Lancho, A., et al. Saddlepoint Approximations for Noncoherent Single-Antenna Rayleigh Block-Fading Channels, *in 2019 IEEE International Symposium on Information Theory (ISIT), 7-12 July 2019, Paris, France, 5 pp.*
DOI: https://doi.org/10.1109/ISIT.2019.8849659

Saddlepoint Approximations for Noncoherent Single-Antenna Rayleigh Block-Fading Channels

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***Abstract*—This paper presents saddlepoint approximations of state-of-the-art converse and achievability bounds for noncoherent, single-antenna, Rayleigh block-fading channels. These approximations can be calculated efficiently and are shown to be accurate for SNR values as small as 0 dB, blocklengths of 168 channel uses or more, and when the channel's coherence interval is not smaller than two. It is demonstrated that the derived approximations recover both the normal approximation and the reliability function of the channel.**

I. Introduction

New services in next-generation's wireless systems will require low latency and high reliability; see [1] and references therein. Under such constraints, *capacity* and *outage capacity* are not accurate benchmarks, and more refined metrics on the maximum coding rate, which take into account the short packet size required in low-latency applications, are called for.

Several techniques can be used to characterize the finite-blocklength performance. One possibility is to fix a reliability constraint and study the maximum coding rate as the blocklength grows. This approach, sometimes referred to as *normal approximation*, was followed *inter alia* by Polyanskiy *et al.* [2] and has been generalized to several wireless communication channels; see, e.g., [3]–[8]. Particularly relevant to the present paper is the recent work by Lancho *et al.* [7], [8], who derived a high-SNR normal approximation for noncoherent single-antenna Rayleigh block-fading channels. Alternatively, one can fix the coding rate and study the exponential decay of the error probability as the blocklength increases. The resulting error exponent is usually referred to as *reliability function* [9, Ch. 5]. Both the exponential and sub-exponential behavior of the error probability can be characterized via the *saddlepoint method* [10, Ch. XVI]. This method has been applied in [11]–[13] to obtain approximations of the random coding union (RCU) bound [2, Th. 16], the RCU bound with parameter s (RCU_s) [14, Th. 1], and the meta-converse (MC) bound [2, Th. 31] for some memoryless channels.

A. Lancho, G. Vazquez-Vilar and T. Koch have received funding from the European Research Council (ERC) under the European Union's Horizon 2020 research and innovation programme (714161), the Spanish Ministerio de Economía y Competitividad (RYC-2014-16332, TEC2016-78434-C3-3-R (AEI/FEDER, EU) and IJCI2015-27020), the Spanish Ministerio de Educación, Cultura y Deporte (FPU014/01274), and the Comunidad de Madrid (S2103/ICE-2845). G. Durisi and J. Östman have been supported by the Swedish Research Council under Grants 2016-03293 and 2014-6066.

In this paper, we apply the saddlepoint method to derive approximations of the MC and the RCU_s bounds for noncoherent single-antenna Rayleigh block-fading channels. While these approximations must be evaluated numerically, the computational complexity is independent of the number of diversity branches L. This is in stark contrast to the nonasymptotic MC, RCU and RCU_s bounds, whose evaluation has a computational complexity that grows linearly in L. Numerical evidence suggests that the saddlepoint approximations, although developed under the assumption of large L, are accurate even for $L = 1$ if the SNR is greater than or equal to 0 dB. Furthermore, the proposed expansions are shown to recover the normal approximation and the reliability function of the channel, thus providing a unifying tool for the two regimes, which are usually considered separately in the literature.

In our analysis, the saddlepoint method is applied to the tail probabilities appearing in the nonasymptotic MC and RCU_s bounds. These probabilities often depend on a set of parameters, such as the SNR. Existing saddlepoint expansions do not consider such dependencies. Hence, they can only characterize the behavior of the expansion error in function of L, but not in terms of the remaining parameters. In contrast, we derive in Section II saddlepoint expansions for random variables whose distribution depends on a parameter θ, carefully analyze the error terms, and demonstrate that they are uniform in θ. We then apply the expansions to the Rayleigh block-fading channel introduced in Section III. As shown in Sections IV–VI, this results in accurate performance approximations, in which the error terms depend only on the blocklength and are uniform in the remaining parameters.

II. Saddlepoint Expansion

Let $\{Z_k\}_{k=1}^n$ be a sequence of independent and identically distributed (i.i.d.), real-valued, zero-mean, random variables, whose distribution depends on $\theta \in \Theta$, where Θ denotes the set of possible values of θ.

