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STABLE LIMITS FOR EMPIRICAL PROCESSES ON VAPNIK-ČERVONENKIS CLASSES OF FUNCTIONS

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Abstract

Alexander's (1987) central limit theorem for empirical processes on Vapnik-Červonenkis classes of functions is extended to the case with non-Gaussian stable limits. The corresponding weak laws of large numbers are also established.

Key Words.
Empirical processes; Vapnik-Červonenkis classes of functions; stable domains of attraction.

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1. Introduction

Pollard (1982) obtained sufficient conditions for the central limit theorem for empirical processes on Vapnik-Červonenkis classes of functions which was completely characterized by Alexander (1987). Our main goal in this paper is to extend Alexander's result to the case of non-Gaussian stable limit.

Let $(S,\mathcal{S})$ be a measurable space and let P be a probability measure on it. Let $\{X_n: n \in \mathbb{N}\}$ be the sequence of coordinate functions in $(S^{\mathbb{N}},\mathcal{S}^{\mathbb{N}},P^{\mathbb{N}})$; so, they are independent and identically distributed random variables with law P. By $\{\varepsilon_n : n \in \mathbb{N}\}$ we will always represent a Rademacher sequence, i.e., a sequence of independent and identically distributed random variables such that $P\{\varepsilon_n=+1\} = P\{\varepsilon_n=-1\} = \frac{1}{2}$.

From now on, we will use the sample space $(\Omega,\mathcal{A},\Pr) = (S^{\mathbb{N}}x[0,1],\mathcal{S}^{\mathbb{N}}x\mathcal{B}_{[0,1]},P^{\mathbb{N}}x\lambda)$, where $\mathcal{B}_{[0,1]}$ is the Borel σ-algebra for the usual topology in [0,1] and λ is the Lebesgue measure. This space is rich enough to support the sequence $\{X_n : n \in \mathbb{N}\}$ and a Rademacher sequence $\{\varepsilon_n: n \in \mathbb{N}\}$ independent from each other.

$\mathcal{L}_p(S,\mathcal{S},P)$ (or simply $\mathcal{L}_p$), $0\le p<\infty$, will be the class of real measurable functions f on S such that $\int_S |f|^p dP < \infty$. On $\mathcal{L}_p(S,\mathcal{S},P)$ we will define the pseudometric

$$\rho_p(f,g) = \rho_p^P(f,g) = \left(\int_S|f-g|^p\, dP\right)^{1/p} , \quad f,g \in \mathcal{L}_p(S,\mathcal{S},P).$$

If $f \in \mathcal{L}_1(S,\mathcal{S},P)$, we will write Pf instead of $\int_S f\, dP$. For $0 < p < \infty$, $\Lambda_p = \Lambda_p(S,\mathcal{S},P)$ will denote the space of real measurable functions f on S such that $\sup_{t>0} t^p P\{|f|>t\}<\infty$ and we will consider on it the function

$$\Lambda_{p,\infty}(f,g)=\Lambda^P_{p,\infty}(f,g)= \left(\sup_{t>0} t^p P\{|f-g|>t\}\right)^{1/p}, \quad f,g\in \Lambda_p(S,\mathcal{S},P).$$

$\Lambda_{p,\infty}(f,0)$ is equivalent to a norm for $p>1$.

We will denote by $\mathcal{F}$ a class of real measurable functions on S and we will assume that the envelope of $\mathcal{F}$, $F(s)= \sup_{f\in\mathcal{F}} |f(s)|$, is finite for all $s\in S$. Although F is not necessarily measurable, there always exists a measurable $F^*:S\longrightarrow \overline{\mathbb{R}}$ (see Dudley (1984), Theorem 3.1.1) such that $F\le F^*$ and for all measurable functions h such that $F \le h$, we have that $F^*\le h$ almost surely with respect to P. We will always assume that $(S,\mathcal{S},P)$ is complete and that $\mathcal{S}$ is countably generated (i.e., there exists $\{B_n : n\in\mathbb{N}\}\subset\mathcal{S}$ such that $\mathcal{S}=\sigma\{B_n : n\in \mathbb{N}\}$).

If $(\mathcal{X},\mathcal{E},\mu)$ is a probability space, μ^* will be the outer probability measure for μ, i.e.,

$$\mu^*(A) = \inf \{\mu(B):\ B \in \mathcal{E},\ A \subset B\}\ ,\ A \subset \mathcal{X}.$$

From Proposition 2.2 in Andersen (1985) it follows that

$$P^*\{F > t\} = P\{F^* > t\}\ ,\ \text{fo all } t \in \mathbb{R}.$$

Since the coordinate functions $X_n : S^{\mathbb{N}} \longrightarrow S$ are perfect random variables with respect to Pr(see Andersen (1985), Chapter II, section II.2 and II.3 for the definition and a detailed study of this property) it also holds that

$$\Pr^*\{F(X_n) > t\} = P\{F^* > t\}\ ,\ \text{for all } t \in \mathbb{R}.$$

Let $\{P_n : n \in \mathbb{N}\}$ be the sequence of empirical probability measures constructed from $\{X_n : n \in \mathbb{N}\}$, i.e.,$P_n = n^{-1} \sum_{i=1}^{n} \delta_{X_i}$. For $1\le p\le 2$, we will write

$$\nu_n^{(p)} = n^{1-1/p}(P_n - P)\ ,\ n \in \mathbb{N}.$$

Since F is finite for all $s \in S$, we will have that for all $n \in \mathbb{N}$ and for all $p \in [1,2]$,

$$\{\nu^{(p)}(f):\ f \in \mathcal{F}\} = \{n^{-1/p} \sum_{i=1}^{n} (f(X_i) - Pf):\ f \in \mathcal{F}\}$$

is a stochastic process with sample paths in the space $\ell^\infty(\mathcal{F})$ of real bounded functions defined on $\mathcal{F}$. We will endow $\ell^\infty(\mathcal{F})$ with the sup norm,

$$\|H\|_{\mathcal{F}} = \sup_{f\in\mathcal{F}} |H(f)|\ ,\quad H \in \ell^\infty(\mathcal{F})$$

(sometimes, with some notational abuse, we will write $\|H(f)\|_{\mathcal{F}}$).

If ρ is a pseudometric on $\mathcal{F}$,$C_u(\mathcal{F},\rho)$ will be the space of uniformly ρ-continuous bounded real functions.

