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STRATEGY-PROOF ALLOCATION MECHANISMS FOR ECONOMIES WITH PUBLIC GOODS

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Abstract

This paper provides a characterization of the class of incentive compatible (i.e., strategy-proof) allocation mechanisms for decision problems associated with classical economic environments. It is shown that when at least one public good is provided, then only dictatorial allocation mechanisms are incentive compatible. Dictatorial mechanisms are very unsatisfactory, as any conflict of interest is always resolved in favor of a single individual (the dictator). This result reveals a basic incompatibility between incentive compatibility and any other desirable property (e.g., any kind of efficiency, fairness, etc.) of an allocation mechanism. In particular, incentive compatible allocation mechanisms typically produce inefficient outcomes.

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Introduction

The question of incentive compatibility was first posed in economics in relation to public goods provision. Consider, for example, a situation in which there are m public goods (e.g., education, transportation, police, etc.) that can be produced from a single private good (e.g., money). It is well known that if each individual is to decide his contribution for the provision of public goods based only on his own preferences, the resulting allocation will typically be inefficient. Lindahl[8] proposed a scheme for allocating public goods which, if implemented, produces efficienct outcomes. However, the viability of his scheme was questioned by Samuelson[12], who pointed out that individuals will generally have an incentive to misrepresent their preferences in order to avoid some of the financial burden. Hurwicz [5] showed that the problem of incentive compatibility is not exclusive to public goods economies, but is also present in a standard private goods-only economy.

As decentralized market-like institutions are not immune to manipulation by individuals, it is important to determine which institutions (i.e., *allocation mechanisms*) are compatible with individual incentives. An allocation mechanism can be thought of as a mapping that associates a feasible allocation with every profile of individuals' preferences[1]. Incentive compatible allocation mechanisms are those for which an individual is always best off reporting his true preferences (i.e., where reporting one's true preferences must be a dominant strategy in the game form defined by the allocation mechanism). Allocation mechanisms having this property are referred to as *strategy-proof.*

Although the notion of incentive compatibility associated with strategy-proofness is very strong, it is the appropriate condition if one is to consider allocation problems in which individuals have imperfect and asymmetric information about other individuals' preferences. Alternatively, one could introduce explicitly each individual's information and beliefs about the other individuals' preferences, and model the situation as a game of incomplete information. In this context, incentive compatibility would require that reporting one's true type be a Bayesian-Nash equilibrium. Individuals beliefs, however, are usually unknown. Thus, if an allocation mechanism is to satisfy this requirement for all possible profiles of individuals' beliefs, then an individual must be best off *almost always* reporting his true preferences (see Ledyard[7]). Hence requiring that *true reporting* be a Bayesian-Nash equilibrium for all possible beliefs is virtually equivalent to requiring that the mechanism be strategy-proof.

[1]Only *direct revelation mechanisms* are considered here. However, the *revelation principle* tells us that this can be done without loss of generality.

This paper provides a characterization of the class of strategy-proof allocation mechanisms for decision problems in classical economic environments. It is shown that when at least one public good is provided, then only *dictatorial* allocation mechanisms are incentive compatible. Dictatorial mechanisms are very unsatisfactory as any conflict of interests is always resolved in favor of a single individual (the dictator). This result reveals a basic incompatibility between individual incentives and any other desirable property (e.g., any kind of efficiency, fairness, etc.) of an allocation mechanism. In particular, incentive compatible allocation mechanisms typically produce inefficient outcomes.

Previous results have established the impossibility of designing incentive compatible allocation mechanisms that produce Pareto optimal outcomes. For example, Hurwicz[5] establishes this impossibility for a standard two person-two goods pure exchange private economy when the allocation mechanisms must produce also individually rational outcomes. Zhou[16] shows that the impossibility remains even if the requirement of individual rationality is removed. For economies including one private good (on which individuals' preferences are assumed to be linear) and one or more public goods, Green and Laffont[4] characterize the strategy-proof mechanisms that allocate public goods efficiently as Groves mechanisms; since mechanisms in this class are generally *unbalanced,* their characterization implies that strategy-proof allocation mechanisms cannot always yield Pareto efficient outcomes. Also under the strong assumption of quasi-linear preferences, Hurwicz and Walker[6] show that strategy-proof allocation mechanisms will generally produce inefficient outcomes (their results apply to pure exchange private goods economies as well as economies with public goods).

Since the set of Pareto optimal outcomes is typically a *small* subset of the set of feasible outcomes, the results just described give rise to the question whether it might be possible to find strategy-proof allocation mechanisms that are not *too far* (in some sense) from being efficient. The characterization presented here shows that for most economies such allocation mechanisms do not exist, as dictatorial allocation mechanisms produce generally inefficient outcomes[2] . Moreover, these results are obtained without restricting individuals' preferences to be representable by quasi-linear utility functions, which is a *small* set of preferences in the domain of preferences usually associated with economic environments (e.g., preferences that are continuous, monotonic, convex,

[2]Notable exceptions are economies with only public goods. For private goods-only economies, the (very unsatisfactory) allocation mechanisms that give the existing amounts of goods to a single individual (always the same) are dictatorial and produce Pareto optimal outcomes. (Zhou[16] shows that for a two-person pure exchange economy this is the unique strategy-proof allocation mechanism that produces Pareto optimal outcomes.)

etc.).

