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ON THE POLICY FUNCTION IN CONTINUOUS TIME ECONOMIC MODELS

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Abstract

In this paper, I consider a general class of continuous-time economic models with unbounded horizon. I study the sets of conditions under which the policy function is continuous, Lipschitz continuous, and C^1 differentiable. I also single out certain postulates which may prevent higher-order differentiability. The analysis provides, therefore, a firm foundation to the use of dynamic programming methods in continuous time models with unbounded horizon.

Key Words
Policy Functions; Continuous Time Models; Continuity; Lipschitz Continuity; Differentiability.

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1. INTRODUCTION

A long standing problem in continuous time optimization is the C^2 differentiability of the value function. As illustrated in Pontryagin et al. (1962), from the C^2 differentiability of the value function it is easy to obtain a fairly simple proof of the maximum principle and the Euler equations, and clarify the relation between dynamic programming and the canonical Hamiltonian equations. Also, if the value function is C^2 differentiable, then the policy function is C^1 differentiable. This is particularly relevant for analyzing the dynamics of optimal solutions, and for linearization procedures.

It is well known that in general the value function is not C^2 differentiable [see, for instance, Pontryagin et al. (1962)]. The usual counterexamples found in the literature stem from a lack of concavity of the return function, and from noninteriority of optimal solutions. However, for general economic models, under first-order differentiability and concavity of the return function, and a mild interiority condition on optimal solutions, Benveniste and Scheinkman (1979) have shown that the value function is first-order differentiable. Moreover, to overcome the differentiability problem, generalized concepts of solutions for Bellman's equation have been studied [see, for instance, Crandall, Evans and Lions (1984), and Lions (1982)]. The main goal of these approaches is to define a weaker solution concept under which the value function is usually characterized as the unique solution of the associated Bellman equation.

This paper is concerned with the classical methods of variational analysis. The main purpose is to show that for continuous-time economic models with unbounded horizon, under the hypotheses of C^2 differentiability and concavity of the return function, and interiority of the optimal solutions, the value function is C^2 differentiable –and the optimal feedback control or policy is C^1 differentiable.

For finite-horizon, continuous-time optimization problems (under rather strong assumptions) the C^2 differentiability of the value function was established in the early 70's by Fleming (1971) solving the Bellman equation by the "method of characteristics." Under this procedure, the value function is obtained as the solution of a partial differential equation from a given terminal condition. The "method of characteristics," however, cannot be applied to optimization models with unbounded time horizon, since the "characteristics" are simply undefined at infinity. Of course, for such time unbounded models one could consider as in Santos (1990) succesive truncations of the horizon. Then show that a sequence of corresponding policy functions for finite horizon models converges in the C^1 topology. Nevertheless, Section 4 will provide a more straight-forward proof to the C^1 differentiability of the optimal policy (and the C^2 differentiability of the value function), which is fairly specific to the continuous time model. The proof considers directly the entire, infinite-horizon optimization problem, and assumes the Lipschitz continuity of the policy function.

In discrete-time growth models with unbounded horizon, the Lipschitz continuity of the policy function has been established under general conditions by Montrucchio (1987). Montrucchio's method of proof, however, does not generalize to the continuous time case. The approach followed here is fairly different. It is based on an approximation of the optimal policy by a sequence of C^1 functions with uniform Lipschitz constant.

Another important topic addressed in this paper is the higher-order differentiability of the optimal policy. In the discrete time case, Araujo (1989) has constructed an example of a C^3 optimization model in which the optimal policy is C^1 but fails to be C^2 differentiable. It is unknown, however, which are the sources of non differentiability, and whether counterexamples of this sort are robust to small pertubations of the model. Further, it is as yet unclear whether such counterexamples may be constructed from simple return functions. Section 5 shows that even for third-degree-polynomial utilities, the optimal policy may fail to be C^2 differentiable. From some standard results from the theory of dynamical systems, we isolate a set of conditions which may prevent higher-order differentiability. It will follow from these conditions that in one-dimensional models C^∞ policy functions are the rule rather than the exception. The higher-order differentiability of the optimal policy is useful to the study of chaotic dynamics, endogenous cycles, and bifurcations [e.g., see Boldrin and Woodford (1990, Sect. 2)].

The outline of the paper is as follows. Section 2 is concerned with a formal description of the the model, along with some preliminary results. Section 3 is devoted to establish the Lipschitz continuity of the optimal policy, and Section 4 to establish the C^1 differentiability. The higher-order differentiability of the optimal policy is analyzed in Section 5.

2. THE MODEL

We begin with a continuous time version of the model of optimal growth. Given $\delta > 0$ and $T \subset R^{2n}$, consider the following optimization problem: Find an absolutely continuous path $\{x^*(t)\}_{t\geq 0}$ as a solution to

$$V(x_0) = \sup \int_0^\infty L(x(t), \dot{x}(t))\, e^{-\delta t}\, dt \tag{2.1}$$

$$\text{s. t. } (x(t), \overset{o}{x}(t)) \in T \text{, a. e., with } x(0) = x_0 \text{ and } t \geq 0,$$

where $\overset{o}{x}(t)$ denotes the time derivative of $x(\cdot)$, whose existence is guaranteed almost everywhere (a. e.).

<u>Assumption A</u>: The mapping $L : T \to R$ is continuous and on the interior of its domain it is C^1 differentiable. Moreover, there is some constant $\alpha > 0$ such that the function $L(x, \overset{o}{x}) + \frac{\alpha}{2}||\overset{o}{x}||^2$ is concave for all $(x, \overset{o}{x})$ in T.

<u>Assumption B</u>: The set T is convex, and with non empty interior.

Both assumptions are entirely standard. The norm $||\cdot||$ is the usual Euclidean one. The concavity requirement asserted in Assumption A is termed $\alpha_{\overset{o}{x}}$-concavity [see, for instance, Montrucchio (1987)].

The question of the existence of an optimal solution to problem (2.1) has been amply analyzed in the literature [e.g., Carlson and Haurie (1987) and Toman (1986)]. Following Benveniste and Scheinkman (1979), we shall assume locally the existence of an optimal solution which satisfies a certain mild interiority condition.

<u>Assumption C</u>: There exists an open set U in R^n such that for every x_0 in U, there is an optimal solution $\{x^*(t)\}_{t\geq 0}$ to problem (2.1), with $x^*(0) = x_0$, and the value function $V(x_0)$ is finitely valued; moreover, for all x_0 in U there is a given time $h > 0$ such that $(x^*(t), \overset{o}{x}^*(t)) \in \text{int}(T)$ for all $t \leq h$.