The cumulant generating function (CGF) is defined as

$$\psi_\theta(\zeta) \triangleq \log \mathsf{E}\left[e^{\zeta Z_k}\right] \tag{1}$$

and the characteristic function is defined as

$$\varphi_\theta(\zeta) \triangleq \mathsf{E}\left[e^{i\zeta Z_k}\right] \tag{2}$$

where $i \triangleq \sqrt{-1}$. We denote by $\psi_\theta^{(k)}$ the k-th derivative of $\zeta \mapsto \psi_\theta(\zeta)$. For the first, second, and third derivatives we sometimes use also the notation ψ'_θ, ψ''_θ, and ψ'''_θ.

A real-valued random variable Z_k is said to be *lattice* if it is supported on the points b, $b \pm h$, $b \pm 2h \ldots$ for some b and h. A random variable that is not lattice is said to be *nonlattice*. It can be shown that a random variable is nonlattice if, and only if, for every $\delta > 0$ [10, Ch. XV.1, Lemma 4]

$$|\varphi_\theta(\zeta)| < 1, \quad |\zeta| \geq \delta. \tag{3}$$

We shall say that a family of random variables Z_k (parametrized by θ) is nonlattice if for every $\delta > 0$

$$\sup_{\theta \in \Theta} |\varphi_\theta(\zeta)| < 1, \quad |\zeta| \geq \delta. \tag{4}$$

Similarly, we shall say that a family of distributions (parametrized by θ) is nonlattice if the corresponding family of random variables is nonlattice.

Proposition 1: Let the family of i.i.d. random variables $\{Z_k\}_{k=1}^n$ (parametrized by θ) be nonlattice. Suppose that there exists a $\zeta_0 > 0$ such that

$$\sup_{\theta \in \Theta, |\zeta| < \zeta_0} \left|\psi_\theta^{(k)}(\zeta)\right| < \infty, \quad k = 0, \ldots, 4 \tag{5}$$

and

$$\inf_{\theta \in \Theta, |\zeta| < \zeta_0} \left|\psi''_\theta(\zeta)\right| > 0. \tag{6}$$

1) If for a given $\gamma \geq 0$ there exists a $\tau \in [0, \zeta_0)$ such that $n\psi'_\theta(\tau) = \gamma$, then

$$\mathsf{P}\left[\sum_{k=1}^n Z_k \geq \gamma\right] = e^{n[\psi_\theta(\tau) - \tau\psi'_\theta(\tau)]}\left[f_\theta(\tau,\tau) + \frac{K_\theta(\tau,n)}{\sqrt{n}} + o\left(\frac{1}{\sqrt{n}}\right)\right] \tag{7}$$

where $o(1/\sqrt{n})$ comprises terms that vanish faster than $1/\sqrt{n}$ and are uniform in τ and θ. Here,

$$f_\theta(u,\tau) \triangleq e^{n\frac{u^2}{2}\psi''_\theta(\tau)} Q\left(u\sqrt{n\psi''_\theta(\tau)}\right) \tag{8a}$$

$$K_\theta(\tau,n) \triangleq \frac{\psi'''_\theta(\tau)}{6\psi''_\theta(\tau)^{3/2}}\left(-\frac{1}{\sqrt{2\pi}} + \frac{\tau^2\psi''_\theta(\tau)n}{\sqrt{2\pi}} - \tau^3\psi''_\theta(\tau)^{3/2}n^{3/2}f_\theta(\tau,\tau)\right) \tag{8b}$$

with $Q(\cdot)$ standing for the Gaussian Q-function.

2) Let U be uniformly distributed on $[0,1]$. If for a given $\gamma \geq 0$ there exists a $\tau \in [0, \min\{\zeta_0, \eta\})$ (for some arbitrary $\eta < 1$ independent of n and θ) such that $n\psi'_\theta(\tau) = \gamma$, then

$$\mathsf{P}\left[\sum_{k=1}^n Z_k \geq \gamma + \log U\right] \leq e^{n[\psi_\theta(\tau) - \tau\psi'_\theta(\tau)]} \times \left[f_\theta(\tau,\tau) + f_\theta(1-\tau,\tau) + \frac{\hat{K}_\theta(\tau)}{\sqrt{n}} + o\left(\frac{1}{\sqrt{n}}\right)\right] \tag{9}$$

where

$$\hat{K}_\theta(\tau) \triangleq \frac{2}{\sqrt{2\pi}}\frac{\psi'''_\theta(\tau)}{6\psi''_\theta(\tau)^{3/2}} \tag{10}$$

and $o(1/\sqrt{n})$ is uniform in τ and θ.