The problem of incentive compatibility also arises in political environments (e.g., when a group of public officials must be chosen from a pool of candidates, or a public project from a set of feasible projects must be selected). In this context, it has been shown that when individuals' preferences are *unrestricted*, strategy-proof decision mechanisms are dictatorial (Gibbard[3], Satterthwaite[13], Barberà and Peleg[1], Zhou[15]).

Using this approach, Satterthwaite and Sonnenschein[14] have established that strategy-proof *nonbossy*[3] differentiable allocation mechanisms are *locally serially dictatorial*[4]. However, the implications of these results are unclear (the hierarchies of dictators might differ for different profiles, each individual's choice set might be independent of the other individuals choices), and the conditions under which allocation mechanisms are *globally* serially dictatorial are hard to check, and in fact they require one to know the specific allocation mechanism that is to be used[5]. Moreover, nonbossiness is not an inexcusable condition which cannot be abandoned if a mechanism has other desirable properties.

The present theorems improve upon Satterthwaite and Sonnenschein results in that they establish that for economies with at least a public good all strategy-proof allocation mechanisms (bossy or nonbossy, differentiable or non differentiable) are (globally) dictatorial. The theorems are silent, however, about economies with private goods only, as nondictatorial and strategy-proof allocation mechanisms do exist in this case: For example, consider an economy in which there are two private goods, x and y, and three individuals. The economy is endowed with three units of x and nothing of y; but y can be produced with constant returns to scale using x as input. Let the allocation mechanism f assign to individuals 1, 2 and 3 their most preferred consumption bundle whose cost in term of x does not exceed, respectively, $1 + m_2 - m_3$, $1 + m_3 - m_1$ and $1 + m_1 - m_2$ units of x, where m_i is zero if Individual i's utility at zero is positive, and is one

[3]A mechanism is nonbossy if no individual can change some other individual's outcome (by changing the utility function he reports) and maintain his own.

[4]Serially dictatorial mechanism are those which determine the allocation by fixing a hierarchy of individual where the highest ranked individuals chooses his outcome from a set of outcomes exogenously given, the second ranked individual chooses his outcome from a set determined by the first individual's choice, and so on.

[5]This is so even for allocation mechanisms on public goods only economies: For example, when individual preferences are representable by continuous convex preferences, then medium voter type mechanisms are strategy-proof and locally dictatorial, but they are not dictatorial when their range is a one dimensional set (see Moulin[10]). Zhou[14] has shown that when the range is a higher dimensional set, then all strategy-proof mechanisms on this domain of preferences are dictatorial.

otherwise. This allocation mechanism is strategy-proof (no individual can influence his own outcome), and, if individuals' preferences are monotonic, it produces Pareto optimal outcomes. (This is a slightly modified version of an example discussed in Satterthwaite and Sonnenschein[14]. For pure exchange private goods-only economies, Barberà and Jackson[2] have recently characterized the class of strategy-proof anonymous and nonbossy allocation mechanisms as those for which individuals trade according to fixed exogenously given proportions.)

2. The Model

The set of individuals is $N = \{1, ..., n\}$, where $n \geq 1$. Each individual consumption set is of the form $X \times Y_i \subset R_+^m \times R_+^{l_i}$. The set of feasible allocations is denoted by Z, a compact subset of $X \times \prod_1^n Y_i$. Thus, the coordinate members of X are public goods, while the other coordinates are private goods or externalities which might or might not be fully public goods[6]. This representation allows consideration of all possible allocation problems: pure public goods economies (for which the sets Y_i are singletons), economies with only private goods (for which the set X is a singleton), and mixed economies (economies with public goods as well as private goods and other externalities).

Individual i's preferences are represented by utility functions (i.e., real-valued functions on $X \times Y_i$). Individuals utility functions might be known to have certain properties (e.g., to be continuous, or to be increasing in certain dimensions). Thus, for each individual $i \in N$, let U_i denote the set of a priori *admissible* utility functions. (When admissible utility functions are assumed to be quasi-concave, then the set $X \times Y_i$ is assumed to be convex.) The set of admissible utility profiles is therefore $\mathbf{U} = \prod_1^n U_i$. Utility profiles are denoted by $\mathbf{u} = (u_1, ..., u_n)$. For $\mathbf{u} \in \mathbf{U}$ and $S \subset N$, $\mathbf{u}_{-S}$ is the profile obtained from $\mathbf{u}$ by deleting the utility functions of the members of S.

For each point or subset A of Z, write A_i and A_x for the projections of A into, respectively, $X \times Y_i$ and X. Hence for each $z \in Z$, z_i is the bundle of goods received by Individual i, and z_x is the provision of public goods. Also, for an arbitrary set $A \subset R^n$, $\#A$ denotes its cardinality, and *dimA* its dimension (the dimension of the smallest affine subspace that contains it).

An *allocation mechanism* is a mapping $f : \mathbf{U} \rightarrow Z$. An allocation mechanism f is *manipulable by* $i \in N$ *at* $\mathbf{u} \in \mathbf{U}$ if there is $u_i' \in U_i$ such that $u_i(f_i(\mathbf{u}_{-i}, u_i')) > u_i(f_i(\mathbf{u}))$. An allocation mechanism is *strategy proof* if it is not manipulable by any $i \in N$ at any $\mathbf{u} \in \mathbf{U}$. Given an allocation mechanism f, write Z^f for its range. An allocation mechanism is *dictatorial* if there is an individual $i \in N$ such that for each $\mathbf{u} \in \mathbf{U}$, $f_i(\mathbf{u})$ maximizes u_i on Z_i^f (and Individual i is referred to as *the dictator*

[6]In the framework provided here, an allocation problem might be representable in different ways, depending on which of the public goods we choose to include in the set X.

for f).

