UNIVERSIDAD CARLOS III DE MADRID working papers

Working Paper 02-56
Economics Series 20
November 2002

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ON THE GENERIC STRATEGIC STABILITY OF NASH EQUILIBRIA IF VOTING IS COSTLY *

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Abstract

We prove that for generic plurality games with positive cost of voting, the number of Nash equilibria is finite. Furthermore all the equilibria are regular, hence stable sets as singletons.

Keywords: Plurality Rule, Regular Equilibria, Stable Sets.

JEL Classification: C72, D72.

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* We would like to thank Luisa Bassotto for helpful comments.

1 Introduction

Plurality games without cost of voting are a typical example of a game structure in which, also for generic preferences, there is a infinite number of Nash equilibria and where a concept like Nash equilibrium is completely inadequate.[1] Moreover, also perfect, and even proper, equilibrium falls far short of the traditional concepts in this area that call for "sophisticated voting".

In this note we prove that these facts are not anymore true with positive cost of voting.

Under plurality rule each voter can vote for any candidate or abstain. The candidate receiving the largest total amount of votes wins the election. Any voter receives a payoff from the election of a candidate, and, unless he abstains, pays a positive cost. We prove that, in the class of plurality games with positive cost of voting above described, the number of Nash equilibria is generically finite. Furthermore, all the equilibria are regular, hence stable sets as singletons. This implies that "sophisticated voting" cannot eliminate them.

The motivation to undertake this analysis is twofold. First of all, a basic tool in applying non-cooperative game theory is to have finiteness of the set of equilibrium distribution. Since the conjecture that equilibrium distribution is generically finite for every game-form has been proved to be incorrect (see Govindan and McLennan, 2001), one has to prove such a result for each class of games, and we prove it for the game-form arising from plurality costly voting. Second, the result that all the equilibria are regular directly implies that they cannot be eliminated by the usual refinements based either on perturbation of utilities or on perturbation of strategies, and hence the Nash solution concept appears to be completely adequate, differently from the case of plurality games without cost of voting. We describe the model in Section 2, and we present the results in Section 3.

2 The Model

Let $K = (1, ..., k)$ be the finite set of candidates and $N = (1, ..., n)$ the finite set of voters. Under plurality rule every voter has $k + 1$ pure strategies, namely voting for each candidate or abstaining. Given a pure strategy vector, the candidate receiving the largest amount of votes is elected, while in case of a tie we assume an equal probability lottery among the winners. Hence, the set of candidates K and the set of voters N define a family of games; each game in this family is identified by the utility vectors $\left\{u^i\right\}_{i\in N}$, where $u^i = \left(u^i_1, ..., u^i_k\right)$ and each u^i_c represents the player i's utility for the election of candidate c, and by the vector of costs of voting $\delta = (\delta^1, ..., \delta^n)$. Hence, every plurality game with costly voting with n voters and k candidates can be seen as a point $(u, \delta) \in \Re^{nk} \times \Re^n_{++}$. Because voting for any candidate costs δ^i, for every mixed strategy combination

[1] Obviously, the election of any candidate can be a Nash equilibrium outcome, if there are at least three voters.

σ, the expected payoff of player i is simply

$$U^i(\sigma) = \delta^i(\sigma_0^i - 1) + \sum_{c \in K} p(c \mid \sigma) u_c^i$$

or in vector notation

$$U^i(\sigma) = \langle p(\sigma), u^i \rangle - \delta^i(1 - \sigma_0^i),$$

where $p(c \mid \sigma)$ is the probability that candidate c is elected under σ, σ_0^i is the probability that i abstains and $p(\sigma) = (p(1 \mid \sigma), p(2 \mid \sigma), ..., p(k \mid \sigma))$.

In the following $K_0 = K \cup \{0\}$ denotes the strategy space of each player and we will use a superscript to indicate the voter and a subscript to denote the pure strategy, i.e. σ_c^i is the probability that player i votes for c while σ_0^i is the probability that he abstains.

3 The Result

In this section we prove that for generic plurality games with cost of voting every Nash equilibrium is regular, in the Harsany's definition.[2] Our proof follows the lines of De Sinopoli (2001) where it is proved that, without cost of voting, generically, all the equilibria which induce a mixed distribution over candidates are regular. To prove our result, we extensively use the vector $\pi^c(\sigma^{-i})$ that expresses the difference in the probability distribution of the electoral outcomes if player i votes for $c \in K$ or if he abstains, under the fixed strategy σ^{-i} of the others.

