ON NON REPRESENTABLE PREFERENCES

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Abstract
In this note, we prove that for every non-separable metric space there is a continuous preference ordering which is non representable by an utility function.

Key words
Preference Ordering; Utility Function; Non Separable Metric Space.

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1. Introduction

This work is concerned with the numerical representation of all continuous preference orderings on a topological space. As it is well known, if $X$ is a connected and separable topological space, then continuous preference orderings on $X$ always have utility representations (see Eilenberg (1941) and Debreu (1954)). The assumption of connectedness is not necessary in the setting of metric spaces: if $X$ is perfectly separable, every continuous preference ordering is representable by an utility function (Debreu (1954)).

However, we show here that separability is also a necessary condition for the representability of all continuous preference orderings on a metric space. That is, if $X$ is a non separable metric space, there exists a continuous preference ordering which does not admit an utility representation. This is relevant since consumption sets in infinite dimensional commodity spaces are not separable, in general.

2. Definitions

A preference ordering on the set $X$ is, to be precise, a binary relation on $X$, say $\preceq$, which is reflexive, transitive and complete.

An utility representation for the preference ordering $\preceq$ on $X$ is a function $u : X \to \mathbb{R}$ such that $x \preceq y$ if and only if $u(x) \leq u(y)$.

Let $X$ be a topological space. We say that $X$ is separable if it contains a countable subset whose closure is $X$. We say that $X$ is perfectly separable (or that $X$ satisfies the second countability axiom) if there is a countable class of open subsets such that every open subset in $X$ is the union of sets of that class. Every perfectly separable topological space is separable. Every separable metric space is perfectly separable. A topological space $X$ is connected if there is no partition of $X$ into two disjoint, non-empty closed sets. We say that $X$ is path connected if for all $x, y$ in $X$ there is a continuous function $f : [0, 1] \to X$ with $f(0) = x$ and $f(1) = y$. Note that every path connected space is connected and every convex set in a linear topological space is path connected.

A preference ordering $\preceq$ on a topological space $X$ is continuous if the sets $\{x \in X ; x \preceq x'\}, \{x \in X ; x' \preceq x\}$ are closed for all $x' \in X$. A subset $B \subseteq X$ bounds $\preceq$ if for every $x \in X$ there are $a, b$ in $B$ with $a \preceq x \preceq b$. A preference ordering $\preceq$ is countably bounded if there exists a countable set $B \subseteq X$ that bounds $\preceq$. Any preference ordering which has an utility representation is countably bounded.
3. The existence theorem

THEOREM: Let \( X \) be a non separable metric space. Then there is a continuous preference ordering on \( X \) which cannot be represented by an utility function.

To prove the theorem we shall make use of an auxiliar space \( L \) called the long line (see Monteiro (1987), example 5, p. 151). Let \( \Omega_1 \) be the least non-countable ordinal. We denote by \( \Omega \) the set of all ordinals \( \alpha \) such that \( \alpha < \Omega_1 \). That is to say that \( \Omega \) is the set of all countable ordinals. Note that \( \Omega \) is a well ordered set, non-countable and such that for all \( \alpha \in \Omega \), \( \{ \beta \in \Omega ; \beta \leq \alpha \} \) is countable.

Between each \( \alpha \in \Omega \) and its follower \( \alpha + 1 \) put one copy of the real interval \((0,1)\). The space \( L \) that we get, ordered in the obvious way, is called the long line. We consider on \( L \) the order topology. The details on the topological space \( L \) can be seen in Steen and Seebach (1970, pp. 71, 72).

LEMMA: For each \( \alpha \in L \), \( \alpha \neq 0 \) the order interval \([0, \alpha] = \{ x \in L ; 0 \leq x \leq \alpha \} \) is a compact set homeomorphic to the real interval \([0,1] \).