Some of the calculations that will be carried out need certain measurability conditions on the class of functions $\mathcal{F}$. The following definition comes from Alexander (1987). We say that $\mathcal{F}$ is *supremum measurable* (for P) if $\sup_{f\in\mathcal{F}} |Q\{f(X_n) : n\in \mathbb{N}\}|$ is Pr-completion measurable for each function Q which is a linear or quadratic function of finitely many of the $f(X_n)$.

Recall that $U = \{U(f) : f \in \mathcal{F}\}$ is a *p-stable stochastic process*, $1\le p\le 2$, if for all $k \in N$ and for all $\{f_1,\dots,f_k\} \subset \mathcal{F}$, $(U(f_1),\dots,U(f_k))$ has a p-stable distribution in R^k. Also, $\{Z(f): f \in \mathcal{F}\}$ is a subgaussian process (or a process with subgaussian increments) if there exists $\tau>0$ such that for all $f,g \in \mathcal{F}$ and for all $\lambda \in R$,

$$E\exp\{\lambda(Z(f)-Z(g))\} \le \exp\{\lambda^2\tau E(Z(f)-Z(g))^2/2\}$$

(see Jain and Marcus (1978) and Kahane (1968)). If $Z(f) = \sum_{i=1}^{n} \varepsilon_i h_i(f)$ and $\sum_{i=1}^{\infty} h_i^2(f) < \infty$ for all $f \in \mathcal{F}$ then Z is subgaussian with $\tau = 1$. In particular, for fixed $\omega \in S^N$, the process $\{\sum_{i=1}^{n} \varepsilon_i(.)f(X_i(\omega)): f\in\mathcal{F}\}$ is subgaussian with $\tau=1$.

Since $(\ell^\infty(\mathcal{F}), \|\cdot\|_{\mathcal{F}})$ is a nonseparable (except for finite $\mathcal{F}$) Banach space and since $\nu_n^{(p)}: \Omega \longrightarrow \ell^\infty(\mathcal{F})$ is not necessarily measurable with respect to $\mathcal{A}$ and the Borel σ-algebra corresponding to the topology induced by $\|\cdot\|_{\mathcal{F}}$, we need to consider a convenient definition of weak convergence. We will follow here Hoffmann-Jørgensen's (1984) approach which we present next for functions taking values in $(\ell^\infty(\mathcal{F}), \|\cdot\|_{\mathcal{F}})$. The following definitions can be seen in Andersen, Giné and Zinn (1988). For the probability space $(\Omega, \mathcal{A}, Pr)$, we say $\xi:\Omega \longrightarrow \ell^\infty(\mathcal{F})$ is a *random element* in $\ell^\infty(\mathcal{F})$ if $\lim_{M\to\infty} Pr^*\{\|\xi\|_{\mathcal{F}}>M\} = 0$. If ξ is a random element in $\ell^\infty(\mathcal{F})$, we define the set function $\mathcal{L}^*(\xi)$ by $\mathcal{L}^*(\xi)(A) = Pr^*\{\xi \in A\}$, $A \subset \ell^\infty(\mathcal{F})$ and, for any $B \subset \ell^\infty(\mathcal{F})$, we write

$$\mathcal{L}^*(\xi)|_B(A) = Pr^*\{\xi \in A\cap B\}, \quad A \subset \ell^\infty(\mathcal{F}).$$

Let $\{\xi_n: n\in\mathbb{N}\}$ be a sequence of random elements in $\ell^\infty(\mathcal{F})$. Let $\delta>0$. We say that the sequence of set functions $\left\{\sum_{i=1}^{n} \mathcal{L}^*(\xi_i)|_{\{\|x\|_{\mathcal{F}}>\delta\}} : n\in\mathbb{N}\right\}$ is *eventually tight* if

$$\sup_{n\in N} \sum_{i=1}^{n} Pr^*\{\|\xi_i\|_{\mathcal{F}} >\delta\} < \infty$$

and for every $\varepsilon>0$ there exists a compact set $K \subset \ell^\infty(\mathcal{F})$ such that

$$\lim_{n\to\infty}\sup \sum_{i=1}^{n} Pr^*\{\xi_i \in G^c, \ \|\xi_i\|_{\mathcal{F}}>\delta\} < \varepsilon,$$

for any open set $G \supset K$.

To establish the concept of weak convergence (or convergence in law) we recall the definitions of upper integral and Radon measure in $\ell^\infty(\mathcal{F})$. If $(\mathcal{X}, \mathcal{E}, \mu)$ is a probability space and $f:\mathcal{X}\longrightarrow\mathbb{R}$ is a function then the *upper integral* of f with respect to μ is $E^*f = \int^* f d\mu = \inf\{\int h d\mu: f\le h,\ h:\mathcal{X}\longrightarrow\mathbb{R}$ measurable$\}$ (with $\inf \emptyset = +\infty$). We say that a Borel finite measure γ on $\ell^\infty(\mathcal{F})$ is a *Radon measure* if for any Borel set B in $\ell^\infty(\mathcal{F})$ we have that

$$\gamma(B) = \sup \{\gamma(K): K \subset B,\ K \text{ compact}\}.$$

For a sequence $\{\xi_n : n \in \mathbb{N}\}$ of random elements in $\ell^\infty(\mathcal{F})$ we say that it *converges in law to a Radon limit* if there exists a probability Radon measure γ in $\ell^\infty(\mathcal{F})$ such that for all bounded and $\|.\|_{\mathcal{F}}$-continuous $H: \ell^\infty(\mathcal{F}) \longrightarrow \mathbb{R}$ we have

$$\lim_{n\to\infty} E^* H(\xi_n) = \int H d\gamma.$$

$\{\xi_n : n\in\mathbb{N}\}$ is *eventually tight* if for any $\varepsilon > 0$ there exists a compact $K \subset \ell^\infty(\mathcal{F})$ such that

$$\limsup \Pr^* \{\xi_n \in G^c\} < \varepsilon$$

for any open set $G \supset K$.