Under the above standard assumptions, it follows from Fleming and Rishel (1975, Corollary III.6.1) that every optimal path $\{x^*(t)\}_{t\geq 0}$ with $x^*(0) = x_0$ in U is C^1 differentiable on the interval [0, h]. Moreover, for all $t \leq h$ the optimal path satisfies Euler's equation

$$(2.2) \qquad D_1L(x^*(t), \overset{o}{x}^*(t))\, e^{-\delta t} - \frac{d}{dt}[D_2L(x^*(t), \overset{o}{x}^*(t))\, e^{-\delta t}] = 0,$$

where $\frac{d}{dt}$ is the time derivative, and $D_1L(x^*(t), \overset{o}{x}^*(t))$ and $D_2L(x^*(t), \overset{o}{x}^*(t))$ are the first-order partial derivatives of L.[1] Benveniste and Scheinkman (1979) have shown that the value function V, defined in (2.1), is C^1 differentiable on U. Therefore, this function must obey the functional equation of dynamic programming

$$(2.3) \qquad \delta V(x_0) = \max_{\overset{o}{x}} L(x_0, \overset{o}{x}) + DV(x_0)\cdot\overset{o}{x}.$$

solution to problem (2.1) if and only if it obeys at all times equation (2.3). The dynamics of the optimization problem (2.1) are, therefore, fully characterized by Bellman's equation. In virtue of Assumption C, the optimal feedback control or policy $\overset{o}{x}(0) = g(x(0))$ must satisfy the first-order conditions

$$D_2L(x_0, \overset{o}{x}) + DV(x_0) = 0. \tag{2.4}$$

By the concavity of L, it follows from equation (2.4) that the function $\overset{o}{x}(0) = g(x(0))$ is continuous on U.

Remark 2.1: As shown in Benveniste and Scheinkman (1979), the conditions that insure the differentiability of V are: strict concavity and first-order differentiability of L, and Assumptions B and C. Such conditions are the only required to establish the continuity of the policy function g. The $\alpha_{\overset{o}{x}}$-concavity of L stated in Assumption A will be employed in the sequence.

3. LIPSCHITZ CONTINUITY OF THE POLICY FUNCTION

The purpose of this section is to show that under the above assumptions, and a Lipschitz condition on the derivative of the return function L, the policy function g is a Lipschitz mapping on U.[2] In the discrete time case, the Lipschitz continuity of the policy function has been established by Montrucchio (1987).

To summarize the main ideas underlying the Lipschitz continuity of the policy function in discrete-time differentiable models, consider the following simple optimization problem

$$\max_{x_1} F(x_0, x_1),$$

where F is a C^2, α_{x_1}-concave function defined on R^{2n}. In that case, an optimal solution $x_1 = g(x_0)$ exists, and must satisfy the first-order conditions

$$D_2F(x_0, x_1) = 0.$$

By the implicit function therorem, the derivative $Dg(x_0) = -[D_{22}F(x_0,x_1)]^{-1}\cdot D_{21}F(x_0,x_1)$, where $[D_{22}F(x_0,x_1)]^{-1}$ is the inverse matrix. Note that the α_{x_1}-concavity of F imposes a uniform bound on the matrix norm $||[D_{22}F(x_0, x_1)]^{-1}||$. Therefore, in the case in which $D_{21}F(x_0, x_1)$ is uniformly bounded the optimal policy g is a Lipschitz mapping. Under general conditions, Montrucchio (1987) has shown that these results hold for the Bellman equation of a discrete-time, infinite-horizon optimization model even if the return and value functions are not C^2 differentiable.

One way to weaken the C^2 differentiability requirement is to approximate the given function by a sequence of C^2 mappings. (Such method of proof will be employed subsequently.) Assume, for instance, that F is a C^1, α_{x_1}-concave mapping and that the partial derivative $D_2F(x_0, x_1)$ is Lipschitz continuous with respect to x_0. Consider a sequence of C^2, α_{x_1}-concave functions $\{F_n\}_{n\geq 0}$, which converge to F in the C^1 topology. Such sequence may be chosen with the property that the cross-partial derivative $D_{21}F_n$ is uniformly bounded for all $n \geq 0$. Then the corresponding sequence of optimal policies $\{g_n\}_{n\geq 0}$ is Lipschitz continuous, with a uniform Lipschitz constant. Also $\{g_n\}_{n\geq 0}$ converges point-wise to g. Therefore, the policy function g is Lipschitz continuous.

Unlike the discrete time case, one cannot apply directly this method of proof to the Bellman equation (2.3). That is, one cannot longer assume that $D_2F(x, \overset{o}{x}) = D_2L(x, \overset{o}{x}) + DV(x)\cdot\overset{o}{x}$ is Lipschitz continuous with respect to x. Indeed, this is only true if the derivative $DV(x)$ is a Lipschitz function. (By equations (2.3) and (2.4), this would amount to assume that g is already a Lipschitz function.) Also, there does not seem to be a reasonable method to construct a discrete-time approximation to problem (2.1), and via a limiting argument to make valid uniform Lipschitz bounds for the continuous time case.

As already suggested, the approach taken in this paper is to consider a sequence of continuous-time optimization problems in which the optimal policies are shown to be C^1 differentiable with uniformly bounded derivatives. Such sequence of differentiable policies converges point-wise to the optimal policy g. The C^1 differentiability of the optimal policies is established following the method of proof of Santos (1990).

Theorem 3.1: *Assume that the derivative* DL *is a Lipschitz mapping. Under Assumptions* A *to* C, *the policy function* g *is Lipschitz continuous on the set* U.

Proof: As pointed out above, assume first that L is C^2 differentiable. Consider now the following optimization problem

$$(3.1) \qquad V_n(x_0, 0) = \max \int_0^h L(x(t), \overset{o}{x}(t))\, e^{-\delta t}\, dt + W_n(x(h))\, e^{-\delta h}$$

$$\text{s. t. } (x(t), \overset{o}{x}(t)) \in T, \text{ with } x(0) = x_0 \text{ and } 0 \leq t \leq h,$$

where W_n is a C^2 concave function. Let $g_n(x_n(t), t) = \overset{o}{x}_n(t)$ denote the optimal policy to problem (3.1) at time t $(0 \leq t \leq h)$. Let W_n approach V, as $n\to\infty$. Then $g_n(x, t)$

converges uniformly to g(x) on the set [0, h]. Therefore, by Assumption C, for every x_0 in U there is $n' > 0$ such that the optimal solution $\{x_n^*(t)\}_{0 \le t \le h}$ to (3.1) has the property that $(x_n^*(t), \mathring{x}_n^*(t)) \in \text{int}(T)$ for all $0 \le t \le h$ and all $n \ge n'$.

