontieff utility function is a member of U^Q. Given $x, y \in \Re^m$, let $[x, y]$ denote the (closed) segment connecting them. The notation $[x, y) \equiv [x, y] \setminus \{y\}$, $(x, y] \equiv [x, y] \setminus \{x\}$, and $(x, y) \equiv [x, y] \setminus \{x, y\}$ will be also used. A set $A \subset \Re^m$, is *star-shaped (relative to* $B \subset \Re^m$*) with respect to* $x \in A$, if for each $y \in A$ one has $[x, y] \cap B \subset A$. Lemma A.5 establishes that whenever $u_n \in U^*$, the set (-function) $O(u_n)$ depends only of the maximizer of u_n.

Lemma A.5. *If* $u_n, u_n' \in U^*$ *are such that* $\arg\max(u_n, X^f) = \arg\max(u_n', X^f)$, *then* $O(u_n) = O(u_n')$.

Proof: Lemma A.5 is proven in two steps.

STEP 1: First it is shown that $O(u_n)$ is star-shaped (relative to $\bar{X}^f$) with respect to $\arg\max(u_n, X^f)$. Suppose not; let $u_n \in U^*$ and $x, y \in \bar{X}^f$ be such that $x \in O(u_n)$, $y \in \left(x, \arg\max(u_n, X^f)\right)$, and $y \notin O(u_n)$. Without loss of generality, assume that $(x, y] \cap O(u_n) = \emptyset$, and since $O(u_n)$ is closed, let $z \in (x, y)$, and write $x^* = \frac{1}{2}(x + z)$. Now let $\left\{u^k\right\}$ be a sequence of generalized Leontieff utility functions such that for each k $\arg\max(u^k, X^f) = x^*$, and such that its indifference curves through x^* approach the line passing x and y (see Figure 1).

Since $x \in O(u_n)$, Lemma 1 implies that for each k, $u^k(f(u^k, ..., u^k, u_n)) \geq u^k(x)$, and since $O(u_n)$ is compact, some subsequence of $\left\{f(u^k, ..., u^k, u_n)\right\}$ converges to x. Let $u_n' \in U^*$ be such that $\arg\max(u_n', X^f) = y$. Then again Lemma 1 implies $u^k(f(u^k, ..., u^k, u_n')) \geq u^k(y) > u^k(x)$, and therefore a subsequence of $\left\{f(u^k, ..., u^k, u_n')\right\}$ converges to some point $x' \in (x, y]$.

Strategy-proofness of f yields $u_n(f(u^k, ..., u^k, u_n)) \geq u_n(f(u^k, ..., u^k, u_n'))$ for each k, and therefore $u_n(x) \geq u(x')$, which is a contradiction since u_n is strictly concave and $x' \in (x, \arg\max(u_n, X^f))$. This contradiction establishes Step 1.

STEP 2: Now suppose, by way of contradiction, that Lemma A.5 does not hold; i.e., there are u_n, u'_n such that $\arg\max(u_n, X^f) = \arg\max(u'_n, X^f) = \bar{x}$, and such that there is $y \in O(u_n)\backslash O(u'_n)$. By Step 1, there is $z \in (y, \bar{x}]$ such that $[z, \bar{x}] \subset O(u'_n)$. As in Step 1, let $\{u^k\}$ be a sequence of generalized Leontieff utility functions satisfying $\arg\max(u^k, X^f) = z$, and such that its indifference curves approach the line passing y and $\bar{x}$. By an argument similar to the one in Step 1, y is a limit point of the sequence $\{f(u^k, ..., u^k, u'_n)\}$. Moreover, $f(u^k, ..., u^k, u_n) = z$ (Lemma 1). Since f is strategy-proof, for each k one has $u_n(f(u^k, ..., u^k, u_n)) \geq u_n(f(u^k, ..., u^k, u'_n)) = u_n(z)$. Hence $u_n(y) \geq u_n(z)$, which is a contradiction since u_n is strictly quasi-concave thereby establishing Lemma A.5.□

For the remainder of the proof, let f be a mechanism satisfying the assumptions of Theorem 1. Lemma A.6 establishes that whenever Individual n reports $u_n \in U^*$, the set $O(u_n)$ is either the entire set $\bar{X}^f$, or the maximizer of u_n.