Proof: See [15]. ■

III. Rayleigh Block-Fading Channel

We consider a single-antenna Rayleigh block-fading channel with coherence interval T. For this channel model, the input-output relation within the ℓ-th coherence interval is given by

$$\mathbf{Y}_\ell = H_\ell \mathbf{X}_\ell + \mathbf{W}_\ell \tag{11}$$

where $\mathbf{X}_\ell$ and $\mathbf{Y}_\ell$ are T-dimensional, complex-valued, random vectors containing the input and output signals, respectively; $\mathbf{W}_\ell$ is the additive noise with i.i.d., zero-mean, unit-variance, circularly-symmetric, complex Gaussian entries; and H_ℓ is a zero-mean, unit-variance, circularly-symmetric, complex Gaussian random variable. We assume that H_ℓ and $\mathbf{W}_\ell$ are mutually independent and take on independent realizations over successive coherence intervals. Moreover, the joint law of $(H_\ell, \mathbf{W}_\ell)$ does not depend on the channel inputs. We consider a noncoherent setting where the transmitter and the receiver are aware of the distribution of H_ℓ but not of its realization.

We next introduce the notion of a channel code. For simplicity, we shall restrict ourselves to codes whose blocklength n satisfies $n = LT$, where L denotes the number of coherence intervals of length T needed to transmit the entire codeword. An (M, L, ϵ, ρ)-code for the channel (11) consists of:

1) An encoder f: $\{1, \ldots, M\} \to \mathbb{C}^{LT}$ that maps the message A, which is uniformly distributed on $\{1, \ldots, M\}$, to a codeword $\mathbf{X}^L = [\mathbf{X}_1, \ldots, \mathbf{X}_L]$. The codewords satisfy the per-coherence-interval power constraint[1]

$$\|\mathbf{X}_\ell\|^2 = T\rho, \quad \ell = 1, \ldots, L. \tag{12}$$

2) A decoder g: $\mathbb{C}^{LT} \to \{1, \ldots, M\}$ satisfying the average error probability constraint

$$\mathsf{P}\left[g\left(\mathbf{Y}^L\right) \neq A\right] \leq \epsilon \tag{13}$$

where $\mathbf{Y}^L = [\mathbf{Y}_1, \ldots, \mathbf{Y}_L]$ is the channel output induced by the transmitted codeword $\mathbf{X}^L = f(A)$ according to (11).

The *maximum coding rate* and *minimum error probability* are respectively defined as

$$R^*(L, \epsilon, \rho) \triangleq \sup\left\{\frac{\log M}{LT} : \exists (M, L, \epsilon, \rho)\text{-code}\right\} \tag{14a}$$

$$\epsilon^*(L, R, \rho) \triangleq \inf\left\{\epsilon : \exists (2^{LTR}, L, \epsilon, \rho)\text{-code}\right\}. \tag{14b}$$

We shall present our results in terms of error probability and use that upper (lower) bounds on $\epsilon^*(L, R, \rho)$ can be translated into lower (upper) bounds on $R^*(L, \epsilon, \rho)$ and *vice versa*.

[1] In the information theory literature, it is more common to impose a power constraint per codeword $\mathbf{X}^L$. However, practical systems typically require a per-coherence-interval constraint. Although it may be preferable to impose (12) with inequality, since it allows more freedom in optimizing the codebook, it seems plausible that using maximum power is optimal. For the high-SNR normal approximation presented in [7], [8], this turns out to be the case.

IV. Saddlepoint Expansions for RCU$_s$ and MC

Throughout the paper, we shall evaluate the achievability bounds for the capacity-achieving input distribution, for which the inputs are of the form $\mathbf{X}^L = \sqrt{T\rho}\mathbf{U}^L$, where the components of $\mathbf{U}^L = [\mathbf{U}_1, \ldots, \mathbf{U}_L]$ are i.i.d. and uniformly distributed on the unit sphere in $\mathbb{C}^T$. This distribution can be viewed as a single-antenna particularization of *unitary space-time modulation* (USTM).