By equations (2.4) and (3.1), the function $g_n(\cdot, h)$ is C^1 differentiable at the point $x_n^*(t)$. Also, from the Maximum Principle it must hold for each t in [0, h] that

$$\text{(3.2)} \qquad \mathring{x}_n^*(t) = D_2H(x_n^*(t), q_n(t))$$

$$\text{(3.3)} \qquad \mathring{q}_n(t) = - D_1H(x_n^*(t), q_n(t)) + \delta q_n(t)$$

where $q_n(t)$ is equal to the derivative $D_1V_n(x_n^*(t), t)$ [the derivative of the function $V_n(\cdot,t)$ at $x_n^*(t)$], and $D_1H(x_n^*(t), q_n(t))$ and $D_2H(x_n^*(t), q_n(t))$ are the partial derivatives of the Hamiltonian

$$H(x_n^*(t), q_n(t)) = \max_{\mathring{x}} L(x_n^*(t), \mathring{x}) + q_n(t)\cdot\mathring{x}$$

Equations (3.2) and (3.3) define a first-order differential equation on (x, q). As L is assumed to be C^2 differentiable, these equations define by Assumptions A and C a flow $\phi(., t) = \phi_t$ such that

$$(x_n^*(h), DW_n(x_n^*(h))) \overset{\phi_{t-h}}{\rightarrow} (x_n^*(t), D_1V_n(x_n^*(t), t)).$$

Since ϕ_{t-h} is locally C^1 differentiable, then either D_1V_n is a C^1 mapping at $x_n^*(t)$ or it has an unbounded slope. We shall now show that at each time t, D_1V_n has a Lipschitz constant independent of n.

By the C^1 differentiability of ϕ, it follows that D_1V_n is C^1 differentiable at $x_n^*(t)$, whenever t is sufficiently close to h, say t in [t', h]. Also,

$$V_n(x_n^*(t'), t') = \max \int_{t'}^{h} L(x(t), \mathring{x}(t))\, e^{-\delta t}\, dt + W_n(x(h))\, e^{-\delta h}$$

$$\text{s. t. } (x(t), \mathring{x}(t)) \in T, \text{ with } x(t') = x_n^*(t') \text{ and } 0 \le t \le h.$$

Therefore, for a given function $\{z(t)\}_{t\geq t'}$ with $\overset{o}{z}(t) = D_1 g_n(x^*(t), t)\cdot z(t)$ it must hold true that

$$z(t')'\cdot D^2V_n(x_n^*(t'),\ t')\cdot z(t') = \int_{t'}^{h} (z(t),\ \overset{o}{z}(t))'\cdot D^2L(x_n^*(t),\ \overset{o*}{x}_n(t))\cdot(z(t),\ \overset{o}{z}(t))\ e^{\delta t}\,dt\ +$$

$$z(h)'\cdot D^2W_n(x_n^*(h))\cdot z(h)\ e^{-\delta h}$$

where $D^2V_n(x_n^*(t'), t')$ denotes in this case the second-order derivative of the function $V_n(\ \cdot,\ t')$ at $x_n^*(t')$ and z(t')'is a transposed vector. Consider now the following concave, quadratic optimization problem

$$(3.4)\ \psi_n(x_n^*(t),\ t') = \max_{\{y(t)\}_{h\geq t\geq t'}} \int_{t'}^{h} (y(t),\ \overset{o}{y}(t))'\cdot D^2L(x_n^*(t),\ \overset{o*}{x}_n(t))\cdot(y(t),\ \overset{o}{y}(t))\ e^{-\delta t}\,dt\ +$$

$$y(h)'\cdot D^2W_n(x_n^*(h))\cdot y(h)\ e^{-\delta h}$$

$$\text{s. t.}\quad y(t') = z_n(t').$$

A staight-forward computation shows that the path $\{z(t)\}_{h\geq t\geq t'}$ satisfies the necessary and sufficient first-order conditions

$$(3.5)\ [z(t)'\cdot D_{11}L(x_n^*(t),\ \overset{o*}{x}_n(t)) + \overset{o}{z}(t)'\cdot D_{21}L(x_n^*(t),\ \overset{o*}{x}_n(t))]\ e^{-\delta t} - \frac{d}{dt}\,([z(t)'\cdot D_{12}L(x_n^*(t),$$
$$\overset{o*}{x}_n(t)) + \overset{o}{z}(t)'\cdot D_{22}L(x_n^*(t),\ \overset{o*}{x}_n(t))]\ e^{-\delta t}) = 0. \qquad (t' \leq t \leq h)$$

$$(3.6)\ z(h)'\cdot D_{12}L(x_n^*(h),\ \overset{o*}{x}_n(h)) + \overset{o}{z}(h)'\cdot D_{22}L(x_n^*(h),\ \overset{o*}{x}_n(h)) + z(h)'\cdot D^2W_n(x_n^*(h)) = 0.$$

Therefore, $\psi_n(x^*(t'),t') = z(t')'\cdot D^2V_n(x_n^*(t'),\ t')\cdot z(t')$. Hence, for $||\,z(t')\,|| = 1$ the value $z(t')'\cdot D^2V_n(x_n^*(t'),\ t')\cdot z(t')$ is uniformly bounded (independently of n), since this value is (in absolute terms) less than or equal to the value achieved in (3.4) by the constant control $\overset{o}{y}(t) = \frac{z_n(t')}{t' - h}$. Consequently, the functions $DV_n(\cdot,\ t')$ and $Dg_n(\cdot,\ t')$ are Lipschitz continuous and their respective Lipschitz constants are independent of n.

Therefore, if L is C^2 differentiable, then the function g is locally Lipschitz on U. Moreover, if L is C^1 differentiable, and the derivative DL is Lipschitz continuous, then L can be approximated by a sequence of C^2 functions $\{L_n\}_{n\geq 0}$ with uniformly bounded second-order derivatives. Whence, in this case the function g is also a Lipschitz mapping on U. The theorem is proved.

