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Bias Assessment and Reduction for the 2SLS Estimator in General Dynamic Simultaneous Equations Models

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Abstract

We consider the bias of the 2SLS estimator in general dynamic simultaneous equation models with g endogenous regressors. By using asymptotic expansion techniques we approximate 2SLS coefficient estimation bias under innovation errors, p lagged-dependent variables and strongly-exogenous explanatory variables. Large-T approximations bias of the structural form is then used to construct corrected estimators for the parameters of interest in the general DSEM (C2SLS). Simulations show that the C2SLS gives almost unbiased estimators and low mean squared errors. Alternatively, the numerical bootstrap method results suggest that the non-parametric bootstrap could be used in 2SLS for improving estimation in general DSEM.

Keywords:General Dynamic simultaneous equations model;Asymptotic approximations; Bias correction; Bootstrap; Monte Carlo simulations; 2SLS; C2SLS

JEL classification: C13; C32

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1 Introduction

To explore the finite sample properties of estimators in the static simultaneous equations model(SEM), Nagar (1959) found the bias approximation for k-class estimators to the order of T^{-1}, and also derived an approximation for the second moment to order T^{-2}, by using asymptotic expansions essentially based on employing Taylor expansions. Later work in this area included Phillips (2000), Mikhail (1972), Hahn and Hausman (2002), and Bun and Windmeijer (2011) examined bias approximation and reduction in the static simultaneous equation models.

In the dynamic regression models, a number of researchers show that least squares estimators can be seriously biased in small samples. They include Grubb and Symons (1987), Hoque and Peters (1986), and Peters (1989). Kiviet, Phillips, and Schipp (1999), Kiviet and Phillips (1993), Kiviet and Phillips (1995) while Phillips and Liu-Evans (2015) show that the bias in 2SLS in a dynamic simultaneous equation model (DSEM) can be expressed in two parts, a part which derives from simultaneity and a part which is due to the dynamics. However, this latter paper only focuses on the first order DSEM rather than the general DSEM(p lagged dependent variables). In the high order dynamic case, Kiviet and Phillips (1994) present the small sample bias of OLS for the standard ARMAX $(p, 0, k)$ model; however, this is a single equation regression model rather than a DSEM.

In this paper, we are interested in extending the Phillips and Liu-Evans (2015), and Kiviet, Phillips, and Schipp (1999) analysis for the first order DSEM to the general order DSEM assuming that the structural disturbances are normally and independently distributed with mean vector $0'$ and fixed covariance matrix $\Sigma = (\sigma_{ij})$. This general dynamic simultaneous equations model includes the endogenous variables which are lagged p time periods, and strongly exogenous $I(0)$ regressors lagged q time period.

With this model, we analyse the behaviour of 2SLS when sample size is small. Analytically, we derive the bias approximation of 2SLS to order T^{-1}, and confirm the evidence which has been observed in Kiviet, Phillips, and Schipp (1999), Kiviet and Phillips (1993) and Phillips and Liu-Evans (2015), i.e the bias comes from the simultaneity and dynamics respectively. Interestingly, the numerical results show that these two parts actually have opposite signs. In this case, bias correction methods which effectively reduce the bias in the static case (Kiviet and Phillips (1989), Sawa

(1973) and Iglesias and Phillips (2012), etc.) may not be suitable for our dynamic models. However, if we subtract the observed bias approximation in estimation from the corresponding estimator, the bias corrected estimator may be unbiased to order T^{-1} theoretically. Kiviet and Phillips (2005) show that $O(\sigma^2)$ bias approximation can be used for corrected 2SLS (C2SLS) estimation of dynamic models. Kiviet, Phillips, and Schipp (1999) and Liu-Evans and Phillips (2012) use the $O(T^{-1})$ bias approximation in COLS estimation of autoregressive models, and it presents almost unbiased estimators. Phillips and Liu-Evans (2015) show in Monte carlo simulations that by using the C2SLS in the first order DSEM, the new C2SLS method gives almost unbiased estimation. Hence, we develop the bias corrected estimator by employing the estimated bias approximation applied to the traditional 2SLS estimator. Ideally, using the large-T approximation in this paper directly for a reduced-bias estimator may tend to yield more accurate numerical results than any existing approximation. Hence we would expect the $O(T^{-1})$ bias approximation in our paper to yield a substantial improvement over the uncorrected 2SLS estimator.

Our numerical results show that the bias approximation may tend to overstate the magnitude of the "true" bias as given by the Monte Carlo estimates in 2SLS. However, importantly, the bias corrected estimator, based upon $O(T^{-1})$ approximation, very substantially reduces the Monte Carlo 2SLS bias. Moreover, in most cases, it does not inflate the MSE. Hence, the bias corrected estimator, based upon $O(T^{-1})$ bias approximation, can be recommended as a bias reduction technique for practical use. The other alternative bias reduction method is also considered. Freedman (1984) pointed out that the residual bootstrap method could be useful in bias reduction in 2SLS estimation, since it may have some effect in eliminating the bias that comes from the dynamic part. Ip (1991) provides strong support that the bootstrap 2SLS can correct bias for both static and dynamic parts to order T^{-1}. In our experiments, the bootstrap method is not as good as C2SLS, but it may still effectively reduce the bias in the 2SLS. When L , the order of over-identification is large, the estimates of endogenous and exogenous coefficients may have small MSE when using the bootstrap 2SLS.

The next section will introduce the general model. Section 3 evaluates the bias approximation for the first equation in the structural form. Section 4 introduces the new bias correction method C2SLS. The numerical experiments and the associated results are present in section 5, and 6. In these two sections we also employ the non-parametric residual bootstrap 2SLS estimator. The last section is our conclusion part.

2 The Model

The complete system we are interested in:

$$YB + \sum_{i=1}^{p} Y_{-i}A^{(i)} + \sum_{j=0}^{q} X_{-j}C^{(j)} = \tilde{U}, \tag{1}$$

where Y is a $T \times G$ matrix of T observations on G endogenous variables, X is a $T \times K$ matrix of observations on K stationary (we will relax this assumption in our further work) and strongly exogenous variables, Y_{-i} is a $T \times G$ matrix of observations on the endogenous variables lagged i time periods (G lagged endogenous explanatory variables) and we assume that the initial values $(Y_{1-p,}, ..., Y_0)$ are non-stochastic. The model also involves K current exogenous variables in the matrix X which is assumed to be of full rank K, and has q lags X_{-j}, while $\tilde{U}$ is a $T \times G$ matrix of structural disturbances. The matrices $B, A^{(i)}$ and $C^{(j)}$ are of dimension $G \times G, G \times G$ and $K \times G$, respectively, and B is assumed to be non-singular. The rows of $\tilde{U}$ are assumed to be normally and independently distributed with zero mean and fixed covariance matrix $\tilde{\Sigma} = (\tilde{\sigma}_{mn})$.

Furthermore, we assume that the eigenvalues (real or complex values) of the system of difference equations are inside the unit circle which ensures the stability for our system. Thus the roots (real or complex values) of the determinantal equation
$det|B\varpi^p + A^{(1)}\varpi^{p-1} + A^{(2)}\varpi^{p-2} + ... + A^{(p)}| = 0$
are smaller than unity in absolute value: $|\varpi|^h < 1, h = 1, 2, ..., p$. This statement of the system essentially follows that of Dhrymes (1970), Chapter 12; Davidson (2000), Section 4.3.2.

The reduced form of the model is:

$$\begin{aligned} Y &= -\sum_{i=1}^{p} Y_{-i}A^{(i)}B^{-1} - \sum_{j=0}^{q} X_{-j}C^{(j)}B^{-1} + \tilde{U}B^{-1} \\ &= \sum_{i=1}^{p} L^i Y\Gamma^{(i)} + \sum_{j=0}^{q} L^j X\Pi^{(j)} + \tilde{V} \end{aligned} \tag{2}$$

$$= ZA^* + \tilde{V},$$

where $\Gamma^{(i)} = -A^{(i)}B^{-1}$, $\Pi^{(j)} = -C^{(j)}B^{-1}$ and $\tilde{V} = \tilde{U}B^{-1}$. The rows of $\tilde{V}$ are normally and independently distributed with zero mean and covariance matrix $\tilde{\Omega} = (\tilde{\omega}_{mn}) = E(\tilde{V}'\tilde{V})/T$. Also $Z = [R : S]$ is a $T \times (P+Q)$ matrix where $P = \sum_{m=1}^{G} p(m)$ and $Q = \sum_{n=1}^{K} q(n)$. Here the $T \times P$ matrix R includes all the observations for the (stochastic) lagged endogenous variables, and the $T \times Q$ matrix S includes the observations for all the other regressors. A^* is the $(P+Q) \times G$ coefficients matrix.

The stochastic part $\tilde{W}$ of $Y = \bar{Y} + \tilde{W}$ from equation (2) has rows $\tilde{w}_t'$, $t = 1, 2...T$, which can be written as follows.

$$\begin{aligned}
\tilde{w_1}' &= \tilde{v_1}', \\
\tilde{w_2}' &= \tilde{v_2}' + \tilde{w_1}'\Gamma^{(1)}, \\
\tilde{w_3}' &= \tilde{v_3}' + \tilde{w_1}'\Gamma^{(2)} + \tilde{w_2}'\Gamma^{(1)}, \\
\tilde{w_4}' &= \tilde{v_4}' + \tilde{w_1}'\Gamma^{(3)} + \tilde{w_2}'\Gamma^{(2)} + \tilde{w_3}'\Gamma^{(1)}, \\
&\vdots \\
\tilde{w_p}' &= \tilde{v_p}' + \tilde{w_1}'\Gamma^{(p-1)} + \tilde{w_2}'\Gamma^{(p-2)} + \tilde{w_3}'\Gamma^{(p-3)} + ... + \tilde{w}_{p-1}'\Gamma^{(1)}, \\
\tilde{w_{p+1}}' &= \tilde{v_{p+1}}' + \tilde{w_1}'\Gamma^{(p)} + \tilde{w_2}'\Gamma^{(p-1)} + \tilde{w_3}'\Gamma^{(p-2)} + ... + \tilde{w}_p'\Gamma^{(1)}, \\
&\vdots \\
\tilde{w_T}' &= \tilde{v_T}' + \tilde{w}_{T-p}'\Gamma^{(p)} + ... + \tilde{w}_{T-2}'\Gamma^{(2)} + \tilde{w}_{T-1}'\Gamma^{(1)}.
\end{aligned}$$

Let the $T \times T$ matrix D be such that,

$$D = \begin{bmatrix} 0 & 0 & . & . & . & . & 0 \\ 1 & 0 & & & & & 0 \\ 0 & 1 & & & & & 0 \\ . & . & . & . & & & . \\ . & . & . & . & & & . \\ 0 & 0 & . & . & 0 & 1 & 0 \end{bmatrix}, \quad D^2 = \begin{bmatrix} 0 & 0 & . & . & . & . & 0 \\ 0 & 0 & & & & & 0 \\ 1 & 0 & & & & & 0 \\ . & . & . & . & & & . \\ . & . & . & . & & & . \\ 0 & 0 & . & . & 1 & 0 & 0 \end{bmatrix},$$

where $D^{T-1}.D = D^T = 0$ and D^0 is I_T. Also we define a $TG \times G$ matrix J formed by stacking the matrices J_t, $t = 0, 1, ..., T-1$, as follows

$$
\begin{aligned}
&J_0 = I_G, \\
&J_1 = \Gamma^{(1)}, \\
&J_2 = \Gamma^{(2)} + \Gamma^{(1)} J_1, \\
&J_3 = \Gamma^{(3)} + \Gamma^{(2)} J_1 + \Gamma^{(1)} J_2, \\
&J_4 = \Gamma^{(4)} + \Gamma^{(3)} J_1 + \Gamma^{(2)} J_2 + \Gamma^{(1)} J_3, \\
&\vdots \\
&J_p = \Gamma^{(p)} + \Gamma^{(p-1)} J^{(1)} + ... + \Gamma^{(1)} J_{p-1}, \\
&J_{p+1} = \Gamma^{(p)} J_1 + ... + \Gamma^{(1)} J_p, \\
&J_{p+2} = \Gamma^{(p)} J_2 + ... + \Gamma^{(1)} J_{p+1}, \\
&\vdots \\
&J_{T-1} = \Gamma^{(p)} J_{T-p} + ... + \Gamma^{(1)} J_{T-2}.
\end{aligned}
$$

The matrix J can be written as:

$$
J = \begin{bmatrix} J_0 \\ J_1 \\ J_2 \\ \vdots \\ J_p \\ J_{p+1} \\ J_{p+2} \\ \vdots \\ \vdots \\ J_{T-1} \end{bmatrix}
= \begin{bmatrix}
I & 0 & 0 & 0 & 0 & ... & ... & 0 & 0 & 0 \\
\Gamma^{(1)} & 0 & 0 & 0 & 0 & ... & ... & 0 & 0 & 0 \\
\Gamma^{(2)} & \Gamma^{(1)} & 0 & 0 & 0 & ... & ... & 0 & 0 & 0 \\
\vdots & \vdots & . & . & & & & & \cdots & \vdots \\
\Gamma^{(p)} & \Gamma^{(p-1)} & ... & \Gamma^{(1)} & 0 & ... & ... & & & . \\
0 & \Gamma^{(p)} & \Gamma^{(p-1)} & ... & \Gamma^{(1)} & 0 & ... & ... & & . \\
0 & 0 & \Gamma^{(p)} & \Gamma^{(p-1)} & ... & \Gamma^{(1)} & 0 & ... & ... & . \\
\vdots & \vdots & . & . & & & & & \cdots & \vdots \\
\vdots & \vdots & . & . & & & & & \cdots & \vdots \\
0 & .... & . & \cdots & \Gamma^{(p)} & \Gamma^{(p-1)} & ... & & \cdots & \Gamma^{(1)}
\end{bmatrix}
\begin{bmatrix} I \\ J_1 \\ J_2 \\ \vdots \\ J_p \\ J_{p+1} \\ J_{p+2} \\ \vdots \\ J_{T-2} \end{bmatrix}.
$$

With these definitions $\tilde{W}$ can be represented in terms of $\tilde{V}$ as follows:

$$
\tilde{W} = \sum_{t=1}^{T-1} D^t \tilde{V} J_t + \tilde{V} = \sum_{t=0}^{T-1} D^t \tilde{V} J_t. \qquad (3)
$$

In equation (3), $\sum_{i=0}^{T-1} D^t \tilde{V} J_t$ is the stochastic part of Y. In equation (2) $Z = [R : S]$, and in accordance with our notation, Z may be decomposed as:

$$Z = \bar{Z} + \tilde{W}^*. \tag{4}$$

Here $\bar{Z} = [\bar{R} : X]$ is taken to be the non-stochastic part of Z, whose component matrix $\bar{R}$ is the non-stochastic part of R. The stochastic part of Z is $\tilde{W}^* = \omega W^*$, and $\tilde{W}^*$ can be expressed as:

$$\begin{aligned} \tilde{W}^* &= [\tilde{R} : 0] = [L\tilde{W} : L^2\tilde{W} : .... : L^p\tilde{W} : 0] \\ &= [\sum_{t=1}^{T-1} D^t\tilde{V}J_{t-1} : \sum_{t=2}^{T-1} D^t\tilde{V}J_{t-2} : ... : \sum_{t=p}^{T-1} D^t\tilde{V}J_{t-p} : 0]. \end{aligned}$$

The standardized form of $\tilde{W}^*$ can be presented as:

$$\tilde{W^*} = \left[\sum_{i=1}^{p} \sum_{t=i}^{T-1} D^t\tilde{V}J_{t-i}\Psi_i^{'} : 0 \right], \tag{5}$$

where $\Psi_i^{'} = e_i^{'} \otimes I_G$ is $G \times P$ matrix with $P = \sum_{m=1}^{G} p(m)$ and where all component $G \times G$ matrices are zero except the i^{th} which is an identity matrix. e_i is the $p \times 1$ unit vector with all elements equal to zero except the i^{th} which is unity.

3 Structural Form Estimation–Two Stage Least Square Estimation

In this section we derive the large T approximations to the bias of 2SLS estimators when estimating the structural coefficients of the first equation which forms part of the complete system equation (1), and we shall write the equation as:

$$y_1 = Y_2\beta_1 + \sum_{i=1}^{p} L^iY_1a_1^{(i)} + \sum_{j=0}^{q} L^jX_1c_1^{(j)} + \tilde{u}_1 = \Upsilon\delta_1 + \tilde{u}_1, \tag{6}$$

where

$$\Upsilon = [Y_2 : R_1 : S_1] \quad and \quad \delta_1^{'} = (\beta_1, a_1^{(1)}, ..., a_1^{(p)}, c_1^{(1)}, ..., c_1^{(q)}).$$

Here $Y_1 = [y_1 : Y_2]$ is a $T \times (g+1)$ matrix of observations on $g+1$ included endogenous variables. L^iY_1 is the i period lagged version of Y_1, X_1 is a $T \times k$ matrix of observations on k stationary exogenous variables. Υ is a $T \times (g + P^* + Q^*)$ matrix which includes the $T \times g$ matrix Y_2, the $T \times P^*$ matrix R_1 contains the lagged endogenous regressor values and the $T \times Q^*$ matrix S_1 contains the exogenous regressor values which are taken as fixed. $P^* = \sum_{m=1}^{g+1} p(m)$ and $Q^* = \sum_{n=1}^{k} q(n)$ which allows for the equations to contain different numbers of lagged endogenous and exogenous regressors respectively. δ_1 is a $(g + P^* + Q^*) \times 1$ vector which contains all the structural form parameters. We shall denote:

$$\bar{\Upsilon} = [\bar{Y}_2 : \bar{R}_1 : \bar{S}_1] \quad and \quad \tilde{F} = [\tilde{W}_2 : \tilde{R}_1 : 0] \tag{7}$$

as, respectively, the non-stochastic and stochastic parts of Υ which will be used in later analysis. Notice that the non-stochastic part of Y contains $\bar{Y}_2$ and $\bar{R}_1$ which are the unconditional expectations of Y_2 and R_1 respectively. Note also that in $\tilde{F}$, $\tilde{W}_2$ is the relevant stochastic part of $\tilde{W}$ for Y_2 as given in equation (3).

The standard 2SLS estimator of δ_1 can be written as:

$$\begin{aligned}\hat{\delta_1} &= (\hat{\Upsilon}'\hat{\Upsilon})^{-1}\hat{\Upsilon}' y_1 \\ &= \delta_1 + (\hat{\Upsilon}'\hat{\Upsilon})^{-1}\hat{\Upsilon}'\tilde{u}_1\end{aligned} \tag{8}$$

where

$$\hat{\Upsilon} = [\hat{Y}_2 : R_1 : S_1] \quad and \quad \hat{Y}_2 = \sum_{i=1}^{p} L^i Y \hat{\Gamma}_2^{(i)} + \sum_{j=0}^{q} L^j X \hat{\Pi}_2^{(j)}$$

and $\hat{Y}_2$ is obtained when the reduced form equation (2) is estimated by OLS. The matrix R_1 which refers to $L^i Y_1$ is $T \times P^*$, where $P^* = \sum_{m=1}^{g+1} p(m)$. $\hat{\Gamma}_2^{(i)}$, $i = 1, 2, ..., p$ and $\hat{\Pi}_2^{(j)}$, $j = 1, 2, ..., q$ are respectively, $G \times g$ and $K \times g$ matrices of estimated reduced form coefficients in equation (2). $\hat{\Upsilon}$ can be also decomposed into non-stochastic part $\bar{\Upsilon}$ and stochastic part $(\hat{\Upsilon} - \bar{\Upsilon})$, hence:

$$\hat{\Upsilon} = \bar{\Upsilon} + (\hat{\Upsilon} - \bar{\Upsilon}).$$

Using equation (7) and (8), and $\bar{Y_2} = \sum_{i=1}^{p} L^i \bar{Y} \Gamma_2^{(i)} + \sum_{j=0}^{q} L^j X \Pi_2^{(j)}$, the stochastic part of $\hat{\Upsilon}$ can be written as:

$$\begin{aligned}\hat{\Upsilon}-\bar{\Upsilon} &= \left[\sum_{i=1}^{p} L^i\bar{Y}(\hat{\Gamma}_2^i-\Gamma_2^i)+\sum_{j=1}^{q} L^jX(\hat{\Pi}_2^j-\Pi_2^j)+\sum_{i=1}^{p} L^i\tilde{W}\hat{\Gamma}_2^i : \sum_{i=1}^{p} L^i\tilde{W}_1 : 0\right] \quad (9)\\ &= \left[\sum_{i=1}^{p} L^i\bar{Y}(\hat{\Gamma}_2^{(i)}-\Gamma_2^{(i)})+\sum_{j=1}^{q} L^jX(\hat{\Pi}_2^j-\Pi_2^j)+\sum_{i=1}^{p} L^i\tilde{W}(\hat{\Gamma}_2^{(i)}-\Gamma_2^{(i)}) : 0 : 0\right]\\ &\quad +[\sum_{i=1}^{p} L^i\tilde{W}\Gamma_2^{(i)} : \tilde{R}_1 : 0].\end{aligned}$$

We define:

$$\Delta_1 = \left[\sum_{i=1}^{p} L^i\bar{Y}(\hat{\Gamma}_2^{(i)}-\Gamma_2^{(i)})+\sum_{j=1}^{q} L^jX(\hat{\Pi}_2^j-\Pi_2^j)+\sum_{i=1}^{p} L^i\tilde{W}(\hat{\Gamma}_2^{(i)}-\Gamma_2^{(i)}) : 0 : 0\right], \quad (10)$$
$$\Delta_2 = [\sum_{i=1}^{p} L^i\tilde{W}\Gamma_2^{(i)} : \tilde{R}_1 : 0].$$

Then $\hat{\Upsilon} = \bar{\Upsilon}+\Delta_1+\Delta_2$ and it is possible to write

$$\begin{aligned}\hat{\Upsilon}'\hat{\Upsilon} &= \bar{\Upsilon}'\bar{\Upsilon}+\Delta_1'\Delta_1+\Delta_2'\Delta_2+\bar{\Upsilon}'\Delta_1+\bar{\Upsilon}'\Delta_2+\Delta_1'\bar{\Upsilon}+\Delta_2'\bar{\Upsilon}+\Delta_1'\Delta_2+\Delta_2'\Delta_1 \quad (11)\\ &= \bar{\Upsilon}'\bar{\Upsilon}+E(\Delta_2'\Delta_2)+\Delta_1'\Delta_1+(\Delta_2'\Delta_2-E(\Delta_2'\Delta_2))+(\bar{\Upsilon}'\Delta_1+\Delta_1'\bar{\Upsilon})\\ &\quad +(\bar{\Upsilon}'\Delta_2+\Delta_2'\bar{\Upsilon})+(\Delta_1'\Delta_2+\Delta_2'\Delta_1).\end{aligned}$$

Let $H^{-1} = \bar{\Upsilon}'\bar{\Upsilon}+\mathbb{E}(\Delta_2'\Delta_2)$ which is $O(T)$, then put the $O_p(T^{1/2})$ component of $\hat{\Upsilon}'\hat{\Upsilon}$ as J_1^* [1] and the $O_p(1)$ component as J_2^* [2]. We can then express $(\hat{\Upsilon}'\hat{\Upsilon})^{-1}$ from equation (11)as follows:

$$\begin{aligned}(\hat{\Upsilon}'\hat{\Upsilon})^{-1} &= (H^{-1}+J_1^*+J_2^*)^{-1} = H(I+J_1^*H+J_2^*H)^{-1} \quad (12)\\ &= H-HJ_1^*H+o_p(T^{-3/2})\end{aligned}$$

and noting that $\hat{\Upsilon} = \bar{\Upsilon}+\Delta_1+\Delta_2$, we have

$$\hat{\Upsilon}'\tilde{u}_1 = \bar{\Upsilon}'\tilde{u}_1+\Delta_1'\tilde{u}_1+\Delta_2'\tilde{u}_1. \quad (13)$$

Here $\Delta_1'\tilde{u}_1$ is $O_p(1)$, $\hat{\Upsilon}'\tilde{u}_1$ and $\Delta_2'\tilde{u}_1$ are $O_p(T^{1/2})$. Combining equation (12) with

[1] J_1^* includes $(\Delta_2'\Delta_2-\mathbb{E}(\Delta_2'\Delta_2))$, $((\bar{\Upsilon}'\Delta_1+\Delta_1'\bar{\Upsilon})$, and $(\bar{\Upsilon}'\Delta_2+\Delta_2'\bar{\Upsilon})$ $(\Delta_1'\Delta_2+\Delta_2'\Delta_1)$

[2] The component of J_2^* is $\Delta_1'\Delta_1$.

(13) gives:

$$\hat{\delta}_1 - \delta_1 = (\hat{\Upsilon}'\hat{\Upsilon})^{-1}\hat{\Upsilon}'\tilde{u}_1 = H\bar{\Upsilon}\tilde{u}_1 + H\Delta_1'\tilde{u}_1 + H\Delta_2'\tilde{u}_1 - HJ_1^*H\bar{\Upsilon}'\tilde{u}_1 - HJ_1^*H\Delta_2\tilde{u}_1 + o_p(T^{-1}). \tag{14}$$

Taking expectations term by term yields the 2SLS bias, and this is given in *Theorem 1* below. Defining $H^{*-1} = \mathbb{E}(Z'Z)$, where recall that $Z = [\tilde{R} : S]$ which includes all the lagged endogenous variables and all the exogenous variables and let $I_2 = \begin{bmatrix} I_P \\ 0 \end{bmatrix}$ which is $(P+Q) \times P$ selection matrix, then $I_2' H^* I_2 = H^{**}$, a sub-matrix of H^*. We also define the matrix $C^* = \begin{bmatrix} \Gamma_2^{(*)} & : & I_1 & : & 0 \end{bmatrix}$ which is $P \times (g + P^* + Q^*)$ matrices which contains the $P \times g$ matrix $\Gamma_2^{(*)} = (\Gamma_2^{(1)}, \Gamma_2^{(2)}, ..., \Gamma_2^{(p)})'$, the $P \times P^*$ selection matrix I_1, and the $P \times Q^*$ matrix $(0_{P\times Q^*})$. It then follows that we may write $\Delta_2 = [\sum_{i=1}^{p} L^i \tilde{W}\Gamma_2^{(i)} : \tilde{R}_1 : 0] = \tilde{R}C^*$, where $\tilde{R} = [LW : L^2W : ... : L^pW]$ includes all the stochastic part of lagged dependent variables. We will use this expression for further calculations in the appendix. Assume $\tau = \sigma_1^2\phi$ and $\vartheta = \Lambda^{**'}\tau$, ϕ is defined by using, Nagar (1959), the decomposition for $\tilde{V}$, $\tilde{V} = S^* + \tilde{u}_1\phi'$, where S^* and $\tilde{u}_1$ are normally and independently distributed. Then $\phi\sigma_1^2 = \mathbb{E}\left(\frac{1}{T}\tilde{V}'\tilde{u}_1\right)$. We define $\Lambda^{**} = \begin{bmatrix} I_g : 0 \\ 0 \end{bmatrix}$ which is a $G \times (g + P^* + Q^*)$ dimension selection matrix, then $\tilde{V}_2$ is the $T \times g$ submatrix of matrix $\tilde{V}$, which can be expressed as $[\tilde{V}_2 : 0 : 0] = \tilde{V}\begin{bmatrix} I_g : 0 \\ 0 \end{bmatrix}$.