We define the *generalized information density* as

$$i_s(\mathbf{x}_\ell;\mathbf{y}_\ell) \triangleq \log \frac{\mathrm{p}_{\mathbf{Y}_\ell|\mathbf{X}_\ell}(\mathbf{y}_\ell|\mathbf{x}_\ell)^s}{\mathsf{E}\big[\mathrm{p}_{\mathbf{Y}_\ell|\mathbf{X}_\ell}(\mathbf{y}_\ell|\mathbf{X}_\ell)^s\big]}. \tag{15}$$

For brevity, let $i_{\ell,s}(\rho) \triangleq i_s(\mathbf{X}_\ell;\mathbf{Y}_\ell)$ and $I_s(\rho) \triangleq \mathsf{E}[i_{\ell,s}(\rho)]$. The CGF of the zero-mean random variable $I_s(\rho) - i_{\ell,s}(\rho)$ is

$$\psi_{\rho,s}(\tau) = \log \mathsf{E}\Big[e^{\tau(I_s(\rho)-i_{\ell,s}(\rho))}\Big] \tag{16}$$

which depends on the parameters $\theta = \{\rho, s\}$. For some arbitrary $0 < \underline{s} < \overline{s}$, $0 < \underline{\rho} < \overline{\rho}$, $0 < a < 1$, and $0 < b < \min\Big\{\frac{T}{T-1}, \frac{1+T\overline{\rho}}{T\overline{\rho}\overline{s}}\Big\}$, we denote by $\mathcal{S}_\psi$ the set of all $(\tau,\rho,s) \in \mathbb{R}^3$ satisfying $-a \le \tau \le b$, $\underline{\rho} \le \rho \le \overline{\rho}$, and $\underline{s} \le s \le \overline{s}$. In [15], we show that $\mathcal{S}_\psi$ is in the region of convergence of $\psi_{\rho,s}$, i.e.,

$$\sup_{(\tau,\rho,s)\in\mathcal{S}_\psi} \psi^{(k)}_{\rho,s}(\tau) < \infty, \quad k \in \mathbb{Z}_0^+. \tag{17}$$

A. RCU$_s$ Bound

As upper bound on $\epsilon^*(L,R,\rho)$, we use the RCU$_s$ bound [14, Th. 1], which states that, for every $s > 0$,

$$\epsilon^*(L,R,\rho) \le \mathsf{P}\left[\sum_{\ell=1}^{L} i_{\ell,s}(\rho) \le LTR - \log(U)\right] \tag{18}$$

where U is uniformly distributed on the interval $[0,1]$.

Theorem 2 (Saddlepoint Expansion RCU$_s$): The coding rate R and minimum error probability ϵ^* can be parametrized by $(\tau,\rho,s) \in \mathcal{S}_\psi$ as

$$R(\tau,s) = \frac{1}{T}(I_s(\rho) - \psi'_{\rho,s}(\tau)) \tag{19a}$$

$$\epsilon^*(\tau,s) \le e^{L[\psi_{\rho,s}(\tau)-\tau\psi'_{\rho,s}(\tau)]} \times\left[f_{\rho,s}(\tau,\tau) + f_{\rho,s}(1-\tau,\tau) + \frac{\hat{K}_{\rho,s}(\tau)}{\sqrt{L}} + o\left(\frac{1}{\sqrt{L}}\right)\right] \tag{19b}$$

where $f(\cdot,\cdot)$ is defined in (8a), $\hat{K}_{\rho,s}(\cdot)$ is defined in (10), and $o(1/\sqrt{L})$ is uniform in τ, s and ρ.

Proof: The desired result follows by applying Proposition 1, Part 2) to (18). For details see [15]. ■

Remark 1: The set $\mathcal{S}_\psi$ with $\overline{s} = 1$ includes $0 \le \tau < 1$. In this case, the identity (19a) with $s \in (0,1]$ and $\tau \in [0,1)$ characterizes all rates R between the critical rate

$$R_s^{\mathrm{cr}}(\rho) \triangleq \frac{1}{T}\big(I_s(\rho) - \psi'_{\rho,s}(1)\big) \tag{20}$$

and $I_s(\rho)$. Solving (19a) for τ, we obtain from Theorem 2 an upper bound on the minimum error probability $\epsilon^*(L,R,\rho)$ as a function of the rate $R \in (R_s^{\mathrm{cr}}(\rho), I_s(\rho)]$, $s \in (0,1]$.

B. Meta-Converse Bound

Let the auxiliary output probability density function (pdf)

$$\mathrm{q}_{\mathbf{Y}_\ell,s}(\mathbf{y}_\ell) \triangleq \frac{1}{\mu(s)}\mathsf{E}\big[\mathrm{p}_{\mathbf{Y}_\ell|\mathbf{X}_\ell}(\mathbf{y}_\ell|\mathbf{X}_\ell)^s\big]^{1/s} \tag{21}$$

where $\mu(s)$ is a normalizing factor. We define the *generalized mismatched information density* as

$$j_s(\mathbf{x}_\ell;\mathbf{y}_\ell) \triangleq \log \frac{\mathrm{p}_{\mathbf{Y}_\ell|\mathbf{X}_\ell}(\mathbf{y}_\ell|\mathbf{x}_\ell)}{\mathrm{q}_{\mathbf{Y}_\ell,s}(\mathbf{y}_\ell)}. \tag{22}$$