It should be noticed that the definition of dictatorial mechanisms given here is weaker than the one usually encountered in the social choice framework. Here the dictator preferences need only determine the dictator's consumption bundle. Of course, any *conflict of interests* between the dictators and other individuals is always solved in favor of the dictator. If for example there are public goods whose provision has to be decided, the decision will be made based solely on the dictator's preferences. Thus, for economies with only public goods, this notion is the usual one.

Barberà and Peleg[1], and Zhou[15] have established that a strategy-proof allocation mechanism for a pure public good economy must be dictatorial. In the present framework these theorems can be formulated as follows:

Theorem (Barberà and Peleg[1]): *Let $f : \mathbf{U} \to Z$ be a strategy-proof allocation mechanism. If $\#Z^f \geq 3$ and for each $i \in N$, Y_i is a singleton and U_i contains all the continuous utility functions, then f is dictatorial.*

Theorem (Zhou[15][7]): *Let $f : \mathbf{U} \to Z$ be a strategy-proof allocation mechanism. If $dim Z^f \geq 2$ and for each $i \in N$, Y_i is a singleton and U_i contains all the continuous quasi-concave utility functions, then f is dictatorial.*

For the general class of allocation problems considered here, there might be certain natural restrictions on individuals' admissible utility functions. For example, if the *non-public* dimensions of the allocation problem are private *goods* that are always desirable or freely disposable, then admissible utility functions must be increasing in those dimensions. Thus, for each $i \in N$, a utility function u_i is said to be *y_i-increasing* if for all $y_i, y_i' \in Y_i$ with $y_i > y_i'$, one has $u_i(x, y_i) \geq u_i(x, y_i')$. When the inequality is strict, u_i is said to be *strictly y_i-increasing* (The convention used for vector inequalities is as follows: For $a, b \in R^n$, $a \geq b$ iff $a_i \geq b_i$ for all $i = 1, ..., n$; $a > b$ iff $a \geq b$ and $a \neq b$.) Note that utility functions u_i that are constant in the non-public dimensions are y_i-increasing according to this definition.

3. The results

Theorems 1 and 2 bellow apply to allocation mechanisms for economies with public goods. Theorem 1 establishes that every strategy-proof allocation mechanism which has to decide the provision of one or several public goods must be dictatorial, even restricting the sets of admissible utility

[7]This formulation of Zhou's theorem is not the most general possible. In fact, his theorem only requires that all quadratic utility functions be admissible. The formulation presented here, however, makes it easier to relate the theorem with the assumptions on preferences most common in economic environments.

functions to contain only functions that are increasing in the non-public dimensions. Theorem 2 establishes an identical result for *continuous* allocation mechanisms, when admissible utility functions are required to be strictly y_i-increasing. Both these theorems contain Barberà and Peleg's and Zhou's theorems as particular cases—however, these theorems play a fundamental role in their proofs—as well as Moreno and Walker's Theorem[9], and Theorem 2 in Moreno[10].

Theorem 1: *Let $f : \mathbf{U} \to Z$ be a strategy-proof allocation mechanism. If either*

(1.1) $\#Z_x^f \geq 3$ and for each $i \in N$, U_i contains all the continuous y_i-increasing utility functions, or

(1.2) $dim Z_x^f \geq 2$, and for each $i \in N$, U_i contains all the continuous quasi-concave y_i-increasing utility functions,

then f is dictatorial.

It should be noticed that Theorem 1 does not require that the non-public dimensions be private goods. Goods can be of any nature (private goods, public goods or other externalities): It is only required that utility functions increasing in specified dimensions be admissible. Also note that the conditions on the set Z_x^f effectively require the presence of public goods in the allocation problem. As the example in the introduction shows, strategy-proof nondictatorial allocation mechanisms do exist when public goods are absent from the allocation problem.

Theorem 1 is proved by first showing that under the assumptions of the theorem, the restriction of any strategy-proof allocation mechanism to the set of utility profiles for which all of its coordinate utility functions are constant on the non-public goods is dictatorial (Lemma 1). The theorem is then proved by showing that the dictator in this subdomain is also a dictator in the whole domain.

For each $i \in N$, write U_i^x for the set of utility functions $u_i^x \in U_i$ of the form $u_i^x(x, y_i) = v(x)$, where v is a real valued function on X (i.e., the set of utility functions that are constant in the non-public dimensions), and let $\mathbf{U}^x$ denote the set $\prod_1^n U_i^x$.

Lemma 1: *Let f be a strategy-proof allocation mechanism satisfying the assumptions of Theorem 1. Then the restriction of f to $\mathbf{U}^x$ is dictatorial.*

Proof: Let $\bar{\mathbf{y}} \in \prod_1^n Y_i$ arbitrary, and for each $i \in N$, let $\bar{U}_i$ denote the set of utility functions on $X \times \{\bar{y}_i\}$ that are restrictions of functions on U_i. Note that the sets $\bar{U}_i$ and U_i^x can be put in a one to one correspondence; i.e., for each $\bar{u}_i \in \bar{U}_i$, there is one and only one $u_i^x \in U_i^x$ such that $\bar{u}_i(x, \bar{y}_i) = u_i^x(x, y_i)$, for each $(x, y_i) \in X \times Y_i$. Write $\bar{u}_i \approx u_i^x$ when the functions $\bar{u}_i$ and u_i^x are related in this way.