With these and earlier definitions of terms we may state the following:

Theorem 1 . The bias of the 2SLS estimator of the first structural equation parameters to order T^{-1} is given by:

$$\begin{aligned}\mathbb{E}(\hat{\delta}_1 - \delta_1) =& H(tr\{\bar{Z}H^*\bar{Z}'\}.I)\vartheta - H\bar{\Upsilon}'\bar{Z}H^*\bar{Z}'\bar{\Upsilon}H\vartheta \\ & - H(tr\{\bar{Z}H^*\bar{Z}'\bar{\Upsilon}H\bar{\Upsilon}'\}.I)\vartheta \\ & + H\sum_{i=1}^{p}\sum_{j=1}^{p}\sum_{t=i,j}^{T-1}(T-t)tr\{\Omega J_{t-i}\Psi_i' H^{**}\Psi_j J_{t-j}'\}\vartheta \\ & - H\sum_{l=1}^{p}\sum_{j=1}^{p}\sum_{r=l,j}^{T-1}(T-r)\left(tr\left\{\bar{Z}H^*I_2\Psi_l J_{r-l}'\Omega J_{s-j}\Psi_j' C^* H\bar{\Upsilon}'\right\}.I\right)\vartheta\end{aligned} \tag{15}$$

$$- H \sum_{i=1}^{p} \sum_{l=1}^{p} \sum_{t=i,j}^{T-1} (T-t) C^{*'} \Psi_i J_{t-i}^{'} \Omega J_{t-j} \Psi_j^{'} I_2^{'} H^* \bar{Z}^{'} \bar{\Upsilon} H \vartheta$$

$$- H \sum_{i=1}^{p} \sum_{j=1}^{p} \sum_{t=i,j}^{T-1} (T-t) (tr\{\Omega J_{t-i} \Psi_i^{'} I_2^{'} H^* \bar{Z}^{'} \bar{\Upsilon} H C^{*'} \Psi_l J_{t-l}^{'}\}.I) \vartheta$$

$$- H \bar{\Upsilon}^{'} \bar{Z} H^* \sum_{i=1}^{p} \sum_{j=1}^{p} \sum_{t=i,j}^{T-1} I_2 (T-t) \Psi_i J_{t-i}^{'} \Omega J_{t-j} \Psi_j^{'} C^{*'} H \vartheta$$

$$- H \sum_{i=1}^{p} \sum_{l=1}^{p} \sum_{t=i,l}^{T-1} (T-t) \left(tr \left\{ \Omega J_{t-i} \Psi_i^{'} H^{**} \left[\sum_{j=1}^{p} \sum_{m=1}^{p} \sum_{s=j,m}^{T-1} (T-s) \Psi_j \right. \right. \right.$$

$$\left. \left. \left. \times J_{s-j} \Omega J_{s-m} \Psi_m^{'} C^* \right] H C^{*'} \Psi_l J_{s-l}^{'} \right\} .I \right) \vartheta$$

$$- H \sum_{i=1}^{p} \sum_{j=1}^{p} \sum_{t=i,j}^{T-1} (T-t) C^{*'} \Psi_i J_{t-i}^{'} \Omega J_{t-j} \Psi_j^{\prime} H^{**}$$

$$\times \sum_{l=1}^{p} \sum_{m=1}^{p} \sum_{r=l,m}^{T-1} (T-l) \Psi_l J_{r-l}^{'} \Omega J_{r-m} \Psi_m^{'} C^* H \vartheta$$

$$- H \bar{\Upsilon}^{'} \sum_{i=1}^{p} \sum_{t=i}^{T-1} \sum_{j=1}^{p} \sum_{s=j}^{T-1} D^t D^{s'} \bar{\Upsilon} H (tr\{\Omega J_{s-j} \Psi_j^{'} H^{**} \Psi_i J_{t-i}^{'}\}.I) \vartheta$$

$$- H \bar{\Upsilon}^{'} \sum_{i=1}^{p} \sum_{t=i}^{T-1} \sum_{j=1}^{p} \sum_{s=j}^{T-1} D^t D^s \bar{\Upsilon} H \Lambda^{**'} \Omega J_{t-i} \Psi_i^{\prime} H^{**} \Psi_j J_{s-j} \tau$$

$$- H \sum_{i=1}^{p} \sum_{t=i}^{T-1} \sum_{j=1}^{p} \sum_{s=j}^{T-1} (tr\{\Psi_j J_{s-j}^{'} \Omega J_{t-i} \Psi_i^{'} H^{**}\}.I)(tr\{D^t D^{s'} \bar{\Upsilon} H \bar{\Upsilon}^{'}\}.I) \vartheta$$

$$- H \Lambda^{**'} \sum_{i=1}^{p} \sum_{t=i}^{T-1} \sum_{j=1}^{p} \sum_{s=j}^{T-1} \Omega J_{s-j} \Psi_j^{'} H^{**} \Psi_i J_{t-i}^{'} (tr\{D^t \bar{\Upsilon} H \bar{\Upsilon}^{'} D^s\}.I) \tau$$

$$- H \bar{\Upsilon}^{'} \sum_{i=1}^{p} \sum_{t=i}^{T-1} D^t \bar{\Upsilon} H^{'} C^{*'} \Psi_i J_{t-i}^{'} \tau$$

$$- H \sum_{i=1}^{p} \sum_{t=i}^{T-1} C^{*'} \Psi_i J_{t-i}^{'} (tr\{\bar{\Upsilon}^{'} D^{t'} \bar{\Upsilon} H\}.I) \tau$$

$$- H \sum_{i=1}^{p} \sum_{t=i}^{T-1} \sum_{j=1}^{p} \sum_{s=j}^{T-1} (tr\{D^t D^{s'} \bar{Z} H^* I_2 \Psi_i J_{t-i}^{'} \Omega J_{s-j} \Psi_j^{'} C^* H \bar{\Upsilon}^{'}\}.I) \vartheta$$

$$- H \sum_{i=1}^{p} \sum_{t=i}^{T-1} \sum_{j=1}^{p} \sum_{s=j}^{T-1} \Lambda^{**'} \Omega J_{s-j} \Psi_j^{'} C^* H \bar{\Upsilon}^{'} D^{t'} D^{s'} \bar{Z} H^* I_2 \Psi_i J_{t-i}^{'} \tau$$

$$- H \Lambda^{**} \sum_{l=1}^{p} \sum_{r=l}^{T-1} \sum_{j=1}^{p} \sum_{s=j}^{T-1} \Omega J_{s-j} \Psi_j^{'} C^* H \bar{\Upsilon}^{'} D^{r'} D^s \bar{Z} H^* I_2 \Psi_l J_{r-l}^{'} \tau$$

$$- H \Lambda^{**} \sum_{l=1}^{p} \sum_{r=l}^{T-1} \sum_{j=1}^{p} \sum_{s=j}^{T-1} \Omega J_{r-l} \Psi_l^{\prime} I_2^{\prime} H^* \bar{Z}^{\prime} D^{r'} D^s \bar{\Upsilon} H C^{*'} \Psi_j J_{s-j}^{\prime} \tau$$

$$
\begin{aligned}
&- H\sum_{i=1}^{p}\sum_{t=i}^{T-1}\sum_{j=1}^{p}\sum_{s=j}^{T-1} C^{*'}\Psi_i J^{'}_{t-i}\Omega J_{s-j}\Psi^{'}_j I^{'}_2 H^{*}\bar{Z}^{'} D^{t} D^{s'}\bar{\Upsilon} H\vartheta \\
&- H\sum_{i=1}^{p}\sum_{t=i}^{T-1}\sum_{j=1}^{p}\sum_{s=j}^{T-1} C^{*'}\Psi_i J^{'}_{t-i}\Omega\Lambda^{**} H\bar{\Upsilon}^{'} D^{r'} D^{t'}\bar{Z} H^{*} I_2\Psi_j J^{'}_{s-j}\tau \\
&- H\sum_{i=1}^{p}\sum_{t=i}^{T-1}\sum_{j=1}^{p}\sum_{s=j}^{T-1} C^{*'}\Psi_i J^{'}_{t-i}(tr\{D^{t'} D^{s}\bar{Z} H^{*} I_2\Psi_j J^{'}_{s-j}\Omega\Lambda^{**} H\bar{\Upsilon}^{'}\}.I)\tau \\
&- H\sum_{i=1}^{p}\sum_{t=i}^{T-1}\sum_{j=1}^{p}\sum_{s=j}^{T-1} C^{*'}\Psi_i J^{'}_{t-i}\Omega\Lambda^{**} H\bar{\Upsilon}^{'} D^{s'} D^{t}\bar{Z} H^{*} I_2\Psi_j J^{'}_{s-j}\tau \\
&- H\bar{\Upsilon}^{'}\sum_{i=1}^{p}\sum_{t=i}^{T-1}\sum_{j=1}^{p}\sum_{s=j}^{T-1} D^{t} D^{s}\left(tr\left\{\Omega J_{t-i}\Psi^{'}_i C^{*} H\Lambda^{**'}\right\}.I\right)\bar{Z} H^{*} I_2\Psi_j J^{'}_{s-j}\tau \\
&- H\bar{\Upsilon}^{'}\sum_{i=1}^{p}\sum_{t=i}^{T-1}\sum_{j=1}^{p}\sum_{s=j}^{T-1} D^{t} D^{s'}\bar{Z} H^{*} I_2\Psi_j J^{'}_{s-j}\Omega J_{t-i}\Psi^{'}_i C^{*} H\vartheta \\
&- H\sum_{i=1}^{p}\sum_{t=i}^{T-1}\sum_{j=1}^{p}\sum_{s=j}^{T-1}\Lambda^{**'}\left(tr\left\{\bar{Z} H^{*} I_2\Psi_i J^{'}_{t-i}\Omega J_{s-j}\Psi^{'}_j C^{*} H\bar{\Upsilon}^{'} D^{t} D^{s'}\right\}.I\right)\tau \\
&- H\sum_{i=1}^{p}\sum_{t=i}^{T-1}\sum_{j=1}^{p}\sum_{s=j}^{T-1}\Lambda^{**'}\Omega J_{s-j}\Psi^{'}_j C^{*} H\bar{\Upsilon}^{'} D^{t} D^{s}\bar{Z} H^{*} I_2\Psi_i J^{'}_{t-i}\tau \\
&- H\sum_{i=1}^{p}\sum_{t=i}^{T-1}\sum_{j=1}^{p}\sum_{s=j}^{T-1} C^{*'}\Psi_i J^{'}_{t-i}(tr\{\Omega\Lambda^{**} H C^{*'}\Psi_j J^{'}_{s-j}\}.I) \\
&\qquad\qquad\qquad\times (tr\{D^{t'}\bar{Z} H^{*}\bar{Z}^{'} D^{s'}\}.I)\tau \\
&- H\sum_{i=1}^{p}\sum_{t=i}^{T-1}\sum_{j=1}^{p}\sum_{s=j}^{T-1}(tr\{D^{t'}\bar{Z} H^{*}\bar{Z}^{'} D^{s'}\}.I) C^{*'}\Psi_i J^{'}_{t-i}\Omega H C^{*'}\Psi_j J^{'}_{s-j}\vartheta \\
&- H\sum_{i=1}^{p}\sum_{t=i}^{T-1}\sum_{j=1}^{p}\sum_{s=j}^{T-1}(tr\{\Omega J_{t-i}\Psi^{'}_i C^{*} H C^{*'}\Psi_j J^{'}_{s-j}\}.I)(tr\{\bar{Z} H^{*}\bar{Z}^{'} D^{t} D^{s'}\}.I)\vartheta \\
&- H\Lambda^{**'}\sum_{i=1}^{p}\sum_{t=i}^{T-1}\sum_{j=1}^{p}\sum_{s=j}^{T-1}\Omega J_{s-j}\Psi^{'}_j C^{*} H C^{*'}\Psi_i J^{'}_{t-i}(tr\{\bar{Z} H^{*}\bar{Z}^{'} D^{t} D^{s}\}.I)\tau \\
&- H C^{*'}\sum_{i=1}^{p}\sum_{t=i}^{T-1}\sum_{j=1}^{p}\sum_{s=j}^{T-1}\sum_{l=1}^{p}\sum_{r=l}^{T-1}\Psi_i J^{'}_{t-i}\Omega J_{r-l}\Psi^{'}_l C^{*'} H^{'} C^{*'}\Psi_j J^{'}_{s-j}(tr(D^{t'} D^{s} D^{r'}).I)\tau \\
&- H C^{*'}\sum_{i=1}^{p}\sum_{t=i}^{T-1}\sum_{j=1}^{p}\sum_{s=j}^{T-1}\sum_{l=1}^{p}\sum_{r=l}^{T-1}\Psi_i J^{'}_{t-i}(tr(D^{t'} D^{s} D^{r'}).I)(tr(\phi^{'} J_{s-j}\Psi^{'}_j C^{*} H C^{*'}\Psi_l J^{'}_{r-l}\Omega).I)\tau \\
&+ o(T^{-1}).
\end{aligned}
$$

A proof of this result is given by Appendix A and it is obtained by evaluating the expectations of each term.

From the result, we note that the bias of 2SLS to order T^{-1} of the first structural form equation has two distinct parts: a part is due to the simultaneity of the system which is represented by the first three terms in the above, and a part which is due to the dynamic nature of the structural equation which is represented by all the remaining terms. Here, $H = (\bar{\Upsilon}'\bar{\Upsilon} + \mathbb{E}(\Delta_2'\Delta_2))^{-1}$, $H^* = (\bar{Z}'\bar{Z} + \mathbb{E}(\tilde{W}^{*'}\tilde{W}^*)^{-1})$ and $H^{**} = I_2'H^*I_2 = I_2'(\bar{Z}'\bar{Z} + \mathbb{E}(\tilde{W}^{*'}\tilde{W}^*)^{-1})I_2$, from which, we observe that the bias that comes solely from the simultaneity terms should not include the expected stochastic parts in the first three terms. In fact, the expression in Theorem 1 should reduce to the Nagar (1959) bias approximation in static models when any terms that result from the inclusion of lagged endogenous regressors are removed. This means that a reduction of the above result to that for the static case will obtain with the removal of any terms involving the "D" matrix and the expected stochastic parts in the first three terms, and this may be shown to be the case. Note that the first ten items without D terms, will be removed by using the FLIML which will be analysed in the future work. The numerical results will be discussed in section 6.

4 Bias corrected 2SLS Estimator

Biased corrected 2SLS estimator for structural form equations parameters can be obtained by estimating the approximating bias and then subtracting this bias estimate from the corresponding estimator. As we shown in section 3, the bias approximations depend upon the reduced form coefficient matrices $\Gamma^{(1)}, \Gamma^{(2)}..., \Gamma^{(p)}, \Pi^{(1)}, \Pi^{(2)}..., \Pi^{(q)}$, the non-stochastic matrices $X, LX, ...L^qX$, the starting values y_0', Ly_0', L^2y_0' and L^3y_0' vectors. The $G \times 1$ column vector τ, is equal to $\sigma_1^2\phi = \mathbb{E}(\frac{1}{T}\tilde{V}'\tilde{u}_1)$.

To obtain the estimated bias terms, the reduced form parameter matrices $\Gamma^{(1)}, \Gamma^{(2)}...$, $\Gamma^{(p)}, \Pi^{(1)}, \Pi^{(2)}..., \Pi^{(q)}$ are replaced by their OLS estimates. The $G \times 1$ column vector τ is estimated from $[Y - \sum_{i=1}^{p} L^iY\hat{\Gamma}^{(i)} - \sum_{j=0}^{q} L^{(j)}X\Pi^{(j)}]'(y_1 - \Upsilon\hat{\delta}_1)/T$, the inner product of the G reduced form residuals vectors and the first equation of structural form residuals vector, which is obtained when equation 6 is estimated by 2SLS. Then ϑ is replaced by estimated ϑ , where $\hat{\vartheta} = \Lambda^{**'}\hat{\tau}$, where $\Lambda^{**'} = \begin{bmatrix} I_g : 0 \\ 0 \end{bmatrix}$ which is $G \times (g + P^* + Q^*)$ dimension selection matrix.

Definition 1. ***Given*** $\hat{\hat{\delta}}_{1,b(2SLS)}$ ***is estimated 2SLS bias approximations for the coefficient bias***

$\delta_{1,b(2SLS)} \in \left(\beta_{1,b(2SLS)}, \alpha^{(1)}_{1,b(2SLS)}, ..., \alpha^{(p)}_{1,b(2SLS)}, c^{(0)}_{1,b(2SLS/FLIML)}, c^{(1)}_{1,b(2SLS)}, ..., c^{(q)}_{1,b(2SLS)}\right)$, ***and given that*** $\hat{\delta}_{1,2SLS}$ ***is the 2SLS estimator of*** δ_1***, the C2SLS bias corrected estimator*** $\hat{\delta}_{1,C2SLS}$ ***is as following:***

$$\hat{\delta}_{1,C2SLS} = \hat{\delta}_{1,2SLS} - \hat{\delta}_{1,b(2SLS)}. \tag{16}$$

To examine how well the C2SLS works for practical bias correction, a set of Monte Carlo experiments were conducted and the results are discussed in section 6.

5 Numerical Experiments Design

5.1 Numerical Model

The experiments were conducted using a three equation dynamic simultaneous equation model with four lagged endogenous variables based on sample sizes 50 and 100. Hence the matrix of endogenous variables is $Y = (y_1, y_2, y_3)$. Under the condition for the existence of the moments for the 2SLS estimator [3], in our experiment the degree of over-identification L is greater or equal to 2, so that 2SLS estimates possess a finite mean and variance. In our experiments, we chose $L = 2,\ 4$ and 6 . To commence, we generated two exogenous variables in each equation respectively. L is varied by augmenting the exogenous variables in both second and third equations. Hence, when $L = 2$, the exogenous variable matrix $X = (x_1, x_2, x_3, x_4, x_5, x_6)$; when $L = 4$, $X = (x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8)$; and when $L = 6$, $X = (x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9, x_{10})$. Each exogenous variable is generated as Gaussian autoregressive process with mean zero and with an autoregressive coefficient of 0.9, and they are independent of each other.

$$x_{jt} = 0.9x_{j(t-1)} + \varsigma_{jt} \qquad \varsigma_{jt} \sim \mathcal{N}(0,\,1).$$

The coefficient matrices are as follows:

$$B = \begin{bmatrix} 1 & -1.11 & -3 \\ -\beta_{21} & 1 & -4.6 \\ -\beta_{31} & -8 & 1 \end{bmatrix}, \qquad A^{(1)} = \begin{bmatrix} \alpha^{(1)}_{11} & 0.56 & -0.45 \\ -\alpha^{(1)}_{21} & -0.62 & 0.28 \\ -\alpha^{(1)}_{31} & -0.90 & -0.32 \end{bmatrix},$$

[3] Sargan (1974) showed that the moments of the 2SLS exist up to the order of over-identification in the static SEM. We shall assume the result is valid for the DSEM also.

$$A^{(2)} = \begin{bmatrix} \alpha_{11}^{(2)} & -0.80 & -0.82 \\ -\alpha_{21}^{(2)} & 0.72 & -0.90 \\ -\alpha_{31}^{(2)} & -0.50 & 0.78 \end{bmatrix}, \qquad A^{(3)} = \begin{bmatrix} -\alpha_{11}^{(3)} & -0.46 & -0.80 \\ -\alpha_{21}^{(3)} & -0.72 & 0.31 \\ -\alpha_{31}^{(3)} & -0.31 & 0.74 \end{bmatrix},$$

$$A^{(4)} = \begin{bmatrix} \alpha_{11}^{(4)} & -0.36 & -0.2 \\ -\alpha_{21}^{(4)} & -0.46 & 0.58 \\ -\alpha_{31}^{(4)} & 0.58 & 0.70 \end{bmatrix}.$$

$$L = 2 \quad C' = \begin{bmatrix} c_{11} & c_{21} & c_{31} & 0.00 & 0.00 & 0.00 & 0.00 \\ -1.00 & 0.00 & 0.00 & 0.75 & -0.24 & 0.00 & 0.00 \\ 1.00 & 0.00 & 0.00 & 0.00 & 0.00 & -0.15 & 0.86 \end{bmatrix};$$

$$L = 4 \quad C' = \begin{bmatrix} c_{11} & c_{21} & c_{31} & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 \\ -1.00 & 0.00 & 0.00 & 0.75 & -0.24 & 0.35 & 0.00 & 0.00 & 0.00 \\ 1.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & -0.15 & 0.86 & -0.58 \end{bmatrix};$$

$$L = 6 \quad C' = \begin{bmatrix} c_{11} & c_{21} & c_{31} & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 \\ -1.00 & 0.00 & 0.00 & 0.75 & -0.24 & 0.35 & 0.68 & 0.00 & 0.00 & 0.00 & 0.00 \\ 1.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & -0.15 & 0.86 & -0.58 & 0.33 \end{bmatrix}.$$

There are 17 coefficients in the first equation to be estimated and they are given below:

$\beta_{21} = 2.00$	$\beta_{31} = 5.00$	$\alpha_{11}^{(1)} = 0.50$	$\alpha_{21}^{(1)} = 0.36$	$\alpha_{31}^{(1)} = 0.40$	$\alpha_{11}^{(2)} = 1.20$
$\alpha_{21}^{(2)} = 0.60$	$\alpha_{31}^{(2)} = -0.38$	$\alpha_{11}^{(3)} = 0.65$	$\alpha_{21}^{(3)} = 1.20$	$\alpha_{31}^{(3)} = 0.38$	$\alpha_{11}^{(4)} = 0.50$
$\alpha_{21}^{(4)} = 0.60$	$\alpha_{31}^{(4)} = -0.20$	$c_{11} = 1.00$	$c_{21} = 0.60$	$c_{31} = -0.50$	NaN

This model disturbances are generated as standard normal random variables. The reduced form of the model is:

$$Y = LY\Gamma^{(1)} + L^2Y\Gamma^{(2)} + L^3Y\Gamma^{(3)} + L^4Y\Gamma^{(4)} + X\Pi + \tilde{V},$$

where $\tilde{V} = (\tilde{v_1}, \tilde{v_2}, \tilde{v_3})$ is a $T \times 3$ matrix of reduced form disturbances. We use a matrix P from a Choleski factorisation of Ω to generate the reduced form errors. Hence each row of $\tilde{V}$ is obtained from the transpose of

$$\begin{bmatrix} \tilde{v}_{1,t} \\ \tilde{v}_{2,t} \\ \tilde{v}_{3,t} \end{bmatrix} = P \begin{bmatrix} \tilde{e}_{1,t} \\ \tilde{e}_{2,t} \\ \tilde{e}_{3,t} \end{bmatrix},$$

where $\tilde{e}_{1,t}$, $\tilde{e}_{2,t}$ and $\tilde{e}_{3,t}$ denote the standardised disturbances. Each row of $\tilde{U}$ has mean $0'$ and covariance matrix I, and is i.i.d. The distribution of the structural disturbances can be evaluated from

$$B^{'}\tilde{v}_t = \tilde{u}_t \Rightarrow \tilde{u}_t \sim \mathcal{N}(0, \Sigma) \qquad where \quad \Sigma = B^{'}\Omega B.$$

We set the structural covariance matrix is as follows:

$$\Sigma = \begin{bmatrix} 0.3524 & 0.3448 & 0.3112 \\ 0.3448 & 0.3668 & 0.2984 \\ 0.3112 & 0.2984 & 0.4064 \end{bmatrix},$$

from which the reduced form covariance is:

$$\Omega = \begin{bmatrix} 0.0055 & 0.0054 & 0.0030 \\ 0.0054 & 0.0844 & 0.0085 \\ 0.0030 & 0.0085 & 0.0069 \end{bmatrix}.$$

Based on the above parameters, the relevant eigenvalues of the reduced form equations which determine the stationarity condition can be calculated from the following determinantal equation.

$$det \mid \Gamma^{(4)} + \varpi\Gamma^{(3)} + \varpi^2\Gamma^{(2)} + \varpi^3\Gamma^{(1)} - \varpi^4 I_3 \mid = 0.$$

All the roots ϖ are complex , but they are inside the unit circle Holmgren (2000), which ensures the stability of this system.

$$\varpi_1 = 0.6947 + 0.4789i, \quad \varpi_2 = 0.6947 - 0.4789i, \quad \varpi_3 = -0.0561 + 0.5955i,$$
$$\varpi_4 = -0.0561 - 0.5955i, \quad \varpi_5 = 0.0996 + 0.7235i, \quad \varpi_6 = 0.0996 - 0.7235i,$$

$$\varpi_7 = -0.5847 + 0.0909i, \quad \varpi_8 = -0.5847 - 0.0909i \quad \varpi_9 = -0.2039 + 0.4099i,$$

$$\varpi_{10} = -0.2039 - 0.4099i, \quad \varpi_{11} = 0.4688 + 0.0000i, \quad \varpi_{12} = -0.2651 + 0.0000i.$$

This system above is slightly different from the general model equation (1). We have normalized with respect to $\beta_{11} = 1 = \beta_{22} = \beta_{33}$. To achieve the general case (with high lag order), we choose 4 lags (most finance data are quarterly data). While many of the simulations conducted in the literature focus on two equation models, we decided to simulate a three equations model in this paper.

The initial values, $y_0', Ly_0', L^2y_0', L^3y_0'$ are generated by averaging the simulated reduced form 1000 times. We first take the expectation of the reduced form, where $\mathbb{E}(y') = \mathbb{E}(y')_{-1} = \mathbb{E}(y')_{-2} = \mathbb{E}(y')_{-3} = \mathbb{E}(y')_{-4}$, and $\bar{x}' = \mathbb{E}(X)$, which is as follows:

$$\mathbb{E}(y') = \mathbb{E}(y')\Gamma^{(1)} + \mathbb{E}(y')\Gamma^{(2)} + \mathbb{E}(y')\Gamma^{(3)} + \mathbb{E}(y')\Gamma^{(4)} + \bar{x}'\Pi.$$

From it we obtain $\mathbb{E}(y')$. Then using this 1×3 vector $\mathbb{E}(y')$ as the starting value in the reduced form to generate $T \times G$ matrix $(Y_0)_1$ which is $\left((y_{0_1}')_1, (y_{0_2}')_1 ... (y_{0_{T-1}}')_1, (y_{0_T}')_1\right)'$. Following this procedure, we generate the $M = 1,000$ sets of $T \times G$ matrices Y_0 which is $(Y_0)_1, (Y_0)_2 ... (Y_0)_{M-1}, (Y_0)_M$. Then the pool of initial value Y_0 is $Y_0 = \sum_{m=1}^{M} (Y_0)_m$ which is $T \times G$ matrix. Hence, the initial value in this four lagged dependent variables model is given by $y_0' = \sum_{m=1}^{M} (y_{0_T}')_m / M$, $Ly_0' = \sum_{m=1}^{M} (y_{0_{T-1}}')_m / M$, $L^2 y_0' = \sum_{m=1}^{M} (y_{0_{T-2}}')_m / M$ and $L^3 y_0' = \sum_{1}^{M} (y_{0_{T-3}}')_m / M$.

5.2 The Simulation model

The number of Monte Carlo replications is 20,000, while 199 bootstrap replicates are used when constructing the bias corrected bootstrap.

Bootstrap

Based on Freedman (1984), Ip (1991) provides support for the asymptotic validity of the 2SLS bootstrap in static and dynamic models where errors are normal, and MacKinnon (2002) conducted hypothesis testing in static model which also supports the asymptotic validity of the 2SLS bootstrap.

The residual bootstrap 2SLS is simulated by first estimating the equation of interest using 2SLS. Then by using the estimates and resampling the estimated residuals, pseudo-date (B sets) are generated. Bootstrap replicates are obtained by implementing 2SLS on each of B sets. The bias corrected bootstrap estimate of δ_1 can be calculated as $2\hat{\delta}_1 - \hat{\delta}_{1,\bar{b}}$, where $\hat{\delta}_1$ is the original estimate, and $\hat{\delta}_{1,\bar{b}}$ is the mean of the bootstrap replicates.

Freedman's bootstrap remains the same steps as the usual residual bootstrap, except the generation of the pseudo data.