Andersen, Giné and Zinn (1988) present sufficient conditions for eventual tightness of the sequence of partial sums $\left\{\sum_{i=1}^{n} \xi_i : n \in \mathbb{N}\right\}$: it is enough to have eventual tightness of the sequence of law sums outside neighborhoods of zero and convergence in probability to zero of the sums of truncated variables, uniformly in n, as the truncation levels tend to zero. These hypotheses are based on the characterization of relative compactness for sums of triangular arrays of symmetric random variables with infinitely divisible non-Gaussian limit points given by Mandrekar and Zinn (1980), Theorem 2.10.

In our study of empirical processes associated to the sequence $\{X_n : n\in\mathbb{N}\}$, the random elements $\xi_{ni} : \Omega \longrightarrow \ell^\infty(\mathcal{F})$ are of the form $\xi_{ni} = n^{-1/p}\delta_{X_i}$, $i=1,\dots,n$, and we are interested in the convergence in law of either centered sequences

$$(1.1)\qquad \{\nu_n^{(p)}(f) : f \in \mathcal{F}\} = \{n^{-1/p} \sum_{i=1}^{n} (\delta_{X_i} - P)(f) : f \in \mathcal{F}\} =$$

$$= \{n^{-1/p} \sum_{i=1}^{n} (f(X_i) - Pf) : f \in \mathcal{F}\},\quad 1 \le p \le 2$$

or symmetrized sums as

$$(1.2)\qquad \{n^{-1} \sum_{i=1}^{n} \varepsilon_i f(X_i) : f \in \mathcal{F}\}.$$

The following proposition particularizes Corollary 2.3 in Andersen, Giné and Zinn (1988) to the case of random elements in (1.1) with $1<p<2$.

PROPOSITION 1.1. Let $\mathcal{F}$ be a class of functions in $\mathcal{L}_1(S, \mathcal{S}, P)$ with envelope F finite everywere. Let $1<p<2$. Assume

(i) the sequence of set functions

$$\{n\,\mathcal{L}^*(\{n^{-1/p}f(X_1):f\in\mathcal{F}\})|_{\{\|x\|_{\mathcal{F}}>\delta\}}:n\in\mathbb{N}\}$$

is eventually tight for all $\delta > 0$,

(ii) the finite-dimensional distributions of the sequence

$$\{\{n^{-1/p}\sum_{i=1}^{n}(f(X_i)-Pf):f\in\mathcal{F}\}:n\in\mathbb{N}\}$$

converge in law to p-stable probability measures, and

(iii) for all $\eta > 0$,

$$\lim_{\delta\to0}\limsup_{n\to\infty}\Pr^*\{\|n^{-1/p}\sum_{i=1}^{n}\varepsilon_i fI_{\{F^*\le\delta n^{1/p}\}}(X_i)\|_{\mathcal{F}}>\eta\}=0.$$

Then (1.1) converges in law to a p-stable Radon probability measure in $\ell^\infty(\mathcal{F})$ determined by (ii).

The proof can be seen in Andersen, Giné and Zinn (1988). The same proof allows for sufficient conditions for convergence of the sequence (1.2).

PROPOSITION 1.2. Let $\mathcal{F}$ be a class of functions in $\mathcal{L}_0(S,\mathcal{S},P)$ with envelope F finite everywhere. Assume

(i) the sequence of set functions

$$\{n\mathcal{L}^*(\{n^{-1}f(X_1):f\in\mathcal{F}\})|_{\{\|x\|_{\mathcal{F}}>\delta\}}:n\in\mathbb{N}\}$$

is eventually tight for all $\delta>0$,

(ii) the finite-dimensional distributions of the sequence

$$\{\{n^{-1}\sum_{i=1}^{n}\varepsilon_i f(X_i):f\in\mathcal{F}\}:\ n\in\mathbb{N}\}$$

converge in law to 1-stable probability measures, and

(iii) for all $\eta > 0$,

$$\lim_{\delta\to0}\limsup_{n\to\infty}\Pr^*\{\|n^{-1}\sum_{i=1}^{n}\varepsilon_i fI_{\{F^*\le\delta n\}}(X_i)\|_{\mathcal{F}}>\eta\}=0.$$

Then (1.2) converges in law to a 1-stable Radon probability measure in $\ell^\infty(\mathcal{F})$ determined by (ii).

Next, we state Ottaviani's inequality for random elements in a convenient way to use it below. We say that $\{\xi_n:\ n\in\mathbb{N}\}$ is a sequence of *independent random elements* if they can be written as functions of the coordinates X_n:

$S^N \longrightarrow S$, i.e., i.e. $\xi_n = h(X_n)$ for some function h. Let $S_n = \sum_{i=1}^{n} \xi_i$. Since $\max_{1\le k\le n} \|\xi_k\|_{\mathcal{F}} \le 2 \max_{1\le k\le n} \|S_k\|_{\mathcal{F}}$, we have the following proposition (see Lemma 2.7 in Dudley and Philipp (1983)).

PROPOSITION 1.3. Let $\{\xi_n: n \in \mathbb{N}\}$ be a sequence of independent random elements. If

$$\max_{1\le k\le n} \Pr^*\{\|S_n - S_k\|_{\mathcal{F}} > \alpha\} = c < 1$$

then

$$\Pr^*\{\max_{1\le k\le n} \|\xi_k\|_{\mathcal{F}} > 4\alpha\} \le (1-c)^{-1}\Pr^*\{\|S_n\|_{\mathcal{F}} > \alpha\}.$$

We introduce now the concept of metric entropy. Let $(\mathcal{F},\rho)$ be a pseudometric space and let $\varepsilon>0$. The *covering number* of $\mathcal{F}$ using balls of ρ-radius ε is

$$N(\varepsilon,\rho,\mathcal{F})=\min\{n\in\mathbb{N}: \text{there exist } f_1,\dots,f_n\in\mathcal{F} \text{ such that } \sup_{f\in\mathcal{F}} \min_{i\le n} \rho(f,f_i)\le\varepsilon\}.$$

The function $H(\varepsilon,\rho,\mathcal{F})=\log N(\varepsilon,\rho,\mathcal{F})$ is called *metric entropy*. We will use a form of entropy introduced by Pollard (1982). For any finite set $T \subset S$, let

$$d_T^{(2)}(f,g) = \left(\frac{\sum_{x\in T} (f-g)^2(x)}{\sum_{x\in T} F^2(x)}\right)^{1/2}, \quad f,g \in \mathcal{F},$$

and for $\varepsilon>0$, let

$$N^{(2)}(\varepsilon,\mathcal{F}) = \sup_{\substack{T\subset S\\ T \text{ finite}}} N(\varepsilon,d_T^{(2)},\mathcal{F}).$$