It holds that

$$j_s(\mathbf{x}_\ell;\mathbf{y}_\ell) = \log\mu(s) + \frac{1}{s}i_s(\mathbf{x}_\ell;\mathbf{y}_\ell). \tag{23}$$

For brevity, let $j_{\ell,s}(\rho) \triangleq j_s(\mathbf{X}_\ell;\mathbf{Y}_\ell)$ and $J_s(\rho) \triangleq \mathsf{E}[j_{\ell,s}(\rho)]$. When $s = 1$, $j_{\ell,1}(\rho) = i_{\ell,1}(\rho)$ and $J_1(\rho) = I_1(\rho)$, in which case we write $i_\ell(\rho) \triangleq i_{\ell,1}(\rho)$ and $C(\rho) \triangleq I_1(\rho)$.[2]

A lower bound on $\epsilon^*(L,R,\rho)$ follows by evaluating the MC bound [2, Th. 31] for the auxiliary pdf $\mathrm{q}_{\mathbf{Y}_\ell,s}$ and using [2, Eq. (102)]. This yields, for every $\xi > 0$ and $s > 0$,

$$\epsilon^*(L,R,\rho) \ge \mathsf{P}\left[\sum_{\ell=1}^{L}(I_s(\rho) - i_{\ell,s}(\rho)) \ge sLJ_s(\rho) - s\xi\right] - e^{(\xi - LTR)} \tag{24}$$

where we have used (23) to express $j_{\ell,s}(\rho)$ in terms of $i_{\ell,s}(\rho)$.

Theorem 3 (Saddlepoint Expansion MC): For every rate R and $(\tau,\rho,s) \in \mathcal{S}_\psi$

$$\epsilon^*(L,R,\rho) \ge -e^{L\left[J_s(\rho) - \frac{\psi'_{\rho,s}(\tau)}{s} - TR\right]} + e^{L[\psi_{\rho,s}(\tau)-\tau\psi'_{\rho,s}(\tau)]}\left[f_{\rho,s}(\tau,\tau) + \frac{K_{\rho,s}(\tau,L)}{\sqrt{L}} + o\left(\frac{1}{\sqrt{L}}\right)\right] \tag{25}$$

where $f(\cdot,\cdot)$ is defined in (8a), $K_{\rho,s}(\cdot,\cdot)$ is defined in (10), and $o(1/\sqrt{L})$ is uniform in τ, s and ρ.

Proof: The inequality (25) follows by applying Proposition 1, Part 1) to the probability term in (24) with $\xi = LJ_s(\rho) - L\psi'_{\rho,s}(\tau)/s$. For details see [15]. ■

The expansions (19b) and (25) can be evaluated more efficiently than the nonasymptotic bounds (18) and (24). Indeed, (18) and (24) require the evaluation of the L-dimensional distribution of $\sum_{\ell=1}^{L} i_{\ell,s}(\rho)$, whereas (19b) and (25) depend only on the cumulants $\psi_{\rho,s}(\tau)$, $\psi'_{\rho,s}(\tau)$, $\psi''_{\rho,s}(\tau)$ and $\psi'''_{\rho,s}(\tau)$, which can be obtained by solving one-dimensional integrals.

V. Normal Approximation

The maximum coding rate can be expanded as

$$R^*(L,\epsilon,\rho) = \frac{C(\rho)}{T} - \sqrt{\frac{V(\rho)}{LT^2}}Q^{-1}(\epsilon) + \mathcal{O}\left(\frac{\log L}{L}\right) \tag{26}$$

where $V(\rho) \triangleq \mathsf{Var}\big[i_\ell(\rho)\big]$. This is usually referred to as *normal approximation*. As we shall outline next, (26) can also be recovered from the expansions (19b) and (25).

[2]Recall that we chose the input distribution to be USTM, which for the power constrain (12) is capacity achieving.

To prove that the right-hand side (RHS) of (26) is achievable, we evaluate (19a) for $s = 1$ and

$$\tau_L = \frac{Q^{-1}\Big(\epsilon - \frac{k_{1,\rho}}{\sqrt{L}}\Big)}{\sqrt{L\psi''_{\rho,1}(0)}} \tag{27}$$

where $k_{1,\rho} > 0$ is independent of L and bounded in ρ. We show in [15] that with this choice of τ_L, we can make the RHS of (19b) less than ϵ by choosing $k_{1,\rho}$ sufficiently large. Hence, by evaluating $R(\tau_L, 1)$, we characterize $R^*(L, \epsilon, \rho)$.