Write $\bar{\mathbf{U}} = \prod_1^n \bar{U}_i$, and $\bar{Z} = Z_x \times \{\bar{\mathbf{y}}\}$, and define the allocation mechanism $\bar{f} : \bar{\mathbf{U}} \to \bar{Z}$ by $\bar{f}(\bar{\mathbf{u}}) = (f_x(\mathbf{u}^x), \bar{\mathbf{y}})$, where $\mathbf{u}^x \in \mathbf{U}^x$ is such that for each $i \in N$, $\bar{u}_i \approx u_i^x$. Clearly, $\bar{f}$ is strategy-proof. Suppose not; let $j \in N$, $\bar{\mathbf{u}} \in \bar{\mathbf{U}}$, and $\bar{u}_j' \in \bar{\mathbf{U}}$ be such that $\bar{u}_j(\bar{f}_j(\bar{\mathbf{u}}_{-i}, \bar{u}_i')) > \bar{u}_j(\bar{f}_j(\bar{\mathbf{u}}))$, and let $\mathbf{u}^x \in \mathbf{U}^x$, $u_j'^x \in U_j^x$ be such that for each $i \in N$, $\bar{u}_i \approx u_i^x$, and $\bar{u}_j' \approx u_j'^x$. Then one has

$$u_i^x(f_j(\mathbf{u}_{-i}^x, u_i'^x)) = \bar{u}_j(\bar{f}_j(\bar{\mathbf{u}}_{-i}, \bar{u}_i')) > \bar{u}_j(\bar{f}_j(\bar{\mathbf{u}})) = u_j(f_j(\mathbf{u}^x)),$$

and therefore f is manipulable by Individual j at $\mathbf{u}^x \in \mathbf{U}^x$, which contradicts that f is strategy-proof. Hence $\bar{f}$ is strategy-proof.

Next, it is shown that $Z_x^{\bar{f}} = Z_x^f$. Obviously $Z_x^{\bar{f}} \subset Z_x^f$. To show that $Z_x^{\bar{f}} \supset Z_x^f$, let $\bar{x} \in Z_x^f$, and let $\mathbf{u} \in \mathbf{U}$ be such that $f_x(\mathbf{u}) = \bar{x}$. Let $\mathbf{u}^x \in \mathbf{U}^x$ be such that for each $i \in N$, $u_i^x(x, y_i) = v(x)$, where v has $\bar{x}$ as its unique maximizer. Since f is strategy-proof, one has $f_x(\mathbf{u}_{-1}, \bar{u}_1) = \bar{x}$ (otherwise Individual 1 can manipulate f at $\mathbf{u}$). Similarly,

$$f_x(\mathbf{u}_{-1}, u_1^x) = f_x(\mathbf{u}_{-\{1,2\}}, u_1^x, u_2^x) = ... = f_x(u_n, \mathbf{u}_{-n}^x) = f_x(\mathbf{u}^x) = \bar{x}.$$

Now let $\bar{\mathbf{u}} \in \bar{\mathbf{U}}$ be such that for each $i \in N$, $\bar{u}_i \approx u_i^x$; then one has $\bar{f}_x(\bar{\mathbf{u}}) = f_x(\mathbf{u}^x) = \bar{x}$. Hence $\bar{x} \in Z_x^{\bar{f}}$.

If assumption (1.1) of Theorem 1 holds, then the Barberà and Peleg's Theorem implies that $\bar{f}$ is dictatorial. If (1.2) of Theorem 1 holds, then Zhou's Theorem implies that $\bar{f}$ is dictatorial. Therefore in either case, $\bar{f}$ is dictatorial; i.e., there is $j \in N$ such that for each $\bar{\mathbf{u}} \in \bar{\mathbf{U}}$, $\bar{f}_j(\bar{\mathbf{u}})$ maximizes $\bar{u}_j$ on $Z_x^{\bar{f}}$. This in fact implies that for each $\mathbf{u}^x \in \mathbf{U}^x$, $f_j(\mathbf{u}^x)$ maximizes u_j^x on Z; i.e., that the restriction of f to $\mathbf{U}^x$ is dictatorial.□

In order to establish Lemma 2, some additional notation has to be introduced. For each $i \in N$ and $A_i \subset X \times Y_i$, denote by $\bar{A}_i$ the set of y_i-maximal points—i.e., the points $(x, y_i) \in A_i$ for which there is no $(x, y_i') \in A_i$ such that $y_i' > y_i$. Lemma 2 establishes that when an allocation mechanism is dictatorial, the set of y_i-maximal points of the projection of its range into the dictator's consumption set must be closed.

Lemma 2: *Let $f : \mathbf{U} \to Z$ be an allocation mechanism. If Individual i is a dictator for f and U_i contains all the continuous concave y_i-increasing utility functions on Z_i, then $\bar{Z}_i^f$ is closed.*

Proof: Let $(\bar{x}, \bar{y}_i)$ in the closure of $\bar{Z}_i^f$. Let $\mathbf{u} \in \mathbf{U}$ be such that u_i is given for each $(x, y_i) \in Z_i$ by

$$u_i(x, y_i) = - \parallel x - \bar{x} \parallel + \min\{\alpha_1 y_i^1, ..., \alpha_{l_i} y_i^{l_i}\},$$

where $\alpha_1, ..., \alpha_{l_i}$ are such that $\alpha_1 \bar{y}_i^1 = ... = \alpha_{l_i} \bar{y}_i^{l_i}$.