If $1 \le p < 2$, for any finite set $T = \{x_1,x_2,\dots,x_n\} \subset S$, let $P_T=n^{-1}\sum_{i=1}^{n} \delta_{x_i}$,

and define

$$d_T^{(p)}(f,g) = \frac{\Lambda_{p,\infty}^{P_T}(f-g)}{\Lambda_{p,\infty}^{P_T}(F)}, \quad f,g \in \mathcal{F}.$$

For $\varepsilon>0$, let

$$N^{(p)}(\varepsilon,\mathcal{F}) = \sup_{\substack{T\subset S\\ T \text{ finite}}} N(\varepsilon,d_T^{(p)},\mathcal{F}).$$

Let $Z = \{Z(f): f \in \mathcal{F}\}$ be a subgaussian process with $\tau=1$. Consider

$\sigma(f,g) = E((Z(f)-Z(g))^2)^{1/2}$, $f,g \in \mathcal{F}$ and let $|\mathcal{F}|_\sigma = \sup_{f,g\in\mathcal{F}} \sigma(f,g)$. Marcus and Pisier (1978) proved that if

$$\int_0^{|\mathcal{F}|_\sigma} (\log N(\varepsilon,\sigma,\mathcal{F}))^{1/2}\, d\varepsilon < \infty$$

then Z has a version $\tilde{Z}$ with σ-uniformly continuous sample paths and, for $f_0 \in \mathcal{F}$,

$$(1.3) \qquad E \sup_{f\in\mathcal{F}} |\tilde{Z}(f)| \le E|\tilde{Z}(f_0)| + C \left[\int_0^{|\mathcal{F}|_\sigma} (\log N(\varepsilon,\sigma,\mathcal{F}))^{1/2} + \varphi(|\mathcal{F}|_\sigma) \right]$$

where $\varphi(\delta) = 4\delta(\log\log 4\, |\mathcal{F}|_\sigma\, \delta^{-1})^{1/2}$, $0<\delta\le|\mathcal{F}|_\sigma$. If $Z(f) = \sum_{i=1}^{n} \varepsilon_i h_i(f)$ with h_i continuous in $(\mathcal{F},\sigma)$ then $Z = \tilde{Z}$ in (1.3).

Next, we present the property introduced by Vapnik and Červonenkis (1971) in their study of Glivenko-Cantelli's theorem for classes of sets more general that semiintervals in $\mathbb{R}$. Let S be a non-empty set and let $A \subset S$. If $\mathcal{C}$ is a class of subsets of S, we will write

$$\Delta^{\mathcal{C}}(A) = \#\,\{C\cap A:\ C \in \mathcal{C}\}$$

and

$$m^{\mathcal{C}}(n) = \max\,\{\Delta^{\mathcal{C}}(A):\ A \subset S,\ \#A = n\}\quad,\ n\in \mathbb{N}.$$

We define the (Vapnik-Červonenkis) *index of* $\mathcal{C}$ as $V(\mathcal{C}) = \min\,\{n \in \mathbb{N}:\ m^{\mathcal{C}}(n) < 2^n\}$, or $V(\mathcal{C}) = +\infty$ if $m^{\mathcal{C}}(n) = 2^n$ for all $n \in \mathbb{N}$. Say that $\mathcal{C}$ is a *Vapnik-Červonenkis class of sets* if $V(\mathcal{C}) < \infty$, i.e., if there exists an integer n such that for every set $A \subset S$ with n elements, $\{C \cap A: C \in \mathcal{C}\}$ does not contain all the subsets of A. If f is a real function on S, its graph is $G(f)=\{(s,t)\in S\times\mathbb{R}:\ 0\le t\le f(s) \text{ or } f(s)\le t\le 0\}$. Say that the class $\mathcal{F} \in \mathbb{R}^S$ is a *Vapnik-Červonenkis class of functions* if $\{G(f):\ f \in \mathcal{F}\}$ is a Vapnik-Červonenkis class of sets in $S\times\mathbb{R}$. The following lemma expresses the basic property of the entropy (with respect to ρ_P) of Vapnik-Červonenkis classes of functions. The original result for the bound presented here goes back to Lemma 7.13 in Dudley (1978) (for classes of sets). The same proof of Lemma II.36 of Pollard (1984) allows to extend his result to the case where $1<p<2$.

LEMMA 1.4. Let $\mathcal{F}$ be a class of measurable real functions defined on the measurable space $(S,\mathcal{S})$ with envelope F and let P be a probability measure on $(S,\mathcal{S})$ such that $PF^p < \infty$, for $1\le p\le 2$. If $\mathcal{F}$ is a Vapnik-Červonenkis class then there exist constants B_p and w, independent of F and P, such that

$$N(\varepsilon\|F\|_p, \rho_p, \mathcal{F}) \le B_p\varepsilon^{-w} \quad , \text{ for } 0<\varepsilon<1.$$

PROOF. Without loss of generality, we can assume that $\mathcal{F}$ is strictly positive. Let $\mathcal{G} = \{ \frac{f}{F}\colon f\in \mathcal{F}\}$ and $Q(\cdot) = \dfrac{P(\cdot F^p)}{PF^p}$.

For each pair $f_1, f_2 \in \mathcal{F}$, since $\left|\frac{f_1}{F}\right| \le 1$ and $\left|\frac{f_2}{F}\right| \le 1$, we have

$$(PF^p)^{-1}P|f_1-f_2|^p = Q\left|\frac{f_1}{F} - \frac{f_2}{F}\right|^p \le 2^{p-1}Q\left|\frac{f_1}{F} - \frac{f_2}{F}\right|$$

and, so

$$(1.4) \qquad N(\varepsilon\|F\|_p, \rho_p^P, \mathcal{F}) \le N(2^{1-p}\varepsilon^p, \rho_1^Q, \mathcal{G}).$$

The point (s,t) belongs to the graph of f if, and only if, the point $(s, \frac{t}{F(s)})$ is in the graph of $\frac{f}{F}$; it follows that $\mathcal{G}$ is also a Vapnik-Červonenkis class. Applying Lemma II.36 in Pollard (1984) to $\mathcal{G}$, whose envelope is 1, and using (1.4) we obtain

$$N(\varepsilon\|F\|_p, \rho_p^P, \mathcal{F}) \le A(2^{1-p}\varepsilon^p)^{-v} = B_p\varepsilon^{-w},$$

where $B_p = A2^{v(p-1)}$ and $w=vp$. This proves the lemma.