Definition 1 $\pi^c(\sigma^{-i}) = p(\sigma, c^i) - p(\sigma, 0^i)$.

The next results (see De Sinopoli, 2001) on the vector $\pi^c(\sigma^{-i})$ are quite obvious.

Lemma 2 (α) $\sum_{k \in K} \pi_k^c(\sigma^{-i}) = 0$;
(β) $\pi_c^c(\sigma^{-i}) \geq 0$ *and for every* $k \neq c$, $\pi_k^c(\sigma^{-i}) \leq 0$.

Furthermore, the fact that voting is costly immediately implies:

Lemma 3 $\forall \sigma^i \in BR(\sigma^{-i}),\ \sigma_c^i > 0 \Longrightarrow \langle \pi^c(\sigma^{-i}), u^i \rangle > 0$.

[2] More precisely, we use the modified version of regularity, as proposed in van Damme (1991). The definition of van Damme differs from Harsanyi's one in requiring that the strategy used as reference point is contained in the support of the equilibrium, while Harsanyi uses the first strategy for each player. However Harsanyi assumes, in the various proofs, that his reference point belongs to the support of the equilibrium, hence he actually works with the same definition. For the definition of regularity and its properties we refer to van Damme (1991).

Let $C_i, B_i \subset K_0$, $C = \prod_i C_i$ and $B = \prod_i B_i$. Let $IR(C,B)$ denotes the set of games that have an irregular equilibrium with support C and with pure best replies (PBR) B.

Fix C, B such that $\emptyset \neq C \subseteq B$ (obviously, if C and B do not satisfy these conditions, $IR(C,B)$ is empty).

Let N^B be the set of players for which $0 \in B_i$ and let us consider a pure strategy vector $v_* \in B$ such that $v_*^i = 0$ if $i \in N^B$. Let $L_i = \left\{c^i \in K_0 : c^i \in B_i \backslash v_*^i\right\}$.

Consider a game (u, δ) which has an equilibrium σ with $C(\sigma) = C$ and $PBR(\sigma) = B$. Then the following equalities hold:

$$\left[U^i\left(\sigma, c^i\right) - U^i\left(\sigma, v_*^i\right)\right] = 0 \quad \forall i \in N,\ \forall c^i \in L_i. \tag{1}$$

Denoting $l_i = \#L_i = \#B_i - 1$, $l = \sum_{i=1}^{n} l_i$ and $n^b = N^B$, define the following subvectors of $(u, \delta) \in \Re^{nk} \times \Re^n_{++}$: $u_* \in \Re^{n-n^b}$, which specifies the utility that a player $i \notin N^B$gets if the candidate that he votes for in v_* is elected, $u_{**} \in \Re^{nk-l-n+n^b}$, which specifies each voter's utility for the candidates voting for whom it is not a best reply, and $u^\circ \in \Re^l$, which specifies each player i's utility for the candidates in L_i. To simplify the notation, let $u^* = (u_*, u_{**}, \delta)$. Clearly $(u, \delta) = (u^*, u^\circ)$.

Given σ, v_* and u^*, (1) is a linear system of l equations in l unknowns, the u°. Since for every player i we have l_i equations, the system can be written as:

$$\begin{aligned}
\Pi_*^\circ\left(\sigma^{-i}\right) u^{\circ i} &= -\Pi_*^*\left(\sigma^{-i}\right) u^{*i} + \vec{\delta}^i \quad \forall i \in N^B \\
\Pi_*^\circ\left(\sigma^{-i}\right) u^{\circ i} &= -\Pi_*^*\left(\sigma^{-i}\right) u^{*i} \quad \forall i \notin N^B
\end{aligned}$$

where $\Pi_*^\circ\left(\sigma^{-i}\right)$ and $u^{\circ i}$ are, respectively, a square matrix and a column vector of dimension l_i, while $-\Pi_*^*\left(\sigma^{-i}\right) u^{*i}$ is a column vector. The whole system has the following matrix representation:

$$\Pi_*^\circ(\sigma)\, u^\circ = -\Pi_*^*(\sigma)\, u^* + \tilde{\delta} \tag{2}$$

where $\tilde{\delta}$ is the obvious column vector.