Proof. It is clear that it suffices to prove the result when \( \alpha = \alpha \in \Omega \). As \( \{ \beta \in \Omega ; \beta \leq \alpha \} \) is a well ordered countable set, there is an order preserving \( f : \{0,1,\ldots,\alpha\} \rightarrow [0,1] \) such that \( f(0) = 0 \) and \( f(\alpha) = 1 \). We define \( \tilde{f} : [0, \alpha] \rightarrow [0,1] \) by

\[
\tilde{f}(b) = \begin{cases} 
  f(b) & \text{if } b \in \Omega \\
  f(\beta) + t(f(\beta + 1) - f(\beta)) & \text{if } b = \beta + t, \beta \in \Omega, t \in (0,1).
\end{cases}
\]

It is clear that \( \tilde{f} \) is an isomorphism of the order structures.

PROOF OF THE THEOREM: Let \( X \) be a non separable metric space. Non separable metric spaces are characterized by the following property:

There are \( \varepsilon > 0 \) and an uncountable set \( D \subset X \) such that

\[
\text{for all } x, y \in D, x \neq y \text{ implies } d(x, y) \geq 3\varepsilon. \quad (1)
\]

Otherwise, for each \( \varepsilon = \frac{1}{n}, n \in N \), there exists a countable set \( D_n \) verifying (1) such that \( X = \bigcup_{n \in D_n} B(a, \frac{1}{n}) \), where \( B(a, \frac{1}{n}) = \{ x \in X ; d(x, a) < \frac{1}{n} \} \). Then the set \( D = \bigcup D_n \) will be countable and dense.

As \( D \) is uncountable, for each \( \alpha \in \Omega \) we can choose an \( x_\alpha \in D \) in such a way that \( \alpha \neq \beta \) implies \( x_\alpha \neq x_\beta \). By the lemma, for each \( \alpha \in \Omega \) there exist \( \varphi_\alpha : [0, \varepsilon] \rightarrow L \), which is an isomorphism between the order structures of \([0, \varepsilon] \subset R \) and \([\varphi_\alpha(0), \varphi_\alpha(\varepsilon)] = [0, \alpha] \subset L \).
Let $U : X \to L$ be defined by

$$
U(x) = \begin{cases} 
0 & \text{if } x \notin \bigcup_{\omega \in \Omega} B(x_0, \varepsilon) \\
\varphi_\omega(\varepsilon - d(x_0, x)) & \text{if } x \in B(x_0, \varepsilon)
\end{cases}
$$

It is clear that $U$ is continuous in $B(x_0, \varepsilon)$ because $\varphi_\omega$ and $d$ are continuous. If $x \in X$ is such that $d(x_0, x) = \varepsilon$, we have $U(x) = 0$, and $\varphi_\omega(0) = 0$, then $U$ is continuous in $x$. As the intersection of two different balls is empty and $U$ is constant in the exterior of $\bigcup_{\omega \in \Omega} B(x_0, \varepsilon)$, we have that $U$ is continuous in $X$.

For $x, y \in X$, we define $x \preceq y$ if and only if $U(x) \leq U(y)$. It is clear that $\preceq$ is a continuous preference ordering on $X$, but has no utility representation because is not countably bounded. To see it, note that given a countable set $B \subseteq X$ there exists $\Omega B \subseteq \Omega$ such that $\sup_{b \in B} U(b) < \alpha$ and then there is not a countable set $B \subseteq X$ that bounds $\preceq$.

4. Final remark

We remark that separability is not a necessary condition for the representability of all preference orderings on a general topological space $X$. Monteiro (1987) proves that a continuous preference ordering on a path connected topological space $X$ is representable if and only if it is numerably bounded. A continuous preference ordering on a compact topological space has one best and one worst point. Then any continuous preference ordering on a compact or $\sigma$-compact (an union of a countable family of compact sets) path connected topological space is representable by utility functions. Note that any compact or $\sigma$-compact metric space is separable but compact topological spaces in general need not to be separable.
References