Vapnik and Červonenkis (1981) characterized, under certain measurability conditions, the uniformly bounded classes of functions $\mathcal{F}$ verifying the strong law of large numbers uniformly on $\mathcal{F}$: $\|P_n-P\|_{\mathcal{F}} \longrightarrow 0$ almost surely if, and only if,

$$\frac{EH(\varepsilon, \rho_\infty^{P_n}, \mathcal{F})}{n} \longrightarrow 0$$

as $n\to\infty$ for all $\varepsilon>0$, where $\rho_\infty^{P_n}(f,g) = \max_{i\le n} |f(X_i) - g(X_i)|$, $f,g \in \mathcal{F}$. With respect to the central limit theorem, Giné and Zinn (1984) obtained conditions (also random) which are sufficient for bounded classes of functions and that become necessary and sufficient for classes of sets. We say that $\mathcal{F} \subset \mathcal{L}_2(S,\mathcal{S},P)$ is *P-pregaussian* if the Gaussian process $\{G_p(f)\colon f \in \mathcal{F}\}$ with $EG_p(f) = 0$ and $EG_p(f_1)G_p(f_2) = P(f_1f_2)-(Pf_1)(Pf_2)$ admits a version with almost all its sample paths bounded and ρ_2-uniformly continuous. We say that $\mathcal{F}$ is a *P-Donsker class* if it is P-pregaussian and $\{\{\nu_n^{(2)}(f)\colon f \in \mathcal{F}\}\colon n \in \mathbb{N}\}$ converges in law to G_p. Giné and Zinn (1984, 1986) showed, under certain measurability conditions, that if $\mathcal{F}$ is uniformly bounded, P-pregaussian and for all $\varepsilon>0$ and for all $\eta>0$ we have that

$$\lim_{n\to\infty} \Pr^{*} \left\{ \frac{\log N\left(\frac{\varepsilon}{n^{1/2}},\ \rho_1^{P_n}, \mathcal{F}\right)}{n^{1/2}} > \eta \right\} = 0$$

then $\mathcal{F}$ is a P-Donsker class. Finally, Talagrand (1987) and Ledoux and Talagrand (1988) have given necessary and sufficient conditions for $\mathcal{F}$ to be a P-Donsker class in terms of ρ_2^P, $\rho_2^{P_n}$ and ρ_1^P.

Although the random conditions elucidate the nature of the problem, from the point of view of applications in Statistics it seems interesting to get results under hypotheses based only on P and not on the sample joint distribution P^N.

With regard to the Vapnik-Červonenkis property, its relevance for the central limit theorem follows from a result by Durst and Dudley (1981) which states that, under measurability conditions, a class of sets is P-Donsker for all probability measures P if, and only if, it is a Vapnik-Červonenkis class. Pollard (1982) established that if $\mathcal{F} \subset \mathcal{L}_2(S,\mathcal{S},P)$ is a Vapnik-Červonenkis class of functions with envelope F satisfying $PF^2 < \infty$ then $\mathcal{F}$ is a P-Donsker class. The definitive central limit theorem for Vapnik-Červonenkis classes of functions has been obtained by Alexander (1987). In this paper we extend Alexander's theorem to the case of non-Gaussian stable limit. This is done in the next section where we obtain necessary and sufficient conditions for the convergence in law of $\{\nu_n^{(p)} : n \in \mathbb{N}\}$ to a p-stable Radon law in $\ell^{\infty}(\mathcal{F})$, $1\le p<2$, for Vapnik-Červonenkis classes of functions $\mathcal{F}$. The sufficient conditions work also for classes $\mathcal{F}$ whose entropy is larger than the entropy of Vapnik-Červonenkis classes of functions $\mathcal{F}$.

2. Main results

The necessary conditions that we state now for non-Gaussian stable domains of attraction (eventual tightness of the sequence of law sums outside neighborhoods of zero and finite dimensional convergence) turn out to be sufficient for Vapnik-Červonenkis classes of functions.

THEOREM 2.1. Let $\mathcal{F} \subset \mathcal{L}_1(S,\mathcal{S},P)$ be a class of functions with envelope F finite everywhere and such that $\sup_{f\in\mathcal{F}} |Pf| < \infty$. Let $1<p<2$. If

$$(2.1)\qquad \{\nu_n^{(p)}: n \in \mathbb{N}\} = \left\{ \left\{ n^{-1/p} \sum^{n} (f(X_i)-Pf): f \in \mathcal{F} \right\} : n \in \mathbb{N} \right\}$$

is eventually tight in $\ell\,(\mathcal{F})$ then

(i) the sequence of set functions

$$\{n\mathcal{L}^{*}(n^{-1/p}\delta_{X_1})\big|_{\{\|x\|_{\mathcal{F}}>\delta\}} : n \in \mathbb{N}\}$$

is eventually tight in $\ell^{\infty}(\mathcal{F})$ for all $\delta>0$.

PROOF. To simplify the notation, we will write ν_n instead of $\nu_n^{(p)}$. Since $\{\nu_n: n \in \mathbb{N}\}$ is eventually tight, by Example 7.28 in Hoffmann-JØrgensen (1984), there exists a totally bounded pseudometric ρ on $\mathcal{F}$ such that

$$(2.2)\qquad \lim_{\tau\to 0} \limsup_{n\to\infty} \Pr^{*} \{\omega_{\rho}(\nu_n,\tau) > \eta\} = 0 \text{ , for all } \eta>0$$

(where $\omega_{\rho}(\nu_n,\tau) = \sup_{\rho(f,g)<\tau} |\nu_n(f)-\nu_n(g)|$) , and

$$(2.3)\qquad \lim_{M\to\infty} \limsup_{n\to\infty} \Pr^{*} \{\|\nu_n\|_{\mathcal{F}} > M\} = 0.$$

Let us first show that

$$(2.4)\qquad \lim_{M\to\infty} \limsup_{n\to\infty} n \Pr^{*} \{F(X_1) > Mn^{1/p}\} = 0.$$