4. C^1 DIFFERENTIABILITY OF THE POLICY FUNCTION

In order to show the C^1 differentiability of the policy function, we shall need to strengthen the differentiability of L and the interiority of optimal paths. We shall maintain Assumption B.

Assumption A': The mapping L is continuous, and on the interior of of its domain it is C^2 differentiable with uniformly bounded second-order derivatives. Moreover, there is some constant $\alpha > 0$ such that $L(x(t), \overset{o}{x}(t)) + \frac{\alpha}{2} || \overset{o}{x}(t) ||^2$ is a concave function on T.

Assumption C': There exists an open set U in R^n such that for every x_0 in U, the value function $V(x_0)$ is finitely valued and there is an optimal solution $\{x^*(t)\}_{t \geq 0}$ to problem (2.1), with $x^*(0) = x_0$, such that $(x^*(t), \overset{o}{x}{}^*(t)) \in int(T)$ for all $t \geq 0$.

The boundedness of the second-order derivatives required in Assumption A' can be weakened in the same way as in Santos (1990). Assumption C' is a strengthening of Assumption C, since the optimal path must now lie in the interior at every moment in time. This interiority requirement is generally assumed in economic models, and can be obtained from restrictions on the return function [cf. Stokey, Lucas and Prescott (1989, p. 134)]. Observe that in order to guarantee the Lipschitz continuity of g at x_0 is only necessary that the optimal path $x^*(\cdot)$ be initially in the interior. However, as will become clear from the development below [see also the example in Santos (1990, p. 7)], the C^1 differetiability of g at x_0 requires that the optimal path $x^*(\cdot)$, and its derivative $\overset{o}{x}{}^*(\cdot)$, be always in the interior.[3]

Theorem 4.1: *Under Assumptions* A', B *and* C' *the policy function* g *is* C^1 *differentiable on* U. *Moreover, for every* x_0 *in* U *there exists a constant* $K > 0$ *such that over the optimal path* $\{x^*(t)\}_{t \geq 0}$, *with* $x^*(0) = x_0$, *it must hold that* $\int_0^\infty || \overset{o}{z}(t) ||^2 e^{-\delta t} dt \leq K$, *for every function* $\{z(t)\}_{t \geq 0}$ *with* $\overset{o}{z}(t) = Dg(x^*(t)) \cdot z(t)$, *for all* $t \geq 0$, *and* $|| z(0) || = 1$.

Corollary 4.2: *Under Assumptions* A', B *and* C', *the value function* V *is* C^2 *differentiable on* U.

The asserted upper bound on the integral $\int_0^\infty || \overset{o}{z}(t) ||^2 e^{-\delta t} dt$ implies that the dynamical system generated by the derivative of the policy function, Dg, cannot grow at an exponential rate higher than $\frac{\delta}{2}$. An analogous result holds in discrete time models [Santos (1990, Prop. 2.3)]. In fact, following the methodology outlined by Santos (1989) this property of the derivative can be used to establish the joint differentiability of the policy function with respect to the initial state and a vector of parameters. Under such framework, the results on local determinacy of equilibria of Kehoe, Levine and Romer (1990) can be extended to continuous time economies with general equilibrium dynamics.

Another useful implication of this property of the derivative Dg pertains to the characteristic roots (resp. exponents) associated to fixed points (resp. periodic orbits) of the Euler equation (2.3). It is well known [see Levhari and Liviatan (1972) and Benhabib and Nishimura (1979)] that at a given stationary point or closed orbit if λ is a characteristic exponent then so is $-\lambda + \delta$. It follows that the Euler equation contains n characteristic exponents λ_i such that $\mathrm{re}\,\lambda_i \geq \frac{\delta}{2}$, and n characteristic exponents λ_j such that $\mathrm{re}\,\lambda_j \leq \frac{\delta}{2}$. Moreover, in the case where L is $\alpha_{\overset{o}{x}}$-concave there is no exponent with real part equal to $\frac{\delta}{2}$ [cf. Santos (1990, Lemma 3.9)]. The asserted upper bound on the integral $\int_0^\infty || \overset{o}{z}(t) ||^2 e^{-\delta t} dt$ implies then that at a given stationary point or closed orbit the n characteristic exponents associated to the dynamical system generated by g are those n exponents λ_i of the Euler equation such that $\mathrm{re}\,\lambda_i < \frac{\delta}{2}$ (cf. op. cit. Prop. 2.3).

From the Lipschitz continuity of DV and g, one can now derive the following preliminary result.

Lemma 4.3: Let $\{x^*(t)\}_{t\geq 0}$ be an optimal interior solution. If the policy function g is differentiable at the point $x^*(0)$, then it is differentiable at every point of the orbit $\{x^*(t)\}_{t\geq 0}$.

Proof: Consider the mapping $x_t = \psi(x_0)$ given by

$$x_0 \to (x_0, DV(x_0)) \overset{\phi_t}{\to} (x_t, DV(x_t)) \to x_t.$$

where ϕ_t denotes the flow induced by equations (3.2) and (3.3), and $DV(x_t) = q(t)$. Since DV and ϕ_t are differentiable mappings, the mapping ψ is also differentiable at x_0. Furthermore, since the derivative $D\phi_t$ is invertible and, by Theorem 3.1, DV is a

Lipschitz function at $x^*(t)$, it must hold true that $D\psi(x_0)$ is also invertible. By the inverse function theorem, the mapping ψ has a local inverse ψ^{-1} which is differentiable at the point $x^*(t)$.

Therefore, $(x^*(t), DV(x^*(t))) = \phi_t(x_0, DV(x_0)) = \phi_t(\psi^{-1}(x^*(t)), DV(\psi^{-1}(x^*(t)))$. As all these functions are differentiable, an application of the chain rule shows then that the mapping DV is differentiable at $x^*(t)$. Furthermore, by the implicit function theorem applied to equation (2.4) the mapping g is also differentiable at $x^*(t)$, and $x^*(t)$ is an arbitrary point of the optimal orbit $\{x^*(t)\}_{t\geq 0}$. The lemma is thus established.

Following Santos (1990) we introduce the following condition which will play a fundamental role in the method of proof.