In [15], we further show that all the derivatives of $\psi_{\rho,s}$ exist uniformly in s and ρ. Consider the Taylor series

$$\psi'_{\rho,1}(\tau) = \tau\psi''_{\rho,1}(0) + \frac{\tau^2}{2}\psi'''_{\rho,1}(\tilde{\tau}) \tag{28}$$

for some $\tilde{\tau} \in (0, \tau)$. Combining (28) with (19a) this yields

$$R^*(L,\epsilon,\rho) \geq \frac{C(\rho)}{T} - \sqrt{\frac{\psi''_{\rho,1}(0)}{LT^2}}Q^{-1}\bigg(\epsilon - \frac{k_{1,\rho}}{\sqrt{L}}\bigg) - \frac{Q^{-1}\Big(\epsilon - \frac{k_{1,\rho}}{\sqrt{L}}\Big)^2 \psi'''_{\rho,1}(\tilde{\tau})}{2L\psi''_{\rho,1}(0)}. \tag{29}$$

Using that $\psi''_{\rho,1}(0) = V(\rho)$, and expanding $Q^{-1}(\epsilon - k_{1,\rho}/\sqrt{L})$ around ϵ, we can write (29) as

$$R^*(L,\epsilon,\rho) \geq \frac{C(\rho)}{T} - \sqrt{\frac{V(\rho)}{LT^2}}Q^{-1}(\epsilon) + \mathcal{O}\bigg(\frac{1}{L}\bigg) \tag{30}$$

demonstrating that the RHS of (26) is achievable.

To show that the RHS of (26) is also a converse bound, we evaluate (25) for $s = 1$ and

$$\tau_L = \frac{Q^{-1}\left(\epsilon + \frac{k_{2,\rho}}{\sqrt{L}}\right)}{\sqrt{L\psi''_{\rho,1}(0)}} - \frac{\psi'''_{\rho,1}(0)\bigg(1 + Q^{-1}\left(\epsilon + \frac{k_{2,\rho}}{\sqrt{L}}\right)^2\bigg)}{3L\psi''_{\rho,1}(0)^2} \tag{31}$$

where $k_{2,\rho}$ is independent of L and bounded in ρ. A Taylor series expansion of (25) then yields

$$\begin{aligned} R^*(L,\epsilon,\rho) \leq{}& \frac{C(\rho)}{T} - \sqrt{\frac{\psi''_{\rho,1}(0)}{LT^2}}Q^{-1}(\epsilon) \\ &+ \frac{1}{2T}\frac{\log L}{L} - \frac{\sqrt{2\pi\psi''_{\rho,1}(0)}}{LT}e^{\frac{Q^{-1}(\epsilon)^2}{2}} \\ &- \frac{\left(1 - Q^{-1}(\epsilon)^2\right)\psi'''_{\rho,1}(0)}{6\psi''_{\rho,1}(0)LT} + \mathcal{O}\bigg(\frac{1}{L^{3/2}}\bigg). \end{aligned} \tag{32}$$

Using that $\psi''_{\rho,1}(0) = V(\rho)$, and collecting terms of order $(\log L)/L$, we conclude that the RHSs of (32) and (26) coincide.

Finally note that [7, Eq. (16)]

$$\begin{aligned} C(\rho) ={}& (T-1)\log(T\rho) - \log\Gamma(T) \\ &- (T-1)\bigg[\log(1+T\rho) + \frac{T\rho}{1+T\rho} - \psi(T-1)\bigg] \\ &+ {}_2F_1\bigg(1, T-1; T; \frac{T\rho}{1+T\rho}\bigg) + o_\rho(1) \end{aligned} \tag{33a}$$

$$V(\rho) = (T-1)^2\frac{\pi^2}{6} + (T-1) + o_\rho(1) \tag{33b}$$

where $\Gamma(\cdot)$ denotes the gamma function, $\psi(\cdot)$ the digamma function, ${}_2F_1(\cdot,\cdot;\cdot;\cdot)$ the Gauss hypergeometric function, and $o_\rho(1)$ comprises terms that are uniform in L and vanish as $\rho \to \infty$. We thus recover from (26) the high-SNR normal approximation [7, Th. 1], [8, Th. 1].