We have

$$\frac{n\Pr^{*}\{\|f(X_1)-Pf\|_{\mathcal{F}} > Mn^{1/p}\}}{1+n\Pr^{*}\{\|f(X_1)-Pf\|_{\mathcal{F}} > Mn^{1/p}\}}$$

$$\le 1-\exp\{-n\Pr^{*}\{\|f(X_1)-Pf\|_{\mathcal{F}}\} > Mn^{1/p}\}$$

$$\text{(since } (\max_{k\le n} \|f(X_k)-Pf\|_{\mathcal{F}})^{*} = \max_{k\le n} \|f(X_k)-Pf\|_{\mathcal{F}}^{*})$$

$$(2.5)\qquad \le \Pr^{*}\{\max_{k\le n} \|f(X_i)-Pf\|_{\mathcal{F}} > Mn^{1/p}\}$$

(since the condition in Proposition 1.3 holds for $c = \frac{1}{2}$ and n large enough by (2.3))

$$\le 2\Pr^{*}\left\{ \left\| \sum_{i=1}^{n} (f(X_i)-Pf)\right\|_{\mathcal{F}} > \frac{M}{4}\, n^{1/p} \right\}$$

$$= 2\Pr^{*}\{\|\nu_n\|_{\mathcal{F}} > \frac{M}{4}\}.$$

Recalling that $\sup_{f\in\mathcal{F}} |Pf| < \infty$, (2.4) follows from (2.5) and (2.3). To obtain (i), we have to show that for any $\varepsilon>0$ there exists a compact set $K \subset \ell^{\infty}(\mathcal{F})$ so that

(2.6) $\limsup_{n\to\infty} n\, Pr^* \{\xi_{n1} \in G^c, \|\xi_{n1}\| > \delta\} < \varepsilon$, for any open set $G \supset K$,

where $\xi_{n1} = n^{-1/p}\delta_{X_1}$. By convexity of $\omega_\rho(\cdot,\tau)$ and (2.2),

$$\lim_{\tau\to 0}\limsup_{n\to\infty} n\, Pr^*\{\omega_\rho(n^{-1/p}(f(X_1)-Pf),\tau > \eta\} = 0,$$

for all $\eta>0$. Since $\omega_\rho(\cdot,\tau)$ is positively linear and $\sup_{f\in\mathcal{F}} |Pf| < \infty$, we get

$$\lim_{\tau\to 0}\limsup_{n\to\infty} n\, Pr^*\{\omega_\rho(n^{-1/p}(f(X_1),\tau > \eta\} = 0, \text{ for all } \eta>0.$$

Now, given $\varepsilon>0$, for each $q \in \mathbb{N}$ there exists a sequence $\{\tau_{qj}: j \in \mathbb{N}\}$, $\tau_{qj}\downarrow 0$ as $j\to\infty$, and there exists $\tau>0$ such that

(2.7) $$\limsup_{n\to\infty} n\, Pr^*\{\omega_\rho(n^{-1/p}f(X_1),\tau_{pq}) > 2^{-j}\} < \varepsilon 2^{-j-q}$$

and, by (2.3),

(2.8) $$Pr^*\{\|n^{-1/p}f(X_1)\|_{\mathcal{F}} > \tau\} < \varepsilon/2.$$

Let $K_m = \bigcap_{q=1}^{m}\bigcap_{j=1}^{\infty} \{H \in \ell^\infty(\mathcal{F}): \omega_\rho(H,\tau_{qj}) \le 2^{-j},\ \|H\|_{\mathcal{F}} \le \tau\}$. By Arzela-Ascoli's theorem, each K_m is compact in $(\ell^\infty(\mathcal{F}), \|\cdot\|_\infty)$. The set $K = \bigcap_{m=1}^{\infty} K_m$ is also compact and for any open set $G \supset K$ there exist $m \in \mathbb{N}$ such that $G \supset K_m$, by the finite intersection property. Hence (2.7), (2,8) and (2.4) show that (2.6) hold, and the proof is complete.

The following sufficient conditions are fulfilled by classes larger than the Vapnik-Červonenkis ones.

THEOREM 2.2. Let $\mathcal{F} \subset \mathcal{L}_1(S,\mathcal{S},P)$ be a supremum measurable class of functions with envelope F finite everywhere. Let $1 < p < 2$. Assume (i) in Theorem 2.1,

(ii) for each $k \in \mathbb{N}$ and each $(f_1,\dots,f_k) \in \mathcal{F}^k$

$$\left\{ n^{-1/p}\sum_{i=1}^{n} (f_1(X_i)-Pf_1,\dots,f_k(X_i)-Pf_k):\ n \in \mathbb{N}\right\}$$

converges in law in R^k, and

(iii) $$\int_0^1 (\log N^{(2)}(\varepsilon,\mathcal{F}))^{1/2}\, d\varepsilon < \infty.$$

Then $\{\{n^{-1/p}\sum_{i=1}^{n}(f(X_i)-Pf): f \in \mathcal{F}\}: n \in \mathbb{N}\}$ converges in law in $\ell^\infty(\mathcal{F})$ to the p-stable Radon measure determined by (ii).

PROOF. By Proposition 1.1, it is enough to show that

$$(2.9) \qquad \lim_{\delta\to 0}\limsup_{n\to\infty} \Pr{}^* \left\{ \left\| n^{-1/p}\sum_{i=1}^{n} \varepsilon_i f I_{\{F^* \le \delta n^{1/p}\}}(X_i) \right\|_{\mathcal{F}} > \eta \right\} = 0,$$

for any $\eta>0$.