Condition D: Let $\{x^*(t)\}_{t\geq 0}$ be an optimal path. Then a function $z(\cdot)$ defined on $[0, \infty)$ with $||z(0)|| = 1$ is said to satisfy Condition D if

D.1. For all $t \geq 0$

$$[z(t)'\cdot D_{11}L(x^*(t), \overset{o}{x}{}^*(t)) + \overset{o}{z}(t)'\cdot D_{21}L(x^*(t), \overset{o}{x}{}^*(t))]\, e^{-\delta t} - \frac{d}{dt}([z(t)'\cdot D_{12}L(x^*(t), \overset{o}{x}{}^*(t)) + \overset{o}{z}(t)'\cdot D_{22}L(x^*(t), \overset{o}{x}{}^*(t))]\, e^{-\delta t}) = 0.$$

D.2. There exists a uniform constant $M > 0$ such that

$$-\int_0^\infty \frac{1}{2}(z(t), \overset{o}{z}(t))'\cdot D^2L(x^*(t), \overset{o}{x}{}^*(t))\cdot(z(t), \overset{o}{z}(t))\, e^{-\delta t}\, dt \leq M.$$

Remark 4.4: As in the proof of Theorem 3.1, Condition D.1 corresponds to the first-order variational conditions of a quadratic expansion along an optimal orbit $\{x^*(t)\}_{t\geq 0}$ of the infinite horizon problem (2.1). Condition D.2 will play the role of a transversality condition. By the $\alpha_{\overset{o}{x}}$-concavity of L, Condition D.2 implies that $\int_0^\infty \frac{1}{2}\, ||\overset{o}{z}(t)||^2\, e^{-\delta t}\, dt \leq \frac{M}{\alpha}$. Therefore, $\lim_{t\to\infty} ||z(t)||^2\, e^{-\delta t} = 0$.

Lemma 4.5: Let $\{x^*(t)\}_{t\geq 0}$ be an optimal path. If a sequence $\{z(t)\}_{t\geq 0}$ satisfies condition D, then $\{z(t)\}_{t\geq 0}$ is an optimal solution to the quadratic optimization problem

$$\text{(4.1)} \qquad \max_{\{y(t)\}_{t\geq 0}} \int_0^\infty \frac{1}{2}(y(t), \overset{o}{y}(t))'\cdot D^2L(x^*(t), \overset{o}{x}{}^*(t))\cdot(y(t), \overset{o}{y}(t))\, e^{-\delta t}\, dt$$

$$\text{s. t.} \quad y(0) = z(0).$$

Proof: Assume that there is another path $\{z'(t)\}_{t\geq 0}$ which solves problem (4.1). Then $\{z'(t)\}_{t\geq 0}$ must obey all times the Euler equation given by Condition D.1. Also, there is a given constant $M' > 0$ for which $\{z'(t)\}_{t\geq 0}$ must obey Conditon D.2. Let $\eta(t) = z'(t) - z(t)$. Then following a standard argument [cf. Arrow and Kurz (1970, pp. 44-45)]

$$\int_0^T \frac{1}{2}(\eta(t),\ \mathring{\eta}(t))'\cdot D^2L(x^*(t),\ \mathring{x}^*(t))\cdot(\eta(t),\ \mathring{\eta}(t))\ e^{-\delta t}\,dt\ \geq$$

$$\int_0^T((\eta(t)'\cdot D_{11}L(x^*(t),\ \mathring{x}^*(t)) + \mathring{\eta}(t)'\cdot D_{21}L(x^*(t),\ \mathring{x}^*(t))\ e^{-\delta t} -$$

$$\frac{d}{dt}\,[(\eta(t)'\cdot D_{12}L(x^*(t),\ \mathring{x}^*(t)) + \mathring{\eta}(t)'\cdot D_{22}L(x^*(t),\ \mathring{x}^*(t)))\ e^{-\delta t}])\cdot\eta(t)\ dt$$

$$+\ (\eta(t)'\cdot D_{12}L(x^*(t),\ \mathring{x}^*(t)) + \mathring{\eta}(t)'\cdot D_{22}L(x^*(t),\ \mathring{x}^*(t)))\cdot\eta(t)\ e^{-\delta t}\,\Big|_{t=0}^{t=T}\ \geq$$

(4.2) $$(\eta(T)'\cdot D_{12}L(x^*(T),\ \mathring{x}^*(T)) + \mathring{\eta}(T)'\cdot D_{22}L(x^*(T),\ \mathring{x}^*(T)))\cdot\eta(T)\ e^{-\delta T},$$

where the first inequality follows from an integration by parts. The second inequality follows from the fact that $\{\eta(t)\}_{t\geq 0}$ satisfies Condition D.1, and $\eta(0) = 0$.

Since $\eta(t) = z(t) - z'(t)$, $\{z(t)\}_{t\geq 0}$ and $\{z'(t)\}_{t\geq 0}$ satisfy conditon D.2, and the second-order derivatives of L are assumed to be uniformly bounded, for some T arbitrarily large the value in (4.2) must be near zero (cf. Remark 4.4). Hence, $\lim_{T\to\infty} \int_0^T \frac{1}{2}((\eta(t),\ \mathring{\eta}(t))'\cdot D^2L(x^*(t),\ \mathring{x}^*(t))\cdot(\eta(t),\ \mathring{\eta}(t)))\ e^{-\delta t}\,dt = 0$. As $L(x^*(t), \mathring{x}^*(t))$ is $\alpha_{\mathring{x}}$-concave, it must be the case that $\eta(t) = 0$ for all t. Therefore, $\{z(t)\}_{t\geq 0}$ is an optimal solution to problem (4.1). The proof is complete.

Lemma 4.6: Let $\{x^*(t)\}_{t\geq 0}$ be an optimal solution to problem (2.1). Assume that g is differentiable at every point of the orbit $\{x^*(t)\}_{t\geq 0}$. Then every sequence $\{z(t)\}_{t\geq 0}$ with $\mathring{z}(t) = Dg(x^*(t))\cdot z(t)$, for all $t \geq 0$, and $\|z(0)\| = 1$, satisfies Condition D.

Proof: Note that every optimal interior solution $\{x^*(t)\}_{t\geq 0}$ must obey at all times the Euler equation $D_1L(x^*(t), \mathring{x}^*(t))e^{-\delta t} - \frac{d}{dt}[D_2L(x^*(t), \mathring{x}^*(t))\ e^{-\delta t}] = 0$ where $\mathring{x}^*(t) = g(x^*(t))$. Then, it is readily seen [cf. equation (3.5)] that any solution $\{z(t)\}_{t\geq 0}$ with $\mathring{z}(t) = Dg(x^*(t))\cdot z(t)$ must satisfy Condition D.1.