Our target is to estimate

$$y_1 = Y_2\beta_1 + LY_1\alpha_1^{(1)} + L^2Y_1\alpha_1^{(2)} + L^3Y\alpha_1^{(3)} + L^4Y_1\alpha_1^{(4)} + X_1c_1 + \tilde{u}_1 \tag{17}$$

and we would like to generate the pseudo data y_1^* , LY_1^* , $L^2Y_1^*$, $L^3Y_1^*$, $L^4Y_1^*$ and Y_2^* from equation (17) by resampling the residuals $\hat{u}_{1,2SLS}$. However, the first element y_1^* cannot be obtained without knowing the first element of Y_2^*. Hence, we use the reduced form of Y_2, which is estimated by OLS as,

$$Y_2 = LY\hat{\Gamma}_2^{(1)} + L^2Y\hat{\Gamma}_2^{(2)} + L^3Y\hat{\Gamma}_2^{(3)} + L^4Y\hat{\Gamma}_2^{(4)} + X\hat{\Pi}_2 + \hat{\tilde{V}}_2. \tag{18}$$

Equation (18) is used in conjunction with the 2SLS estimate of equation (17), which will become,

$$y_1 = Y_2\hat{\beta}_1 + LY_1\hat{\alpha}_1^{(1)} + L^2Y_1\hat{\alpha}_1^{(2)} + L^3Y_1\hat{\alpha}_1^{(3)} + L^4Y_1\hat{\alpha}_1^{(4)} + X_1\hat{c}_1 + \hat{\tilde{u}}_1. \tag{19}$$

Then, we can resample the $\hat{\tilde{u}}_1$ in equation (19) to generate $\hat{\tilde{u}}_1^*$ and then resample the $\hat{\tilde{V}}_2$ in equation (18) to give $\hat{\tilde{V}}_2^*$. Note that the disturbances are resampled from the rows of $(\hat{\tilde{u}}_1, \hat{\tilde{V}}_2)$, so that the elements in the resampled residuals $\hat{\tilde{u}}_1^*$ and $\hat{\tilde{V}}_2^*$ correspond to each other.

Based on the resampled residuals, we can generate the pseudodata which we need. Here we use the same procedure as we defined in section 5 to get the initial values of $y_0^{'*}, Ly_0^{'*}, L^2y_0^{'*}, L^3y_0^{'*}$, but the parameters now are replaced by the estimated value in this 2SLS-bootstrap method. Then, it is possible for us to generate $y_{2_1}^{'*}$ from equation (18) by using $\hat{\tilde{v}}_{2_1}^{'*}$. This can be then substituted into equation (19) and used with $Ly_{1_1}^{'*}$, $L^2y_{1_1}^{'*}$,$L^3y_{1_1}^{'*}$, $L^4y_{1_1}^{'*}$ and $\hat{\tilde{u}}_{11}^*$ to generate y_{11}^*. Then y_{11}^* can be put into equation (18) to generate the second vector element of (Y_2^*) which can be used in (19) to give the next element (y_1^*) to put in equation (18). Continuing this iteration gives the full vectors of

y_1^* , LY_1^* , $L^2Y_1^*$, $L^3Y_1^*$, $L^4Y_1^*$ and Y_2^*.

Finally the actual data are replaced by pseudodata to estimate the equation of interest by using the traditional 2SLS estimation method. Thus, Y_2^* is regressed on $(LY^* : L^2Y^* : L^3Y^* : L^4Y^* : X)$ in order to generate the fitted values $\hat{Y}_2^*$, the y_1^* is regressed on $(\hat{Y}_2^* : LY^* : L^2Y^* : L^3Y^* : L^4Y^* : X_1)$ to give the bootstrap 2SLS replicates $\hat{\beta}_{1,b}^*$, $\hat{\alpha}_{1,b}^{(1)*}$, $\hat{\alpha}_{1,b}^{(2)*}$, $\hat{\alpha}_{1,b}^{(3)*}$, $\hat{\alpha}_{1,b}^{(4)*}$ and $\hat{c}_{1,b}^*$. For each $\delta_1 \in (\beta_1, \alpha^{(1)}, \alpha^{(2)}, \alpha^{(3)}, \alpha^{(4)}, c_1)$, the bias corrected bootstrap 2SLS estimate is given by $\hat{\delta}_{1,b} = 2\hat{\delta}_{1,2SLS} - \hat{\delta}_{1,\bar{b}}^*$,where $\hat{\delta}_{1,\bar{b}}^* = \frac{1}{B}\sum_{b=1}^{B} \hat{\delta}_{1,b}^*$.

The bias corrected bootstrap based on our numerical design is as follows:

Definition 1. Given $\hat{\delta}_{1,\bar{b}}$ as the mean of the bootstrap 2SLS replicates for the coefficient $\delta_1 \in (\beta_1, \alpha^{(1)}, \alpha^{(2)}, \alpha^{(3)}, \alpha^{(4)}, c_1)$, and given $\hat{\delta}_{1,2SLS}$ as the 2SLS estimator of δ_1, the bootstrap bias corrected estimator $\hat{\delta}_{1,b}$ is as follows:

$$\hat{\delta}_{1,b} = 2\hat{\delta}_{1,2SLS} - \hat{\delta}_{1,\bar{b}}^*.$$

6 Numerical Results

The numerical results show a comparison of the performance of Monte Carlo 2SLS, and the residual bootstrap 2SLS and C2SLS, which is summarized in Table 1 to Table 3. Table 1 report the overall bias approximation, simultaneity bias, and dynamic biases, respectively. Table 2 presents the bias of Monte Carlo 2SLS, bias of Bootstrap 2SLS, and the bias of C2SLS respectively. Table 3 presents the MSE of Monte Carlo 2SLS, Bootstrap 2SLS, and C2SLS respectively. β_{21}, β_{31} are the coefficients of endogenous variables of the first structural form equation. α_{11}^1 to α_{31}^4 are the coefficients of the lagged endogenous variables (4 lagged endogenous variables). c_{11} is the constant, and c_{21}, c_{31} are the parameters of exogenous regressors.

Table 1 shows that the bias approximation may tend to overstate the magnitude of the "true" bias as given by the Monte Carlo estimates in 2SLS. For example, when $\beta_{21} = 2.00$, $L = 2$, and $T = 50$, the 2SLS bias is -0.3042, whilst the bias approximation slightly higher than the MC 2SLS bias of -0.3229. Moreover, when we numerically evaluate the dynamic bias and the simultaneity bias separately, the results show that they have opposite signs. If we still look at the coefficient above, the approximated bias is -0.3229 where -0.5322 comes from simultaneity, and 0.2093 comes from the dynamics. It implies that if the bias correction method can only eliminate either the simultaneity bias or the dynamic bias but not both, then instead of decreasing the overall bias, the bias correction method could possibly provide more biased estimates. Hence, a bias correction method which effectively reduces the bias in the static case may not do so in the dynamic case.

When the sample size increases, both approximated bias and the bias of Monte Carlo 2SLS decreases. At the same time, when the order of over-identification L increases, this is followed by an increase in the 2SLS bias and a corresponding increase in the approximation.

The results for the corrected 2SLS(C2SLS) estimator which was constructed by subtracting the bias estimate are presented in Table 2. This bias corrected estimator, based upon $O(T^{-1})$ approximation, significantly reduces the Monte Carlo 2SLS bias. For α_{31}^2, the coefficient of $L^2 y_3$ in the first equation, in fact when $L = 2$ and sample size is $T = 50$, by using the new C2SLS estimator, the bias reduced from +61% to +9%. Generally, C2SLS gives almost unbiased estimators in both sample size 50 and 100, when over-identification level is $L = 2, 4$ and 6. The alternative approach based on the non-parametric residual bootstrap applied to 2SLS also reduces the bias effectively; in most cases the bootstrap 2SLS gives almost unbiased estimates when sample size is 100. However, in general, compared with C2SLS, the performance of bootstrap 2SLS is weaker in reducing the bias. As we have shown for α_{31}^2, when the bootstrap method is used the bias is reduced to +30%, and when the sample size increases to $T = 100$ and over-identification level is still $L = 2$, both these two bias correction methods yield almost unbiased results eliminating around a 15% bias from the 2SLS estimator. It is clear that, generally, these two bias corrected estimators have a substantially smaller bias than their uncorrected counterparts.

Table 3 reports the MSE of Monte Carlo 2SLS, C2SLS, and Bootstrap 2SLS respectively . Generally, the MSE for C2SLS is smaller than the corresponding MSE for the Monte Carlo 2SLS while both are smaller than the bootstrap 2SLS MSE. Interestingly, the MSE of the bootstrap 2SLS is lower than that of the Monte Carlo 2SLS for the coefficient of endogenous variables and exogenous variables when $L = 4, 6$, in both sample size sets. In few cases, the MSE of C2SLS is slightly larger than that of 2SLS because of the almost unbiasedness estimates of 2SLS itself when sample size is large. However, this increasing is trivial. For α_{21}^2, which is the coefficient of L^2y_2 in the first equation of the structural form, when sample size is 100 and the over-identification level is $L = 2, 4$ and 6, the percentage of bias for 2SLS is -2% , -3%, -2% and the MSE is 0.0375, 0.0294, 0.0250, while for C2SLS, the MSE is 0.0398, 0.0294, 0.0272. It is clear that the C2SLS has the smallest MSE, and the bootstrap 2SLS has the largest MSE . However,the MSE of the bootstrap 2SLS is not far from the results for 2SLS, and when L increases, the difference becomes smaller.

7 Conclusion

The $O(T^{-1})$ bias in 2SLS estimation of a general DSEM can be decomposed into two parts, which come from the simultaneity and dynamics respectively. These two bias components may be of opposite signs which indicates that the bias correction used should be able to reduce the bias that comes from both components; otherwise the overall bias could become absolutely larger. Notice that the bias approximation tends to overstate the magnitude of the "true" bias as given by the Monte Carlo estimates in 2SLS. Even so, the bias corrected estimator, based upon the $O(T^{-1})$ approximation, very substantially reduces the Monte Carlo 2SLS bias. In addition, it was found to be better overall in terms of MSE, as there is no inflation of the 2SLS MSE. Hence, from the theoretical and analytic analysis, the bias corrected estimator, based upon $O(T^{-1})$ can be recommended as a bias reduction technique.

The bootstrap simulation results in this paper provide evidence in support of the alternative bias correction technique based on the bootstrap. It performs particularly well in bias correction. While the bias correction is not as effective as C2SLS, the computer cost is less which may be a consideration. The bootstrap also reduces the MSE in 2SLS for both endogenous and exogenous variables when L is large.

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Appendix

A The Evaluation for Theorem 1

A.1 Lemmas

The following lemmas will be used in later evaluations.

Lemma 1: **The expectation of a product of three normal (symmetric) / non-normal(zero mean symmetric) random variables is zero.** i.e

$$\mathbb{E}(\Xi A \Psi B \Phi) = 0$$

where Ξ, Ψ, and Φ are three normal (symmetric) / non-normal (zero mean symmetric) random variables.

Lemma 2: $(Z'Z)^{-1} = [\mathbb{E}(Z'Z)]^{-1} + O_p(T^{\frac{-3}{2}})$

Proof:

$$\begin{aligned} Z'Z &= \mathbb{E}(Z'Z) + (Z'Z - \mathbb{E}(Z'Z)) \\ &= \mathbb{E}(Z'Z)\left[I + [\mathbb{E}(Z'Z)]^{-1}(Z'Z - \mathbb{E}(Z'Z))\right]. \\ (Z'Z)^{-1} &= \left[I + [\mathbb{E}(Z'Z)]^{-1}(Z'Z - \mathbb{E}(Z'Z))\right]^{-1}(\mathbb{E}(Z'Z))^{-1} \end{aligned}$$

where,

$$\left[I + [\mathbb{E}(Z'Z)]^{-1}(Z'Z - \mathbb{E}(Z'Z))\right]^{-1} = \left[I - [\mathbb{E}(Z'Z)]^{-1}(Z'Z - \mathbb{E}(Z'Z))\right] + o_p(T^{-1/2}).$$

Hence,

$$\begin{aligned} (Z'Z)^{-1} &= \left[I - [\mathbb{E}(Z'Z)]^{-1}(Z'Z - \mathbb{E}(Z'Z))\right](\mathbb{E}(Z'Z))^{-1} + o_p(T^{-3/2}) \\ &= \left([\mathbb{E}(Z'Z)]^{-1} - [\mathbb{E}(Z'Z)]^{-1}(Z'Z - \mathbb{E}(Z'Z))\right)[\mathbb{E}(Z'Z)]^{-1} + o_p(T^{-3/2}) \\ &= [\mathbb{E}(Z'Z)]^{-1} + O_p(T^{-3/2}). \end{aligned}$$

Lemma 3: **Based on Nagar (1959), $\tilde{V}$ is a $T \times G$ reduced disturbances**

matrix for a G equation system, and $\tilde{U}$ is a $T \times G$ matrix of structural disturbances and $\tilde{u}_1$ is a $T \times 1$ disturbance vector which is the first column of $\tilde{U}$. Assumptions in Nagar (1959) which refer to the appropriate ranks and the transposed rows of $\tilde{U}$ are $NID(0, \Sigma)$ are also held here. Using Nagar (1959)'s decomposition $\tilde{V} = S^* + \tilde{u}_1 \phi^{'}$, where $\tilde{u}_1$ and S^* are normally distributed but independent, $\phi \sigma_1^2 = \mathbb{E}\left(\frac{1}{T}\tilde{V}^{'}\tilde{u}_1\right)$. Then Nagar (1959)'s finding becomes as:

$$\mathbb{E}(S^* A S^{*'}) = tr(C_2^* A).I,$$
$$\mathbb{E}(S^{*'} A S) = tr(A).IC_2^*,$$
$$\mathbb{E}(S^* A S^*) = A^{'} C_4^*,$$
$$\mathbb{E}(S^{*'} A S^{*'}) = C_2^* A,$$

where A is a corresponding and constant matrix, $C_2^* = \Omega - \sigma_1^2 \phi \phi^{'}$, Ω is the covariance matrix of $\tilde{V}$.

Lemma 4: **Mikhail (1972) Suppose also that U, V, W and X are matrices, with the same number of rows, whose elements are normally distributed random variables with the properties that if ϕ_{ri} and Ψ_{sj} are elements of any of these matrices**

$$\mathbb{E}(\phi_{ri}\Psi_{sj}) = 0, \qquad r \neq s$$
$$= \omega_\phi \Psi_{ij}, \qquad r = s$$

and denote the matrix whose elements are $\omega_\phi \Psi_{ij}$ by $\Omega_{\phi\Psi}$ for ϕ, $\Psi = U$, V, W and X.

Suppose also that A, B and C are constant matrices of such dimensions that the various products considered below exist, then:

1. $\mathbb{E}(UAVBWCX) = A'\Omega_{uv}BC'\Omega_{wx} + B'\Omega_{vx}\,\mathrm{tr}(\Omega_{uw}CA') + C'\Omega_{wv}BA'\Omega_{ux}$,
2. $\mathbb{E}(U'AVBWCX) = \Omega_{uv}BC'\Omega_{wx}\,\mathrm{tr}(A) + \Omega_{uw}CA'B'\Omega_{vx} + \Omega_{ux}\,\mathrm{tr}(AB'\Omega_{vw}C)$,
3. $\mathbb{E}(UAV'BWCX) = BC'\Omega_{wx}tr(\Omega_{uv}A') + B'C'\Omega_{wu}A\Omega_{vx} + C'\Omega_{wv}A'\Omega_{ux}tr(B)$,
4. $\mathbb{E}(UAVBW'CX) = A'\Omega_{uv}B\Omega_{wx}tr(C) + CA'\Omega_{uw}B'\Omega_{vx} + C'A'\Omega_{ux}tr(B\Omega_{wv})$,
5. $\mathbb{E}(U'AV'BWCX) = \Omega_{uv}A'BC'\Omega_{wx} + \Omega_{uw}CBA\Omega_{vx} + \Omega_{ux}tr(A\Omega_{vw}C)tr(B)$,

6. $\mathbb{E}(U'AVBW'CX) = \Omega_{uv}B\Omega_{wx}tr(C)tr(A) + \Omega_{uw}B'\Omega_{vx}tr(AC') + \Omega_{ux}tr(AC)tr(B'\Omega_{vw})$,
7. $\mathbb{E}(UAV'BW'CX) = B\Omega_{wx}tr(A'\Omega_{uv})trC + CB\Omega_{wu}A\Omega_{vx} + C'B\Omega_{wv}A'\Omega_{ux}$,
8. $\mathbb{E}(U'AV'BW'CX) = \Omega_{uv}A'B\Omega_{wx}tr(C) + \Omega_{uw}B'C'A\Omega_{vx} + \Omega_{ux}tr(A\Omega_{vw}B'C)$.

A.2 Evaluating the Expectations

From equation 14

$$\mathbb{E}(\hat{\alpha}_1 - \alpha_1) = \mathbb{E}\left\{H\bar{\Upsilon}\tilde{u}_1 + H\Delta_1^{'}\tilde{u}_1 + H\Delta_2^{'}\tilde{u}_1 - HJ_1^{*}H\bar{\Upsilon}^{'}\tilde{u}_1 - HJ_1^{*}H\Delta_2\tilde{u}_1\right\} + o(T^{-1}), \tag{A.1}$$

evaluating the expectation for each term.

The first term,

$$(i) \qquad \mathbb{E}\{H\bar{\Upsilon}\tilde{u}_1\} = H\bar{\Upsilon}\mathbb{E}\{\tilde{u}_1\} = 0. \tag{A.2}$$

The second term,

$$(ii) \qquad \mathbb{E}\{H\Delta_1^{'}\tilde{u}_1\} = H\mathbb{E}\{\Delta_1^{'}\tilde{u}_1\}.$$

Recalling equation 10 for the definition of Δ_1, we have:

$$\begin{aligned} H\Delta_1^{'}\tilde{u}_1 &= H\left(\bar{Z}(Z^{'}Z)^{-1}Z^{'}[\tilde{V}_2 : 0 : 0] + \left[\tilde{R}I_2^{'}(Z^{'}Z)^{-1}Z^{'}\tilde{V}_2 : 0 : 0)\right]\right)^{'}\tilde{u}_1 \\ &= H\Lambda^{**'}\tilde{V}^{'}\bar{Z}(\mathbb{E}(Z^{'}Z))^{-1}\bar{Z}^{'}\tilde{u}_1 + H\Lambda^{**'}\tilde{V}^{'}\bar{Z}(\mathbb{E}(Z^{'}Z))^{-1}I_2\tilde{R}^{'}\tilde{u}_1 \\ &\quad + H\Lambda^{**'}\tilde{V}^{'}\tilde{W}^{*'}(\mathbb{E}(Z^{'}Z))^{-1}\bar{Z}^{'}\tilde{u}_1 + H\Lambda^{**'}\tilde{V}^{'}\tilde{W}^{*}(\mathbb{E}(Z^{'}Z))^{-1}I_2\tilde{R}^{'}u_1 + o_p(T^{-1}) \end{aligned}$$

where, $\tilde{V}_2$ is the $T \times g$ submatrix of matrix $\tilde{V}$, which can be expressed as $[\tilde{V}_2 : 0 : 0] = \tilde{V}\begin{bmatrix} I_g : 0 \\ 0 \end{bmatrix}$, and we define $\Lambda^{**} = \begin{bmatrix} I_g : 0 \\ 0 \end{bmatrix}$ which is with $G \times (g + P^{*} + Q^{*})$ dimension selection matrix. Also,

$$\begin{bmatrix} \hat{\Gamma}_2^{(1)} - \Gamma_2^{(1)} \\ \vdots \\ \hat{\Gamma}_2^{(p)} - \Gamma_2^{(p)} \\ \hat{\Pi}_2^{(1)} - \Pi_2^{(1)} \\ \vdots \\ \hat{\Pi}_2^{(q)} - \Pi_2^{(q)} \end{bmatrix} = (Z^{'}Z)^{-1}Z^{'}\tilde{V}_2{}^{4} = (\mathbb{E}\{Z^{'}Z\})^{-1}\bar{Z}^{'}\tilde{V}_2 + (\mathbb{E}\{Z^{'}Z\})^{-1}\tilde{W}^{*'}\tilde{V}_2 + o_p(T^{-1/2})$$

and

$$\begin{bmatrix} \hat{\Gamma}_2^{(1)} - \Gamma_2^{(1)} \\ \vdots \\ \hat{\Gamma}_2^{(p)} - \Gamma_2^{(p)} \end{bmatrix} = I_2^{\prime}(Z^{'}Z)^{-1}Z^{'}\tilde{V}_2 = I_2^{\prime}(\mathbb{E}\{Z^{'}Z\})^{-1}\bar{Z}^{'}\tilde{V}_2 + I_2^{\prime}(\mathbb{E}\{Z^{'}Z\})^{-1}\tilde{W}^{*'}\tilde{V}_2 + o_p(T^{-1/2}),$$

and

$$[\tilde{V}_2 : 0 : 0] = \tilde{V}\Lambda^{**}.$$

Taking expectation, the last two terms are zero, then this gives

$$\begin{aligned} \mathbb{E}\{H\Delta_1^{'}\tilde{u}_1\} &= H\Lambda^{**'}\mathbb{E}\left\{\tilde{V}^{'}\bar{Z}(\mathbb{E}(Z^{'}Z))^{-1}\bar{Z}^{'}\tilde{u}_1\right\} \\ &\quad + H\Lambda^{**'}\mathbb{E}\left\{\tilde{V}^{'}\tilde{W}^{*}(\mathbb{E}(Z^{'}Z))^{-1}I_2\tilde{R}^{\prime}u_1\right\} + o(T^{-1}). \end{aligned} \tag{A.3}$$

The first term can be expressed as:

(1)

$$\begin{aligned} H\Lambda^{**'}\mathbb{E}\left\{\tilde{V}^{'}\bar{Z}(\mathbb{E}(Z^{'}Z))^{-1}\bar{Z}^{'}\tilde{u}_1\right\} &= H\Lambda^{**'}\mathbb{E}\left\{(S^{*} + \tilde{u}_1\phi^{'})\bar{Z}(\mathbb{E}(Z^{'}Z))^{-1}\bar{Z}^{'}\tilde{u}_1\right\} \\ &= H(tr\{\bar{Z}(\mathbb{E}(Z^{'}Z))^{-1}\bar{Z}^{'}\Lambda^{**'}\}.I)(\sigma_1^2\phi). \end{aligned}$$

The second term can be evaluated as:

(2)

$$H\Lambda^{**'}\mathbb{E}\left\{\tilde{V}^{'}\tilde{W}^{*}(\mathbb{E}(Z^{'}Z))^{-1}I_2\tilde{R}^{\prime}u_1\right\}^{5}$$

[4] By using Lemma 2.

$$= H\Lambda^{**'}\mathbb{E}\left\{(S^* + \tilde{u}_1\phi')'\left[\sum_{i=1}^{p}\sum_{t=i}^{T-1} D^t\tilde{V}J_{t-i}\Psi_i^{'} : 0\right][\mathbb{E}(Z'Z)]^{-1}I_2\sum_{j=1}^{p}\sum_{s=j}^{T-1}\Psi_j J_{s-j}^{\prime}\tilde{V}'D^{s'}\tilde{u}_1\right\}$$

$$= H\Lambda^{**'}\sum_{i=1}^{p}\sum_{t=i}^{T-1}\sum_{j=1}^{p}\sum_{s=j}^{T-1}\mathbb{E}\left\{\phi\tilde{u}_1^{\prime}D^t\tilde{u}_1\phi' J_{t-i}\Psi_i^{\prime}I_2^{\prime}[\mathbb{E}(Z'Z)]^{-1}I_2\Psi_j J_{s-j}^{\prime}\phi\tilde{u}_1^{\prime}D^{s'}\tilde{u}_1\right\}$$

$$+ H\Lambda^{**'}\sum_{i=1}^{p}\sum_{t=i}^{T-1}\sum_{j=1}^{p}\sum_{s=j}^{T-1}\mathbb{E}\left\{\phi\tilde{u}_1^{\prime}D^t S^* J_{t-i}\Psi_i^{\prime}I_2^{\prime}[E(Z'Z)]^{-1}I_2\Psi_j J_{s-j}^{\prime}S^{*'}D^{s'}\tilde{u}_1\right\}$$

$$+ H\Lambda^{**'}\sum_{i=1}^{p}\sum_{t=i}^{T-1}\sum_{j=1}^{p}\sum_{s=j}^{T-1}\mathbb{E}\left\{S^{*'}D^t S^* J_{t-i}\Psi_i^{\prime}I_2^{\prime}[\mathbb{E}(Z'Z)]^{-1}I_2\Psi_j J_{s-j}^{\prime}\phi\tilde{u}_1^{\prime}D^{s'}\tilde{u}_1\right\}$$

$$+ H\Lambda^{**'}\sum_{i=1}^{p}\sum_{t=i}^{T-1}\sum_{j=1}^{p}\sum_{s=j}^{T-1}\mathbb{E}\left\{S^{*'}D^t\tilde{u}_1\phi' J_{t-i}\Psi_i^{\prime}I_2^{\prime}[\mathbb{E}(Z'Z)]^{-1}I_2\Psi_j J_{s-j}^{\prime}S^{*'}D^{s'}\tilde{u}_1\right\}$$

$$= (T-t)H\Lambda^{**'}\phi\sigma_1^2 tr\left\{\Omega J_{t-i}\Psi_i^{\prime}I_2^{\prime}[\mathbb{E}(Z'Z)]^{-1}I_2\Psi_j J_{s-j}^{\prime}\right\}.$$ [6]

Combining these two terms together, the result for equation (A.3) is:

$$\begin{aligned}&\mathbb{E}\{H\Delta_1^{'}\tilde{u}_1\} \\ &= H(tr\{\bar{Z}(\mathbb{E}(Z^{'}Z))^{-1}\bar{Z}^{'}\Lambda^{**'}\}.I)(\sigma_1^2\phi) \\ &\quad + (T-t)H\Lambda^{**'}\phi\sigma_1^2 tr\left\{\Omega J_{t-i}\Psi_i^{\prime}I_2^{\prime}[\mathbb{E}(Z'Z)]^{-1}I_2\Psi_j J_{s-j}^{\prime}\right\} + o(T^{-1}).\end{aligned} \tag{A.4}$$

The third term in equation A.1 is:

$$(iii) \qquad \mathbb{E}\{H\Delta_2^{'}\tilde{u}_1\} = H\mathbb{E}(\Delta_2^{'}\tilde{u}_1) = 0. \tag{A.5}$$

Recalling equation 10 for the definition of Δ_2, then clearly $\mathbb{E}\{\tilde{R}^{'}\tilde{u}_1\} = 0$.

The forth term of equation A.1 is:

$$\begin{aligned}(iv) \qquad -\mathbb{E}\{HJ_1^* H\bar{\Upsilon}^{'}\tilde{u}_1\} &= -\mathbb{E}\{H[(\Delta_2^{'}\Delta_2 - \mathbb{E}(\Delta_2^{'}\Delta_2)) + ((\bar{\Upsilon}^{'}\Delta_1 + \Delta_1^{'}\bar{\Upsilon}) + (\bar{\Upsilon}^{'}\Delta_2 \\ &\quad + \Delta_2^{'}\bar{\Upsilon}) + (\Delta_1^{'}\Delta_2 + \Delta_2^{'}\Delta_1)]H\bar{\Upsilon}^{'}\tilde{u}_1\} \\ &= -\mathbb{E}\left\{H\bar{\Upsilon}^{'}\left[\sum_{i=1}^{p}L^i\bar{Y}(\hat{\Gamma}_2^{(i)} - \Gamma_2^{(i)}) + \sum_{j=1}^{q}L^j X(\hat{\Pi}_2^{(i)} - \Pi_2^{(i)})\right.\right. \\ &\quad \left.\left. + \sum_{i=1}^{p}L^i\tilde{W}(\hat{\Gamma}_2^{(i)} - \Gamma_2^{(i)}) : 0 : 0\right]H\bar{\Upsilon}^{'}\tilde{u}_1\right\}\end{aligned}$$

[5] Recalling equation 5 in section 2, $\tilde{W}^* = \left[\sum_{i=1}^{p}\sum_{t=i}^{T-1}D^t\tilde{V}J_{t-i}\Psi_i^{'} : 0\right]$, and $\tilde{R} = \sum_{i=1}^{p}\sum_{t=i}^{T-1}D^t\tilde{V}J_{t-i}\Psi_i^{'}$.