The system in (2) has an unique solution if the square matrix $\Pi_*^\circ(\sigma)$ of dimension l is not singular. The following lemma proves this.

Lemma 4 *The matrix $\Pi_*^\circ(\sigma)$ has a strictly positive determinant, hence it is not singular.*

Proof. The matrix $\Pi_*^\circ(\sigma)$ is block diagonal, where each block coincides with a matrix $\Pi_*^\circ\left(\sigma^{-i}\right)$. Hence, if each $\Pi_*^\circ\left(\sigma^{-i}\right)$ has a strictly positive determinant, $\Pi_*^\circ(\sigma)$ is not singular. As mentioned, $\Pi_*^\circ\left(\sigma^{-i}\right)$ is a square matrix of dimension l_i. The (c, m) entry of the square matrix $\Pi_*^\circ\left(\sigma^{-i}\right)$ is the probability that candidate m is elected if player i votes for c minus the probability that m is elected if i plays v_*^i, with $c, m \in L_i$. By definition:

$$\pi_*^c\left(\sigma^{-i}\right) = p\left(\sigma, c^i\right) - p\left(\sigma, v_*^i\right) = \pi^c\left(\sigma^{-i}\right) - \pi^{v_*^i}\left(\sigma^{-i}\right)$$

where, with abuse of notation, $\pi^0\left(\sigma^{-i}\right) = \vec{0}$. Then the (c,m) entry of the matrix $\Pi^{\circ}_{*}\left(\sigma^{-i}\right)$ is:

$$\Pi^{\circ}_{*}\left(\sigma^{-i}\right)_{cm} = (a_{cm} + b_m) \qquad c, m \in L_i$$

where:

$$a_{cm} = \pi^{c}_{m}\left(\sigma^{-i}\right) \text{ and } b_m = -\pi^{v^i_*}_{m}\left(\sigma^{-i}\right)$$

Lemma 2 implies that the matrix $A = (a_{cm})$ is an improper Minkowsky-matrix and Lemma 3 implies that all the diagonal elements of A are strictly positive. The next step is to show that the matrix A is dominant diagonal. To this end, let us define the following functions:

$$s_c(d) = a_{cc}d_c - \sum_{m \in L_i/c} |a_{cm}|\, d_m = \sum_{m \in L_i} a_{cm} d_m \quad c \in L_i$$

We have to show that a $d^* \gg 0$ exists such that $s_c(d^*) > 0\ \forall\, c \in L_i$. By lemma $2(\alpha)$ we know that $\sum_{m \in K} a_{cm}$ is equal to zero and by Lemma 3 we know that $\sum_{m \in K} a_{cm} u^i_m > 0$. Hence for every $\varepsilon > 0$, $\sum_{m \in K} a_{cm} \varepsilon u^i_m > 0$. For ε sufficiently close to zero: $1 + \varepsilon u^i_m > 0\ \ \forall m \in K$. Choose an $\tilde{\varepsilon}$ that satisfies these conditions: $d^* = \vec{1} + \tilde{\varepsilon} u^i \gg 0$ and $s_c(d^*) > 0\ \ \forall c \in L_i$. In fact

$$s_c(d^*) \geq \sum_{m \in K} a_{cm}(1 + \tilde{\varepsilon} u^i_m) > 0 \quad \forall c \in L_i$$

where the first inequality follows from lemma $2(\beta)$ and the positivity of each $1+\tilde{\varepsilon} u^i_m$, and the second one from lemma $2(\alpha)$ and the positivity of $\sum_{m \in K} a_{cm} \tilde{\varepsilon} u^i_m$. This proves that the matrix A has a (positive) dominant diagonal and implies that all the principal minors of A are strictly positive. The matrix A is then an M-matrix, in the definition of Ostrowsky (1955, p.95). In Ostrowsky (1955, p.97) the following result is contained: if Q is an $n \times n$ M-matrix then:

$$|q_{ij} + d_j| \geq |q_{ij}| > 0 \quad for\ d_j \geq 0 \ \ (j = 1, \ldots, n)$$

This result with the fact that $b_m \geq 0$ (Lemma $2(\beta)$) implies the claim. ■

Now we can prove our main result.