Also, by the Bellman principle $V(x^*(0)) = \int_0^T L(x^*(t), \mathring{x}^*(t))\ e^{-\delta t}\,dt + V(x^*(T))\ e^{-\delta T}$.

The statement of the Lemma implies that V is twice differentiable at the points $x^*(0)$ and $x^*(T)$. Therefore, for $\{z(t)\}_{t\geq 0}$ with $\mathring{z}(t) = Dg(x^*(t))\cdot z(t)$ it follows that

$$z(0)'\cdot D^2V(x^*(0))\cdot z(0) - \int_0^T (z(t), \mathring{z}(t))'\cdot D^2L(x^*(t), \mathring{x}^*(t))\cdot(z(t), \mathring{z}(t))\, e^{-\delta t}\, dt -$$

$$z(T)'\cdot D^2V(x^*(T))\cdot z(T)\, e^{-\delta T} = 0.$$

By the concavity of V, we have for every $T \geq 0$ that

$$- \int_0^T (z(t), \mathring{z}(t))'\cdot D^2L(x^*(t), \mathring{x}^*(t))\cdot(z(t), \mathring{z}(t))\, e^{-\delta t}\, dt \leq - z(0)'\cdot D^2V(x^*(0))\cdot z(0).$$

As DV is a Lipschitz mapping, there is a uniform constant $M > 0$ such that every sequence $\{z(t)\}_{t\geq 0}$ with $\mathring{z}(t) = Dg(x^*(t))\cdot z(t)$, for all $t \geq 0$, and $|| z(0) || = 1$, must satisfy Condition D.2. This proves the lemma.

Proof of Theorem 4.1: By Theorem 3.1, the function g is locally Lipschitz continuous on the open neighborhood U. Therefore, by Rademacher's theorem [cf. Stein (1970)] the function g is differentiable almost everywhere on U. Moreover, following Clarke (1975, Prop. 1.13) the function g is C^1 differentiable on U if every sequence of differentiable points $\{x_n\}_{n\geq 0}$ converging to $x(0)$ in U has the property that $\lim_{n\to\infty} Dg(x_n)$ is uniquely defined.

Assume, therefore, that g is differentiable at a point $x_n \in U$. Let $\{x_n^*(t)\}_{t\geq 0}$ be the optimal solution to problem (2.1) with $x_n^*(0) = x_n$. Then the mapping g is differentiable at every point of the path $\{x_n^*(t)\}_{t\geq 0}$, by Lemma 4.3. Let $\mathring{z}_{ni}(0)$ denote the i^{th} column of the derivative $Dg(x_n^*(0))$. Let z_i be the canonical basis element of R^n with 1 in the i^{th} coordinate. Then by Lemmas 4.5 and 4.6 the path $\{z_n(t)\}_{t\geq 0}$ with $\mathring{z}_n(t) = Dg(x_n^*(t))\cdot z_n(t)$ and $z_n(0) = z_i$ is an optimal solution to the quadratic optimization problem

$$\text{(4.3)} \qquad \max_{\{y_n(t)\}_{t\geq 0}} \int_0^\infty \frac{1}{2} (y_n(t), \mathring{y}_n(t))'\cdot D^2L(x_n^*(t), \mathring{x}_n^*(t))\cdot(y_n(t), \mathring{y}_n(t))\, e^{-\delta t}\, dt$$

$$\text{s. t. } y_n(0) = z_i.$$

Since by Theorem 3.1 the function g is Lipschitz continuous, the sequence $\{\mathring{z}_n(0)\}_{n\geq 0}$ is bounded. Hence, it follows from the $\alpha_{\mathring{x}}$-concavity of L and Euler's equation [cf.

Condition D.1] that for every T the sequence of functions $[\{(z_n(t), \overset{o}{z}_n(t))\}_{0 \le t \le T}]_{n \ge 0}$ is equicontinuous. Therefore, we can extract a subsequence of $[\{(z_n(t), \overset{o}{z}_n(t))\}_{t \ge 0}]_{n \ge 0}$ which converges point-wise to a given limit, say $\{(z(t), \overset{o}{z}(t))\}_{t \ge 0}$. Moreover, as g is a continuous function it is readily seen that $\{(z(t), \overset{o}{z}(t))\}_{t \ge 0}$ must satisfy Conditon D, and by Lemma 4.5 it is therefore an optimal solution to The quadratic optimization problem (4.1).

However, every converging subsequence of $\{(z_n(t), \overset{o}{z}_n(t))\}$ must converge to a unique limit. For, if not, the strictly concave, quadratic problem (4.1) would contain several optimal solutions. Therefore, the entire sequence $[\{(z_n(t), \overset{o}{z}_n(t))\}_{t \ge 0}]_{n \ge 0}$ converges point-wise to $\{(z(t), \overset{o}{z}(t))\}_{t \ge 0}$. Hence, the sequence $\{\overset{o}{z}_n(0)\}_{n \ge 0}$ is convergent, where $\overset{o}{z}_n(0)$ is the i^{th} column of $Dg(x_n)$. Since the choice of $i = 1, \ldots, n$ is arbitrary, it follows that for every sequence of differentiable points $\{x_n\}_{n \ge 0}$ converging to $x(0)$ in U, $\lim_{n \to \infty} Dg(x_n)$ is uniquely defined. Consequently, the function g is C^1 differentiable on U. Moreover, it follows from Lemma 4.6 and Remark 4.4 that there is $K = \frac{M}{\alpha}$ such that $\int_0^\infty \| \overset{o}{z}(t) \|^2 e^{-\delta t} dt \le K$ for $\overset{o}{z}(t) = Dg(x^*(t)) \cdot z(t)$, for all $t \ge 0$, and $\| z(0) \| = 1$. The theorem is proved.

5. HIGHER-ORDER DIFFERENTIABILITY OF THE POLICY FUNCTION

In the preceding section we showed that under general conditions if the variational integrand L is C^2 differentiable, then the policy function g is a C^1 mapping. Under the previous assumptions, our goal here is to explore the higher-order differentiability of the policy function g, as the degree of differentiability of the integrand L is increased. The higher-order differentiability of optimal paths is often useful to analyze the behavior of chaotic dynamics, endogenous cycles and bifurcations [see, for instance, Boldrin and Woodford (1990, Sect. 2)].

In the discrete time case, Araujo (1989) has presented an example of a C^3 optimization problem in which the policy function is only C^1 differentiable at an unstable stationary point. It is unknown, however, whether examples of this sort arise from very simple return functions, and if these examples are robust to small perturbations of the objective.