[6] Using Lemma 4.

$$
\begin{aligned}
&- \mathbb{E}\left\{ H \left[\sum_{i=1}^{p} L^i \bar{Y} (\hat{\Gamma}_2^{(i)} - \Gamma_2^{(i)}) + \sum_{j=1}^{q} L^j X (\hat{\Pi}_2^{(i)} - \Pi_2^{(i)}) \right.\right. \\
&\qquad \left.\left. + \sum_{i=1}^{p} \right] L^i \tilde{W} (\hat{\Gamma}_2^{(i)} - \Gamma_2^{(i)}) : 0 : 0]' \bar{\Upsilon} H \bar{\Upsilon}' \tilde{u}_1 \right\} \\
&- \mathbb{E}\left\{ H \left[\bar{\Upsilon}' \sum_{i=1}^{p} L^i \tilde{W} C^* + C^{*'} \left(\sum_{i=1}^{p} L^i \tilde{W} \right)' \bar{\Upsilon} \right] H \bar{\Upsilon}' \tilde{u}_1 \right\} \\
&- \mathbb{E}\left\{ H \left[\sum_{i=1}^{p} L^i \bar{Y} (\hat{\Gamma}_2^{(i)} - \Gamma_2^{(i)}) + \sum_{j=1}^{q} L^j X (\hat{\Pi}_2^{i} - \Pi_2^{i}) \right.\right. \\
&\qquad \left.\left. + \sum_{i=1}^{p} L^i \tilde{W} (\hat{\Gamma}_2^{(i)} - \Gamma_2^{(i)}) : 0 : 0 \right]' \left[\sum_{i=1}^{p} L^i \tilde{W} \Gamma_2^{(i)} : \tilde{R}_1 : 0 \right] H \bar{\Upsilon}' \tilde{u}_1 \right\} \\
&- \mathbb{E}\left\{ H \left[\sum_{i=1}^{p} L^i \tilde{W} \Gamma_2^{(i)} : \tilde{R}_1 : 0 \right]' \left[\sum_{i=1}^{p} L^i \bar{Y} (\hat{\Gamma}_2^{(i)} - \Gamma_2^{(i)}) \right.\right. \\
&\qquad \left.\left. + \sum_{j=1}^{q} L^j X (\hat{\Pi}_2^{i} - \Pi_2^{i}) + \sum_{i=1}^{p} L^i \tilde{W} (\hat{\Gamma}_2^{(i)} - \Gamma_2^{(i)}) : 0 : 0 \right] H \bar{\Upsilon}' \tilde{u}_1 \right\} \\
&+ o(T^{-1}),
\end{aligned}
$$

where the definition of Δ_1 and Δ_2 is from equation 10, and the expression of J_1^* is in the footnote 1 in section 3.

By using the expression

$$
\begin{bmatrix} \hat{\Gamma}_2^{(1)} - \Gamma_2^{(1)} \\ \vdots \\ \hat{\Gamma}_2^{(p)} - \Gamma_2^{(p)} \\ \hat{\Pi}_2^{(1)} - \Pi_2^{(1)} \\ \vdots \\ \hat{\Pi}_2^{(q)} - \Pi_2^{(q)} \end{bmatrix} = (Z'Z)^{-1} Z' \tilde{V}_2 = (\mathbb{E}\{Z'Z\})^{-1} \bar{Z}' \tilde{V}_2 + (\mathbb{E}\{Z'Z\})^{-1} \tilde{W}^{*'} \tilde{V}_2 + o_p(T^{-1/2})
$$

and

$$
\begin{aligned}
\begin{bmatrix} \hat{\Gamma}_2^{(1)} - \Gamma_2^{(1)} \\ \vdots \\ \hat{\Gamma}_2^{(p)} - \Gamma_2^{(p)} \end{bmatrix} = I_2' (Z'Z)^{-1} Z' \tilde{V}_2 &= I_2' (\mathbb{E}\{Z'Z\})^{-1} \bar{Z}' \tilde{V}_2 + I_2' (\mathbb{E}\{Z'Z\})^{-1} \tilde{W}^{*'} \tilde{V}_2 \\
&\quad + o_p(T^{-1/2})
\end{aligned}
$$

and

$$[\tilde{V}_2 : 0 : 0] = \tilde{V}\Lambda^{**},$$

the above (iv) expression can be written as:

$$\begin{aligned}
&-\mathbb{E}\{H\bar{\Upsilon}'\bar{Z}[\mathbb{E}(Z'Z)]^{-1}\bar{Z}'\tilde{V}\Lambda^{**}H\bar{\Upsilon}'\tilde{u}_1\} - \mathbb{E}\{H\bar{\Upsilon}'\tilde{R}I_2'[\mathbb{E}(Z'Z)]^{-1}\tilde{W}^{*'}\tilde{V}\Lambda^{**}H\bar{\Upsilon}'\tilde{u}_1\}\\
&-\mathbb{E}\left\{H\Lambda^{**'}\tilde{V}'\bar{Z}[\mathbb{E}(Z'Z)]^{-1}\bar{Z}'\bar{\Upsilon}H\bar{\Upsilon}'\tilde{u}_1\right\} - \mathbb{E}\{H\Lambda^{**'}\tilde{V}'\tilde{W}^{*}[\mathbb{E}(Z'Z)]^{-1}I_2\tilde{R}'\bar{\Upsilon}H\bar{\Upsilon}'\tilde{u}_1\}\\
&-\mathbb{E}\{H\bar{\Upsilon}'\tilde{R}CH\bar{\Upsilon}'\tilde{u}_1\} - \mathbb{E}\{HC^{*'}\tilde{R}'\bar{\Upsilon}H\bar{\Upsilon}'\tilde{u}_1\}\\
&-\mathbb{E}\{H\Lambda^{**'}\tilde{V}'\tilde{W}^{*}(\mathbb{E}(Z'Z))^{-1}\bar{Z}'\tilde{R}CH\bar{\Upsilon}'\tilde{u}_1\} - \mathbb{E}\{H\Lambda^{**'}\tilde{V}'\bar{Z}(\mathbb{E}(Z'Z))^{-1}I_2\tilde{R}'\tilde{R}CH\bar{\Upsilon}'\tilde{u}_1\}\\
&-\mathbb{E}\{HC'\tilde{R}'\bar{Z}(\mathbb{E}(Z'Z))^{-1}\tilde{W}^{*'}\tilde{V}\Lambda^{**}H\bar{\Upsilon}'\tilde{u}_1\} - \mathbb{E}\{HC'\tilde{R}'\tilde{R}I_2'(\mathbb{E}(Z'Z))^{-1}\bar{Z}'\tilde{V}\Lambda^{**}H\bar{\Upsilon}'\tilde{u}_1\}.
\end{aligned}$$

Using $\tilde{R}$ ($\tilde{R} = \sum_{i=1}^{p}\sum_{t=i}^{T-1} D^t\tilde{V}J_t\Psi_i'$, and $\tilde{W}^* = \left[\sum_{i=1}^{p}\sum_{t=i}^{T-1} D^t\tilde{V}J_t\Psi_i' : 0\right]$ from equation 5, and the decomposition of $\tilde{V}$, $\tilde{V} = S^* + \tilde{u}_1\phi'$, then (iv) can be obtained from the sum of $(1) - (8)$ below:

(1)

$$-\mathbb{E}\{H\bar{\Upsilon}'\bar{Z}[\mathbb{E}(Z'Z)]^{-1}\bar{Z}'\tilde{V}\Lambda^{**}H\bar{\Upsilon}'\tilde{u}_1\} = -\mathbb{E}\{H\bar{\Upsilon}'\bar{Z}[\mathbb{E}(Z'Z)]^{-1}\bar{Z}'\tilde{u}_1\phi'\Lambda^{**}H\bar{\Upsilon}'\tilde{u}_1\} \tag{A.6}$$

$$= -H\bar{\Upsilon}'\bar{Z}[\mathbb{E}(Z'Z)]^{-1}\bar{Z}'\bar{\Upsilon}H'\Lambda^{**'}(\sigma_1^2\phi).$$

(2)

$$-\mathbb{E}\{H\bar{\Upsilon}'\tilde{R}I_2'[\mathbb{E}(Z'Z)]^{-1}\tilde{F}^{*'}\tilde{V}\Lambda^{**}H\bar{\Upsilon}'\tilde{u}_1\} \tag{A.7}$$

$$= -\mathbb{E}\left\{H\bar{\Upsilon}'\sum_{i=1}^{p}\sum_{t=i}^{T-1} D^t\tilde{V}J_{t-i}\Psi_i'I_2'[\mathbb{E}(Z'Z)]^{-1}\begin{bmatrix}\sum_{j=1}^{p}\sum_{s=j}^{T-1}\Psi_jJ_{s-j}'\tilde{V}'D^{t'}\\ 0\end{bmatrix}\tilde{V}\Lambda^{**}H\bar{\Upsilon}'\tilde{u}_1\right\}.$$

For the moment, we shall focus on the the following equation (Moving the summations and first three fixed terms H, $\bar{\Upsilon}'$, and D^t outside of expectation symbol):

$$\mathbb{E}\left\{\tilde{V}J_{t-i}\Psi_i'I_2'[\mathbb{E}(Z'Z)]^{-1}\begin{bmatrix}\Psi_jJ_{s-j}'\tilde{V}'D^{s'}\\ 0\end{bmatrix}\tilde{V}\Lambda^{**}H\bar{\Upsilon}'\tilde{u}_1\right\} \tag{A.8}$$

$$= \mathbb{E}\{\tilde{u}_1\phi'J_{t-i}\Psi_i'I_2'[\mathbb{E}(Z'Z)]^{-1}I_2\Psi_jJ_{s-j}'\phi\tilde{u}_1'D^{s'}\tilde{u}_1\phi'\Lambda^{**}H\bar{\Upsilon}'\tilde{u}_1\}$$

$$+\,\mathbb{E}\{S^* J_{t-i}\Psi_i^{'} I_2^{'}[\mathbb{E}(Z'Z)]^{-1} I_2 \Psi_j J_{s-j}^{'} S^{*'} D^{s'} \tilde{u}_1 \phi^{'} \Lambda^{**} H \bar{\Upsilon}^{'} \tilde{u}_1\}$$
$$+\,\mathbb{E}\{\tilde{u}_1 \phi^{'} J_{t-i}\Psi_i^{'} I_2^{'}[\mathbb{E}(Z'Z)]^{-1} I_2 \Psi_j J_{s-j}^{'} S^{*'} D^{s'} S^* \Lambda^{**} H \bar{\Upsilon}^{'} \tilde{u}_1\}$$
$$+\,\mathbb{E}\{S^* J_{t-i}\Psi_i^{'} I_2^{'}[\mathbb{E}(Z'Z)]^{-1} I_2 \Psi_j J_{s-j}^{'} \phi \tilde{u}_1^{'} D^{s'} S^* \Lambda^{**} H \bar{\Upsilon}^{'} \tilde{u}_1\}\}.$$

Then equation (A.8) will be calculated from $(a) - (d)$ below:

(a)

$$\begin{aligned}
&\mathbb{E}\{\tilde{u}_1 \phi^{'} J_{t-i}\Psi_i^{'} I_2^{'}[\mathbb{E}(Z'Z)]^{-1} I_2 \Psi_j J_{s-j}^{'} \phi \tilde{u}_1^{'} D^{s'} \tilde{u}_1 \phi^{'} \Lambda^{**} H \bar{\Upsilon}^{'} \tilde{u}_1\} \\
&\quad = \mathbb{E}\left\{ \tilde{u}_1 \tilde{u}_1^{'} \bar{\Upsilon} H^{'} \Lambda^{**'} \phi \phi^{'} J_{t-i}\Psi_i^{'} I_2^{'}[\mathbb{E}(Z'Z)]^{-1} \begin{bmatrix} \Psi_j J_{s-j}^{'} \\ 0 \end{bmatrix} \phi \tilde{u}_1^{'} D^{s'} \tilde{u}_1 \right\} \\
&\quad = \sigma_1^4 (tr(D^{s'}) I + D^s + D^{s'}) \bar{\Upsilon} H^{'} \Lambda^{**'} \phi \phi^{'} J_{t-i}\Psi_i^{'} I_2^{'}[\mathbb{E}(Z'Z)]^{-1} I_2 \Psi_j J_{s-j}^{'} \phi^{7} \\
&\quad = \sigma_1^4 (D^s + D^{s'}) \bar{\Upsilon} H^{'} \Lambda^{**'} \phi \phi^{'} J_{t-i}\Psi_i^{'} I_2^{'}[\mathbb{E}(Z'Z)]^{-1} I_2 \Psi_j J_{s-j}^{'} \phi.
\end{aligned}$$

(b)

$$\begin{aligned}
&\mathbb{E}\{S^* J_{t-i}\Psi_i^{'} I_2^{'}[\mathbb{E}(Z'Z)]^{-1} I_2 \Psi_j J_{s-j}^{'} S^{*'} D^{s'} \tilde{u}_1 \phi^{'} \Lambda^{**} H \bar{\Upsilon}^{'} \tilde{u}_1\} \\
&\quad = \mathbb{E}\left\{ S^* J_{t-i}\Psi_i^{'} I_2^{'}[\mathbb{E}(Z'Z)]^{-1} \begin{bmatrix} \Psi_j J_{s-j}^{'} \\ 0 \end{bmatrix} S^{*'} D^{s'} \tilde{u}_1 \tilde{u}_1^{'} \bar{\Upsilon} H^{'} \Lambda^{**'} \phi \right\} \\
&\quad = \sigma_1^2 tr\{C_2^* J_{t-i}\Psi_i^{'} I_2^{'}[\mathbb{E}(Z'Z)]^{-1} I_2 \Psi_j J_{s-j}^{'}\} D^{s'} \bar{\Upsilon} H^{'} \Lambda^{**'} \phi \\
&\quad = \sigma_1^2 tr\{(\Omega - \phi\phi^{'}\sigma^2) J_{t-i}\Psi_i^{'} I_2^{'}[\mathbb{E}(Z'Z)]^{-1} I_2 \Psi_j J_{s-j}^{'}\} D^{s'} \bar{\Upsilon} H^{'} \Lambda^{**'} \phi.
\end{aligned}$$

Using Lemma 3, $\mathbb{E}\{S^* A S^{*'}\} = tr\{C_2^* A\} I$.

(c)

$$\begin{aligned}
&\mathbb{E}\{\tilde{u}_1 \phi^{'} J_{t-i}\Psi_i^{'} I_2^{'}[\mathbb{E}(Z'Z)]^{-1} I_2 \Psi_j J_{s-j}^{'} S^{*'} D^{s'} S^* \Lambda^{**} H \bar{\Upsilon}^{'} \tilde{u}_1\} \\
&\quad = \mathbb{E}\left\{ \tilde{u}_1 \tilde{u}_1^{'} \bar{\Upsilon} H^{'} \Lambda^{**'} S^{*'} D^s S^* \begin{bmatrix} \Psi_j J_{s-j}^{'} \\ 0 \end{bmatrix}^{'} I_2^{'}[\mathbb{E}(Z'Z)]^{-1} I_2 \Psi_i J_{t-i}^{'} \phi \right\} \\
&\quad = \sigma_1^2 \bar{\Upsilon} H \Lambda^{**'} \mathbb{E}(S^{*'} D^{s'} S^*) J_{s-j} \Psi_j^{'} I_2^{'}[\mathbb{E}(Z'Z)]^{-1} I_2 \Psi_i J_{t-i}^{'} \phi
\end{aligned}$$

[7] Using Lemma 4.

$$= \sigma_1^2 \bar{\Upsilon} H \Lambda^{**'} tr\{D^s\} C_2^* J_{s-j} \Psi_j' I_2' [\mathbb{E}(Z'Z)]^{-1} I_2 J_{t-i}' \Psi_i \phi = 0.$$

Using Lemma 3, $\mathbb{E}\{S^{*'} A S^*\} = tr\{A\} C_2^*$.

(d)

$$\begin{aligned}
&\mathbb{E}\{S^* J_{t-i} \Psi_i' I_2' [\mathbb{E}(Z'Z)]^{-1} I_2 \Psi_j J_{s-j}' \phi \tilde{u}_1' D^{s'} S^* \Lambda^{**} H \bar{\Upsilon}' \tilde{u}_1\} \\
&\quad = \mathbb{E}\{D^s \tilde{u}_1 \phi' J_{s-j} \Psi_j' I_2' [\mathbb{E}(Z'Z)]^{-1} I_2 \Psi_i J_{t-i}' C_2^* \Lambda^{**} H \bar{\Upsilon}' \tilde{u}_1\}.
\end{aligned}$$

Using the definition of S^* and $\tilde{u}_1$: S^* and $\tilde{u}_1$ are independent, and Lemma 3 that $\mathbb{E}\{S^* A S^*\} = A' C_2^*$.

Then, we have:

$$\begin{aligned}
&\mathbb{E}\{D^s \tilde{u}_1 \phi' J_{s-j} \Psi_j' I_2' [\mathbb{E}(Z'Z)]^{-1} I_2 \Psi_i J_{t-i}' C_2^* \Lambda^{**} H \bar{\Upsilon}' \tilde{u}_1\} \\
&\quad = \mathbb{E}\{D^s \tilde{u}_1 \tilde{u}_1' \bar{\Upsilon} H' \Lambda^{**'} C_2^* J_{t-i} \Psi_i' I_2' [\mathbb{E}(Z'Z)]^{-1} I_2 \Psi_j J_{s-j}' \phi\} \\
&\quad = \sigma_1^2 D^s \bar{\Upsilon} H \Lambda^{**'} C_2^* J_{t-i} \Psi_i' I_2' [\mathbb{E}(Z'Z)]^{-1} I_2 \Psi_j J_{s-j}' \phi \\
&\quad = \sigma_1^2 D^s \bar{\Upsilon} H \Lambda^{**'} \Omega J_{t-i} \Psi_i' I_2' [\mathbb{E}(Z'Z)]^{-1} I_2 \Psi_j J_{s-j}' \phi \\
&\qquad - \sigma_1^4 2 D^s \bar{\Upsilon} H \Lambda^{**'} \phi \phi' J_{t-i} \Psi_i' I_2' [\mathbb{E}(Z'Z)]^{-1} I_2 \Psi_i J_{s-j}' \phi.
\end{aligned}$$

Putting $(a) - (d)$ together, we have:

$$\begin{aligned}
&\mathbb{E}\left\{ \tilde{V} J_{t-i} \Psi_i' I_2' [\mathbb{E}(Z'Z)]^{-1} \begin{bmatrix} \Psi_j J_{s-j}' \tilde{V}' D^{s'} \\ 0 \end{bmatrix} \tilde{V} \Lambda^{**} H \bar{\Upsilon}' \tilde{u}_1 \right\} \\
&\quad = \sigma_1^2 D^{s'} \bar{\Upsilon} H \Lambda^{**'} \phi tr\{\Omega J_{t-i} \Psi_i' I_2' [\mathbb{E}(Z'Z)]^{-1} I_2 \Psi_i J_{s-j}'\} \\
&\qquad + \sigma_1^2 D^s \bar{\Upsilon} H \Lambda^{**'} \Omega J_{t-i} \Psi_i' I_2' [\mathbb{E}(Z'Z)]^{-1} I_2 \Psi_j J_{s-j}' \phi.
\end{aligned}$$

Then, equation A.7 becomes:

$$\begin{aligned}
&- \mathbb{E}\{H \bar{\Upsilon}' \tilde{R} I_2' [\mathbb{E}(Z'Z)]^{-1} \tilde{F}^{*'} \tilde{V} \Lambda^{**} H \bar{\Upsilon}' \tilde{u}_1\} \qquad \text{(A.9)} \\
&\quad = -H \bar{\Upsilon}' \sum_{i=1}^{p} \sum_{t=i}^{T-1} \sum_{j=1}^{p} \sum_{s=j}^{T-1} D^t D^{s'} \bar{\Upsilon} H tr\left\{ \Omega \left[J_{s-j} \Psi_j' : 0 \right] [\mathbb{E}(Z'Z)]^{-1} I_2 \Psi_i J_{t-i}' \right\} \Lambda^{**'} (\sigma^2 \phi) \\
&\quad - H \bar{\Upsilon}' \sum_{i=1}^{p} \sum_{t=i}^{T-1} \sum_{j=1}^{p} \sum_{s=j}^{T-1} D^t D^s \bar{\Upsilon} H \Lambda^{**'} \Omega J_{t-i} \Psi_i' I_2' [\mathbb{E}(Z'Z)]^{-1} I_2 \Psi_j J_{s-j}' (\sigma^2 \phi).
\end{aligned}$$

(3)

$$-\mathbb{E}\{H\Lambda^{**'}\tilde{V}'\bar{Z}[\mathbb{E}(Z'Z)]^{-1}\bar{Z}'\bar{\Upsilon}H\bar{\Upsilon}'\tilde{u}_1\} \tag{A.10}$$
$$= -\mathbb{E}\{H\Lambda^{**'}\phi\tilde{u}_1'\bar{Z}[\mathbb{E}(Z'Z)]^{-1}\bar{\Upsilon}H\bar{\Upsilon}'\tilde{u}_1\}$$
$$= -H\Lambda^{**'}\phi\sigma_1^2 tr\{\bar{Z}[\mathbb{E}(Z'Z)]^{-1}\bar{\Upsilon}H\bar{\Upsilon}'\}$$
$$= -Htr\{\bar{Z}[\mathbb{E}(Z'Z)]^{-1}\bar{\Upsilon}H\bar{\Upsilon}'\}\Lambda^{**'}(\sigma_1^2\phi).$$

(4)

$$-\mathbb{E}\{H\Lambda^{**'}\tilde{V}'\tilde{W}^*[\mathbb{E}(Z'Z)]^{-1}I_2\tilde{R}'\bar{\Upsilon}H\bar{\Upsilon}'\tilde{u}_1\} \tag{A.11}$$
$$= -\mathbb{E}\left\{H\Lambda^{**'}\tilde{V}'\left[\sum_{i=1}^{p}\sum_{t=i}^{T-1}D^t\tilde{V}J_{t-i}\Psi_i' : 0\right][\mathbb{E}(Z'Z)]^{-1}I_2\sum_{j=1}^{p}\sum_{s=j}^{T-1}\Psi_j J_{s-j}'\tilde{V}'D^{s'}\bar{\Upsilon}H\bar{\Upsilon}'\tilde{u}_1\right\}.$$

Here:

$$\mathbb{E}\left\{\tilde{V}'D^t\tilde{V}J_{t-i}\Psi_i' I_2'[\mathbb{E}(Z'Z)]^{-1}I_2\Psi_j J_{s-j}'\tilde{V}'D^{s'}\bar{\Upsilon}H\bar{\Upsilon}'\tilde{u}_1\right\}$$
$$= \mathbb{E}\{\phi\tilde{u}_1'D^t\tilde{u}_1\phi' J_{t-i}\Psi_i' I_2'[E(Z'Z)]^{-1}I_2\Psi_j J_{s-j}'\phi\tilde{u}_1'D^{s'}\bar{\Upsilon}H\bar{\Upsilon}'\tilde{u}_1\}$$
$$+ \mathbb{E}\{\phi\tilde{u}_1'D^tS^*J_{t-i}\Psi_i' I_2'[\mathbb{E}(Z'Z)]^{-1}I_2 v_j J_{s-j}'S^{*'}D^{s'}\bar{\Upsilon}H\bar{\Upsilon}'\tilde{u}_1\}$$
$$+ \mathbb{E}\{S^{*'}D^t\tilde{u}_1\phi' J_{t-i}\Psi_i' I_2'[\mathbb{E}(Z'Z)]^{-1}I_2\Psi_j J_{s-j}'S^{*'}D^{s'}\bar{\Upsilon}H\bar{\Upsilon}'\tilde{u}_1\}$$
$$+ \mathbb{E}\{S^{*'}D^tS^*J_{t-i}\Psi_i' I_2'[\mathbb{E}(Z'Z)]^{-1}I_2\Psi_j J_{s-j}'\phi\tilde{u}_1'D^{s'}\bar{\Upsilon}H\bar{\Upsilon}'\tilde{u}_1\}$$
$$= \sigma_1^4\phi\phi' J_{t-i}\Psi_i' I_2'[\mathbb{E}(Z'Z)]^{-1}I_2\Psi_j J_{s-j}'\phi tr\{(D^t + D^{t'})D^{s'}\bar{\Upsilon}H\bar{\Upsilon}'\}$$
$$+ \sigma_1^2 tr\{\Omega J_{t-i}\Psi_i' I_2'[\mathbb{E}(Z'Z)]^{-1}I_2\Psi_j J_{s-j}'\}tr\{D^tD^{s'}\bar{\Upsilon}H\bar{\Upsilon}'\}\phi$$
$$- \sigma_1^4 tr\{\phi J_{t-i}\Psi_\phi' i' I_2'[\mathbb{E}(Z'Z)]^{-1}I_2\Psi_j J_{s-j}'\}tr\{D^tD^{s'}\bar{\Upsilon}H\bar{\Upsilon}'\}\phi$$
$$+ \sigma_1^2 tr\{D^t\bar{\Upsilon}H\bar{\Upsilon}'D^s\}\Omega J_{s-j}\Psi_j' I_2'[\mathbb{E}(Z'Z)]^{-1}I_2 v_i J_{t-i}'\phi$$
$$- \sigma_1^4 tr\{D^t\bar{\Upsilon}H\bar{\Upsilon}'D^s\}\phi\phi' J_{s-j}\Psi_j' I_2'[\mathbb{E}(Z'Z)]^{-1}I_2\Psi_i J_{t-i}'\phi$$
$$+ 0$$
$$= \sigma_1^2 tr\{\Omega J_{t-i}\Psi_i' I_2'[\mathbb{E}(Z'Z)]^{-1}I_2\Psi_j J_{s-j}'\}tr\{D^tD^{s'}\bar{\Upsilon}H\bar{\Upsilon}'\}\phi$$
$$+ \sigma_1^2 tr\{D^t\bar{\Upsilon}H\bar{\Upsilon}'D^s\}\Omega J_{s-j}\Psi_j' I_2'[\mathbb{E}(Z'Z)]^{-1}I_2\Psi_i J_{t-i}'\phi.$$

Using Lemma 3.