Since Individual i is a dictator for f, and for each $(x, y_i) \in Z_i^f \setminus \{(\bar{x}, \bar{y}_i)\}$ one has $u_i(\bar{x}, \bar{y}_i) > u_i(x, y_i)$, then $f_i(u_i, \mathbf{u}_{-i}) = (\bar{x}, \bar{y}_i)$. Hence $(\bar{x}, \bar{y}_i) \in \bar{Z}_i^f$.□

PROOF OF THEOREM 1: The theorem is proved by induction on the number of individuals. Let f be a strategy-proof allocation mechanism satisfying the assumptions of Theorem 1. It is shown that f is dictatorial.

The case $n = 1$ is simple. Suppose that for some $u_1 \in U_1$ there is $(x, y_1) \in Z^f$ such that $u_1(x, y_1) > u_1(f(u_1))$. Since $(x, y_1) \in Z^f$, let $u_1' \in U_1$ be such that $f(u_1') = (x, y_1)$. Then one has

$$u_1(f(u_1')) = u_1(x, y_1) > u_1(f(u_1)),$$

and therefore f is manipulable by Individual 1 at u_1, which contradicts that f is strategy-proof. Thus, for each $u_1 \in U_1$ and each $(x, y_1) \in Z^f$, one has $u_1(f(u_1)) \geq u_1(x, y_1)$, and therefore Individual 1 is the dictator for f (i.e., f is dictatorial).

Assuming that Theorem 1 holds for $n \leq k - 1$, it remains to be shown that it holds for $n = k$. Henceforth assume, without loss of generality (w.l.o.g.), that Individual 1 is the dictator for the restriction of f to $\mathbf{U}^x$(Lemma 1). We show that he is in fact a dictator for f.

For an arbitrary $u_2 \in U_2$, let the mapping $f^{u_2} : \prod_2^n U_i \to Z$ be given for each $\mathbf{u}_{-2} \in \prod_2^n U_i$ by $f^{u_2}(\mathbf{u}_{-2}) = f(u_2, \mathbf{u}_{-2})$. The mapping f^{u_2} can be regarded as an allocation mechanism, and the notions of strategy-proofness and dictatorship, as well as the previous lemmas, apply to it.

It is easy to show that if f is strategy-proof, then f^{u_2} is strategy-proof also. Suppose not; let $i \in N \setminus \{2\}$, $\mathbf{u}_{-2} \in \prod_2^n U_j$, and $u_i' \in U_i$ be such that $u_i(f_i^{u_2}(\mathbf{u}_{-\{i,2\}}, u_i')) > u_i(f_i^{u_2}(\mathbf{u}_{-2}))$. Hence

$$u_i(f_i(\mathbf{u}_{-\{i,2\}}, u_i', u_2)) = u_i(f_i^{u_2}(\mathbf{u}_{-\{i,2\}}, u_i')) > u_i(f_i^{u_2}(\mathbf{u}_{-2})) = u_i(f_i(\mathbf{u}_{-2}, u_2)),$$

and therefore f is manipulable by Individual i at $(\mathbf{u}_{-2}, u_2) \in \mathbf{U}$, contradicting that f is strategy-proof.

Next, it is shown that $Z_x^{f^{u_2}} = Z_x^f$. Since $Z_x^{f^{u_2}} \subset Z_x^f$, it needs to be shown that $Z_x^{f^{u_2}} \supset Z_x^f$. Let $x \in Z_x^f$, and suppose that $x \notin Z_x^{f^{u_2}}$. Let $u_1^x \in U_1^x$ be such that $u_1^x(x, y_1) = v(x)$, where v has x as its unique maximizer. As $x \notin Z_x^{f^{u_2}}$, then $f^{u_2}(\mathbf{u}_{-\{1,2\}}, u_1^x) = (x', \mathbf{y})$, satisfies $x' \neq x$; i.e., $f_x(\mathbf{u}_{-\{1,2\}}, u_1^x, u_2) = x' \neq x$. For $i \in N \setminus \{1\}$, let $u_i^x \in U_i^x$ be such that $u_i^x(x, y_i) = v'(x)$, where is v' has x' as its unique maximizer. Since f is strategy-proof one has

$$f_x(\mathbf{u}_{-\{1,2\}}, u_1^x, u_2^x) = f_x(\mathbf{u}_{-\{1,2,3\}}, u_1^x, u_2^x, u_3^x) = ... = f_x(\mathbf{u}^x) = x' \neq x.$$

But notice that $\mathbf{u}^x \in \mathbf{U}^x$, and therefore this contradicts that Individual 1 is the dictator for the restriction of f to $\mathbf{U}^x$. Hence $Z_x^{f^{u_2}} \supset Z_x^f$.

Thus, f^{u_2} is a strategy-proof allocation mechanism involving $k-1$ individuals and since $Z_x^{f^{u_2}} = Z_x^f$, it satisfies the assumptions of Theorem 1. The induction hypothesis therefore implies that it is dictatorial. It is now shown that in fact Individual 1 (the dictator for the restriction of f to $\mathbf{U}^x$) must be the dictator for f^{u_2} also.