Lemma A.6. *For each $u_n \in U^*$, either $O(u_n) = \bar{X}^f$ or $O(u_n) = \arg\max(u_n, X^f)$.*

Proof of Lemma A.6 under (1.2): Suppose not; let $u_n \in U^*$ and $y, z \in \bar{X}^f$ be such that $y \in O(u_n) \setminus \arg\max(u_n, X^f)$, and $z \notin O(u_n)$. Let $u \in U$ (recall $U^C \subset U$) be such that $\arg\max(u, X^f) = y$, and $\arg\max(u, X^{f^{u_n}}) = z$.

(A function with these properties can be constructed as follows: Let $\alpha^1, \alpha^2 \in \Re^m$, $\alpha^1, \alpha^2 \gg 0$, be such that for $j, j' \in I(\bar{x})$ and $k, k' \in I(y)$, $\alpha^1_j z_j = \alpha^1_{j'} z_{j'}$, and $\alpha^2_k y_k = \alpha^2_{k'} y_{k'}$, and let $u^1, u^2 \in U$ be defined, respectively, by $u^s(x) = \min\{\alpha^s_j x_j,\ j \in I(z)\}$, $s = 1, 2$. Then $\arg\max(u^2, X^f) = y$, and as $y \notin X^{f^{u^2_n}}$ and $\bar{X}^{f^{u^2_n}}$ is closed by Lemma A.3; hence there is $\hat{x} \in \Re^m_+$ such that $\alpha^2_k \hat{x}_k = \alpha^2_{k'} \hat{x}_{k'}$ for $k, k' \in I(y)$, and such that for each $x \in X^{f^{u^2_n}}$, one has $u^2(x) < u^2(\hat{x}) < u^2(y)$. Now let $\beta > 0$ be such that $u^1(z) = \beta u^2(\hat{x})$. The function $\bar{u}(x) = \max\{u^1(x), \beta u^2(x)\}$ satisfies (1) and (2) above.)

Lemma 1 yields $f^{u_n}(u, ..., u) = f(u, ..., u, u_n) = x$, and $f(u, ..., u, u_n) = y$. As $y \neq \arg\max(u_n, X^f)$ assume w.l.o.g. (Lemma A.5) that $u_n(y) > u_n(x)$. Then

$$u_n(f(u, ..., u)) = u_n(y) > u_n(x) = u_n(f(u, ..., u, u_n)),$$

and therefore f is manipulable by Individual n at $(u, ..., u, u_n)$. This contradiction establishes Lemma A.6 under (1.2).

Proof of Lemma A.6 under (2.2): Suppose not; let $u_n \in U^*$ and $y, z \in \bar{X}^f$ be such that $y \in O(u_n)$, $y \neq \arg\max(u_n, X^f) \equiv x^*$, and $z \notin O(u_n)$. Since $\bar{X}^f$ is plentiful, choose $y, z \in \bar{X}^f$ such that there is no $\lambda \in (0, 1)$ with $z \geq \lambda x^* + (1 - \lambda)y$.

Let $u \in U$ be such that $\arg\max(u, X^f) = z$, and such that there is $\hat{x} \in [z, y] \cap O(u_n)$ such that $u(\hat{x}) > u(x)$, for each $x \in [z, x^*] \cap O(u_n)$ (see Figure 2). Lemma 1 implies that $u(f(u, ..., u, u_n)) \geq$

$u(\hat{x}) > u(x^*)$; hence $f(u, ..., u, u_n) \neq x^*$. Let $u'_n \in U$ be such that $\arg\max(u'_n, X^f) = x^*$, and such that $u'_n(y) > u'_n(x)$ for each $x \in \arg\max(u_n, O(u_n))$. Lemma A.5 yields $O(u'_n) = O(u_n)$, and therefore by Lemma 1 $f(u, ..., u, u'_n) \in \arg\max(u, O(u_n))$. Finally, let $u''_n \in U$ be such that $\arg\max(u''_n, X^f) = y$. Then $y \in O(u''_n)$ by Lemma A.4, and Lemma 1 implies that $f(u, ..., u, u''_n) = y$. Hence Individual n can manipulate f at $(u, ..., u, u'_n)$. □

Lemma A.7 establishes that set (-function) $O(u_n)$ is constant on the set of strictly increasing utility functions with a unique maximizer on $\bar{X}^f$.