Let $\mathcal{F}_{n,\delta} = \{f I_{\{F^* \le \delta n^{1/p}\}}: f \in \mathcal{F}\}$ and $F_{n,\delta} = F I_{\{F^* \le \delta n^{1/p}\}}$. For each $x_1,\dots,x_n$ fixed, consider the subgaussian process $\{Z(f): f\in \mathcal{F}_{n,\delta}\} = \{n^{-1/p}\sum_{i=1}^{n}\varepsilon_i f(x_i): f \in \mathcal{F}_{n,\delta}\}$. Its associated $\mathcal{L}_2$-distance is

$$\sigma(f,g) = \left(\frac{\sum_{i=1}^{n}(f-g)^2(x_i)}{n^{2/p}} \right)^{1/2}.$$

The diameter of $\mathcal{F}_{n,\delta}$ for σ is

$$|\mathcal{F}_{n,\delta}|_\sigma \le 2\left(\frac{\sum_{i=1}^{n} F^{*2}_{n,\delta}(x_i)}{n^{2/p}} \right)^{1/2} = \Delta_{n,\delta}(F).$$

Let $T_n = \{X_1,\dots,X_n\}$. Since

$$\sigma(f,g) = \Delta_{n,\delta}(F) d^{(2)}_{T_n}(f,g)$$

we get that

$$(2.10)\ N(\varepsilon,\sigma,\mathcal{F}_{n,\delta}) = N\left(\frac{\varepsilon}{\Delta_{n,\delta}(F)}, d^{(2)}_{T_n}, \mathcal{F}_{n,\delta}\right) \le N^{(2)}\left(\frac{\varepsilon}{\Delta_{n,\delta}(F)}, \mathcal{F}_{n,\delta}\right).$$

Now, for any $\eta>0$,

$$\Pr{}^*\left\{ \left\| n^{-1/p}\sum_{i=1}^{n}\varepsilon_i f(X_i) \right\|_{\mathcal{F}_{n,\delta}} > \eta \right\}$$

$$\le E_X\left[1 \wedge \frac{1}{\eta} E_\varepsilon \left\| \frac{\sum_{i=1}^{n}\varepsilon_i f(x_i)}{n^{1/p}} \right\|_{\mathcal{F}_{n,\delta}} \right]$$

(using (1.3))

$$\le C\, E_X\left[\int_0^{\Delta_{n,\delta}(F)} (\log N(\varepsilon,\sigma,\mathcal{F}_{n,\delta}))^{1/2} d\varepsilon + \varphi(\Delta_{n,\delta}(F))\right]$$

(by 2.10)

$$= C\, E_X\left[\int_0^{\Delta_{n,\delta}(F)} (\log N^{(2)}\left(\frac{\varepsilon}{\Delta_{n,\delta}(F)}, \mathcal{F}_{n,\delta}\right))^{1/2} d\varepsilon + 4(\log\log 4)^{1/2}\, \Delta_{n,\delta}(F)\right]$$

$$= CE_X\Delta_{n,\delta}(F)\left[\int_0^1 (\log N^{(2)}(\varepsilon,\mathcal{F}_{n,\delta}))^{1/2}\, d\varepsilon + 4(\log\log 4)^{1/2}\right]$$

$$\le CE_X\Delta_{n,\delta}(F)\left[\int_0^1 (\log N^{(2)}(\varepsilon,\mathcal{F}))^{1/2} d\varepsilon + 4(\log\log 4)^{1/2}\right].$$

By (iii), the term between brackets is finite; so, to obtain (2.9) it is enough to show that

$$\lim_{\delta\to 0}\limsup_{n\to\infty} E_X\Delta_{n,\delta}(F) = 0.$$

From (i) we obtained (2.4) and this implies

(2.11) $$\sup t^p \Pr^* \{F > t\} = K < \infty.$$

Thus,

$$E_X\Delta^2_{n,\delta}(F) = 2nE_X\left(\frac{F^{*2}}{n^{2/p}}\, I_{\{F^*\le\delta n^{1/p}\}}\right)$$

$$= 4\int_0^\infty nt\Pr\left\{\frac{F^*}{n^{1/p}}\, I_{\{F^*\le\delta n^{1/p}\}} > t\right\} dt$$

$$\le 4\int_0^\delta nt\Pr\{F^* > tn^{1/p}\}\, dt$$

(by (2.11))

$$\le 2\int_0^\delta Kt^{1-p}dt = 2K\,\frac{\delta^{2-p}}{2-p},$$

(since $1\le p<2$). We get that

$$\lim_{\delta\to 0}\limsup_{n\to\infty} E_X\Delta_{n,\delta}(F) = 0$$

and the result follows.

REMARKS. (a) The proof of Proposition 1.1 gives that the result also holds if we replace the corresponding sets in conditions (i) and (ii) by the sets

$\{H(x) > \delta\}$ and $\{H(\xi_1) \le \delta\}$ for any function $H(\cdot) \ge \|\cdot\|_{\mathcal{F}}$. So, the proof of the Theorem 2.2 shows that the same conclusion is valid if we substitute the hypotheses (i) and (iii) by

(i') the sequence of set functions

$$\{n\mathcal{L}^*(n^{-1/p}\delta x_1)|_{\{H(x)>\delta\}} : n \in \mathbb{N}\}$$

is eventually tight in $\ell^\infty(\mathcal{F})$ for all $\delta>0$, and

(iii') $$\int_0^1 (\log N_H^{(2)}(\varepsilon,\mathcal{F}))^{1/2}\, d\varepsilon < \infty$$

where

$$N_H^{(2)}(\varepsilon,T) = \sup_{\substack{T\subset S \\ T \text{ finite}}} N(\varepsilon, d_{H,T}^{(2)}, \mathcal{F})$$

with

$$d_{H,T}^{(2)}(f,g) = \left(\frac{\sum_{x\in T} (f-g)^2(x)}{\sum_{x\in T} H^2(x)} \right)^{1/2},$$

for any function $H \ge F$ such that $\sup_{t>0} t^p P\{H^* > t\} < \infty$.

(b) Note that in the proof of Theorem 2.2. we have only used that

$$\int_0^1 (\log N(\varepsilon, \rho_2^{P_n}, \mathcal{F}_{n,\delta}))^{1/2}\, d\varepsilon < \infty$$

for all $n \in \mathbb{N}$ and all $\delta > 0$.

(c) If we replace (i) by

$$\sup_{t>0} t^p P\{F^* > t\} < \infty,$$

the same proof gives stochastic boundedness of the sequence

$$\left\{ \|n^{-1/p} \sum_{i=1}^{n} (f(X_i) - Pf\|_{\mathcal{F}} : n \in \mathbb{N} \right\}.$$

(d) The technique used in the proof cannot be used to establish Alexander's central limit theorem.

Combining Theorem 2.1 and Theorem 2.2 we extend Alexander's central limit theorem for empirical process on Vapnik-Červonenkis classes of functions to the case with p-stable Radon limit, $1<p<2$.