Suppose not; since f^{u_2} is dictatorial, w.l.o.g. assume that Individual 3 is the dictator for f^{u_2}. Let $x, x' \in Z_x^{f^{u_2}} = Z_x^f$, $x \neq x'$, and let $u_1^x \in U_1^x$, be such that $u_1^x(x, y_1) = v(x)$, where v has x as its unique maximizer, and for $i \in N \setminus \{1\}$, let $u_i^x \in U_i^x$ be such that $u_i^x(x, y_i) = v'(x)$, where v' has x' as its unique maximizer. As Individual 3 is the dictator for f^{u_2}, then $f_x^{u_2}(u_1^x, u_3^x, ..., u_n^x) = f_x(u_1^x, u_2, u_3^x, ..., u_n^x) = x'$, and since f is strategy-proof, one has $f_x(u_1^x, u_2^x, u_3^x, ..., u_n^x) = f_x(\mathbf{u}^x) = x'$. But notice that $\mathbf{u}^x \in \mathbf{U}^x$, which contradicts that Individual 1 is the dictator for the restriction of f to $\mathbf{U}^x$. Hence Individual 1 is the dictator for f^{u_2}.

Finally, it is shown that Individual 1 is the dictator for f. Suppose not; let $\mathbf{u} \in \mathbf{U}$, $(\bar{x}, \bar{y}_1) \in Z_1^f$ be such that $u_1(\bar{x}, \bar{y}_1) > u_1(f_1(\mathbf{u}))$. Since Individual 1 is the dictator for f^{u_2}, then $(\bar{x}, \bar{y}_1) \notin Z_1^{f^{u_2}}$. Choose $(\bar{x}, \bar{y}_1)$ be y_1-maximal in the closure of Z_1^f (note that at this point it is unknown whether or not this set is closed). Let $(\bar{x}, \bar{y}_1') \in \bar{Z}_1^{f^{u_2}}$ be such that $\bar{y}_1 > \bar{y}_1'$ (recall $Z_1^{f^{u_2}}$ is closed from Lemma 2), and let $\hat{u}_1 \in U_1$ be a concave utility function satisfying (see Figure 1)

(a.1) $\hat{u}_1(\bar{x}, \bar{y}_1) > \hat{u}_1(x, y_1)$, for all $(x, y_1) \in Z_1^{f^{u_2}}$, and

(a.2) $\hat{u}_1(x, y_1) > \hat{u}_1(\bar{x}, \bar{y}_1')$, for some $(x, y_1) \in Z_1^{f^{u_2}}$ with $x \neq \bar{x}$.

Write $f_x^{u_2}(\mathbf{u}_{-\{1,2\}}, \hat{u}_1) = \hat{x}$. Since Individual 1 is the dictator for f^{u_2}, (a.2) implies $\hat{x} \neq \bar{x}$. Now let $\hat{u}_2(x, y_2) = \hat{v}(x)$, where $\hat{v}$ is an arbitrary function which unique maximizer on Z_x^f is $\hat{x}$. Since f is strategy-proof, then $f_x(\mathbf{u}_{-\{1,2\}}, \hat{u}_1, \hat{u}_2) = \hat{x}$, and since Individual 1 is the dictator for $f^{\hat{u}_2}$, (a.1) implies $(\bar{x}, \bar{y}_1) \notin Z_1^{f^{\hat{u}_2}}$.

As $(\bar{x}, \bar{y}_1)$ is in the closure of Z_1^f and $(\bar{x}, \bar{y}_1) \notin \bar{Z}_1^{f^{\hat{u}_2}}$, let $\tilde{u}_2 \in U_2$ be such that there is $(\bar{x}, \bar{y}_1'') \in \bar{Z}_1^{f^{\tilde{u}_2}}$ (recall that $Z_x^{f^{\tilde{u}_2}} = Z_x^f$), and such that there is no $(\bar{x}, y_1) \in Z_1^{f^{\tilde{u}_2}}$ for which $y_1 \geq \bar{y}_1''$ (hence $(\bar{x}, \bar{y}_1'') \notin Z_1^{f^{\tilde{u}_2}}$). Also let $(\tilde{x}, \tilde{y}_1) \in \bar{Z}_1^{f^{\hat{u}_2}}$ such that $\tilde{x} \notin \{\bar{x}, \hat{x}\}$. (If assumption (1.2) of Theorem 1 holds, then $\tilde{x}, \bar{x}, \hat{x}$ can be chosen in general position, and such that $\lambda(\bar{x}, \bar{y}_1'') + (1-\lambda)(\tilde{x}, \tilde{y}_1) \notin Z_1^{f^{\hat{u}_2}}$ for each $\lambda \in [0, 1)$—see Figure 1.) Finally, let $\tilde{u}_1 \in U_1$ satisfying

(b.1) $(\bar{x}, \bar{y}_1'')$ uniquely maximizes $\tilde{u}_1$ on $Z_1^{f^{\tilde{u}_2}}$, and

(b.2) $(\tilde{x}, \tilde{y}_1)$ uniquely maximizes $\tilde{u}_1$ on $Z_1^{f^{\hat{u}_2}}$.

(If assumption (1.2) of Theorem 1 holds, then the choice of $(\bar{x}, \bar{y}_1'')$ and $(\tilde{x}, \tilde{y}_1)$ allows one to find a concave utility function with properties (b.1) and (b.2).)