The final expression for equation (A.11) is :

$$
\begin{aligned}
&-\mathbb{E}\{H\Lambda^{**'}\tilde{V}'\tilde{W}^*[\mathbb{E}(Z'Z)]^{-1}I_2\tilde{R}'\bar{\Upsilon}H\bar{\Upsilon}'\tilde{u}_1\} \qquad \text{(A.12)}\\
&= -H\sum_{i=1}^{p}\sum_{t=i}^{T-1}\sum_{j=1}^{p}\sum_{s=j}^{T-1}(tr\{\Omega J_{t-i}\Psi_i' I_2'[\mathbb{E}(Z'Z)]^{-1}I_2\Psi_j J_{s-j}'\}.I)\\
&\qquad \times (tr\{D^t D^{s'}\bar{\Upsilon}H\bar{\Upsilon}'\}.I)\Lambda^{**'}(\sigma_1^2\phi)\\
&\quad - H\Lambda^{**'}\sum_{i=1}^{p}\sum_{t=i}^{T-1}\sum_{j=1}^{p}\sum_{s=j}^{T-1}\Omega J_{s-j}\Psi_j' I_2'[\mathbb{E}(Z'Z)]^{-1}I_2\Psi_i J_{t-i}'(tr\{D^t\bar{\Upsilon}H\bar{\Upsilon}'D^s\}.I)(\sigma_1^2\phi).
\end{aligned}
$$

(5)

$$
\begin{aligned}
&-\mathbb{E}\left\{H\bar{\Upsilon}'\tilde{R}C^*H\bar{\Upsilon}'\tilde{u}_1\right\} \qquad \text{(A.13)}\\
&= -\mathbb{E}\left\{H\bar{\Upsilon}'\sum_{i=1}^{p}\sum_{t=i}^{T-1}D^t S^* J_{t-i}\Psi_i' C^* H\bar{\Upsilon}'\tilde{u}_1\right\} - \mathbb{E}\left\{H\bar{\Upsilon}'\sum_{i=1}^{p}\sum_{t=i}^{T-1}D^t\tilde{u}_1\phi' J_{t-i}\Psi_i' C^* H\bar{\Upsilon}'\tilde{u}_1\right\}\\
&= -\mathbb{E}\left\{H\bar{\Upsilon}'\sum_{i=1}^{p}\sum_{t=i}^{T-1}D^t\tilde{u}_1\tilde{u}_1'\bar{\Upsilon}H'C^{*'}\Psi_i J_{t-i}'\phi\right\}\\
&= -H\bar{\Upsilon}'\sum_{i=1}^{p}\sum_{t=i}^{T-1}D^t\bar{\Upsilon}H'C^{*'}\Psi_i J_{t-i}'(\sigma_1^2\phi).
\end{aligned}
$$

(6)

$$
\begin{aligned}
-\mathbb{E}\left\{HC^{*'}\tilde{R}'\bar{\Upsilon}H\bar{\Upsilon}'\tilde{u}_1\right\} &= -\mathbb{E}\left\{HC^{*'}\sum_{i=1}^{p}\sum_{t=i}^{T-1}\Psi_i J_{t-i}'\tilde{V}'D^{t'}\bar{\Upsilon}H\bar{\Upsilon}'\tilde{u}_1\right\} \qquad \text{(A.14)}\\
&= -\mathbb{E}\left\{H\sum_{i=1}^{p}\sum_{t=i}^{T-1}C^{*'}\Psi_i J_{t-i}'\phi\tilde{u}_1' D^{t'}\bar{\Upsilon}H\bar{\Upsilon}'\tilde{u}_1\right\}\\
&= -H\sum_{i=1}^{p}\sum_{t=i}^{T-1}C^{*'}\Psi_i J_{t-i}'(tr\{\bar{\Upsilon}'D^{t'}\bar{\Upsilon}H\}.I)(\sigma_1^2\phi).
\end{aligned}
$$

(7)

$$
-\mathbb{E}\{H\Lambda^{**'}\tilde{V}'\tilde{W}^*(\mathbb{E}(Z'Z))^{-1}\bar{Z}'\tilde{R}C^*H\bar{\Upsilon}'\tilde{u}_1\} - \mathbb{E}\{H\Lambda^{**'}\tilde{V}'\bar{Z}(\mathbb{E}(Z'Z))^{-1}I_2\tilde{R}'\tilde{R}C^*H\bar{\Upsilon}'\tilde{u}_1\}
$$

(A.15)

$$
\begin{aligned}
&= -\mathbb{E}\left\{H\Lambda^{**'}\tilde{V}'\left[\sum_{i=1}^{p}\sum_{t=i}^{T-1}D^t\tilde{V}J_{t-i}\Psi_i' : 0\right](\mathbb{E}(Z'Z))^{-1}\bar{Z}'\sum_{j=1}^{p}\sum_{s=j}^{T-1}D^s\tilde{V}J_{s-j}\Psi_j' C^* H\bar{\Upsilon}'\tilde{u}_1\right\}\\
&\quad - \mathbb{E}\left\{H\Lambda^{**'}\tilde{V}'\bar{Z}(\mathbb{E}(Z'Z))^{-1}I_2\left(\sum_{l=1}^{p}\sum_{r=l}^{T-1}D^r\tilde{V}J_{r-l}\Psi_l'\right)'\sum_{j=1}^{p}\sum_{s=j}^{T-1}D^s\tilde{V}J_{s-j}\Psi_j' C^* H\bar{\Upsilon}'\tilde{u}_1\right\}.
\end{aligned}
$$

This is calculated in two parts *(7a)* and *(7b)*:

(7a)

$$
\begin{aligned}
&\mathbb{E}\{\tilde{V}^{'}[D^t\tilde{V}J_{t-i}\Psi_i^{'}:0](\mathbb{E}(Z^{'}Z))^{-1}\bar{Z}^{'}D^s\tilde{V}J_{t-j}\Psi_j^{'}C^*H\bar{\Upsilon}^{'}\tilde{u}_1\}\\
&\quad=\mathbb{E}\{\phi\tilde{u}_1^{'}D^t\tilde{u}_1\phi^{'}J_{t-i}\Psi_i^{'}I_2^{'}(\mathbb{E}(Z^{'}Z))^{-1}\bar{Z}^{'}D^s\tilde{u}_1\phi^{'}J_{s-j}^{'}\Psi_jC^*H\bar{\Upsilon}^{'}\tilde{u}_1\}\\
&\qquad+\mathbb{E}\{\phi\tilde{u}_1^{'}D^tS^*J_{t-i}\Psi_i^{'}I_2^{'}(\mathbb{E}(Z^{'}Z))^{-1}\bar{Z}^{'}D^sS^*J_{s-j}\Psi_j^{'}C^*H\bar{\Upsilon}^{'}\tilde{u}_1\}\\
&\qquad+\mathbb{E}\{S^{*'}D^{t'}S^*J_{t-i}\Psi_i^{'}I_2^{'}(\mathbb{E}(Z^{'}Z))^{-1}\bar{Z}^{'}D^s\tilde{u}_1\phi^{'}J_{s-j}\Psi_j^{'}C^*H\bar{\Upsilon}^{'}\tilde{u}_1\}\\
&\qquad+\mathbb{E}\{S^{*'}D^t\tilde{u}_1\phi^{'}J_{t-i}\Psi_i^{'}I_2^{'}(\mathbb{E}(Z^{'}Z))^{-1}\bar{Z}^{'}D^sS^*J_{s-j}\Psi_j^{'}C^*H\bar{\Upsilon}^{'}\tilde{u}_1\}\\
&\quad=\phi\sigma_1^4tr\{(D^t+D^{t'})D^{s'}\bar{Z}(\mathbb{E}(Z^{'}Z))^{-1}I_2\Psi_iJ_{t-i}^{'}\phi\phi^{'}J_{s-j}\Psi_j^{'}C^*H\bar{\Upsilon}^{'}\}\\
&\qquad+\sigma_1^2\phi tr\{D^tD^{s'}\bar{Z}(\mathbb{E}(Z^{'}Z))^{-1}I_2\Psi_iJ_{t-i}^{'}\Omega J_{s-j}\Psi_j^{'}C^*H\bar{\Upsilon}^{'}\}\\
&\qquad-\sigma_1^4\phi tr\{D^tD^{s'}\bar{Z}(\mathbb{E}(Z^{'}Z))^{-1}I_2\Psi_iJ_{t-i}^{'}\phi\phi^{'}J_{s-j}\Psi_j^{'}C^*H\bar{\Upsilon}^{'}\}\\
&\qquad+0\\
&\qquad+\sigma_1^2\Omega(D^t\bar{\Upsilon}HC^{*'}\Psi_jJ_{s-j}^{'})^{'}D^{s'}\bar{Z}(\mathbb{E}(Z^{'}Z))^{-1}I_2\Psi_iJ_{t-i}^{'}\phi\\
&\qquad-\sigma_1^4\phi\phi^{'}(D^t\bar{\Upsilon}HC^{*'}\Psi_jJ_{s-j}^{'})^{'}D^{s'}\bar{Z}(\mathbb{E}(Z^{'}Z))^{-1}I_2\Psi_iJ_{t-i}^{'}\phi\\
&\quad=(tr\{D^tD^{s'}\bar{Z}(\mathbb{E}(Z^{'}Z))^{-1}I_2\Psi_iJ_{t-i}^{'}\Omega J_{s-j}\Psi_j^{'}cH\bar{\Upsilon}^{'}\}.I)(\sigma_1^2\phi)\\
&\qquad+\Omega J_{s-j}\Psi_j^{'}C^*H\bar{\Upsilon}^{'}D^{t'}D^{s'}\bar{Z}(\mathbb{E}(Z^{'}Z))^{-1}I_2\Psi_iJ_{t-i}^{'}(\sigma_1^2\phi).
\end{aligned}
$$

Using Lemma 3.

The final expression of the first part of equation (A.15) can be written as:

$$
\begin{aligned}
&-\mathbb{E}\{H\Lambda^{**'}\tilde{V}^{'}\tilde{W}^*(\mathbb{E}(Z^{'}Z))^{-1}\bar{Z}^{'}\tilde{R}C^*H\bar{\Upsilon}^{'}\tilde{u}_1\} \qquad \text{(A.16)}\\
&=-\sum_{i=1}^{p}\sum_{t=i}^{T-1}\sum_{j=1}^{p}\sum_{s=j}^{T-1}H(tr\{D^tD^{s'}\bar{Z}(\mathbb{E}(Z^{'}Z))^{-1}I_2\Psi_iJ_{t-i}^{'}\Omega J_{s-j}\Psi_j^{'}C^*H\bar{\Upsilon}^{'}\}.I)\Lambda^{**'}(\sigma_1^2\phi)\\
&\quad-\sum_{i=1}^{p}\sum_{t=i}^{T-1}\sum_{j=1}^{p}\sum_{s=j}^{T-1}H\Lambda^{**'}\Omega J_{s-j}\Psi_j^{'}C^*H\bar{\Upsilon}^{'}D^{t'}D^{s'}\bar{Z}(\mathbb{E}(Z^{'}Z))^{-1}I_2\Psi_iJ_{t-i}^{'}(\sigma_1^2\phi).
\end{aligned}
$$

(7b)

$$
\begin{aligned}
&-\mathbb{E}\left\{\tilde{V}^{'}\bar{Z}(E(Z^{'}Z))^{-1}I_2\Psi_lJ_{r-l}^{'}\tilde{V}'D^{r'}D^s\tilde{V}J_{s-j}\Psi_j^{'}C^*H\bar{\Upsilon}^{'}\tilde{u}_1\right\}\\
&\quad=-\mathbb{E}\left\{\phi u_1^{'}\bar{Z}(E(Z^{'}Z))^{-1}I_2\Psi_lJ_{r-l}^{'}\phi u_1^{'}D^{r'}D^su_1\phi'J_{s-j}\Psi_j^{'}C^*H\bar{\Upsilon}^{'}\tilde{u}_1\right\}\\
&\quad=-E\left\{\phi u_1^{'}\bar{Z}(\mathbb{E}(Z^{'}Z))^{-1}I_2\Psi_lJ_{r-l}^{'}S^{*'}D^{r'}D^sS^*J_{s-j}\Psi_j^{'}C^*H\bar{\Upsilon}^{'}\tilde{u}_1\right\}
\end{aligned}
$$

$$= -E\left\{S^{*'}\bar{Z}(\mathbb{E}(Z'Z))^{-1}I_2\Psi_l J'_{r-l}\phi u'_1 D^{r'}D^s S^* J_{s-j}\Psi'_j C^* H\bar{\Upsilon}'\tilde{u}_1\right\}$$

$$= -\mathbb{E}\left\{S^{*'}\bar{Z}(\mathbb{E}(Z'Z))^{-1}I_2\Psi_l J'_{r-l}S^{*'}D^{r'}D^s u_1\phi' J_{s-j}\Psi'_j C^* H\bar{\Upsilon}'\tilde{u}_1\right\}$$

$$= -\phi\sigma_1^2 tr\left\{\bar{Z}(\mathbb{E}(Z'Z))^{-1}I_2\Psi_l J'_{r-l}\Omega J_{s-j}\Psi'_j C^* H\bar{\Upsilon}'\right\} tr\left\{D^{r'}D^s\right\}$$

$$- \Omega J_{s-j}\Psi'_j C^* H\bar{\Upsilon}' D^{r'}D^s\bar{Z}(\mathbb{E}(Z'Z))^{-1}I_2\Psi_l J'_{r-l}\phi\sigma_1^2$$

$$- \Omega J_{r-l}\Psi'_l I'_2(\mathbb{E}(Z'Z))^{-1}\bar{Z}' D^{r'}D^s\bar{\Upsilon}HC^{*'}\Psi_j J'_{s-j}\phi\sigma_1^2.$$

The final expression of the second part of equation (A.15) can be written as:

$$\begin{aligned}
&- \mathbb{E}\{H\Lambda^{**'}\tilde{V}'\bar{Z}(\mathbb{E}(Z'Z))^{-1}I_2\tilde{R}'\tilde{R}C^* H\bar{\Upsilon}'\tilde{u}_1\} \\
&= -H\sum_{l=1}^{p}\sum_{j=1}^{p}\sum_{r=l,j}^{T-1}(T-r)\left(tr\left\{\bar{Z}(\mathbb{E}(Z'Z))^{-1}I_2\Psi_l J'_{r-l}\Omega J_{s-j}\Psi'_j C^* H\bar{\Upsilon}'\right\}.I\right)\vartheta \\
&- H\Lambda^{**}\sum_{l=1}^{p}\sum_{r=l}^{T-1}\sum_{j=1}^{p}\sum_{s=j}^{T-1}\Omega J_{s-j}\Psi'_j C^* H\bar{\Upsilon}' D^{r'}D^s\bar{Z}(\mathbb{E}(Z'Z))^{-1}I_2\Psi_l J'_{r-l}\phi\sigma_1^2 \\
&- H\Lambda^{**}\sum_{l=1}^{p}\sum_{r=l}^{T-1}\sum_{j=1}^{p}\sum_{s=j}^{T-1}\Omega J_{r-l}\Psi'_l I'_2(\mathbb{E}(Z'Z))^{-1}\bar{Z}' D^{r'}D^s\bar{\Upsilon}HC^{*'}\Psi_j J'_{s-j}\phi\sigma_1^2.
\end{aligned} \quad (A.17)$$

(8)

$$- \mathbb{E}\{HC^{*'}\tilde{R}'\bar{Z}(\mathbb{E}(Z'Z))^{-1}\tilde{W}^{*'}\tilde{V}\Lambda^{**}H\bar{\Upsilon}'\tilde{u}_1\} - \mathbb{E}\{HC^{*'}\tilde{R}'\tilde{R}I'_2(\mathbb{E}(Z'Z))^{-1}\bar{Z}\tilde{V}\Lambda^{**}H\bar{\Upsilon}'\tilde{u}_1\}. \quad (A.18)$$

Equation (A.18) can be written as the sum of two parts ($8a$) and ($8b$):

$$\begin{aligned}
&(8a) \quad (A.19) \\
&- \mathbb{E}\{HC^{*'}\tilde{R}'\bar{Z}(\mathbb{E}(Z'Z))^{-1}\tilde{W}^{*'}\tilde{V}\Lambda^{**}H\bar{\Upsilon}'\tilde{u}_1\} \\
&= -\mathbb{E}\left\{H\sum_{i=1}^{p}\sum_{t=i}^{T-1}C^{*'}\Psi_i J'_{t-i}\tilde{V}' D^{t'}\bar{Z}(\mathbb{E}(Z'Z))^{-1}\begin{bmatrix}\sum_{i=1}^{p}\sum_{s=j}^{T-1}\Psi_j J'_{s-j}\tilde{V}' D^{s'} \\ 0'\end{bmatrix}\right. \\
&\qquad\left.\times\tilde{V}\Lambda^{**}H\bar{\Upsilon}'\tilde{u}_1\right\}.
\end{aligned}$$

Here,

$$\mathbb{E}\left\{\tilde{V}' D^{t'}\bar{Z}(\mathbb{E}(Z'Z))^{-1}\begin{bmatrix}\Psi_j J'_{s-j} \\ 0\end{bmatrix}\tilde{V}' D^{s'}\tilde{V}\Lambda^{**}H\bar{\Upsilon}'\tilde{u}_1\right\}$$

$$
\begin{aligned}
&= \mathbb{E}\left\{\phi\tilde{u}_1^{'}D^{t'}\bar{Z}(\mathbb{E}(Z^{'}Z))^{-1}I_2\Psi_jJ_{s-j}^{'}\phi\tilde{u}_1^{'}D^{s'}\tilde{u}_1\phi^{'}\Lambda^{**}H\bar{\Upsilon}^{'}\tilde{u}_1\right\}\\
&\quad + \mathbb{E}\left\{\phi\tilde{u}_1^{'}D^{t'}\bar{Z}(\mathbb{E}(Z^{'}Z))^{-1}I_2\Psi_jJ_{s-j}^{'}S^{*'}D^{s'}S^{*}\Lambda^{**}H\bar{\Upsilon}^{'}\tilde{u}_1\right\}\\
&\quad + \mathbb{E}\left\{S^{*'}D^{t'}\bar{Z}(\mathbb{E}(Z^{'}Z))^{-1}I_2\Psi_jJ_{s-j}^{'}S^{*'}D^{s'}\tilde{u}_1\phi^{'}\Lambda^{**}H\bar{\Upsilon}^{'}\tilde{u}_1\right\}\\
&\quad + \mathbb{E}\left\{S^{*'}D^{t'}\bar{Z}(\mathbb{E}(Z^{'}Z))^{-1}I_2\Psi_jJ_{s-j}^{'}\phi\tilde{u}_1^{'}D^{s'}S^{*}\Lambda^{**}H\bar{\Upsilon}^{'}\tilde{u}_1\right\}\\
&= \sigma_1^4\phi\phi^{'}J_{s-j}\Psi_j^{'}I_2^{'}(\mathbb{E}(Z^{'}Z))^{-1}\bar{Z}^{'}D^tD^{s'}\bar{\Upsilon}H\Lambda^{**'}\phi\\
&\quad + \sigma_1^4\phi\phi^{'}\Lambda^{**}H\bar{\Upsilon}D^{s'}D^{t'}\bar{Z}(\mathbb{E}(Z^{'}Z))^{-1}I_2\Psi_jJ_{s-j}^{'}\phi\\
&\quad + 0\\
&\quad + \sigma_1^2\Omega J_{s-j}\Psi_j^{'}I_2^{'}(\mathbb{E}(Z^{'}Z))^{-1}\bar{Z}^{'}D^tD^{s'}\bar{\Upsilon}H\Lambda^{**'}\phi\\
&\quad - \sigma_1^4\phi\phi^{'}J_{s-j}\Psi_j^{'}I_2^{'}(\mathbb{E}(Z^{'}Z))^{-1}\bar{Z}^{'}D^tD^{s'}\bar{\Upsilon}H\Lambda^{**'}\phi\\
&\quad + \sigma_1^2\Omega\Lambda^{**}H\bar{\Upsilon}D^{s'}D^{t'}\bar{Z}(\mathbb{E}(Z^{'}Z))^{-1}I_2\Psi_jJ_{s-j}^{'}\phi\\
&\quad - \sigma_1^4\phi\phi^{'}\Lambda^{**}H\bar{\Upsilon}D^{s'}D^{t'}\bar{Z}(\mathbb{E}(Z^{'}Z))^{-1}I_2\Psi_jJ_{s-j}^{'}\phi\\
&= \Omega J_{s-j}\Psi_j^{'}I_2^{'}(\mathbb{E}(Z^{'}Z))^{-1}\bar{Z}^{'}D^tD^{s'}\bar{\Upsilon}H\Lambda^{**'}(\sigma_1^2\phi)\\
&\quad + \Omega\Lambda^{**}H\bar{\Upsilon}^{'}D^{s'}D^{t'}\bar{Z}(\mathbb{E}(Z^{'}Z))^{-1}I_2\Psi_jJ_{s-j}^{'}(\sigma_1^2\phi).
\end{aligned}
$$

Using Lemma 3 and 4.

The final result for equation (A.19) is:

$$
\begin{aligned}
&-\mathbb{E}\{HC^{*'}\tilde{R}^{'}\bar{Z}(\mathbb{E}(Z^{'}Z))^{-1}\tilde{W}^{*'}\tilde{V}\Lambda^{**}H\bar{\Upsilon}^{'}\tilde{u}_1\} \qquad \text{(A.20)}\\
&= -H\sum_{i=1}^{p}\sum_{t=i}^{T-1}\sum_{j=1}^{p}\sum_{s=j}^{T-1}C^{*'}\Psi_iJ_{t-i}^{'}\Omega J_{s-j}\Psi_j^{'}I_2^{'}(\mathbb{E}(Z^{'}Z))^{-1}\bar{Z}^{'}D^tD^{s'}\bar{\Upsilon}H\Lambda^{**'}(\sigma_1^2\phi)\\
&\quad - H\sum_{i=1}^{p}\sum_{t=i}^{T-1}\sum_{j=1}^{p}\sum_{s=j}^{T-1}C^{*'}\Psi_iJ_{t-i}^{'}\Omega\Lambda^{**}H\bar{\Upsilon}D^{r'}D^{t'}\bar{Z}(\mathbb{E}(Z^{'}Z))^{-1}I_2\Psi_jJ_{s-j}^{'}(\sigma_1^2\phi).
\end{aligned}
$$

(8b)

$$
\begin{aligned}
&-\mathbb{E}\{HC^{*'}\tilde{R}^{'}\tilde{R}I_2^{'}(\mathbb{E}(Z^{'}Z))^{-1}\bar{Z}^{'}\tilde{V}\Lambda^{**}H\bar{\Upsilon}^{'}\tilde{u}_1\} \qquad \text{(A.21)}\\
&= -\mathbb{E}\left\{H\sum_{i=1}^{p}\sum_{t=i}^{T-1}C^{*'}\Psi_iJ_{t-i}^{'}\tilde{V}^{'}D^{t'}\sum_{j=1}^{p}\sum_{s=j}^{T-1}D^s\tilde{V}J_{s-j}\Psi_j^{'}I_2^{'}(\mathbb{E}(Z^{'}Z))^{-1}\bar{Z}^{'}\tilde{V}\Lambda^{**}H\bar{\Upsilon}^{'}\tilde{u}_1\right\}.
\end{aligned}
$$

Here,

$$
\mathbb{E}\left\{\tilde{V}^{'}D^{t'}D^s\tilde{V}J_{s-j}\Psi_j^{'}I_2^{'}(\mathbb{E}(Z^{'}Z))^{-1}\bar{Z}^{'}\tilde{V}\Lambda^{**}H\bar{\Upsilon}^{'}\tilde{u}_1\right\}
$$

$$
\begin{aligned}
&= \mathbb{E}\{\phi\tilde{u}_1^{'}D^{t'}D^s\tilde{u}_1\phi^{'}J_{s-j}\Psi_j^{'}I_2^{'}(\mathbb{E}(Z^{'}Z))^{-1}\bar{Z}^{'}\tilde{u}_1\phi^{'}\Lambda^{**}H\bar{\Upsilon}^{'}\tilde{u}_1\} \\
&\quad + \mathbb{E}\{\phi\tilde{u}_1^{'}D^{t'}D^sS^*J_{s-j}\Psi_j^{'}I_2^{'}(\mathbb{E}(Z^{'}Z))^{-1}\bar{Z}^{'}S^*\Lambda^{**}H\bar{\Upsilon}^{'}\tilde{u}_1\} \\
&\quad + \mathbb{E}\{S^{*'}D^{t'}D^sS^*J_{s-j}\Psi_j^{'}I_2^{'}(\mathbb{E}(Z^{'}Z))^{-1}\bar{Z}^{'}\tilde{u}_1\phi^{'}\Lambda^{**}H\bar{\Upsilon}^{'}\tilde{u}_1\} \\
&\quad + \mathbb{E}\{S^{*'}D^{t'}D^s\tilde{u}_1\phi^{'}J_{s-j}\Psi_j^{'}I_2^{'}(\mathbb{E}(Z^{'}Z))^{-1}\bar{Z}^{'}S^*\Lambda^{**}H\bar{\Upsilon}^{'}\tilde{u}_1\} \\
&= \sigma_1^4\phi tr\{\frac{1}{2}(D^{t'}D^s + D^{s'}D^t)\}tr\{\bar{\Upsilon}H\Lambda^{**'}\phi\phi^{'}J_{s-j}\Psi_j^{'}I_2^{'}(\mathbb{E}(Z^{'}Z))^{-1}\bar{Z}^{'}\} \\
&\quad + 2\sigma_1^4\phi tr\{\frac{1}{2}(D^{t'}D^s + D^{s'}D^t)\}\bar{\Upsilon}H\Lambda^{**'}\phi\phi^{'}J_{s-j}\Psi_j^{'}I_2^{'}(\mathbb{E}(Z^{'}Z))^{-1}\bar{Z}^{'} \\
&\quad + \sigma_1^2\phi tr\{D^{t'}D^s\bar{Z}(\mathbb{E}(Z^{'}Z))^{-1}I_2\Psi_jJ_{s-j}^{'}\Omega\Lambda^{**}H\bar{\Upsilon}^{'}\} \\
&\quad - \sigma_1^4\phi tr\{D^{t'}D^s\bar{Z}(\mathbb{E}(Z^{'}Z))^{-1}I_2\Psi_jJ_{s-j}^{'}\phi\phi^{'}\Lambda^{**}H\bar{\Upsilon}^{'}\} \\
&\quad + \sigma_1^2 tr\{D^{t'}D^r\}\Omega J_{s-j}\Psi_j^{'}I_2^{'}(\mathbb{E}(Z^{'}Z))^{-1}\bar{Z}^{'}\bar{\Upsilon}H\Lambda^{**'}\phi \\
&\quad - \sigma_1^4 tr\{D^{t'}D^r\}\phi\phi^{'}J_{s-j}\Psi_j^{'}I_2^{'}(\mathbb{E}(Z^{'}Z))^{-1}\bar{Z}^{'}\bar{\Upsilon}H\Lambda^{**'}\phi \\
&\quad + \sigma_1^2\Omega\Lambda^{**}H\bar{\Upsilon}^{'}D^{s'}D^t\bar{Z}(\mathbb{E}(Z^{'}Z))^{-1}I_2\Psi_jJ_{s-j}^{'}\phi \\
&\quad - \sigma_1^4\phi\phi^{'}\Lambda^{**}H\bar{\Upsilon}^{'}D^{s'}D^t\bar{Z}(\mathbb{E}(Z^{'}Z))^{-1}I_2\Psi_jJ_{s-j}^{'}\phi \\
&= (tr\{D^{t'}D^s\bar{Z}(\mathbb{E}(Z^{'}Z))^{-1}I_2\Psi_jJ_{s-j}^{'}\Omega\Lambda^{**}H\bar{\Upsilon}^{'}\}.I)(\sigma_1^2\phi) \\
&+ \Omega J_{s-j}\Psi_j^{'}I_2^{'}(\mathbb{E}(Z^{'}Z))^{-1}\bar{Z}^{'}\bar{\Upsilon}H(tr\{D^{t'}D^r\}.I)\Lambda^{**'}(\sigma_1^2\phi) \\
&+ \Omega\Lambda^{**}H\bar{\Upsilon}^{'}D^{s'}D^t\bar{Z}(\mathbb{E}(Z^{'}Z))^{-1}I_2\Psi_jJ_{s-j}^{'}(\sigma_1^2\phi).
\end{aligned}
$$

Using Lemma 4.