Theorem 5 *For generic plurality games with cost of voting, any Nash equilibrium is regular.*

Proof. To prove the proposition it is enough to prove that, for every possible C and B, the set $IR(C,B)$ is a semi-algebraic set of dimension less than $n(k+1)$. Lemma 4 implies that given an equilibrium σ with support C and pure best replies B, a "reference" strategy $v_* \in C$ and the corresponding subvector u^*, we can uniquely reconstruct the entire vector (u, δ).

For C, B such that $\emptyset \neq C \subseteq B$, let us define the following set:
Let $E^{C,B}$ be the graph of the correspondence that associates to each game in $\Re^{nk} \times \Re^{n}_{++}$ its equilibria with support equal to C and pure best reply B, i.e., $E^{C,B} = \left\{(u, \delta, \sigma) \mid (u, \delta) \in \Re^{nk} \times \Re^{n}_{++}, C(\sigma) = C, PBR(\sigma) = B, \sigma \in NE(u, \delta)\right\}$
Let $E^{C,B}_{*}$ be the projection of $E^{C,B}$ on the strategy space and on those coordinates of the utility vector not corresponding to L_i, i.e. $E^{C,B}_{*} = proj_{\left(\Sigma^n \times \Re^{nk-l} \times \Re^{n}_{++}\right)} E^{C,B}$.
The above construction implies that there is a function $T^{C,B_*} : E^{C,B}_{*} \to \Re^{nk} \times \Re^{n}_{++}$ that maps (u^*, σ) into $u = (u^*, u^\circ)$, defined by:

$$T^{C,B_*} : \begin{array}{l} u^* = u^* \\ u^\circ = \left(\Pi^\circ_*(\sigma)\right)^{-1} \left(-\Pi^{*\shortmid}_{*}(\sigma) u^* + \tilde{\delta}\right). \end{array}$$

The map T^{C,B_*} is smooth as it involves additions, subtractions, multiplications and only a division by the determinant of $\Pi^\circ_*(\sigma)$, which is greater than zero by lemma 4. Furthermore, the sets $E^{C,B}_{*}$ and $\Re^{nk} \times \Re^{n}_{++}$ as well as the map T^{C,B_*} are semi-algebraic. Hence, the result is trivial if C is strictly included in B, since, in this case, by Bochnak *et. al.* (1987, Th. 2.8.8), we have:

$$\dim\left(T^{C,B_*}(E^{C,B}_{*})\right\} \leq \dim E^{C,B}_{*} = \#C - n + n(k+1) - \#B + n < n(k+1)$$

(i.e. having fixed C and B, $\dim(\Sigma^n \mid C) = \#C - n$ and $\dim(L) = \#B - n$).
Hence, we need to prove the statement for the case $B = C$.
In this case the equilibrium is quasi-strict and, hence, it is regular if and only if it is not singular the Jacobian of the map $\tilde{F}(\sigma \mid C, v_*)$ defined by

$$\tilde{F}^{v_{*i}}_{i}(\sigma \mid C, v_*) = \sigma^i_c \left[U^i\left(\sigma, c^i\right) - U^i\left(\sigma, v^i_*\right)\right] \quad \forall i \in N,\ \forall c^i \in L_i$$

Obviously, the Jacobian of T^{C,B_*} does not change if we cross out the rows and the columns corresponding to "candidates" not belonging to L. The corresponding map $\tilde{T}^{C,B_*}$ is implicitly defined by $\tilde{F}(\sigma \mid C, \bar{p}) = 0$. Then, by Lemma 4, the equilibrium is regular if and only if the Jacobian of T^{C,B_*} is not singular. The semi-algebraic version of Sard's Theorem, cf. Bochnak *et. al.* (1987, Th. 9.5.2, p.205), assuring that the set of critical values of T^{C,B_*} is a semi-algebraic set of dimension strictly less than $n(k+1)$, completes the proof. ■

The next corollary, which is an immediate consequence of the above theorem, summarizes our main findings.

Corollary 6 *For generic plurality games with cost of voting:*
α) *The set of Nash equilibria is finite.*
β) *Any Nash equilibrium is a stable set (Mertens, 1989).*

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