Since Individual 1 is the dictator for both $f^{\tilde{u}_2}$ and $f^{\hat{u}_2}$, one has $f_1^{\hat{u}_2}(\mathbf{u}_{-\{1,2\}}, \tilde{u}_1) = f_1(\mathbf{u}_{-\{1,2\}}, \tilde{u}_1, \hat{u}_2) = (\tilde{x}, \tilde{y}_1)$, and $f_1^{\tilde{u}_2}(\mathbf{u}_{-\{1,2\}}, \tilde{u}_1) = f_1(\mathbf{u}_{-\{1,2\}}, \tilde{u}_1, \tilde{u}_2) = (\bar{x}, \bar{y}_1'')$. Because $\hat{v}$ is an arbitrary function having

$\hat{x}$ as its unique maximizer, it can be chosen such that $\hat{v}(\bar{x}) > \hat{v}(\tilde{x})$. Hence

$$\hat{u}_2(f_2(\mathbf{u}_{-\{1,2\}}, \tilde{u}_1, \tilde{u}_2)) = v(\bar{x}) > v(\tilde{x}) = \hat{u}_2(f_2(\mathbf{u}_{-\{1,2\}}, \tilde{u}_1, \hat{u}_2)).$$

Therefore f is manipulable by Individual 2 at $(\mathbf{u}_{-\{1,2\}}, \tilde{u}_1, \tilde{u}_2)$. This contradiction shows that no point $(\bar{x}, \bar{y}_1)$ exists, thereby establishing Theorem 1.□

Note that the admissibility of utility functions that are constant in the non-public goods is fundamental in the proof of Theorem 1. Thus, it will be of interest to determine whether the conclusions of Theorem 1 arise when only utility function strictly increasing in the non-public dimensions are admissible. Theorem 2 establishes that when the allocation mechanism is continuous, an identical result holds.

Write $\mathbf{U}^c$ for the set of all the utility profiles that are continuous on Z. The set $\mathbf{U}^c$ together with the distance d, given for each $\mathbf{u}, \mathbf{u}' \in \mathbf{U}^c$ by

$$d(\mathbf{u}, \mathbf{u}') = \max\{\mid u_i(z_i) - u_i'(z_i') \mid, z, z' \in Z, i \in N\},$$

form a metric space, and the notion of continuity of an allocation mechanism has the standard formulation.

Theorem 2: *Let $f : \mathbf{U} \to Z$ be a strategy-proof allocation mechanism. If $\mathbf{U} \subset \mathbf{U}^c$, f is continuous on $\mathbf{U}$, and either*

(2.1) $\#Z_x^f \geq 3$ and for each $i \in N$, U_i contains all the continuous and strictly y_i-increasing utility functions, or

(2.2) $dimZ_x^f \geq 2$, and for each $i \in N$, $\dot{U}_i$ contains all the continuous quasi-concave and strictly y_i-increasing utility functions,

then f is dictatorial.

Proof: Let f be an allocation mechanism satisfying the assumptions of Theorem 2. In order to show that f is dictatorial, a strategy-proof extension of f to the set of all continuous profiles whose coordinates are (weakly) y_i-increasing utility functions is. Theorem 1 therefore implies that this extension must be dictatorial, thereby establishing that f must be dictatorial.

For each $i \in N$, denote by W_i the set of all the continuous and y_i-increasing functions on Z_i, and write $\mathbf{W} = \prod_1^n W_i$. For each $i \in N$, and $u_i \in W_i$, define the function $u_i^k \in U_i$ given for each $(x, y_i) \in Z_i$ by

$$u_i^k(x, y_i) = u_i(x, y_i) + \frac{1}{k}\|y_i\|.$$

For any $\mathbf{u} \in \mathbf{W}$, let $L(\mathbf{u})$ be the set of limit points of the sequence $\left\{f(\mathbf{u}^k)\right\}$. Note that since Z is compact, the set $L(\mathbf{u})$ is nonempty (and closed) for each $\mathbf{u} \in \mathbf{U}$. Let ϕ be an arbitrary *selection* function, which for each $\mathbf{u} \in \mathbf{U}$ picks a point z in $L(\mathbf{u})$. Finally, define the mapping $F : \mathbf{W} \to Z$ by $F(\mathbf{u}) = \phi(\mathbf{u})$. Note that since f is continuous on $\mathbf{U}$, one has $F(\mathbf{u}) = f(\mathbf{u})$ for $\mathbf{u} \in \mathbf{U}$ (the set $L(\mathbf{u})$ is a singleton in this case).

It is shown that F is strategy-proof. Suppose it is not; let $i \in N$, $\mathbf{u} \in \mathbf{W}$, and $u'_i \in W_i$ be such that $u_i(F_i(\mathbf{u}_{-i}, u'_i)) > u_i(F_i(\mathbf{u}))$. Since u_i is continuous and Z is compact, there are $\delta, \delta' > 0$ and an integer $\bar{K}$ such that for each $k > \bar{K}$, and $z_i, z'_i \in Z_i$ such that $\|z_i - F_i(\mathbf{u})\| < \delta$ and $\|z'_i - F_i(\mathbf{u}_{-i}, u'_i)\| < \delta'$, one has

$$u_i^k(z'_i) > u_i^k(z_i).$$

From the definition of F, there are $k, k' > \bar{K}$ such that both $\|F_i(\mathbf{u}^k) - F_i(\mathbf{u})\| < \frac{\delta}{2}$ and $\|F_i(\mathbf{u}^k_{-i}, u'^k_i) - F_i(\mathbf{u}_{-i}, u'_i)\| < \delta'$. (If $\mathbf{u}$ or $(\mathbf{u}_{-i}, u'_i)$ are in $\mathbf{U}$, this is possible because f is continuous; otherwise this is possible because $F(\mathbf{u})$ and $F(\mathbf{u}_{-i}, u'_i)$ are limit points of $\left\{F(\mathbf{u}^k)\right\}$ and $\left\{F(\mathbf{u}^k_{-i}, u'^k_i)\right\}$, respectively.) Moreover, k, k' can be chosen sufficiently large that also $\|F_i(\mathbf{u}^{k'}) - F_i(\mathbf{u}^k)\| < \frac{\delta}{2}$ (recall that for all $\mathbf{u}^k$, $F(\mathbf{u}^k) = f(\mathbf{u}^k)$, and that f is continuous on $\mathbf{U}$). Then one has

$$u_i^{k'}(F_i(\mathbf{u}^{k'}_{-i}, u'^{k'}_i)) > u_i^{k'}(F_i(\mathbf{u}^{k'})).$$