By identifying certain conditions which may prevent higher-order differentiability, the subsequent development will provide a fairly satisfactory answer to these questions. These conditions are related to certain results on loss of differentiability in the theory of dynamical systems. A fundamental result in this area is a theorem due to Sternberg

(1959), roughly, the theorem states that for a non-linear C^k function $f : R^n \rightarrow R^n$, if at a stationary point x^* the characteristic roots do not fulfill certain independence conditions (named non-resonance conditions), then it is not always possible to establish around the stationary point x^* a C^k correspondence between the orbits of the dynamical system generated by f and the orbits of the dynamical system generated by the derivative $Df(x^*)$. [More precisely, around the stationary point x^*, the function f is not necessarily C^k conjugate to its derivative $Df(x^*)$.]

We first proceed with a simple example of a cubic return function whose policy is only C^1 differentiable.[4] Then we shall focus on the conditions which generate this result.

Example: Define

(5.1) $$L(k, \dot{k}) = 24(k - 1) - 14(k - 1)^2 - \frac{(k - 1)^3}{3} + 5(k - 1)\dot{k} - \frac{1}{2}\dot{k}^2 - 2\dot{k}.$$

Then around the point $(k, \dot{k}) = (1, 0)$ the Hessian matrix $D^2L(k, \dot{k})$ is negative definite. Therefore, L is locally a strongly concave function. Moreover, assuming that the discount rate $\delta = 12$, the Euler equation is satisfied at the point $(k, \dot{k}) = (1, 0)$. Let $y = k - 1$. Then the Euler equation is in fact defined by

(5.2) $$\ddot{y} = 12\dot{y} - 32y + y^2.$$

Hence, the linear system is

(5.3) $$\ddot{y} = 12\dot{y} - 32y.$$

The characteristic roots for the system are $\lambda_1 = 4, \lambda_2 = 8$. Therefore, the system is unstable. Let $x = \dot{y}$. We now claim that around the point (0, 0) there is no C^2 invariant curve $x = h(y)$ with $h'(0) = 4$. Given that the derivative $Dg(1) = 4$ (see the commentary after Theorem 4.1), this claim yields the required example.

The claim will be established following the method of undetermined coefficients [e.g., van Strien (1979)]. Assume that there is a solution of the form

(5.4) $$x = h(y) = 4y + a_2y^2 + o(y^2)$$

where $o(y^2)$ collects all remaining terms of order lower than y^2 (i.e., $\lim_{y \rightarrow 0} \frac{o(y^2)}{y^2} = 0$).

Then the derivative

(5.5) $$h'(y) = 4 + 2a_2y + o(y).$$

As $x = \overset{o}{y}$, from equation (5.2),

$$\frac{\overset{o}{x}}{\overset{o}{y}} = \frac{12x - 32y + y^2}{x}. \tag{5.6}$$

Substituting out equation (5.4) into (5.6), we obtain

$$\frac{\overset{o}{x}}{\overset{o}{y}} = \frac{12(4y + a_2y^2 + o(y^2)) - 32y + y^2}{4y + a_2y^2 + o(y^2)}. \tag{5.7}$$

Since h is an invariant curve for the flow generated by (5.2), then at every $\overset{o}{y} \neq 0$ it follows that $h'(y) = \frac{\overset{o}{x}}{\overset{o}{y}}$.

Therefore, from equations (5.5) and (5.7)

$$4 + 2a_2y + o(y) = \frac{12(4y + a_2y^2 + o(y^2)) - 32y + y^2}{4y + a_2y^2 + o(y^2)}.$$

Hence,

$$16y + 12a_2y^2 + o(y^2) = 48y - 32y + 12a_2y^2 + y^2 + o(y^2).$$

This implies that

$$y^2 = o(y^2).$$

However, this last identity cannot be true around the point y = 0. Therefore, around the point $(k, \overset{o}{k}) = (1, 0)$ the return function (5.1) cannot contain a C^2 invariannt curve $\overset{o}{k} = g(k)$ with slope equal to 4. Consequently, the policy function is at most C^1 at $k = 1$.

Let us now explore the sources of non-differentiability. In the first place, the point $(k, \overset{o}{k}) = (1, 0)$ is an unstable stationary point. Quoting some results from the theory of dynamical systems, Santos and Vila (1988) pointed out that higher-order differentiability may be lost at unstable stationary points. At stable steady states, and at points in the basin of attraction of a steady state, it follows from an application of the center manifold theorem that if the return function is C^k, then the policy function is C^{k-1}, where $k \geq 2$.[5]

In the above example the characteristic roots $\lambda_1 = 4, \lambda_2 = 8$ have the property $\lambda_2 = 2\lambda_1$. Therefore, the roots do not satisfy the independence (non-resonance) conditions given by Sternberg (1959). According to Sternberg's results, higher-order differentiability may only be lost in the case $\lambda_2 = n\lambda_1$, where n is some positive integer. Moreover, following Hirsch, Pugh and Shub (1977, Th. 5.1), if $\lambda_2 = n\lambda_1$ then there is always a C^{n-1} invariant curve whose slope at the steady state is equal to λ_1. Therefore, the loss of higher-order differentiability in our example is maximal.

Another feature of this example is that the return function is locally a polynomial of degree 3. Note that quadratic return functions give rise to linear policy functions, and consequently such policy functions are always C^∞ differentiable. On the other hand, it is worth pointing out that those terms with an exponent greater than 3 would not have any effect on the C^2 differentiability of the policy function. For example, if we add to a quadratic return function a term of the form $m(k-1)^4$, where m is a real number, such perturbation may have an effect on the C^3 differentiability of the invariant curves, but would not have any effect on the C^2 differentiability of such curves.

The resonance conditions of Sternberg (i.e., $\lambda_2 = n\lambda_1$ for some integer $n \geq 1$) are not preserved under small perturbations of the characteristic roots. Likewise, continuous-time, unidimensional models display fairly simple dynamic behavior. Indeed, stationary points are the only type of recurrent dynamics. Therefore, under the above assumptions for one dimensional models, if return functions are C^∞ differentiable, then optimal policies are generically going to be C^∞ differentiable. Consequently, the above example is pathological in the sense that the result will not be preserved for small C^2 perturbations of the return function.