VI. Error Exponent Analysis

The expansions (19b) and (25) can be written as an exponential term times a subexponential factor. As we show next, the exponential terms of both expansions coincide for rates $R^{\text{cr}}_{1/2}(\rho) < R < C(\rho)$, so they characterize the reliability function

$$E_r(R,\rho) \triangleq \lim_{L\to\infty} -\frac{1}{L}\log\epsilon^*(L,R,\rho). \tag{34}$$

Theorem 4: Let $\underline{\rho} \leq \rho \leq \overline{\rho}$ and $\underline{\tau} < \tau < \overline{\tau}$ for some arbitrary $0 < \underline{\rho} < \overline{\rho}$ and $0 < \underline{\tau} < \overline{\tau} < 1$. Set $s_\tau \triangleq 1/(1+\tau)$. Then, the coding rate R and the minimum error probability ϵ^* can be parametrized by $\tau \in (\underline{\tau}, \overline{\tau})$ as

$$R(\tau) = \frac{1}{T}\left(I_{s_\tau}(\rho) - \psi'_{\rho,s_\tau}(\tau)\right) \tag{35a}$$

$$\underline{\mathsf{A}}_\rho(\tau) \leq \epsilon^*(L,R,\rho)e^{-L\left[\psi_{\rho,s_\tau}(\tau) - \tau\psi'_{\rho,s_\tau}(\tau)\right]} \leq \overline{\mathsf{A}}_\rho(\tau) \tag{35b}$$

where

$$\begin{aligned} \overline{\mathsf{A}}_\rho(\tau) \triangleq{}& \frac{1}{\sqrt{2\pi L\tau^2\psi''_{\rho,s_\tau}(\tau)}} + \frac{|\hat{K}_{\rho,s_\tau}(\tau)|}{\sqrt{L}} \\ &+ \frac{1}{\sqrt{2\pi L(1-\tau)^2\psi''_{\rho,s_\tau}(\tau)}} + o\bigg(\frac{1}{\sqrt{L}}\bigg) \end{aligned} \tag{36a}$$

$$\underline{\mathsf{A}}_\rho(\tau) \triangleq \frac{s_\tau^{\frac{1}{s_\tau}}}{\tau\left(2\pi L\psi''_{\rho,s_\tau}(\tau)\right)^{\frac{1}{2s_\tau}}} + o\bigg(\frac{1}{L^{\frac{1}{2s_\tau}}}\bigg). \tag{36b}$$

The little-o term in (36a) is uniform in ρ and τ. The little-o term in (36b) is uniform in ρ (for every given τ).

Proof: To prove the right-most inequality in (35b), we apply (19b) with $s = s_\tau$ and τ satisfying (35a). The claim follows then by using that

$$1 - \frac{1}{u^2L\psi''_{\rho,s}(\tau)} \leq f_{\rho,s}(u,\tau)\sqrt{2\pi u^2L\psi''_{\rho,s}(\tau)} \leq 1 \tag{37}$$

and by simple algebraic manipulations. For details see [15].

To prove the left-most inequality in (35b), we apply (25) with τ replaced by

$$\tau_L = \tau + \frac{\log\Big(\frac{1}{s_\tau\tau}\sqrt{2\pi L\tau^2\psi''_{\rho,s_\tau}(\tau)}\Big)}{L\frac{\partial^2}{\partial\tau^2}\Big(\psi_{\rho,\frac{1}{1+\tau}}(\tau) - \tau I_{\frac{1}{1+\tau}}(\rho)\Big)} \tag{38}$$

and $s_L = 1/(1+\tau_L)$. The claim follows from the left-most inequality in (37) upon noting that $K_{\rho,s}(\tau,L)$ in (25) is of order $1/L$ and by simple algebraic manipulations. For details see [15]. ■

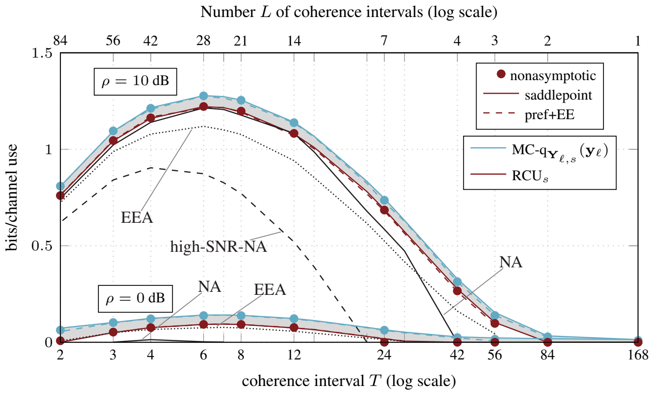

Fig. 1: Bounds on $R^*(L, \epsilon, \rho)$ for $n = 168$, $\epsilon = 10^{-5}$, and $\rho = \{0, 10\}$ dB.