The final result for equation (A.21) is

$$
\begin{aligned}
&- \mathbb{E}\{HC^{*'}\tilde{R}^{'}\tilde{R}I_2^{'}(\mathbb{E}(Z^{'}Z))^{-1}\bar{Z}^{'}\tilde{V}\Lambda^{**}H\bar{\Upsilon}^{'}\tilde{u}_1\} \qquad \text{(A.22)} \\
&\quad = -H\sum_{i=1}^{p}\sum_{t=i}^{T-1}\sum_{j=1}^{p}\sum_{s=j}^{T-1}C^{*'}\Psi_iJ_{t-i}^{'}(tr\{D^{t'}D^s\bar{Z}(\mathbb{E}(Z^{'}Z))^{-1}I_2e_jJ_{s-j}^{'}\Omega\Lambda^{**}H\bar{\Upsilon}^{'}\}.I)(\sigma_1^2\phi) \\
&\quad\quad - H\sum_{i=1}^{p}\sum_{t=i}^{T-1}\sum_{j=1}^{p}\sum_{s=j}^{T-1}C^{*'}\Psi_iJ_{t-i}^{'}\Omega J_{s-j}\Psi_j^{'}I_2^{'}(\mathbb{E}(Z^{'}Z))^{-1}\bar{Z}^{'}\bar{\Upsilon}H(tr\{D^{t'}D^r\}.I)\Lambda^{**'}(\sigma_1^2\phi) \\
&\quad\quad - H\sum_{i=1}^{p}\sum_{t=i}^{T-1}\sum_{j=1}^{p}\sum_{s=j}^{T-1}C^{*'}\Psi_iJ_{t-i}^{'}\Omega\Lambda^{**}H\bar{\Upsilon}^{'}D^{s'}D^t\bar{Z}(\mathbb{E}(Z^{'}Z))^{-1}I_2\Psi_jJ_{s-j}^{'}(\sigma_1^2\phi).
\end{aligned}
$$

Therefore, by combining equations (A.6), (A.9), (A.10), (A.12), (A.13), (A.14), (A.16), (A.17),(A.20), (A.22), we can get the final expression for (iv).

$$
(v) \qquad - \mathbb{E}\{HJ_1^*H\Delta_2^{'}\tilde{u}_1\} = - \mathbb{E}\{H\bar{\Upsilon}^{'}\bar{Z}[\mathbb{E}(Z^{'}Z)]^{-1}\tilde{W}^{*'}\tilde{V}\Lambda^{**}HC^{*'}\tilde{R}^{'}\tilde{u}_1\}
$$

$$- \mathbb{E}\{H\bar{\Upsilon}'\tilde{R}I_2'[\mathbb{E}(Z'Z)]^{-1}\bar{Z}'\tilde{V}\Lambda^{**}HC^{*'}\tilde{R}'\tilde{u}_1\}$$
$$- \mathbb{E}\{H\Lambda^{**'}\tilde{V}'\tilde{W}^{*}[\mathbb{E}(Z'Z)]^{-1}\bar{Z}'\bar{\Upsilon}HC^{*'}\tilde{R}'\tilde{u}_1\}$$
$$- \mathbb{E}\{H\Lambda^{**'}\tilde{V}'\bar{Z}[\mathbb{E}(Z'Z)]^{-1}I_2\tilde{R}'\bar{\Upsilon}HC^{*'}\tilde{R}'\tilde{u}_1\}$$
$$- \mathbb{E}\{HC^{*'}\tilde{R}'\tilde{R}I_2'[\mathbb{E}(Z'Z)]^{-1}\tilde{W}^{*'}\tilde{V}\Lambda^{**}HC^{*'}\tilde{R}'\tilde{u}_1\}$$
$$- \mathbb{E}\{HC^{*'}\tilde{R}'\bar{Z}[\mathbb{E}(Z'Z)]^{-1}\bar{Z}'\tilde{V}\Lambda^{**}HC^{*'}\tilde{R}'\tilde{u}_1\}$$
$$- \mathbb{E}\{H\Lambda^{**'}\tilde{V}'\tilde{W}^{*}(\mathbb{E}(Z'Z))^{-1}I_2\tilde{R}'\tilde{R}C^{*}HC^{*'}\tilde{R}'\tilde{u}_1\}$$
$$- \mathbb{E}\{H\Lambda^{**'}\tilde{V}'\bar{Z}(\mathbb{E}(Z'Z))^{-1}\bar{Z}'\tilde{R}C^{*}HC^{*'}\tilde{R}'\tilde{u}_1\}$$
$$- \mathbb{E}\{HC^{*'}\tilde{R}'\tilde{R}C^{*}HC^{*'}\tilde{R}'\tilde{u}_1\},$$

where the definition of Δ_1 and Δ_2 is from equation 10.

Then v can be obtained from the sum of $(1') - (9')$ below:

$(1')$

$$\begin{aligned} &- \mathbb{E}\{H\bar{\Upsilon}'\bar{Z}[\mathbb{E}(Z'Z)]^{-1}\tilde{W}^{*'}\tilde{V}\Lambda^{**}HC^{*'}\tilde{R}'\tilde{u}_1\} \\ &= -\mathbb{E}\left\{H\bar{\Upsilon}'\bar{Z}[\mathbb{E}(Z'Z)]^{-1}\begin{bmatrix}\sum_{i=1}^{p}\sum_{t=i}^{T-1}\Psi_i J_{t-i}' \\ 0\end{bmatrix}\tilde{V}'D^{t'}\tilde{V}\Lambda^{**}\right. \\ &\qquad \left.\times H\sum_{j=1}^{p}\sum_{s=j}^{T-1}C^{*'}\Psi_j J_{s-j}'\tilde{V}'D^{s'}\tilde{u}_1\right\}. \end{aligned} \tag{A.23}$$

Then, equation (A.23) can be calculated from:

$$\begin{aligned} &\mathbb{E}\left\{\tilde{V}'D^{t'}\tilde{V}\Lambda^{**}HC^{*'}\Psi_j J_{s-j}'\tilde{V}'D^{s'}\tilde{u}_1\right\} \\ &\quad = \mathbb{E}\{\phi\tilde{u}_1'D^{t'}\tilde{u}_1\phi'\Lambda^{**}HC^{*'}\Psi_j J_{s-j}'\phi\tilde{u}_1'D^{s'}\tilde{u}_1\} \\ &\qquad + \mathbb{E}\{\phi\tilde{u}_1'D^{t'}S^{*}\Lambda^{**}HC^{*'}\Psi_j J_{s-j}'S^{*'}D^{s'}\tilde{u}_1\} \\ &\qquad + \mathbb{E}\{S^{*'}D^{t'}S^{*}\Lambda^{**}HC^{*'}\Psi_j J_{s-j}'\phi\tilde{u}_1'D^{s'}\tilde{u}_1\} \\ &\qquad + \mathbb{E}\{S^{*'}D^{t'}\tilde{u}_1\phi'\Lambda^{**}HC^{*'}\Psi_j J_{s-j}'S^{*'}D^{s'}\tilde{u}_1\} \\ &\quad = \sigma_1^4\phi\phi'\Lambda^{**}HC^{*'}\Psi_j J_{s-j}'\phi tr\{(D^t + D^{t'})D^{s'}\} \\ &\qquad + 0 \\ &\qquad + 0 \end{aligned}$$

$$
\begin{aligned}
&+ \sigma_1^2 tr\{D^{t'} D^s\}\Omega J_{s-j}\Psi_j' C^* H\Lambda^{**'}\phi \\
&- \sigma_1^4 tr\{D^{t'} D^s\}\phi\phi' J_{s-j}\Psi_j' C^* H\Lambda^{**'}\phi \\
&= (tr\{D^{t'} D^s\}.I)\Omega J_{s-j}\Psi_j' C^* H\Lambda^{**'}(\phi\sigma_1^2).
\end{aligned}
$$

Then, the final result for equation (A.23) is:

$$
\begin{aligned}
&-\mathbb{E}\{H\bar{\Upsilon}'\bar{Z}[\mathbb{E}(Z'Z)]^{-1}\tilde{W}^{*'}\tilde{V}\Lambda^{**}HC^{*'}\tilde{R}'\tilde{u}_1\} \qquad (A.24)\\
&= -H\bar{\Upsilon}'\bar{Z}I_2[\mathbb{E}(Z'Z)]^{-1}\sum_{i=1}^{p}\sum_{t=i}^{T-1}\sum_{j=1}^{p}\sum_{s=j}^{T-1}\Psi_i J_{t-i}'(tr\{D^{t'}D^s\}.I)\Omega J_{s-j}\Psi_j' C^* H\Lambda^{**'}(\phi\sigma_1^2).
\end{aligned}
$$

(2′)

$$
\begin{aligned}
&-\mathbb{E}\{H\bar{\Upsilon}'\tilde{R}I_2'[\mathbb{E}(Z'Z)]^{-1}\bar{Z}'\tilde{V}\Lambda^{**}HC^{*'}\tilde{R}'\tilde{u}_1\} \qquad (A.25)\\
&= -\mathbb{E}\{H\bar{\Upsilon}'\sum_{i=1}^{p}\sum_{t=i}^{T-1}D^t\tilde{V}J_{t-i}\Psi_i'I_2'[\mathbb{E}(Z'Z)]^{-1}\bar{Z}'\tilde{V}\Lambda^{**}HC^{*'}\sum_{j=1}^{p}\sum_{s=j}^{T-1}\Psi_j J_{s-j}\tilde{V}'D^{t'}\tilde{u}_1\}.
\end{aligned}
$$

Then, equation (A.25) can be calculated from:

$$
\begin{aligned}
&-\mathbb{E}\{\tilde{V}J_{t-i}\Psi_i'I_2'[\mathbb{E}(Z'Z)]^{-1}\bar{Z}'\tilde{V}\Lambda^{**}HC^{*'}\Psi_j J_{s-j}\tilde{V}'D^{s'}\tilde{u}_1\}\\
&= -\mathbb{E}\{u_1\phi' J_{t-i}\Psi_i'I_2'[\mathbb{E}(Z'Z)]^{-1}\bar{Z}'u_1\phi'\Lambda^{**}HC^{*'}\Psi_j J_{s-j}\phi u_1' D^{s'}\tilde{u}_1\}\\
&\quad -\mathbb{E}\{S^* J_{t-i}\Psi_i'I_2'[\mathbb{E}(Z'Z)]^{-1}\bar{Z}'S^*\Lambda^{**}HC^{*'}\Psi_j J_{s-j}\phi u_1' D^{s'}\tilde{u}_1\}\\
&\quad -\mathbb{E}\{S^*\phi' J_{t-i}\Psi_i'I_2'[\mathbb{E}(Z'Z)]^{-1}\bar{Z}'u_1\phi'\Lambda^{**}HC^{*'}\Psi_j J_{s-j}S^{*'}D^{s'}\tilde{u}_1\}\\
&\quad -\mathbb{E}\{u_1\phi' J_{t-i}\Psi_i'I_2'[\mathbb{E}(Z'Z)]^{-1}\bar{Z}'S^*\Lambda^{**}HC^{*'}\Psi_j J_{s-j}S^{*'}D^{s'}\tilde{u}_1\}\\
&= -D^s tr\left\{\Omega J_{t-i}\Psi_i' C^* H\Lambda^{**'}\right\} I\bar{Z}[\mathbb{E}(Z'Z)]^{-1}I_2\Psi_j J_{s-j}'\phi\sigma_1^2\\
&\quad - D^{s'}\bar{Z}[\mathbb{E}(Z'Z)]^{-1}I_2\Psi_j J_{s-j}'\Omega J_{t-i}\Psi_i' C^* H\Lambda^{**'}\phi\sigma_1^2.
\end{aligned}
$$

Then, the final expression of equation (A.25) is:

$$
\begin{aligned}
&-\mathbb{E}\{H\bar{\Upsilon}'\tilde{R}I_2'[\mathbb{E}(Z'Z)]^{-1}\bar{Z}'\tilde{V}\Lambda^{**}HC^{*'}\tilde{R}'\tilde{u}_1\} \qquad (A.26)\\
&= -H\bar{\Upsilon}'\sum_{i=1}^{p}\sum_{t=i}^{T-1}\sum_{j=1}^{p}\sum_{s=j}^{T-1}D^t D^s tr\left\{\Omega J_{t-i}\Psi_i' C^* H\Lambda^{**'}\right\} I\bar{Z}[\mathbb{E}(Z'Z)]^{-1}I_2\Psi_j J_{s-j}'\phi\sigma_1^2\\
&\quad - H\bar{\Upsilon}'\sum_{i=1}^{p}\sum_{t=i}^{T-1}\sum_{j=1}^{p}\sum_{s=j}^{T-1}D^t D^{s'}\bar{Z}[\mathbb{E}(Z'Z)]^{-1}I_2\Psi_j J_{s-j}'\Omega J_{t-i}\Psi_i' C^* H\Lambda^{**'}\phi\sigma_1^2.
\end{aligned}
$$

(3′)

$$-\mathbb{E}\{H\Lambda^{**'}\tilde{V}'\tilde{W}^*[\mathbb{E}(Z'Z)]^{-1}\bar{Z}'\bar{\Upsilon}HC^{*'}\tilde{R}'\tilde{u}_1\} \tag{A.27}$$
$$= -\mathbb{E}\{H\Lambda^{**'}\tilde{V}'\left[\sum_{i=1}^{p}\sum_{t=i}^{T-1} D^t\tilde{V}J_{t-i}\Psi_i' : 0\right][\mathbb{E}(Z'Z)]^{-1}\bar{Z}'\bar{\Upsilon}H\sum_{j=1}^{p}\sum_{s=j}^{T-1} C^{*'}\Psi_j J_{s-j}'\tilde{V}'D^{s'}\tilde{u}_1\}.$$

Here,

$$\begin{aligned}
&\mathbb{E}\{\tilde{V}'D^t\tilde{V}J_{t-i}\Psi_i'I_2'[\mathbb{E}(Z'Z)]^{-1}\bar{Z}'\bar{\Upsilon}HC^{*'}\Psi_j J_{s-j}'\tilde{V}'D^{s'}\tilde{u}_1\}\\
&\quad= \mathbb{E}\{\phi\tilde{u}_1'D^t\tilde{u}_1\phi'J_{t-i}\Psi_i'I_2'[\mathbb{E}(Z'Z)]^{-1}\bar{Z}'\bar{\Upsilon}HC^{*'}\Psi_j J_{s-j}'\phi\tilde{u}_1'D^{s'}\tilde{u}_1\}\\
&\qquad+ \mathbb{E}\{\phi\tilde{u}_1'D^tS^*J_{t-i}\Psi_i'I_2'[\mathbb{E}(Z'Z)]^{-1}\bar{Z}'\bar{\Upsilon}HC^{*'}\Psi_j J_{s-j}'S^{*'}D^{s'}\tilde{u}_1\}\\
&\qquad+ \mathbb{E}\{S^{*'}D^tS^*J_{t-i}\Psi_i'I_2'[\mathbb{E}(Z'Z)]^{-1}\bar{Z}'\bar{\Upsilon}HC^{*'}\Psi_j J_{s-j}'\phi\tilde{u}_1'D^{s'}\tilde{u}_1\}\\
&\qquad+ \mathbb{E}\{S^{*'}D^t\tilde{u}_1\phi'J_{t-i}\Psi_i'I_2'[\mathbb{E}(Z'Z)]^{-1}\bar{Z}'\bar{\Upsilon}HC^{*'}\Psi_j J_{s-j}'S^{*'}D^{s'}\tilde{u}_1\}\\
&\quad= \sigma_1^4\phi\phi'J_{t-i}\Psi_i'I_2'[\mathbb{E}(Z'Z)]^{-1}\bar{Z}'\bar{\Upsilon}HC^{*'}\Psi_j J_{s-j}'\phi tr\{\frac{1}{2}(D^t+D^{t'})D^{r'}\}+\\
&\qquad+0\\
&\qquad+0\\
&\qquad+\sigma_1^2\phi tr\{D^tD^{s'}\}tr\{\Omega J_{t-i}\Psi_i'I_2'[\mathbb{E}(Z'Z)]^{-1}\bar{Z}'\bar{\Upsilon}HC^{*'}\Psi_j J_{s-j}'\}\\
&\qquad-\sigma_1^4\phi tr\{D^tD^{s'}\}tr\{\phi\phi'J_{t-i}\Psi_i'I_2'[\mathbb{E}(Z'Z)]^{-1}\bar{Z}'\bar{\Upsilon}HC^{*'}\Psi_j J_{s-j}'\}\\
&\quad= tr\{D^tD^{s'}\}tr\{\Omega J_{t-i}\Psi_i'I_2'[\mathbb{E}(Z'Z)]^{-1}\bar{Z}'\bar{\Upsilon}HC^{*'}\Psi_j J_{s-j}'\}(\sigma_1^2\phi).
\end{aligned}$$

Then, the final result for equation (A.27) is:

$$\begin{aligned}
&-\mathbb{E}\{H\Lambda^{**'}\tilde{V}'\tilde{W}^*[\mathbb{E}(Z'Z)]^{-1}\bar{Z}'\bar{\Upsilon}HC^{*'}\tilde{R}'\tilde{u}_1\} \qquad (A.28)\\
&\quad= -H\sum_{i=1}^{p}\sum_{t=i}^{T-1}\sum_{j=1}^{p}\sum_{s=j}^{T-1} tr\{D^tD^{r'}\}(tr\{\Omega J_{t-i}\Psi_i'I_2'[\mathbb{E}(Z'Z)]^{-1}\bar{Z}'\bar{\Upsilon}HC^{*'}\Psi_j J_{s-j}'\}.I)\\
&\qquad\qquad\times\Lambda^{**'}(\sigma_1^2\phi).
\end{aligned}$$

(4′)

$$-\mathbb{E}\{H\Lambda^{**'}\tilde{V}'\bar{Z}[\mathbb{E}(Z'Z)]^{-1}I_2\tilde{R}'\bar{\Upsilon}HC^{*'}\tilde{R}'\tilde{u}_1\} \tag{A.29}$$
$$= -\mathbb{E}\{H\Lambda^{**'}\tilde{V}'\bar{Z}[\mathbb{E}(Z'Z)]^{-1}I_2\sum_{i=1}^{p}\sum_{t=i}^{T-1}\Psi_i J_{t-i}'\tilde{V}'D^{t'}\bar{\Upsilon}HC^{*'}\sum_{j=1}^{p}\sum_{s=j}^{T-1}\Psi_j J_{s-j}'\tilde{V}'D^{s'}\tilde{u}_1\}.$$

Then, equation (A.29) can be calculated from:

$$
\begin{aligned}
& -\mathbb{E}\{\tilde{V}'\bar{Z}[\mathbb{E}(Z'Z)]^{-1}I_2\Psi_i J'_{t-i}\tilde{V}'D^{t'}\bar{\Upsilon}HC^{*'}\Psi_j J'_{s-j}\tilde{V}'D^{s'}\tilde{u}_1\} \\
&\quad = -\mathbb{E}\{\phi\tilde{u}_1'\bar{Z}[\mathbb{E}(Z'Z)]^{-1}I_2\Psi_i J'_{t-i}\phi\tilde{u}_1'D^{t'}\bar{\Upsilon}HC^{*'}\Psi_j J'_{s-j}\phi\tilde{u}_1'D^{s'}\tilde{u}_1\} \\
&\qquad - \mathbb{E}\{S^{*'}\bar{Z}[\mathbb{E}(Z'Z)]^{-1}I_2\Psi_i J'_{t-i}S^{*'}D^{t'}\bar{\Upsilon}HC^{*'}\Psi_j J'_{s-j}\phi\tilde{u}_1'D^{s'}\tilde{u}_1\} \\
&\qquad - \mathbb{E}\{S^{*'}\bar{Z}[\mathbb{E}(Z'Z)]^{-1}I_2\Psi_i J'_{t-i}\phi\tilde{u}_1'D^{t'}\bar{\Upsilon}HC^{*'}\Psi_j J'_{s-j}S^{*'}D^{s'}\tilde{u}_1\} \\
&\qquad - \mathbb{E}\{\phi\tilde{u}_1'\bar{Z}[\mathbb{E}(Z'Z)]^{-1}I_2\Psi_i J'_{t-i}S^{*'}D^{t'}\bar{\Upsilon}HC^{*'}\Psi_j J'_{s-j}S^{*'}D^{s'}\tilde{u}_1\} \\
&\quad = -\sigma_1^4\phi\phi' J_{s-j}\Psi_j' C^* H\bar{\Upsilon}'D^t D^s\bar{Z}[\mathbb{E}(Z'Z)]^{-1}I_2\Psi_i J'_{t-i}\phi \\
&\qquad - \phi\sigma_1^2 \operatorname{tr}\left\{\bar{Z}[\mathbb{E}(Z'Z)]^{-1}I_2\Psi_i J'_{t-i}\sigma_1^2\phi\phi' J_{s-j}\Psi_j' C^* H\bar{\Upsilon}'D^t D^{s'}\right\} \\
&\qquad - 0 \\
&\qquad - (\Omega - \sigma_1^2\phi\phi')J_{s-j}\Psi_j' C^* H\bar{\Upsilon}'D^t D^s\bar{Z}[\mathbb{E}(Z'Z)]^{-1}I_2\Psi_i J'_{t-i}\phi\sigma_1^2 \\
&\qquad - tr\left\{\bar{Z}[\mathbb{E}(Z'Z)]^{-1}I_2\Psi_i J'_{t-i}(\Omega - \sigma_1^2\phi\phi')J_{s-j}\Psi_j' C^* H\bar{\Upsilon}'D^t D^{s'}\right\}\phi\sigma_1^2 \\
&\quad = -tr\left\{\bar{Z}[\mathbb{E}(Z'Z)]^{-1}I_2\Psi_i J'_{t-i}\Omega J_{s-j}\Psi_j' C^* H\bar{\Upsilon}'D^t D^{s'}\right\}\phi\sigma_1^2 \\
&\qquad - \Omega J_{s-j}\Psi_j' C^* H\bar{\Upsilon}'D^t D^s\bar{Z}[\mathbb{E}(Z'Z)]^{-1}I_2\Psi_i J'_{t-i}\phi\sigma_1^2.
\end{aligned}
$$

Using Lemma 3 and 4.

The final expression for equation (A.29) is:

$$
\begin{aligned}
& -\mathbb{E}\{H\Lambda^{**'}\tilde{V}'\bar{Z}[\mathbb{E}(Z'Z)]^{-1}I_2\tilde{R}'\bar{\Upsilon}HC^{*'}\tilde{R}'\tilde{u}_1\} \qquad \text{(A.30)} \\
&= -\sum_{i=1}^{p}\sum_{t=i}^{T-1}\sum_{j=1}^{p}\sum_{s=j}^{T-1} H\Lambda^{**'} tr\left\{\bar{Z}[\mathbb{E}(Z'Z)]^{-1}I_2\Psi_i J'_{t-i}\Omega J_{s-j}\Psi_j' C^* H\bar{\Upsilon}'D^t D^{s'}\right\}\phi\sigma_1^2 \\
&\quad - \sum_{i=1}^{p}\sum_{t=i}^{T-1}\sum_{j=1}^{p}\sum_{s=j}^{T-1} H\Lambda^{**'}\Omega J_{s-j}\Psi_j' C^* H\bar{\Upsilon}'D^t D^s\bar{Z}[\mathbb{E}(Z'Z)]^{-1}I_2\Psi_i J'_{t-i}\phi\sigma_1^2.
\end{aligned}
$$

(5′)

$$
\begin{aligned}
& -\mathbb{E}\{HC^{*'}\tilde{R}'\tilde{R}I_2'[\mathbb{E}(Z'Z)]^{-1}\tilde{W}^{*'}\tilde{V}\Lambda^{**}HC^{*'}\tilde{R}'\tilde{u}_1\} \qquad \text{(A.31)} \\
&= -\mathbb{E}\left\{H\mathbb{E}(\tilde{R}'\tilde{R})I_2'[\mathbb{E}(Z'Z)]^{-1}\,{}^{8}\right. \\
&\qquad \left. \times \begin{bmatrix} \sum_{l=1}^{p}\sum_{r=l}^{T-1}\Psi_l J'_{r-l}\tilde{V}'D^{r'} \\ 0 \end{bmatrix} \tilde{V}\Lambda^{**}H\sum_{b=1}^{p}\sum_{h=b}^{T-1}C^{*'}\Psi_b J'_{h-b}\tilde{V}'D^{t'}\tilde{u}_1\right\}
\end{aligned}
$$

$$= -\mathbb{E}\left\{\left(H\sum_{i=1}^{p}\sum_{t=i}^{T-1}(T-t)C^{*'}\Psi_i J_{t-i}^{'}\Omega J_{t-i}\Psi_i^{'}\right) I_2^{'}[\mathbb{E}(Z'Z)]^{-1}\begin{bmatrix}\sum_{l=1}^{p}\sum_{r=l}^{T-1}\Psi_l J_{r-l}^{'}\tilde{V}^{'}D^{r'}\\ 0\end{bmatrix}\right.$$
$$\left.\times\tilde{V}\Lambda^{**}H\sum_{b=1}^{p}\sum_{h=b}^{T-1}C^{*'}\Psi_b J_{h-b}^{'}\tilde{V}^{'}D^{h'}\tilde{u}_1\right\} + o(T^{-1}).$$

Here,

$$\begin{aligned}
&\mathbb{E}\{\tilde{V}^{'}D^{r'}\tilde{V}\Lambda^{**}HC^{*'}\Psi_b J_{h-b}^{'}\tilde{V}^{'}D^{h'}\tilde{u}_1\}\\
&= \mathbb{E}\{\phi\tilde{u}_1^{'}D^{r'}\tilde{u}_1\phi^{'}\Lambda^{**}HC^{*'}\Psi_b J_{h-b}^{'}\phi\tilde{u}_1^{'}D^{h'}\tilde{u}_1\}\\
&\quad + \mathbb{E}\{\phi\tilde{u}_1^{'}D^{r'}S^{*}\Lambda^{**}HC^{*'}\Psi_b J_{h-b}^{'}S^{*'}D^{h'}\tilde{u}_1\}\\
&\quad + \mathbb{E}\{S^{*'}D^{r'}S^{*}\Lambda^{**}HC^{*'}\Psi_b J_{h-b}^{'}\phi\tilde{u}_1^{'}D^{h'}\tilde{u}_1\}\\
&\quad + \mathbb{E}\{S^{*'}D^{r'}\tilde{u}_1\phi^{'}\Lambda^{**}HC^{*'}\Psi_b J_{h-b}^{'}S^{*'}D^{h'}\tilde{u}_1\}\\
&= \sigma_1^4\phi\phi^{'}\Lambda^{**}HC^{*'}\Psi_b J_{h-b}^{'}\phi tr\{(D^t + D^{t'})D^{r'}\}\\
&\quad + 0\\
&\quad + \sigma_1^2 tr\{D^{t'}D^r\}\Omega J_{h-b}\Psi_b^{'}C^{*}H\Lambda^{**'}\phi - \sigma_1^4 tr\{D^{t'}D^r\}\phi\phi^{'}J_{h-b}\Psi_b^{'}C^{*}H\Lambda^{**'}\phi\\
&\quad + 0\\
&= \Omega J_{h-b}\Psi_b^{'}C^{*}H(tr\{D^{t'}D^r\}.I)\Lambda^{**'}(\sigma_1^2\phi).
\end{aligned}$$

Therefore, the final result for equation (A.31) is :

$$\begin{aligned}
&-\mathbb{E}\{HC^{*'}\tilde{R}^{'}\tilde{R}I_2^{'}[\mathbb{E}(Z'Z)]^{-1}\tilde{W}^{*'}\tilde{V}\Lambda^{**}HC^{*'}\tilde{R}^{'}\tilde{u}_1\} \qquad \text{(A.32)}\\
&= -H\sum_{i=1}^{p}\sum_{t=i}^{T-1}(T-t)C^{*'}\Psi_i J_{t-i}^{'}\Omega J_{t-i}\Psi_i)I_2^{'}[\mathbb{E}(Z'Z)]^{-1}\\
&\quad\sum_{l=1}^{p}\sum_{r=l}^{T-1}\sum_{b=1}^{p}\sum_{h=b}^{T-1}I_2\Psi_l J_{r-l}^{'}\Omega J_{h-b}\Psi_b^{'}C^{*}H(tr\{D^{t'}D^r\}.I)\Lambda^{**'}(\sigma_1^2\phi) + o(T^{-1}).
\end{aligned}$$