Lemma A.7. *If there is* $u_n \in \bar{U} \cap U^*$ *such that* $O(u_n) = \arg\max(u_n, X^f)$, *then for each* $u_n \in \bar{U} \cap U^*$, *one has* $O(u_n) = \arg\max(u_n, X^f)$.

Proof: Suppose not; let $u_n, u'_n \in \bar{U} \cap U^*$ be such $O(u_n) = \arg\max(u_n, X^f)$ and $O(u'_n) = \bar{X}^f$ (Lemma A.6). Let $x \in \bar{X}^f \setminus \{\arg\max(u_n, X^f), \arg\max(u'_n, X^f)\}$ (such point exists by assumption), and $u \in \bar{U}$ such that $\arg\max(u, X^f) = x$. Lemma 1 yields $f^{u'_n}(u, ..., u) = x$. Since $O(u_n) = \arg\max(u_n, X^f)$ and $f(u, ...u, u_n) \in \bar{X}^f$(Lemma A.2), then $f(u, ...u, u_n) = \arg\max(u_n, X^f)$. As $\arg\max(u_n, X^f) \neq \arg\max(u'_n, X^f)$ (Lemma A.5), assume w.l.o.g. (again according to Lemma A.5) that $u'_n(\arg\max(u_n, X^f)) > u'_n(x)$. Then f is manipulable at $(u, ..., u, u'_n)$ by Individual n, contradicting that it is strategy-proof, and proving the lemma.□

Proof of Proposition 1: First, it is shown by induction on the number of individuals that f is dictatorial on $(\bar{U} \cap U^*)^n$. The case $n = 1$ is a simple application of Lemma 1. Assuming that f is dictatorial on $(\bar{U} \cap U^*)^n$ for $n \leq k - 1$, it remains to be shown that it is dictatorial on $(\bar{U} \cap U^*)^n$ for $n = k$.

By Lemma A.7, either $O(u_n) = \arg\max(u_n, X^f)$ for each $u_n \in (\bar{U} \cap U^*)$, or $O(u_n) = X^{f^{u_n}}$ for each $u_n \in (\bar{U} \cap U^*)$. If $O(u_n) = \arg\max(u_n, X^f)$ for each $u_n \in (\bar{U} \cap U^*)$, then Lemma A.2 yields $f(\mathbf{u}) = \arg\max(u_n, X^f)$ for each $\mathbf{u} \in (\bar{U} \cap U^*)^n$, and therefore Individual n is a dictator for f on $(\bar{U} \cap U^*)^n$. If $O(u_n) = \bar{X}^f$ for each $u_n \in \bar{U} \cap U^*$, then the induction hypothesis implies that each f^{u_n} is dictatorial on $(\bar{U} \cap U^*)^{n-1}$. In fact, the dictator of each f^{u_n} on $(\bar{U} \cap U^*)^{n-1}$ must be the same.

Suppose not; w.l.o.g., assume Individual 1 is the dictator for f^{u_n} on $(\bar{U} \cap U^*)^{n-1}$, and Individual 2 is the dictator for $f^{u'_n}$ on $(\bar{U} \cap U^*)^{n-1}$. Let $u_1, u_2 \in (\bar{U} \cap U^*)$ be such $\arg\max(u_1, X^f) \neq \arg\max(u_n, X^f)$, and $\arg\max(u_2, X^f) = \arg\max(u_n, X^f)$ (recall that by assumption $\bar{X}^f$ contains at least three points). Then for some $\mathbf{u}_{-\{1,2,n\}} \in (\bar{U} \cap U^*)^{n-3}$, one has $f^{u_n}(u_1, u_2, \mathbf{u}_{-\{1,2,n\}}) = f(u_1, u_2, \mathbf{u}_{-\{1,2,n\}}, u_n) \neq \arg\max(u_n, X^f)$, and $f^{u'_n}(u_1, u_2, \mathbf{u}_{-\{1,2,n\}}) = f(u_1, u_2, \mathbf{u}_{-\{1,2,n\}}, u'_n) = \arg\max(u_n, X^f)$. Hence f is manipulable at $(u_1, u_2, \mathbf{u}_{-\{1,2,n\}}, u_n)$ by Individual n.