COROLLARY 2.3. Let $\mathcal{F} \subset \mathcal{L}_1(S,\mathcal{S},P)$ be a supremum measurable Vapnik-Červonenkis class of functions with envelope F finite everywhere. Assume $\sup_{f\in\mathcal{F}} |Pf| < \infty$. Let $1<p<2$. Then (i) in Theorem 2.1 and (ii) in Theorem 2.2 hold if, and only if,

$$\left\{ \left\{ n^{-1/p} \sum_{i=1}^{n} (f(X_i)-Pf): f \in \mathcal{F} \right\}: n \in \mathbb{N} \right\}$$

converges in law in $\ell^\infty(\mathcal{F})$ to the p-stable Radon measure determined by (ii).

PROOF. Necessity follows from Theorem 2.1. To prove sufficiency it is enough to check condition (iii) in Theorem 2.2. Given $T \subset S$ finite, say $T = \{x_1,\ldots,x_n\}$, let $P_T = n^{-1} \sum_{i=1}^{n} \delta_{x_i}$, and let $\|F\|_{n,2} = \left(\frac{\sum_{i=1}^{n} F^2(x_i)}{n} \right)^{1/2}$. Since

$$\rho_2^{P_T}(f,g) = \|F\|_{n,2}\, d_T^{(2)}(f,g)\ ,\quad f,g \in \mathcal{F}$$

we have that

$$N^{(2)}(\varepsilon, d_T^{(2)}, \mathcal{F}) \le N(\varepsilon\|F\|_{n,2}, \rho_2^{P_T}, \mathcal{F}) \le B_2\varepsilon^{-W},$$

by Lemma 1.4. Hence $N^{(2)}(\varepsilon,\mathcal{F}) \le B_2\varepsilon^{-W}$. This proves condition (iii) in Theorem 2.2 and the result follows.

The same proof leads to the result in the case p=1:

THEOREM 2.4. (a) Let $\mathcal{F} \subset \mathcal{L}_0(S,\mathcal{S},P)$ be a class of functions with envelope F finite everywhere. If

$$(2.12)\qquad \{\nu_n^{(1)}: n \in \mathbb{N}\} = \{\{n^{-1} \sum_{i=1}^{n} \varepsilon_i f(X_i): f \in \mathcal{F}\}: n \in \mathbb{N}\}$$

is eventually tight in $\ell^\infty(\mathcal{F})$ then

(i) the sequence of set functions

$$\{n\mathcal{L}^*(n^{-1}\delta_{X_1})|_{\{\|x\|_{\mathcal{F}}>\delta\}}: n \in \mathbb{N}\}$$

is eventually tight in $\ell^\infty(\mathcal{F})$ for all $\delta>0$.

(b) Let $\mathcal{F} \subset \mathcal{L}_0(S,\mathcal{S},P)$ be a supremum measurable class of functions with envelope F finite everywhere. Assume (i) in part (a),

(ii) for each $k \in \mathbb{N}$ and each $(f_1,\ldots,f_k) \in \mathcal{F}^k$

$$\{n^{-1}\sum_{i=1}^{n}(\varepsilon_i f_1(X_i),\dots,\varepsilon_i f_k(X_i)):\ n\in\mathbb{N}\}$$

converges in law in $\mathbb{R}^k$, and

(iii) $$\int_0^1 (\log N^{(2)}(\varepsilon,\mathcal{F}))^{1/2}d\varepsilon < \infty.$$

Then $\{\{n^{-1}\sum_{i=1}^{n}\varepsilon_i f(X_i):\ f\in\mathcal{F}\}:\ n\in\mathbb{N}\}$ converges in law in $\ell^\infty(\mathcal{F})$ to the 1-stable Radon measure determined by (ii).

COROLLARY 2.5. Let $\mathcal{F}\subset\mathcal{L}_0(S,\mathcal{S},P)$ be a supremum measurable Vapnik-Červonenkis class of functions with envelope F finite everywhere. Then (i) and (ii) in Theorem 2.4 hold if, and only if,

$$\{\{n^{-1}\sum_{i=1}^{n}\varepsilon_i\ f(X_i):\ f\in\mathcal{F}\}:\ n\in\mathbb{N}\}$$

converges in law in $\ell^\infty(\mathcal{F})$ to the 1-stable Radon measure determined by (ii).

REMARKS. (a) The corresponding weak laws of large numbers also hold. In the set up of Theorem 2.1, if

(2.13) $$\|n^{-1/p}\sum_{i=1}^{n}(f(X_i)-Pf)\|_{\mathcal{F}} \longrightarrow 0 \ \text{ in } Pr^* \ \text{ as } n\to\infty$$

then $t^pP\{F^* > t\}\longrightarrow 0$ as $t\to\infty$. If $\mathcal{F}\subset\mathcal{L}_1(S,\mathcal{S},P)$ is a supremum measurable class of functions with, $t^pP\{F^* >t\}\to 0$ as $t\to\infty$ and $\int_0^1 (\log N^{(2)}(\varepsilon,\mathcal{F}))^{1/2}d\varepsilon<\infty$ then (2.13) holds.

(b) The obvious modifications give the weak laws of large numbers for Vapnik-Červonenkis classes of functions and the corresponding results for p=1.

(c) The ideas in Kuelbs and Zinn (1979) and in De Acosta (1981) about the relationship between the strong and the weak laws of large numbers in Banach spaces give strong laws of large numbers if we substitute weak moment conditions by the strong moments $E^*F^p<\infty$, $1\le p<2$.

(d) If $1<p<2$, we will obtain stochastic boundedness assuming only

$$\int_0^1 (\log N^{(p)}\ (\varepsilon,\mathcal{F}))^{1/p'}\ d\varepsilon < \infty, \qquad \frac{1}{p}+\frac{1}{p'} = 1,$$

instead of condition (iii) in Theorem 2.2. Although this improvement is irrelevant for Vapnik-Červonenkis classes of functions, it is interesting for classes larger than Vapnik-Červonenkis ones. The proof follows the same pattern as the one of Theorem 2.2 using now lemmas 3.1. and 3.2 and theorem 3.3 in Marcus and Pisier (1984). A similar result also holds under integrability of log log $N^{(1)}$.

(e) In Romo (1987) can be seen several applications of this results: namely, a weak law of large numbers for "random weighted empiricals" and a rate of convergence for a result by Pollard (1981) on the approximation between the sample variation and the theoretical one in the k-means problem in clustering analysis.

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