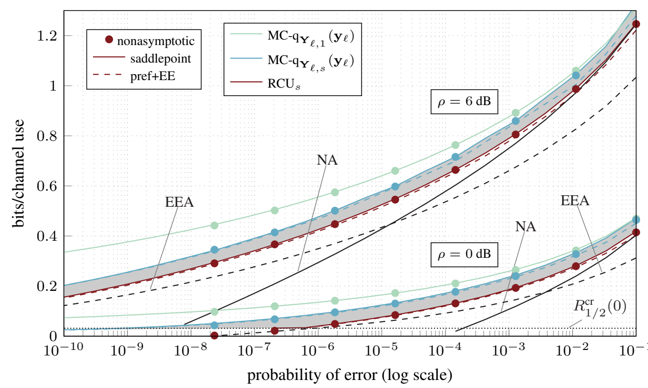

Fig. 2: Bounds on $R^*(L, \epsilon, \rho)$ for $n = 168$, $T = 12$, and $\rho = \{0, 6\}$ dB.

The first three terms of $\overline{\mathsf{A}}_\rho(\tau)$ are positive and dominate the $o(1/\sqrt{L})$ term. Similarly, the first term of $\underline{\mathsf{A}}_\rho(\tau)$ is positive and of order $L^{-\frac{1+\tau}{2}}$. It follows from Theorem 4 that the reliability function $E_r(R, \rho)$ can be parametrized by $\tau \in (0, 1)$ as

$$E_r(R, \rho) = \tau \psi'_{\rho, \frac{1}{1+\tau}}(\tau) - \psi_{\rho, \frac{1}{1+\tau}}(\tau) \tag{39a}$$

$$R = \frac{1}{T}\left(I_{\frac{1}{1+\tau}}(\rho) - \psi'_{\rho, \frac{1}{1+\tau}}(\tau)\right). \tag{39b}$$

VII. Numerical Results and Discussion

In Fig. 1, we study $R^*(L, \epsilon, \rho)$ as a function of L for $n = LT = 168$ (hence T is inversely proportional to L), $\epsilon = 10^{-5}$, and the SNR values of 0 dB and 10 dB. We plot approximations of the $\mathrm{RCU_s}$ bound in red and approximations of the MC bound in blue, which can be obtained by disregarding the $o(1/\sqrt{L})$ terms. Straight lines ("saddlepoint") depict the saddlepoint approximations (19b) and (25), and dashed lines ("pref+EE") depict (35b). We further plot the nonasymptotic bounds (18) and (24) with dots. Finally, we plot the normal approximation (26) ("NA"), the high-SNR normal approximation [7, Th. 1], [8, Th. 1] ("high-SNR-NA"), as well as the error exponent approximation that follows by solving $\epsilon^*(L, R, \rho) \approx \exp\{-LE_r(R, \rho)\}$ for R ("EEA"). Observe that the approximations (19b), (25), and (35b) are almost indistinguishable from the nonasymptotic bounds. Further observe that the normal approximation "NA" is accurate for 10 dB and $L > 10$, but is loose for 0 dB. In contrast, the error exponent approximation "EEA" is loose for 10 dB, but is accurate for 0 dB.

In Fig. 2, we study $R^*(L, \epsilon, \rho)$ as a function of ϵ for $n = 168$, $T = 12$, and the SNR values 6 dB and 0 dB. We plot approximations of the $\mathrm{RCU_s}$ bound in red and approximations of the MC bound in green ($s = 1$) or in blue (s numerically optimized). Straight lines ("saddlepoint") depict the saddlepoint approximations (19b) and (25), and dashed lines ("pref+EE") show the approximations (35b). We further plot the nonasymptotic bounds (18) and (24) with dots. For $\rho = 0$ dB, we also show the critical rate $R^{\mathrm{cr}}_{1/2}(0)$. We finally plot the normal approximation (26) ("NA") and the error exponent approximation that follows by solving $\epsilon^*(L, R, \rho) \approx \exp\{-LE_r(R, \rho)\}$ for R ("EEA"). Observe that the approximations (19b), (25), and (35b) are almost indistinguishable from the nonasymptotic bounds. Further observe how the normal approximation "NA" becomes accurate for large error probabilities, whereas the error exponent approximation "EEA" becomes accurate for small error probabilities. Finally note that the saddlepoint approximations can be evaluated for error probabilities less than 10^{-8}, where the nonasymptotic bounds cannot be evaluated due to their computational complexity.

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