Models with many goods feature more complicated recurrent dynamics. Such models may contain periodic orbits, limit cycles, and other fairly complex configurations of asymptotic behavior. There is here more scope for non differentiability. Of application to our purposes is a result due to Mañe (1975). Such result establishes that if a C^∞ dynamical system contains a C^∞ invariant manifold which is k-normally hyperbolic, then every nearby C^∞ dynamical system contains an invariant manifold, but such manifold is possibly at most C^k differentiable.

Even if the policy function is not always higher-order differentiable, for a *given* model (satisfying the above assumptions) we could consider how big is the set of points where higher-order differentiability is not achieved. For example, as illustrated in Santos an Vila (1988), for one dimensional models higher-order differentiability is lost at most at a countable number of isolated, unstable stationary points. For multidimensional models, there is an analogous result. A well known theorem in the theory of dynamical systems [cf. Bowen (1975, Th.4.11)] states that under the action of an Axiom A diffeomorphism almost all points are in the basin of an attractor.[6] Whence, systems which feature dynamics of this type have the property that the policy function is almost always higher-order differentiable.

FOOTNOTES

1. For functions $L: R^n \times R^n \rightarrow R$, $DL(k, \dot{k})$ denotes the derivative of L evaluated at the point $(k, \dot{k})$, and $D_1L(k, \dot{k})$ and $D_2L(k, \dot{k})$ denote the partial derivatives of L at $(k, \dot{k})$ with respect to k and $\dot{k}$, resp. Also, if L is C^2 differentiable, then $D^2L(k, \dot{k})$ is the Hessian matrix at $(k, \dot{k})$, and for i,j = 1,2, $D_{ij}L(k, \dot{k})$ is the second-order partial derivative of L at $(k, \dot{k})$ with respect to the i^{th} and j^{th} components.

2. A function $g: U \subset R^n \rightarrow R^m$ is called Lipschitz if there is a constant $K > 0$ such that $||g(x) - g(y)|| \leq K \, ||x-y||$ for all x, y in U.

3. For some technical reasons, an exception to this rule is the one-dimensional case, where Santos and Vila (1988) demonstrate the C^1 differentiability of g under the mild interiority requirement given by Assumption C.

4. Since the tools of this section come from the theory of dynamical systems, the example and remaining results have corresponding analogues in the discrete time case.

5. Araujo and Scheinkman (1977) were the first to show that the policy function is differentiable at a hyperbolic stable steady state.

6. A diffeomorphism satisfies Axiom A if the wandering set $\Omega(f)$ is hyperbolic and $\Omega(f) = \overline{\{x: x \text{ is periodic}\}}$, see op. cit.

REFERENCES

Araujo, A. (1989), "The once but not twice differentiability of the policy function," unpublished manuscript.

Araujo, A. and J.A. Scheinkman (1977) " Smoothness, comparative dynamics, and the turnpike property," *Econometrica* **45**, 601-620.

Arrow, K.J. and M. Kurz (1970), *Public investment, the rate of return, and optimal fiscal policy*. Johns Hopkins University Press.

Benhabib, J. and K. Nishimura (1979), "The Hopf bifurcation and the existence and stability of closed orbits in multisector models of optimal economic growth," *Journal of Economic Theory* **21**, 421-444.

Boldrin, M. and M. Woodford (1990), "Equilibrium models displaying endogenous fluctuations and chaos: A survey," *Journal of Monetary Economics* **25**, 189-223.

Bowen, R. (1975) *Equilibrium states and the ergodic theory of Anosov diffeomorphisms*. Lecture Notes in Mathematics no. **470**, Springer-Verlag.

Carlson, D.A. and A. Haurie (1987), *Infinite horizon optimal control*. Lecture Notes in Economics and Mathematical Systems no. **290**, Springer-Verlag.

Clarke, F. H. (1975), "Generalized gradients and applications," *Transactions of the American Mathematical Society* **205**, 247-262.

Crandall, M.G., L.C. Evans and P.-L. Lions (1984), "Some properties of viscosity solutions of Hamilton-Jacobi equations," *Transactions of the American Mathematical Society* **282**, 487-502.

Fleming, W.L. (1971), "Stochastic control for small noise intensities," *SIAM Journal on Control* **9**, 473-517.

Fleming, W.L. and R.W. Rishel (1975), *Deterministic and stochastic optimal control*. Springer-Verlag.

Hirsch, M.C., C.C. Pugh and M. Shub (1987), *Invariant manifolds*. Springer-Verlag.

Kehoe, T.J., D.K. Levine and P.M. Romer (1990), "Determinacy of equilibrium in dynamic models with finitely many consumers," *Journal of Economic Theory* **50**, 1-21.

Levhari, D. and N. Liviatan (1972), "On stability in the saddle-point sense," *Journal of Economic Theory* **4**, 88-93.

Lions, P.-L. (1982), *Generalized solutions of the Hamilton-Jacobi equations*. Research Notes in Mathematics no. **69**, Pitman, London.

Mañe, R. (1974), "Persistent manifolds are normally hyperbolic," *Bulletin of the American Mathematical Society* **80**, 90-91.

Montrucchio, L. (1987), "Lipschitz continuous policy functions for strongly concave optimization problems," *Journal of Mathematical Economics* **16**, 259-273.

Pontryagin, L.S., V.G. Boltyanskii, R.V. Gamkreledze, and E.F. Mischenko (1962), *The mathematical theory of optimal processes*. Interscience, John Wiley.

Santos, M.S. (1989), "Differentiability and comparative analysis in discrete-time infinite-horizon optimization problems," revised version, submitted to *Journal of Economic Theory*.

Santos, M.S. (1990), "Smoothness of the policy function is discrete time economic models," to appear in *Econometrica*.

Santos, M.S. and J.-L. Vila (1988), "Smoothness of the policy function in continuous time economic models: The one dimensional case," to appear in *Journal of Economic Dynamics and Control*.

Stein, E.M. (1970), *Singular integrals and differentiability properties of functions*. Princenton Mathematical Series no. **30**.

Sternberg, S. (1959). "On the structure of local homeomorphisms of Euclidean n-space II," *American Journal of Mathematics* **80**, 623-631.

Stokey, N.L, R.E. Lucas and E.C. Prescott (1989), *Recursive methods in economic dynamics*. Harvard University Press.

Toman, M.A. (1986), "Optimal control with unbounded horizon," *Journal of Economic Dynamics and Control* **9**, 291-316.

van Strien, S. (1979), "Center manifolds are not C^∞," *Mathematische Zeitschrift* **166**, 143-145.