Thus, some Individual $i \in N$ is the dictator for each f^{u_n} on $(\bar{U} \cap U^*)^{n-1}$; i.e., for each $\mathbf{u} \in (\bar{U} \cap U^*)^n$, $f^{u_n}(\mathbf{u}_{-n}) = f(\mathbf{u}) = \arg\max(u_i, X^f)$. Hence some Individual i is in fact a dictator for f on $(\bar{U} \cap U^*)^n$.

Finally, it is shown that f is dictatorial on $\bar{U}^n$. Without loss of generality assume that Individual 1 is the dictator for f on $(\bar{U} \cap U^*)^n$. Suppose that Individual 1 is not a dictator for f on $\bar{U}^n$; i.e., there are $\mathbf{u} \in \bar{U}^n$, and $x \in \bar{X}^f$ such that $u_1(x) > u_1(f(\mathbf{u}))$. Note that $f(\mathbf{u}) \in \bar{X}^f$ by Lemma A.2. Let $u \in (\bar{U} \cap U^*)$ be such that $\arg\max(u, X^f) = \{f(\mathbf{u})\}$. Strategy-proofness of f yields

$$f(\mathbf{u}) = f(\mathbf{u}_{-n}, u) = ... = f(u_1, u, ..., u).$$

Let $u_1' \in (\bar{U} \cap U^*)$ be such that $\arg\max(u_1', X^f) = \{x\}$. Since $(u_1', u, ..., u) \in (\bar{U} \cap U^*)^n$ and Individual 1 is a dictator for f on $(\bar{U} \cap U^*)^n$, then $f(u_1', u, ..., u) = x$. Hence

$$u_1(f(u_1', u, ..., u)) = u_1(x) > u_1(f(u_1, u, ..., u)),$$

and therefore f is manipulable at $(u_1, u, ..., u)$ by Individual 1, contradicting the fact that f is strategy-proof. Hence f is dictatorial on $\bar{U}^n$.□

Bibliography

[1] S. Barberà, and M. Jackson (1994): "A characterization of strategy-proof voting social choice functions for economies with pure public goods," *Social Choice and Welfare* **11**: 241-252.

[2] S. Barberà, and B. Peleg (1990): "Strategy-proof voting schemes with continuous preferences," *Social Choice and Welfare* **7**: 31-38.

[3] A. Gibbard (1973): "Manipulation of voting schemes: a general result," *Econometrica* **41**: 587-602.

[4] D. Moreno (1994): "Nonmanipulable decision mechanisms for economic environments," *Social Choice and Welfare* **11**: 225-240.

[5] H. Moulin (1980): "On strategy-proofness and single peakedness," *Public Choice* **35**: 437-455.

[6] M. Satterthwaite (1975): "Strategy-proofness and Arrow's conditions: existence and correspondence theorems for voting procedures and social welfare functions," *Journal of Economic Theory* **35**: 1-18.

[7] L. Zhou (1991): "Impossibility of strategy-proof allocation mechanisms in economies with public goods," *Review of Economic Studies* **58**: 107-119.