Note, however, that both $\mathbf{u}^{k'}$ and $(\mathbf{u}^{k'}_{-i}, u'^{k'}_i)$ are members of $\mathbf{U}$; hence $F_i(\mathbf{u}^{k'}) = f_i(\mathbf{u}^{k'})$, and $F_i(\mathbf{u}^{k'}_{-i}, u'^{k'}_i) = f_i(\mathbf{u}^{k'}_{-i}, u'^{k'}_i)$, which implies that f is manipulable by Individual i at $\mathbf{u}^{k'}$. This contradiction shows that F is strategy-proof, thereby establishing Theorem 2.□

4. Conclusions

The only allocation problems that are left out of the scope of theorems 1 and 2 are those in which no public goods have to be provided; but as the example in the introduction shows, the absence of public goods from the allocation problem might allow one to find strategy-proof and nondictatorial allocation mechanisms. The results presented here together with the results provided by Barberà and Jackson[2] show the kinds of strategy-proof allocation mechanisms that one can design in economic environments. It is an open question which allocation mechanisms can be designed for pure private goods economies with production, or for economies with externalities that are not fully public goods. It seems as if no simple characterization results can be hoped for in these cases.

Also it seems as if results similar to theorems 1 and 2 ought to hold even when individuals' utility functions are further restricted to be increasing in the public goods. The method of proof used here might be useful in proving such results, although this would require one to obtain theorems similar to Barberà and Peleg's and Zhou's theorems for public goods-only economies when individuals utility functions are known to be increasing.

Finally, it would be of interest to determine minimal restrictions on domain of preferences and/or minimal dimensionality of the allocation problems for which one can find strategy-proof and nondictatorial allocation mechanisms.

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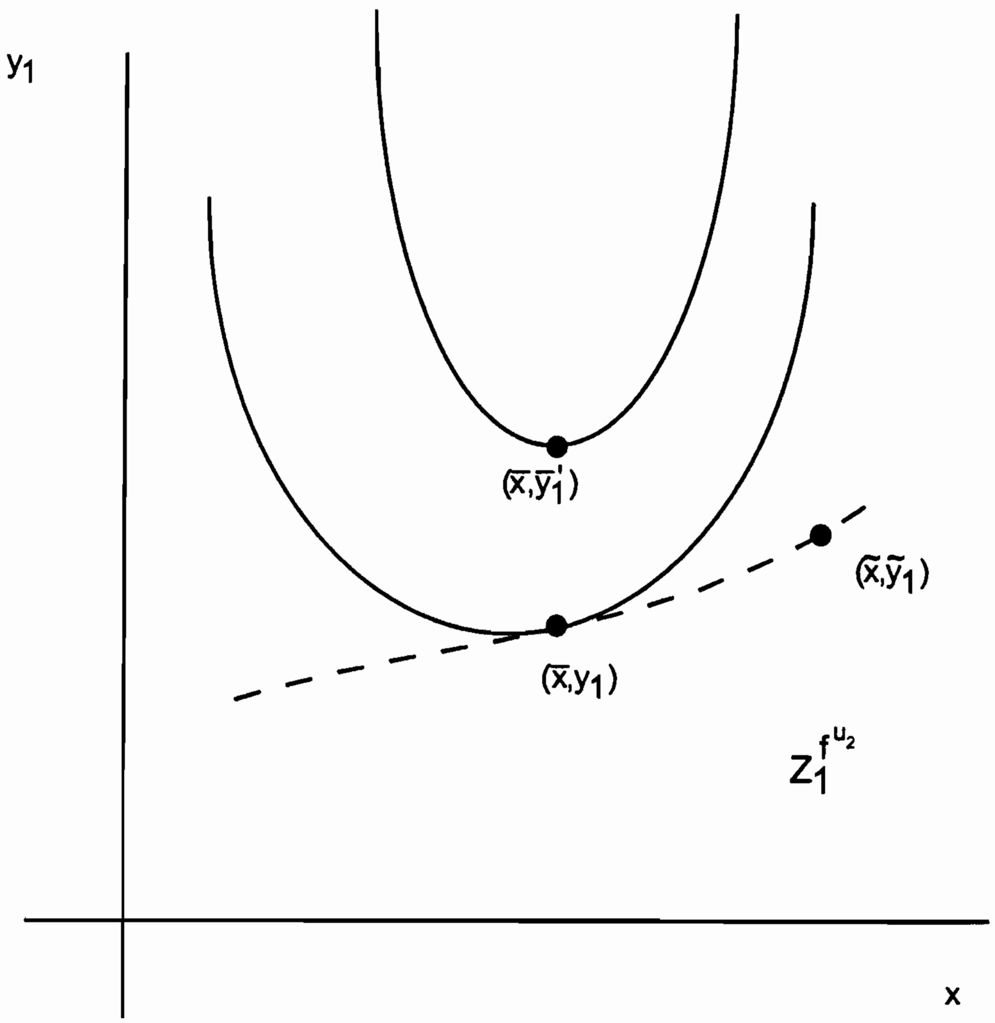

Figure 1