(6′)

$$-\mathbb{E}\{HC^{*'}\tilde{R}^{'}\bar{Z}[\mathbb{E}(Z'Z)]^{-1}\bar{Z}^{'}\tilde{V}\Lambda^{**}HC^{*'}\tilde{R}^{'}\tilde{u}_1\} \qquad \text{(A.33)}$$

---

[8] $\tilde{R}'\tilde{R} = \mathbb{E}(\tilde{R}'\tilde{R}) + (\tilde{R}'\tilde{R} - \mathbb{E}(\tilde{R}'\tilde{R})) \equiv \mathbb{E}(\tilde{R}'\tilde{R}) + O_p(T^{1/2})$, where $\mathbb{E}(\tilde{R}'\tilde{R}) = \sum_{i=1}^{p}\sum_{t=i}^{T-1}\Psi_i J_{t-i}^{'}\tilde{V}^{'}D^{t'}\sum_{j=1}^{p}\sum_{s=j}^{T-1}D^s\tilde{V}J_{s-j}\Psi_j = \sum_{i=1}^{p}\sum_{t=i}^{T-1}(T-t)\Psi_i J_{t-i}^{'}\Omega J_{t-i}\Psi_i^{'})$. In the following calculation I can replace $\tilde{R}'\tilde{R}$ with $\mathbb{E}(\tilde{R}'\tilde{R})$ to the order of the approximation.

$$= -\mathbb{E}\{H\sum_{i=1}^{p}\sum_{t=i}^{T-1} C^{*'}\Psi_i J_{t-i}^{'}\tilde{V}^{'}D^{t'}\bar{Z}[\mathbb{E}(Z'Z)]^{-1}\bar{Z}^{'}\tilde{V}\Lambda^{**}H\sum_{j=1}^{p}\sum_{s=j}^{T-1} C^{*'}\Psi_j J_{s-j}^{'}\tilde{V}^{'}D^{s'}\tilde{u}_1\}.$$

Here,

$$\begin{aligned}
&\mathbb{E}\{\tilde{V}^{'}D^{t'}\bar{Z}[\mathbb{E}(Z'Z)]^{-1}\bar{Z}^{'}\tilde{V}\Lambda^{**}HC^{*'}\Psi_j J_{s-j}^{'}\tilde{V}^{'}D^{s'}\tilde{u}_1\}\\
&= \mathbb{E}\{\phi\tilde{u}_1^{'}D^{t'}\bar{Z}[\mathbb{E}(Z'Z)]^{-1}\bar{Z}^{'}\tilde{u}_1\phi^{'}\Lambda^{**}HC^{*'}\Psi_j J_{s-j}^{'}\phi\tilde{u}_1^{'}D^{s'}\tilde{u}_1\}\\
&\quad + \mathbb{E}\{\phi\tilde{u}_1^{'}D^{t'}\bar{Z}[\mathbb{E}(Z'Z)]^{-1}\bar{Z}^{'}S^{*}\Lambda^{**}HC^{*'}\Psi_j J_{s-j}^{'}S^{*'}D^{s'}\tilde{u}_1\}\\
&\quad + \mathbb{E}\{S^{*'}D^{t'}\bar{Z}[\mathbb{E}(Z'Z)]^{-1}\bar{Z}^{'}S^{*}\Lambda^{**}HC^{*'}\Psi_j J_{s-j}^{'}\phi\tilde{u}_1^{'}D^{s'}\tilde{u}_1\}\\
&\quad + \mathbb{E}\{S^{*'}D^{t'}\bar{Z}[\mathbb{E}(Z'Z)]^{-1}\bar{Z}^{'}\tilde{u}_1\phi^{'}\Lambda^{**}HC^{*'}\Psi_j J_{s-j}^{'}S^{*'}D^{s'}\tilde{u}_1\}\\
&= \sigma_1^4\phi\phi^{'}\Lambda^{**}HC^{*'}\Psi_j J_{s-j}^{'}\phi tr\{(D^{s}+D^{s'})D^{t'}\}\bar{Z}[\mathbb{E}(Z'Z)]^{-1}\bar{Z}^{'}\\
&\quad + \sigma_1^2\phi tr\{\Omega\Lambda^{**}HC^{*'}\Psi_j J_{s-j}^{'}\}tr\{D^{t'}\bar{Z}[\mathbb{E}(Z'Z)]^{-1}\bar{Z}^{'}D^{s'}\}\\
&\quad - \sigma_1^4\phi tr\{\phi\phi^{'}\Lambda^{**}HC^{*'}\Psi_j J_{s-j}^{'}\}tr\{D^{t'}\bar{Z}[\mathbb{E}(Z'Z)]^{-1}\bar{Z}^{'}D^{s'}\}\\
&\quad + \sigma_1^2 tr\{D^{t'}\bar{Z}[\mathbb{E}(Z'Z)]^{-1}\bar{Z}^{'}D^{s'}\}\Omega J_{s-j}\Psi_j^{'}C^{*}H\Lambda^{**'}\phi\\
&\quad - \sigma_1^4 tr\{D^{t'}\bar{Z}[\mathbb{E}(Z'Z)]^{-1}\bar{Z}^{'}D^{s'}\}\phi\phi^{'}J_{s-j}\Psi_j^{'}C^{*}H\Lambda^{**'}\phi\\
&= tr\{\Omega\Lambda^{**}HC^{*'}\Psi_j J_{s-j}^{'}\}(tr\{D^{t'}\bar{Z}[\mathbb{E}(Z'Z)]^{-1}\bar{Z}^{'}D^{s'}.I)(\sigma_1^2\phi)\\
&\quad + (tr\{D^{t'}\bar{Z}[\mathbb{E}(Z'Z)]^{-1}\bar{Z}^{'}D^{s'}\}.I)\Omega J_{s-j}\Psi_j^{'}C^{*}H\Lambda^{**'}(\sigma_1^2\phi).
\end{aligned}$$

Then, the final result for equation (A.33) is:

$$\begin{aligned}
&-\mathbb{E}\{HC^{*'}\tilde{R}^{'}\bar{Z}[\mathbb{E}(Z'Z)]^{-1}\bar{Z}^{'}\tilde{V}\Lambda^{**}HC^{*'}\tilde{R}^{'}\tilde{u}_1\} \qquad (A.34)\\
&\quad = -H\sum_{i=1}^{p}\sum_{t=i}^{T-1}\sum_{j=1}^{p}\sum_{s=j}^{T-1}\Psi_i J_{t-i}^{'}(tr\{\Omega\Lambda^{**}HC^{*'}\Psi_j J_{s-j}^{'}\}.I)\\
&\qquad\qquad \times (tr\{D^{t'}\bar{Z}[\mathbb{E}(Z'Z)]^{-1}\bar{Z}^{'}D^{s'}.I)(\sigma_1^2\phi)\\
&\qquad - H\sum_{i=1}^{p}\sum_{t=i}^{T-1}\sum_{j=1}^{p}\sum_{s=j}^{T-1}(tr\{D^{t'}\bar{Z}H^{*}\bar{Z}^{'}D^{s'}\}.I)C^{*'}\Psi_i J_{t-i}^{'}\Omega HC^{*'}\Psi_j J_{s-j}^{'}H\Lambda^{**'}\sigma_1^2\phi.
\end{aligned}$$

(7′)

$$\begin{aligned}
&-\mathbb{E}\{H\Lambda^{**'}\tilde{V}^{'}\tilde{W}^{*}(\mathbb{E}(Z^{'}Z))^{-1}I_2^{'}\tilde{R}^{'}\tilde{R}C^{*}HC^{*'}\tilde{R}^{'}\tilde{u}_1\} \qquad (A.35)\\
&\quad = -\mathbb{E}\left\{H\Lambda^{**'}\tilde{V}^{'}\left[\sum_{i=1}^{p}\sum_{t=i}^{T-1}D^{t}\tilde{V}J_{t-i}\Psi_i^{'}:0\right](\mathbb{E}(Z^{'}Z))^{-1}I_2^{'}\sum_{j=1}^{p}\sum_{s=j}^{T-1}\Psi_j J_{s-j}\tilde{V}^{'}D^{s'}\right.
\end{aligned}$$

$$\times \sum_{l=1}^{p}\sum_{r=l}^{T-1} D^r \tilde{V} J_{r-l}\Psi_l' C^* H \sum_{b=1}^{p}\sum_{h=b}^{T-1} C^{*'} D^h \tilde{V} J_{h-b}\Psi_b' \tilde{u}_1 \Bigg\}$$

$$= -\mathbb{E}\left\{ H\Lambda^{**'}\tilde{V}' \left[\sum_{i=1}^{p}\sum_{t=i}^{T-1} D^t \tilde{V} J_{t-i}\Psi_i' : 0\right] (\mathbb{E}(Z'Z))^{-1} I_2' \right.$$

$$\left. \times \sum_{j=1}^{p}\sum_{s=j}^{T-1} (T-s)\Psi_j J_{s-j}\Omega J_{s-j}\Psi_j' C^* H \sum_{b=1}^{p}\sum_{h=b}^{T-1} C^{*'}\Psi_b J_{h-b}' \tilde{V}' D^{h'} \tilde{u}_1 \right\}.$$

Here,

$$\mathbb{E}\left\{ \tilde{V}' D^t \tilde{V} J_{t-i}\Psi_i' I_2' (\mathbb{E}(Z'Z))^{-1} I_2' \left[\sum_{j=1}^{p}\sum_{s=j}^{T-1} (T-s)\Psi_j J_{s-j}\Omega J_{s-j}\Psi_j' C^*\right] \right.$$

$$\left. \times HC^{*'}\Psi_b J_{h-b}' \tilde{V}' D^{h'} \tilde{u}_1 \right\}$$

$$= \mathbb{E}\left\{ \phi \tilde{u}_1' D^t \tilde{u}_1 \phi' J_{t-i}\Psi_i' I_2' (\mathbb{E}(Z'Z))^{-1} I_2' \left[\sum_{j=1}^{p}\sum_{s=j}^{T-1} (T-s)\Psi_j J_{s-j}\Omega J_{s-j}\Psi_j' C^*\right] \right.$$

$$\left. \times HC^{*'}\Psi_b J_{h-b}' \phi \tilde{u}_1' D^{h'} \tilde{u}_1 \right\}$$

$$+ \mathbb{E}\left\{ \phi \tilde{u}_1' D^t S^* J_{t-i} e_i' I_2' (\mathbb{E}(Z'Z))^{-1} I_2' \left[\sum_{j=1}^{p}\sum_{s=j}^{T-1} (T-s)\Psi_j J_{s-j}\Omega J_{s-j}\Psi_j' C^*\right] \right.$$

$$\left. \times HC^{*'}\Psi_b J_{h-b}' S^{*'} D^{h'} \tilde{u}_1 \right\}$$

$$+ \mathbb{E}\left\{ S^{*'} D^t S^* J_{t-i}\Psi_i' I_2' (\mathbb{E}(Z'Z))^{-1} I_2' \left[\sum_{j=1}^{p}\sum_{s=j}^{T-1} (T-s) e_j J_{s-j}\Omega J_{s-j}\Psi_j' C^*\right] \right.$$

$$\left. \times HC^{*'}\Psi_b J_{h-b}' \phi \tilde{u}_1' D^{h'} \tilde{u}_1 \right\}$$

$$+ \mathbb{E}\left\{ S^{*'} D^t \tilde{u}_1 \phi' J_{t-i}\Psi_i' I_2' (\mathbb{E}(Z'Z))^{-1} I_2' \left[\sum_{j=1}^{p}\sum_{s=j}^{T-1} (T-s)\Psi_j J_{s-j}\Omega J_{s-j}\Psi_j' C^*\right] \right.$$

$$\left. \times HC^{*'}\Psi_b J_{h-b}' S^{*'} D^{h'} \tilde{u}_1 \right\}$$

$$= \sigma_1^4 \phi\phi' J_{t-i}\Psi_i' I_2' (\mathbb{E}(Z'Z))^{-1} I_2' \left[\sum_{j=1}^{p}\sum_{s=j}^{T-1} (T-s)\Psi_j J_{s-j}\Omega J_{s-j}\Psi_j' C^*\right]$$

$$\times HC^{*'}\Psi_b J_{h-b}' \phi tr\left\{(D^t + D^{t'})D^{h'}\right\}$$

$$+ \sigma_1^2 \phi tr\left\{ \Omega J_{t-i}\Psi_i' I_2' (\mathbb{E}(Z'Z))^{-1} I_2' \left[\sum_{j=1}^{p}\sum_{s=j}^{T-1} (T-s)\Psi_j J_{s-j}\Omega J_{s-j}\Psi_j' C^*\right] \right.$$

$$\left.\times HC^{*'}\Psi_b J_{h-b}^{'}\right\} tr\left\{D^t D^{h'}\right\}$$

$$-\sigma_1^4 \phi tr\left\{\phi\phi^{'} J_{t-i}\Psi_i^{'} I_2^{'}(\mathbb{E}(Z^{'}Z))^{-1} I_2^{'}\left[\sum_{j=1}^{p}\sum_{s=j}^{T-1}(T-s)\Psi_j J_{s-j}\Omega J_{s-j}\Psi_j^{'} C^*\right]\right.$$

$$\left.\times HC^{*'}\Psi_b J_{h-b}^{'}\right\} tr\left\{D^t D^{h'}\right\}$$

$$+0$$

$$+0$$

$$= tr\left\{\Omega J_{t-i}\Psi_i^{'} I_2^{'}(\mathbb{E}(Z^{'}Z))^{-1} I_2^{'}\left[\sum_{j=1}^{p}\sum_{s=j}^{T-1}(T-s)\Psi_j J_{s-j}\Omega J_{s-j}\Psi_j^{'} C^*\right]\right.$$

$$\left.\times HC^{*'}\Psi_b J_{h-b}^{'}\right\} tr\left\{D^t D^{h'}\right\}(\sigma_1^2\phi).$$

Therefore, the final result for equation (A.35) is:

$$-\mathbb{E}\left\{H\Lambda^{**'}\tilde{V}^{'}\tilde{W}^*(\mathbb{E}(Z^{'}Z))^{-1} I_2^{'}\tilde{R}^{'}\tilde{R}C^* HC^{*'}\tilde{R}^{'}\tilde{u}_1\right\} \tag{A.36}$$

$$= -H\sum_{i=1}^{p}\sum_{t=i}^{T-1}\sum_{b=1}^{p}\sum_{h=b}^{T-1}\left(tr\{\Omega J_{t-i}\Psi_i^{'} I_2^{'}(\mathbb{E}(Z^{'}Z))^{-1} I_2^{'}\left[\sum_{j=1}^{p}\sum_{s=j}^{T-1}(T-s)\Psi_j J_{s-j}\Omega J_{s-j}\Psi_j^{'} C^*\right]\right.$$

$$\left.\times HC^{*'}\Psi_b J_{h-b}^{'}\}.I\right)\left(tr\{D^t D^{h'}.I\}\right)\Lambda^{**'}(\sigma_1^2\phi).$$

(8′)

$$-\mathbb{E}\left\{H\Lambda^{**'}\tilde{V}^{'}\bar{Z}(\mathbb{E}(Z^{'}Z))^{-1}\bar{Z}^{'}\tilde{R}C^* HC^{*'}\tilde{R}^{'}\tilde{u}_1\right\} \tag{A.37}$$

$$= -\mathbb{E}\{H\Lambda^{**'}\tilde{V}^{'}\bar{Z}(\mathbb{E}(Z^{'}Z))^{-1}\bar{Z}^{'}\sum_{i=1}^{p}\sum_{t=i}^{T-1} D^t\tilde{V}J_{t-i}\Psi_i^{'} C^* H\sum_{j=1}^{p}\sum_{s=j}^{T-1} C^{*'}\Psi_j J_{s-j}^{'}\tilde{V}^{'} D^{s'}\tilde{u}_1\}.$$

Here,

$$\mathbb{E}\{\tilde{V}^{'}\bar{Z}(\mathbb{E}(Z^{'}Z))^{-1}\bar{Z}^{'} D^t\tilde{V}J_{t-i}v_i^{'} C^* HC^{*'}\Psi_j J_{s-j}^{'}\tilde{V}^{'} D^{s'}\tilde{u}_1\}$$

$$= \mathbb{E}\{\phi\tilde{u}_1^{'}\bar{Z}(\mathbb{E}(Z^{'}Z))^{-1}\bar{Z}^{'} D^t\tilde{u}_1\phi^{'} J_{t-i}\Psi_i^{'} C^* HC^{*'} v_j J_{s-j}^{'}\phi\tilde{u}_1^{'} D^{s'}\tilde{u}_1\}$$

$$+\mathbb{E}\{\phi\tilde{u}_1^{'}\bar{Z}(\mathbb{E}(Z^{'}Z))^{-1}\bar{Z}^{'} D^t S^* J_{t-i}v_i^{'} C^* HC^{*'}\Psi_j J_{s-j}^{'} S^{*'} D^{s'}\tilde{u}_1\}$$

$$+\mathbb{E}\{S^{*'}\bar{Z}(\mathbb{E}(Z^{'}Z))^{-1}\bar{Z}^{'} D^t S^* J_{t-i}\Psi_i^{'} C^* HC^{*'}\Psi_j J_{s-j}^{'}\phi\tilde{u}_1^{'} D^{s'}\tilde{u}_1\}$$

$$+\mathbb{E}\{S^{*'}\bar{Z}(\mathbb{E}(Z^{'}Z))^{-1}\bar{Z}^{'} D^t\tilde{u}_1\phi^{'} J_{t-i}\Psi_i^{'} C^* HC^{*'}\Psi_j J_{s-j}^{'} S^{*'} D^{s'}\tilde{u}_1\}$$

$$= \sigma_1^4\phi\phi' J_{t-i}\Psi_i' C^* H C^{*'}\Psi_j J_{s-j}'\phi tr\{(D^s + D^{s'})\bar{Z}(\mathbb{E}(Z'Z))^{-1}\bar{Z}'D^t\}$$
$$+ \sigma_1^2\phi tr\{\Omega J_{t-i}\Psi_i' C^* H C^{*'}\Psi_j J_{s-j}'\} tr\{\bar{Z}(\mathbb{E}(Z'Z))^{-1}\bar{Z}'D^t D^{s'}\}$$
$$- \sigma_1^4\phi tr\{\phi\phi' J_{t-i}\Psi_i' C^* H C^{*'}\Psi_j J_s'\} tr\{\bar{Z}(\mathbb{E}(Z'Z))^{-1}\bar{Z}'D^t D^{s'}\}$$
$$+ \sigma_1^2 tr\{\bar{Z}(\mathbb{E}(Z'Z))^{-1}\bar{Z}'D^t D^s\}\Omega J_{s-j}\Psi_j' C^* H C^{*'}\Psi_i J_{t-i}'\phi$$
$$- \sigma_1^4 tr\{\bar{Z}(\mathbb{E}(Z'Z))^{-1}\bar{Z}'D^t D^s\}\phi\phi' J_{s-j}\Psi_j' C^* H C^{*'}\Psi_i J_{t-i}'\phi$$
$$+ 0$$
$$= (tr\{\Omega J_{t-i}\Psi_i' C^* H C^{*'}\Psi_j J_{s-j}'\}.I)(tr\{\bar{Z}(\mathbb{E}(Z'Z))^{-1}\bar{Z}'D^t D^{s'}\}.I)(\sigma_1^2\phi)$$
$$+ \Omega J_{s-j}\Psi_j' C^* H C^{*'}\Psi_i J_{t-i}'(tr\{\bar{Z}(\mathbb{E}(Z'Z))^{-1}\bar{Z}'D^t D^s\}.I)(\sigma_1^2\phi).$$

Therefore, the final result of equation (A.37) is:

$$- \mathbb{E}\{H\Lambda^{**'}\tilde{V}'\bar{Z}(\mathbb{E}(Z'Z))^{-1}\bar{Z}'\tilde{R}C^* H C^{*'}\tilde{R}'\tilde{u}_1\} \tag{A.38}$$
$$= -H\sum_{i=1}^{p}\sum_{t=i}^{T-1}\sum_{j=1}^{p}\sum_{s=j}^{T-1}(tr\{\Omega J_{t-i}\Psi_i' C^* H C^{*'}\Psi_j J_{s-j}'\}.I)$$
$$\times (tr\{\bar{Z}(\mathbb{E}(Z'Z))^{-1}\bar{Z}'D^t D^{s'}\}.I)\Lambda^{**'}(\sigma_1^2\phi)$$
$$- H\Lambda^{**'}\sum_{i=1}^{p}\sum_{t=i}^{T-1}\sum_{j=1}^{p}\sum_{s=j}^{T-1}\Omega J_{s-j}\Psi_j' C^* H C^{*'}\Psi_i J_{t-i}'(tr\{\bar{Z}(\mathbb{E}(Z'Z))^{-1}\bar{Z}'D^t D^s\}.I)(\sigma_1^2\phi).$$

(9′)

$$- \mathbb{E}\{H C^{*'}\tilde{R}'\tilde{R}C^* H C^{*'}\tilde{R}'\tilde{u}_1\} \tag{A.39}$$
$$= -\mathbb{E}\left\{H\sum_{i=1}^{p}\sum_{t=i}^{T-1}C^{*'}\Psi_i J_{t-i}'\tilde{V}'D^{t'}\sum_{j=1}^{p}\sum_{s=j}^{T-1}D^s\tilde{V}J_{s-j}\Psi_j' C^* H\sum_{l=1}^{p}\sum_{r=l}^{T-1}C^{*'}\Psi_l J_{r-l}'\tilde{V}'D^{r'}\tilde{u}_1\right\}$$
$$= -\mathbb{E}\left\{HC^{*'}\sum_{i=1}^{p}\sum_{t=i}^{T-1}\sum_{j=1}^{p}\sum_{s=j}^{T-1}\sum_{l=1}^{p}\sum_{r=l}^{T-1}\Psi_i J_{t-i}' S^{*'}D^{t'}D^s S^* J_{s-j}\Psi_j' C^* H C^{*'}\Psi_l J_{r-l}'\phi\tilde{u}_1' D^{r'}\tilde{u}_1\right\}$$
$$- \mathbb{E}\left\{HC^{*'}\sum_{i=1}^{p}\sum_{t=i}^{T-1}\sum_{j=1}^{p}\sum_{s=j}^{T-1}\sum_{l=1}^{p}\sum_{r=l}^{T-1}\Psi_i J_{t-i}' S^{*'}D^{t'}D^s\tilde{u}_1\phi' J_{s-j}\Psi_j' C^* H C^{*'}\Psi_l J_{r-l}' S^{*'}D^{r'}\tilde{u}_1\right\}$$
$$- \mathbb{E}\left\{HC^{*'}\sum_{i=1}^{p}\sum_{t=i}^{T-1}\sum_{j=1}^{p}\sum_{s=j}^{T-1}\sum_{l=1}^{p}\sum_{r=l}^{T-1}\Psi_i J_{t-i}'\phi\tilde{u}_1' D^{t'}D^s S^* J_{s-j}\Psi_j' C^* H C^{*'}\Psi_l J_{r-l}' S^{*'}D^{r'}\tilde{u}_1\right\}$$
$$- \mathbb{E}\left\{HC^{*'}\sum_{i=1}^{p}\sum_{t=i}^{T-1}\sum_{j=1}^{p}\sum_{s=j}^{T-1}\sum_{l=1}^{p}\sum_{r=l}^{T-1}\Psi_i J_{t-i}'\phi\tilde{u}_1' D^{t'}D^s\tilde{u}_1\phi' J_{s-j}\Psi_j' C^* H C^{*'}\Psi_l J_{r-l}'\phi\tilde{u}_1' D^{r'}\tilde{u}_1\right\}$$
$$= 0$$

$$- HC^{*'} \sum_{i=1}^{p} \sum_{t=i}^{T-1} \sum_{j=1}^{p} \sum_{s=j}^{T-1} \sum_{l=1}^{p} \sum_{r=l}^{T-1} \Psi_i J_{t-i}^{'} \left(\Omega - \sigma_1^2 \phi\phi'\right) J_{r-l} \Psi_l' C^{*'} H' C^{*'} \Psi_j J_{s-j}' \phi \sigma_1^2 tr(D^{t'} D^s D^{r'})$$

$$- HC^{*'} \sum_{i=1}^{p} \sum_{t=i}^{T-1} \sum_{j=1}^{p} \sum_{s=j}^{T-1} \sum_{l=1}^{p} \sum_{r=l}^{T-1} \Psi_i J_{t-i}^{'} \phi \sigma_1^2 tr(D^{t'} D^s D^{r'}) tr(\phi' J_{s-j} \Psi_j^{'} C^* H C^{*'} \Psi_l J_{r-l}^{'} (\Omega - \sigma_1^2 \phi\phi'))$$

$$- HC^{*'} \sum_{i=1}^{p} \sum_{t=i}^{T-1} \sum_{j=1}^{p} \sum_{s=j}^{T-1} \sum_{l=1}^{p} \sum_{r=l}^{T-1} \Psi_i J_{t-i}^{'} \phi \sigma_1^2 \phi' J_{r-l} \Psi_l' C^{*'} H' C^{*'} \Psi_j J_{s-j}' \phi \sigma_1^2 tr(D^{t'} D^s D^r)$$

$$- HC^{*'} \sum_{i=1}^{p} \sum_{t=i}^{T-1} \sum_{j=1}^{p} \sum_{s=j}^{T-1} \sum_{l=1}^{p} \sum_{r=l}^{T-1} \Psi_i J_{t-i}^{'} \phi \sigma_1^2 tr(D^{t'} D^s D^{r'}) tr(\phi' J_{r-l} \Psi_l' C^{*'} H' C^{*'} \Psi_j J_{s-j}' \phi \sigma_1^2)$$

$$= -HC^{*'} \sum_{i=1}^{p} \sum_{t=i}^{T-1} \sum_{j=1}^{p} \sum_{s=j}^{T-1} \sum_{l=1}^{p} \sum_{r=l}^{T-1} \Psi_i J_{t-i}^{'} \Omega J_{r-l} \Psi_l' C^{*'} H' C^{*'} \Psi_j J_{s-j}' \phi \sigma_1^2 tr(D^{t'} D^s D^{r'})$$

$$- HC^{*'} \sum_{i=1}^{p} \sum_{t=i}^{T-1} \sum_{j=1}^{p} \sum_{s=j}^{T-1} \sum_{l=1}^{p} \sum_{r=l}^{T-1} \Psi_i J_{t-i}^{'} \phi \sigma_1^2 tr(D^{t'} D^s D^{r'}) tr(\phi' J_{s-j} \Psi_j^{'} C^* H C^{*'} \Psi_l J_{r-l}^{'} \Omega.$$

Therefore, by combining equation (A.24),(A.26), (A.28), (A.30), (A.32), (A.34), (A.36), (A.38), (A.39), we can get the final expression for (v).

Rearranging for the final expression

Recall $H^* = [E(Z^{'}Z)]^{-1}$, set $H^{**} = I_2^{'} H^* I_2$ and assume $\tau = \sigma^2 \phi$ and $\vartheta = \Lambda^{**'} \tau$. We will add all the expectations from $((i) - (v)$ which refer to equation (A.4, (A.6),(A.9), (A.10), (A.12), (A.13), (A.14), (A.16),(A.19) (A.20), (A.22), (A.24),(A.26), (A.28),(A.30), (A.32), (A.34), (A.36), (A.38), (A.39)) we can get the final expression which is our Theorem 1 equation (15).