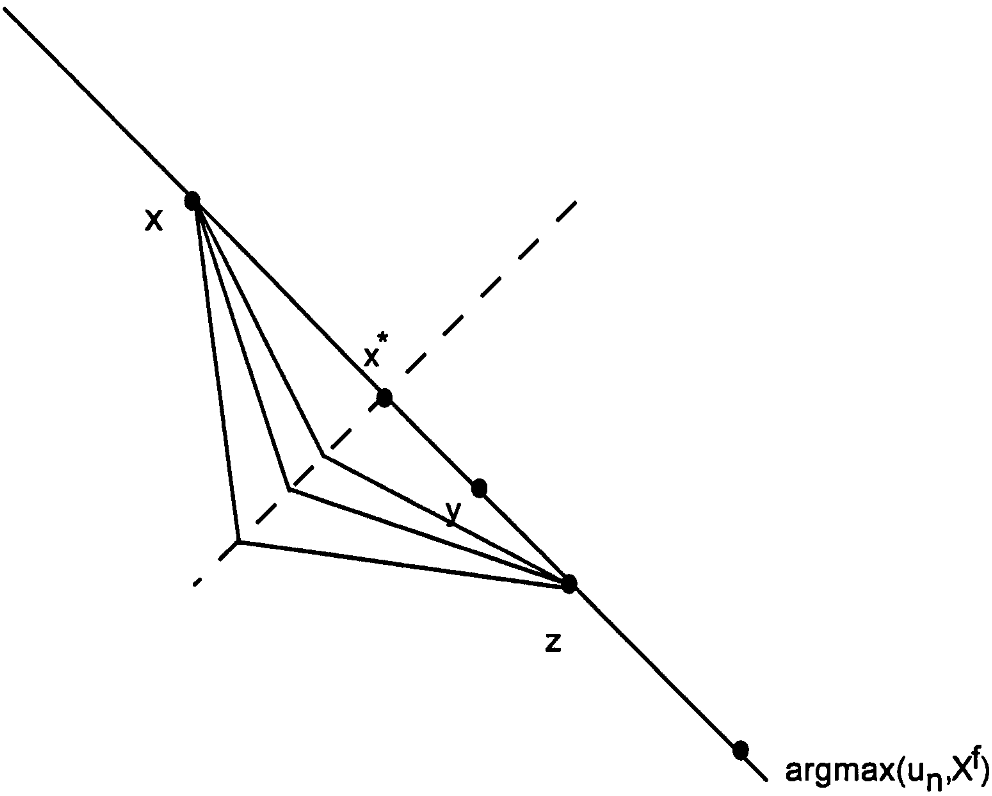

Figure 1

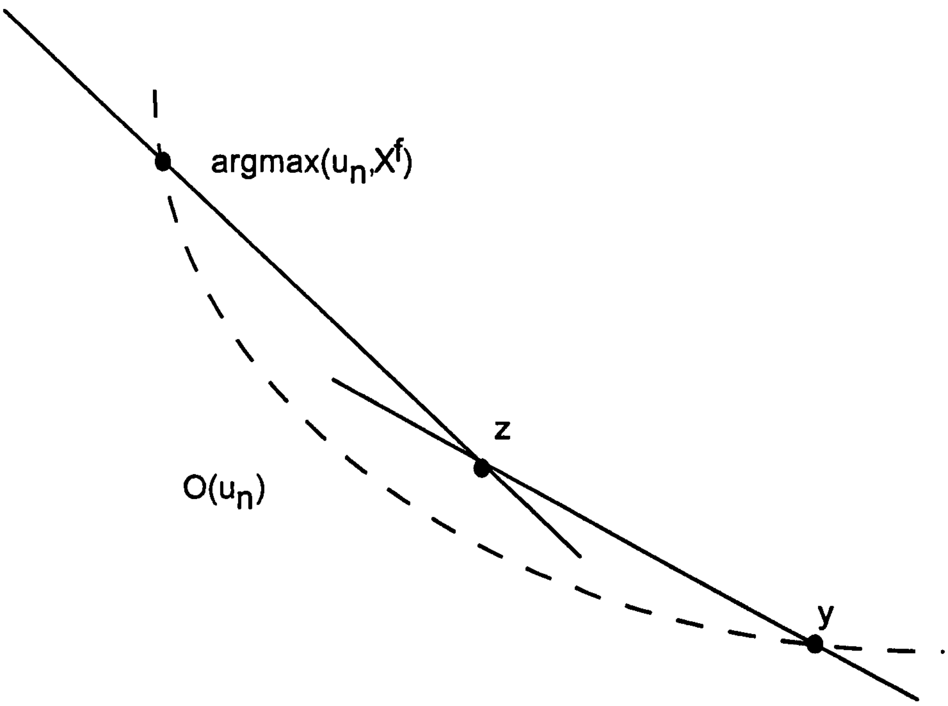

Figure 2