B Numerical Results

Table 1: Approximation bias and MC 2SLS bias, when L=2, 4, 6; T=50, 100

		$T = 50$			$T = 100$		
		$L = 2$	$L = 4$	$L = 6$	$L = 2$	$L = 4$	$L = 6$
$\beta_{21} = 2.00$	MC 2SLS bias	-0.3042	-0.6434	-0.6338	-0.1253	-0.1600	-0.2250
	Approximation bias	-0.3229	-0.7150	-0.8931	-0.1597	-0.1799	-0.3233
	Simultaneity part	-0.5322	-0.8123	-0.9305	-0.2831	-0.3641	-0.5643
	Dynamic Part	0.2093	0.0973	0.0374	0.1234	0.1842	0.2410
$\beta_{31} = 5.00$	MC 2SLS bias	-0.6466	-1.0910	-1.0130	-0.2604	-0.3115	-0.3958
	Approximation bias	-0.6439	-1.1902	-1.4003	-0.3015	-0.2159	-0.4608
	Simultaneity part	-0.9908	-1.4162	-2.7651	-0.6001	-0.4621	-0.7661
	Dynamic Part	0.3469	0.2260	1.3648	0.2986	0.2462	0.3053
$\alpha^1_{11} = 0.50$	MC 2SLS bias	0.0241	-0.0919	-0.0127	0.0082	0.0078	0.0120
	Approximation bias	0.0365	-0.0784	-0.0241	0.0120	0.0106	0.0198
	Simultaneity part	0.1815	-0.0926	-0.1079	0.0310	0.1028	0.0603
	Dynamic Part	-0.1450	0.0142	0.0838	-0.0190	-0.0922	-0.0405
$\alpha^1_{21} = 0.36$	MC 2SLS bias	0.0291	-0.0110	0.0158	0.0134	0.0206	0.0243
	Approximation bias	0.0513	-0.0216	0.0251	0.0252	0.0196	0.0351
	Simultaneity part	0.0501	-0.0285	-0.0732	0.1096	0.0561	0.0571
	Dynamic Part	0.0012	0.0069	0.0481	-0.0844	-0.0365	-0.0220
$\alpha^1_{31} = 0.40$	MC 2SLS bias	0.0297	0.1265	-0.2573	0.0346	0.0343	-0.0264
	Approximation bias	0.0337	0.1593	-0.2149	0.0283	0.0525	-0.0407
	Simultaneity part	0.1247	0.3770	-0.5128	0.0403	0.1698	-0.0700
	Dynamic Part	-0.091	-0.2177	0.2979	-0.012	-0.1173	0.0293
$\alpha^2_{11} = 1.20$	MC 2SLS bias	-0.1651	-0.2898	-0.2569	-0.0636	-0.0688	-0.1028
	Approximation bias	-0.1324	-0.3514	-0.3281	-0.0804	-0.1095	-0.1502
	Simultaneity part	-0.1889	-0.7067	-1.4103	-0.1007	-0.2154	-0.2771
	Dynamic Part	0.0565	0.3553	1.0822	0.0203	0.1095	0.1269
$\alpha^2_{21} = 0.60$	MC 2SLS bias	-0.0613	-0.0793	-0.0952	-0.0152	-0.0197	-0.0114
	Approximation bias	-0.0580	-0.0803	-0.0811	-0.0191	-0.0217	-0.0247
	Simultaneity part	-0.0590	-0.0972	-0.1033	-0.0679	-0.1000	-0.0189
	Dynamic Part	0.0010	0.0169	0.0222	0.0488	0.0783	-0.0058

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Table 1 – continued from previous page

		$T=50$			$T=100$		
		$L=2$	$L=4$	$L=6$	$L=2$	$L=4$	$L=6$
$\alpha_{31}^{2}=-0.38$	MC 2SLS bias	0.2232	0.0391	0.1675	0.0800	0.0688	0.0994
	Approximation bias	0.3746	0.0407	0.1803	0.1018	0.0825	0.1098
	Simultaneity part	0.7055	0.0967	0.4849	0.1164	0.0755	0.3245
	Dynamic Part	-0.3309	-0.0056	-0.3046	-0.0146	0.007	-0.2147
$\alpha_{11}^{3}=0.65$	MC 2SLS bias	-0.0639	-0.0962	-0.2440	-0.0231	-0.0297	-0.0596
	Approximation bias	-0.0702	-0.1208	-0.2921	-0.0259	-0.0540	-0.0732
	Simultaneity part	-0.1840	-0.5007	-0.3786	-0.1027	-0.0708	-0.1102
	Dynamic Part	0.1138	0.3799	0.0865	0.0768	0.0168	0.0370
$\alpha_{21}^{3}=1.20$	MC 2SLS bias	-0.1081	-0.2849	-0.2184	-0.053	-0.0465	-0.0876
	Approximation bias	-0.1399	-0.2087	-0.2034	-0.0507	-0.0603	-0.1280
	Simultaneity part	-0.1539	-0.5886	-0.2733	-0.1497	-0.1010	-0.3024
	Dynamic Part	0.0140	0.3799	0.0699	0.0990	0.0407	0.1744
$\alpha_{31}^{3}=0.38$	MC 2SLS bias	-0.0874	-0.1323	-0.1399	-0.0386	-0.0318	-0.0435
	Approximation bias	-0.1064	-0.2073	-0.1601	-0.0411	-0.0535	-0.0739
	Simultaneity part	-0.1559	-0.3960	-0.2609	-0.0533	-0.0720	-0.1032
	Dynamic Part	0.0495	0.1887	0.1008	0.0122	-0.0185	0.0293
$\alpha_{11}^{4}=0.50$	MC 2SLS bias	-0.0006	-0.1251	-0.0851	0.0017	0.0062	-0.0023
	Approximation bias	-0.0011	-0.0987	-0.1003	0.0020	0.0110	-0.0004
	Simultaneity part	-0.0096	-0.1703	-0.0673	-0.0170	-0.0413	-0.0107
	Dynamic Part	0.0085	0.0716	-0.033	0.0150	0.0303	0.0103
$\alpha_{21}^{4}=0.60$	MC 2SLS bias	-0.0261	-0.0246	-0.1820	-0.0055	-0.0187	-0.0450
	Approximation bias	-0.0258	-0.0208	-0.1921	-0.0068	-0.0335	-0.0410
	Simultaneity part	-0.1456	-0.0736	-0.8031	-0.0108	-0.1024	-0.1599
	Dynamic Part	0.1198	0.0528	0.6110	0.0040	0.0689	0.1189
$\alpha_{31}^{4}=-0.20$	MC 2SLS bias	0.0204	0.2545	-0.0040	0.0076	0.0181	0.0100
	Approximation bias	0.0340	0.3612	-0.0091	0.0030	0.0211	0.0263
	Simultaneity part	0.1723	0.4532	-0.0195	0.0010	0.0760	0.0781
	Dynamic Part	-0.1383	-0.092	0.0104	0.0020	-0.0549	-0.0518
$c_{11}=1.00$	MC 2SLS bias	-0.0944	-0.2015	-0.2674	-0.0373	-0.0437	-0.0721
	Approximation bias	-0.0821	-0.2872	-0.3813	-0.0374	-0.0386	-0.1071
	Simultaneity part	-0.1966	-0.5648	-0.5241	-0.1067	-0.0977	-0.2654
	Dynamic Part	0.1145	0.2776	0.1428	0.0693	0.0591	0.1583

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Table 1 – continued from previous page

		$T = 50$			$T = 100$		
		$L = 2$	$L = 4$	$L = 6$	$L = 2$	$L = 4$	$L = 6$
$c_{21} = 0.60$	MC 2SLS bias	-0.0570	-0.1148	-0.1111	-0.0231	-0.0275	-0.0410
	Approximation bias	-0.0846	-0.1590	-0.1846	-0.0252	-0.0290	-0.0572
	Simultaneity part	-0.1129	-0.3222	-0.4027	-0.1016	-0.0713	-0.1404
	Dynamic Part	0.0283	0.1632	0.2181	0.0764	0.0423	0.0832
$c_{31} = -0.50$	MC 2SLS bias	0.0471	0.1004	0.0973	0.0188	0.0247	0.0325
	Approximation bias	0.0778	0.1264	0.0703	0.0221	0.0240	0.0488
	Simultaneity part	0.1543	0.2651	0.1176	0.0731	0.0381	0.0529
	Dynamic Part	-0.0765	-0.1387	-0.0473	-0.0510	-0.0141	-0.0041

Table 1 presents the bias approximation of the 17 first structural form coefficients in two stage least square estimators and the bias of the Monte Carlo two stage least square estimator. The bias approximation comes from dynamic part and simultaneity part are also reported separately in Table 1.The sample size is 50 and 100 respectively, and for the over-identification level we choose three different cases ($L = 2$, $L = 4$ and $L = 6$).

* Both the Monte Carlo bias and the bias approximation increase when the sample size increases from 50 to 100 in the coefficients $\alpha_{21}^{1} = 0.36$, when $L = 4, 6$; $\alpha_{31}^{2} = -0.38$, when $L = 4$; $\alpha_{31}^{4} = -0.2$, when $L = 6$). It seems abnormal, however,that as in my other experiments, the bias increases when the sample size increases from 50 to 70, then decreases again when the sample size increases. Thus, the trend of the bias of these coefficients decreases when sample size increases. Please see the Note table 4 .

Table 2: Bootstrap and C2SLS bias, when L=2, 4, 6; T=50, 100

		$T = 50$			$T = 100$		
		$L = 2$	$L = 4$	$L = 6$	$L = 2$	$L = 4$	$L = 6$
$\beta_{21} = 2.00$	MC 2SLS bias	-0.3032(-15%)	-0.6444(-32%)	-0.6318(-32%)	-0.1260(-6%)	-0.1600(-8%)	-0.2243(-11%)
	Bootstrap bias	-0.1262(-6%)	-0.4187(-21%)	-0.4490(-22%)	-0.0289(-1%)	-0.0518(-3%)	-0.0851(-4%)
	C2SLS bias	-0.0200(-1%)	0.0303(-2%)	0.0396(+2%)	0.00678(+0%)	0.0188(+0%)	-0.0278(-1%)
$\beta_{31} = 5.00$	MC 2SLS bias	-0.6473(-13%)	-1.0954(-22%)	-1.0068(-20%)	-0.2539(-5%)	-0.3109(-6%)	-0.3947(-8%)
	Bootstrap bias	-0.2788(-6%)	-0.6910(-14%)	-0.7109(-14%)	-0.0505(-1%)	-0.0938(-3%)	-0.14339-3%
	C2SLS bias	0.0973(+2%)	0.0140(+6%)	-0.0016(-0%)	0.0830(+2%)	0.0315(+0%)	-0.0828(-2%)
$\alpha_{11}^{1} = 0.50$	MC 2SLS bias	0.0259(+5%)	-0.0915(-18%)	-0.0136(-3%)	0.0078(+2%)	-0.0001(-0%)	0.0144(+3%)
	Bootstrap bias	0.0208(+4%)	-0.051(-10%)	-0.0056(-1%)	0.0021(+0%)	-0.0014(-0%)	0.0063(+1%)
	C2SLS bias	0.0171(+2%)	0.0057(+3%)	0.0015(+0%)	0.0068(+1%)	0.0004(+0%)	0.0030(-1%)
$\alpha_{21}^{1} = 0.36$	MC 2SLS bias	0.0327(+9%)	-0.0107(-3%)	0.0140(+4%)	0.0151(+4%)	0.0195+5%	0.0282(+8%)
	Bootstrap bias	0.0216(+6%)	0.0013(+0%)	0.0137(+4%)	0.0050(+1%)	0.0048(+1%)	0.01449(+3%)
	C2SLS bias	0.0120(+2%)	0.0033(+0%)	0.0066(+1%)	-0.0050(-1%)	0.0016(+0%)	0.0041(+1%)
$\alpha_{31}^{1} = 0.40$	MC 2SLS bias	0.0287(+7%)	0.1280(+32%)	-0.2583(-65%)	0.0310(+8%)	0.0346(+9%)	-0.0312(-8%)
	Bootstrap bias	0.0297(+7%)	0.0745(+19%)	-0.1949(-49%)	0.0042(+1%)	0.0061(+2%)	-0.0167(-4%)
	C2SLS bias	0.0284(+1%)	0.0331(+6%)	-0.0197(-0%)	-0.0011(-0%)	-0.0030(-1%)	0.0077(+0%)
$\alpha_{11}^{2} = 1.20$	MC 2SLS bias	-0.1680(-14%)	-0.2939(-24%)	-0.2552(-21%)	-0.0615(-5%)	-0.0664(-6%)	-0.1025(-9%)
	Bootstrap bias	-0.0696(-6%)	-0.1845(-15%)	-0.1735(-14%)	-0.0119(-1%)	-0.0195(-2%)	-0.0371(-3%)
	C2SLS bias	0.0292(+2%)	0.0826(+7%)	0.0470(+4%)	0.0024(+0%)	0.0097(+0%)	0.0059(+0%)

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Table 2 – continued from previous page

		$T = 50$			$T = 100$		
		$L = 2$	$L = 4$	$L = 6$	$L = 2$	$L = 4$	$L = 6$
$\alpha^2_{21} = 0.60$	MC 2SLS bias	-0.0646(-11%)	-0.0778(-13%)	-0.0923(-15%)	-0.0147(-2%)	-0.0186(-3%)	-0.0131(-2%)
	Bootstrap bias	-0.0202(-3%)	-0.0514(-9%)	-0.0659(-11%)	-0.0024(-0%)	-0.0051(-1%)	-0.0055(-1%)
	C2SLS bias	0.0168(+0%)	-0.0079(-0%)	0.0204(+4%)	0.0013(+0%)	0.0042(+0%)	-0.0041(-1%)
$\alpha^2_{31} = -0.38$	MC 2SLS bias	0.2324(+61%)	0.0433(+11%)	0.1645(+43%)	0.0793(+21%)	0.0726(+19%)	0.1056(+28%)
	Bootstrap bias	0.1138(+30%)	0.0385(+10%)	0.1209(+32%)	0.0207(+5%)	0.0258(+7%)	0.0414(+11%)
	C2SLS bias	0.0345(+9%)	0.0200/(+5%)	0.0241(+6%)	0.0148(4%)	0.0043(+1%)	-0.0186(-5%)
$\alpha^3_{11} = 0.65$	MC 2SLS bias	-0.0601(-9%)	-0.0919(-14%)	-0.2445(-38%)	-0.0237(-4%)	-0.0304(-5%)	-0.0568(-9%)
	Bootstrap bias	-0.0201(-3%)	-0.0592(+9%)	-0.1830(-28%)	-0.0052(-1%)	-0.0101(-2%)	-0.0224(-3%)
	C2SLS bias	-0.0129(-2%)	-0.0437(-7%)	0.0182(+3%)	0.0020(+0%)	0.0042(+0%)	0.0073(+1%)
$\alpha^3_{21} = 1.20$	MC 2SLS bias	-0.1116(-9%)	-0.2871(-24%)	-0.2161(-18%)	-0.0518(-4%)	-0.0466(-4%)	-0.0834(-7%)
	Bootstrap bias	-0.0476(-4%)	-0.1772(-15%)	-0.1522(-13%)	-0.0109(-1%)	-0.0139(-1%)	-0.0298(-2%)
	C2SLS bias	-0.0192(-1%)	0.0382(+3%)	0.0063(+1%)	0.0024(+0%)	0.0097(+0%)	0.0060(+0%)
$\alpha^3_{31} = 0.38$	MC 2SLS bias	-0.0941(-25%)	-0.1343(-35%)	-0.1425(-38%)	-0.0383(-10%)	-0.0361(-10%)	-0.0520(-14%)
	Bootstrap bias	-0.0467(-12%)	-0.0788(-21%)	-0.0878(-23%)	-0.0102(-3%)	-0.0105(-3%)	-0.0196(-5%)
	C2SLS bias	-0.0060(-1%)	-0.0301(-8%)	-0.0208(-6%)	-0.0027(-0%)	0.0048(+1%)	-0.0268(-7%)
$\alpha^4_{11} = 0.50$	MC 2SLS bias	-0.0005(-0%)	-0.1267(-25%)	-0.0836(-17%)	0.0015(+0%)	0.0053(+1%)	-0.0022(-0%)
	Bootstrap bias	0.0116(+2%)	-0.0740(-15%)	-0.0522(-10%)	0.0007(+0%)	0.0019(+0%)	-0.0006(-0%)
	C2SLS bias	0.0003(+0%)	0.0250(+5%)	0.0084(+0%)	0.0001(+0%)	0.0010(+0%)	0.0010(+0%)

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Table 2 – continued from previous page

		$T = 50$			$T = 100$		
		$L = 2$	$L = 4$	$L = 6$	$L = 2$	$L = 4$	$L = 6$
$\alpha_{21}^4 = 0.60$	MC 2SLS bias	-0.0241(-4%)	-0.0254(-4%)	-0.1816(-30%)	-0.0059(-1%)	-0.0194(-3%)	-0.0454(-8%)
	Bootstrap bias	-0.0064(-1%)	-0.01999(-3%)	-0.1306(-22%)	-0.0012(-0%)	-0.0062(-1%)	-0.0185(-3%)
	C2SLS bias	0.0011(+0%)	-0.0157(-2%)	-0.0135(-2%)	0.0013(-0%)	0.0023(+0%)	0.0038(+0%)
$\alpha_{31}^4 = -0.20$	MC 2SLS bias	0.0201(+10%)	0.2538(+127%)	-0.0084(-4%)	0.0037(+2%)	0.0171(+9%)	0.0134(+7%)
	Bootstrap bias	0.0074(+4%)	0.1500(+75%)	-0.0156(-8%)	-0.0006(-0%)	0.0033(+2%)	0.0031(+2%)
	C2SLS bias	0.0036(+2%)	0.0201(+10%)	-0.0024(-0%)	-0.0001(-0%)	-0.0104(-1%)	0.0031(+2%)
$c_{11} = 1.00$	MC 2SLS bias	-0.0947(-9%)	-0.2024(-20%)	-0.2683(-27%)	-0.0365(-4%)	-0.0440(-4%)	-0.0707(-7%)
	Bootstrap bias	-0.0316(-3%)	-0.1237(-12%)	-0.1898(-19%)	-0.0070(-1%)	-0.0138(-1%)	-0.0267(-3%)
	C2SLS bias	-0.0252(+2%)	-0.0080(-1%)	-0.0091(-1%)	0.0024(+0%)	0.0072(+1%)	0.0081(+1%)
$c_{21} = 0.60$	MC 2SLS bias	-0.0572(-10%)	-0.1154(-19%)	-0.1108(-18%)	-0.0231(-4%)	-0.0273(-5%)	-0.0404(-7%)
	Bootstrap bias	-0.0233(-4%)	-0.0747(-12%)	-0.0799(-13%)	-0.0051(-1%)	-0.0088(-1%)	-0.0154(-3%)
	C2SLS bias	-0.0127(-2%)	-0.0291(-5%)	-0.0076(-1%)	0.0014(+0%)	0.0031(+0%)	0.0040(+1%)
$c_{31} = -0.50$	MC 2SLS bias	0.0471(+9%)	0.1006(+20%)	0.0969(+19%)	0.0189(+4%)	0.0247(+5%)	0.0322(+6%)
	Bootstrap bias	0.0197(+4%)	-0.0424(+13%)	0.0694(+14%)	0.0041(+1%)	0.0079(+2%)	0.0123(+2%)
	C2SLS bias	-0.0087(-2%)	-0.0232(-5%)	-0.0097(-2%)	-0.0008(-0%)	-0.0030(-1%)	-0.0023(-0%)

Table 2presents the bias of the 17 first structural form coefficients in the corrected estimators, i.e.residual bootstrap two stage least square estimator, and corrected two stage least square estimator by subtracting the estimated bias approximation. The sample size is 50 and 100 respectively, and for the over-identification level we choose three different cases ($L = 2$, $L = 4$ and $L = 6$).

The Monte Carlo bias increase when the sample size increases from 50 to 100 in the coefficients $\alpha_{21}^1 = 0.36$, when $L = 4, 6$; $\alpha_{31}^2 = -0.38$, when $L = 4$; $\alpha_{31}^4 = -0.2$, when $L = 6$). It seems abnormal, however,that as in my other experiments, the bias increases when the sample size increases from 50 to 70, then decreases again when the sample size increases. Thus, the trend of the bias of these coefficients decreases when sample size increases. Please see the Note table 4 .

Table 3: The MSE of Bootstrap and C2SLS, when L=2, 4, 6; T=50, 100

		$T = 50$			$T = 100$		
		$L = 2$	$L = 4$	$L = 6$	$L = 2$	$L = 4$	$L = 6$
$\beta_{21} = 2.00$	MSE of MC 2SLS	0.5033	0.6161	0.5510	0.1580	0.0956	0.1169
	MSE of Bootstrap	0.9837	0.5892	0.4768	0.1970	0.0992	0.1077
	MSE of C2SLS	0.4173	0.4360	0.4275	0.1334	0.0829	0.0991
$\beta_{31} = 5.00$	MSE of MC 2SLS	2.0151	1.8829	1.4067	0.6353	0.3847	0.3905
	MSE of Bootstrap	3.6960	1.8449	1.2041	0.7848	0.4005	0.3647
	MSE of C2SLS	1.5251	1.3289	1.0154	0.6003	0.3419	0.3509
$\alpha^1_{11} = 0.50$	MSE of MC 2SLS	0.0399	0.0326	0.0175	0.0152	0.0104	0.0084
	MSE of Bootstrap	0.0544	0.0395	0.0224	0.0172	0.0119	0.0098
	MSE of C2SLS	0.0368	0.0322	0.0180	0.0160	0.0105	0.0097
$\alpha^1_{21} = 0.36$	MSE of MC 2SLS	0.0703	0.0514	0.0438	0.0315	0.0267	0.0254
	MSE of Bootstrap	0.1024	0.0738	0.0561	0.0359	0.0306	0.0294
	MSE of C2SLS	0.0464	0.0477	0.0451	0.0332	0.0290	0.0277
$\alpha^1_{31} = 0.40$	MSE of MC 2SLS	0.3922	0.2388	0.3113	0.1783	0.1446	0.1257
	MSE of Bootstrap	0.6026	0.3167	0.3708	0.2034	0.1654	0.1501
	MSE of C2SLS	0.3392	0.2297	0.3106	0.1542	0.1445	0.1302
$\alpha^2_{11} = 1.20$	MSE of MC 2SLS	0.1794	0.1567	0.1102	0.0585	0.0332	0.0362
	MSE of Bootstrap	0.3237	0.1647	0.1012	0.0713	0.0367	0.0366
	MSE of C2SLS	0.1774	0.1581	0.1012	0.0546	0.0337	0.0308
$\alpha^2_{21} = 0.60$	MSE of MC 2SLS	0.1081	0.0625	0.0577	0.0375	0.0294	0.0250
	MSE of Bootstrap	0.1897	0.0868	0.0713	0.0442	0.0346	0.0298
	MSE of C2SLS	0.1056	0.0610	0.0569	0.0398	0.0294	0.0272
$\alpha^2_{31} = -0.38$	MSE of MC 2SLS	0.4005	0.1734	0.1813	0.1604	0.1238	0.1123
	MSE of Bootstrap	0.6020	0.2404	0.2166	0.1863	0.1413	0.1270
	MSE of C2SLS	0.4003	0.1641	0.1802	0.1409	0.1238	0.1107
$\alpha^3_{11} = 0.65$	MSE of MC 2SLS	0.0940	0.0415	0.1167	0.0331	0.0214	0.0245
	MSE of Bootstrap	0.1662	0.0550	0.1194	0.0394	0.0246	0.0277
	MSE of C2SLS	0.0923	0.0404	0.1069	0.0328	0.0213	0.0270
$\alpha^3_{21} = 1.20$	MSE of MC 2SLS	0.1317	0.1758	0.1044	0.055	0.0348	0.0401
	MSE of Bootstrap	0.2106	0.1893	0.1070	0.0653	0.0392	0.0434
	MSE of C2SLS	0.1300	0.1752	0.0833	0.0545	0.0342	0.0417

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Table 3 – continued from previous page

		$T = 50$			$T = 100$		
		$L = 2$	$L = 4$	$L = 6$	$L = 2$	$L = 4$	$L = 6$
$\alpha_{31}^{3} = 0.38$	MSE of MC 2SLS	0.2642	0.1577	0.1408	0.1273	0.1057	0.0923
	MSE of Bootstrap	0.3739	0.2131	0.1668	0.1469	0.1234	0.1086
	MSE of C2SLS	0.2138	0.1483	0.1357	0.1264	0.1068	0.0920
$\alpha_{11}^{4} = 0.50$	MSE of MC 2SLS	0.0592	0.0523	0.0434	0.0196	0.0124	0.0188
	MSE of Bootstrap	0.0925	0.0631	0.0524	0.0224	0.0141	0.0138
	MSE of C2SLS	0.0584	0.0522	0.0434	0.0203	0.0124	0.0136
$\alpha_{21}^{4} = 0.60$	MSE of MC 2SLS	0.0631	0.0317	0.0791	0.0252	0.0203	0.0216
	MSE of Bootstrap	0.0990	0.0438	0.0842	0.0293	0.0234	0.0246
	MSE of C2SLS	0.0635	0.0301	0.0677	0.0260	0.0199	0.0234
$\alpha_{31}^{4} = -0.20$	MSE of MC 2SLS	0.2089	0.1930	0.1031	0.0879	0.0663	0.0602
	MSE of Bootstrap	0.2914	0.2207	0.1326	0.1009	0.0754	0.706
	MSE of C2SLS	0.2075	0.1852	0.1031	0.0820	0.0661	0.0585
$c_{11} = 1.00$	MSE of MC 2SLS	0.1111	0.0892	0.1125	0.0277	0.0138	0.0169
	MSE of Bootstrap	0.2136	0.100	0.1042	0.0344	0.0153	0.0171
	MSE of C2SLS	0.0982	0.0901	0.1042	0.0254	0.0135	0.0137
$c_{21} = 0.60$	MSE of MC 2SLS	0.0214	0.0209	0.0177	0.007	0.0036	0.0046
	MSE of Bootstrap	0.0425	0.0210	0.0160	0.0088	0.0040	0.0045
	MSE of C2SLS	0.0150	0.0203	0.0086	0.0069	0.0034	0.0038
$c_{31} = -0.50$	MSE of MC 2SLS	0.0139	0.0160	0.01380	0.0049	0.0030	0.0030
	MSE of Bootstrap	0.0272	0.0115	0.01250	0.0062	0.0032	0.0029
	MSE of C2SLS	0.0097	0.0164	0.0115	0.0040	0.0026	0.0031

Table 3 presents the mean squared errors of the 17 first structural form coefficients in the Monte Carlo two stage least squares, corrected two stage least squares and bootstrap two stage least squares respectively. The sample size is 50 and 100 respectively, and for the over-identification level we choose three different cases ($L = 2$, $L = 4$ and $L = 6$) .

C Other Experiments for 2SLS

Table 4: **Percentages of the bias in Monte Caro 2SLS estimation, when L= 4, 6; T=50, 100**

	$L=4$				$L=6$			
	$L=4$				$L=6$			
	$T=50$	$T=70$	$T=90$	$T=100$	$T=50$	$T=70$	$T=90$	$T=100$
$\alpha_{21}^{1}=0.36$	-3%	8%	6%	5%	4%	8%	8%	8%
$\alpha_{31}^{2}=-0.38$	NaN	NaN	NaN	NaN	11%	25%	21%	19%
$\alpha_{31}^{4}=-0.20$	-4%	51%	18%	7%	NaN	NaN	NaN	NaN

Table 4 presents the trend of some related coefficients in certain cases. The over-identification level is $L=4,6$. The sample size is 50 and 100 respectively.