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Universidad Carlos III de Madrid
Calle Madrid, 126
28903 Getafe (Spain)
Fax (34) 916249875

Seasonality Detection in Small Samples using Score-Driven Nonlinear Multivariate Dynamic Location Models*

Szabolcs Blazsek†, Alvaro Escribano‡ and Adrian Licht†

†School of Business, Universidad Francisco Marroquín, Calle Manuel F. Ayau, 01010, Ciudad de Guatemala, Guatemala (e-mails: sblazsek@ufm.edu, adrian.licht@ufm.edu)
‡Department of Economics, Universidad Carlos III de Madrid, Calle Madrid 126, 28903, Getafe (Madrid), Spain (e-mail: alvaroe@eco.uc3m.es)

Abstract

We suggest a new mechanism to detect stochastic seasonality of multivariate macroeconomic variables, by using an extension of the score-driven first-order multivariate t-distribution model. We name the new model as the quasi-vector autoregressive (QVAR) model. QVAR is a nonlinear extension of Gaussian VARMA (VAR moving average). The location of dependent variables for QVAR is updated by the score function, thus QVAR is robust to extreme observations. For QVAR, we present the econometric formulation, computation of the impulse response function (IRF), maximum likelihood (ML) estimation, and conditions of the asymptotic properties of ML that include invertibility. We use quarterly macroeconomic data for the period of 1987:Q1 to 2013:Q2 inclusive, which include extreme observations from three *I(0)* variables: percentage change in crude oil real price, United States (US) inflation rate, and US real gross domestic product (GDP) growth. The sample size of these data is relatively small, which occurs frequently in macroeconomic analyses. The statistical performance of QVAR is superior to that of VARMA and VAR. Annual seasonality effects are identified for QVAR, whereas those effects are not identified for VARMA and VAR. Our results suggest that QVAR may be used as a practical tool for seasonality detection in small macroeconomic datasets.

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I. Introduction

Macroeconomic time series analysis frequently involves stochastic seasonality components in practical applications. The successful detection of those components is useful, for example, for a first-step deseasonalization of macroeconomic data, in order to effectively measure the interaction effects among the deseasonalized macroeconomic variables in a second-step. The detection of seasonality is also useful in those cases, when the effectiveness of a previous seasonality adjustment is verified ex-post. In this paper, we suggest a new mechanism of seasonality detection for multivariate macroeconomic time series, by extending the dynamic conditional score (DCS) model for the multivariate t distribution (Harvey, 2013). We name the extended model as the quasi-vector autoregressive (QVAR) model, and we present that the new seasonality detection mechanism is very useful for a small macroeconomic dataset from the United States (US).

DCS models are observation-driven time series models (Cox, 1981). An example of a DCS model is Beta-t-EGARCH (exponential generalized autoregressive conditional heteroscedasticity) (Harvey and Chakravarty, 2008), which is an outlier-robust alternative to GARCH (Engle, 1982; Bollerslev, 1986). We refer to the recent Beta-t-EGARCH applications of Blazsek and Villatoro (2015), Blazsek and Mendoza (2016), and Blazsek and Monteros (2017). Another example of a DCS model is QAR (Harvey, 2013), which is a nonlinear and outlier-robust alternative to AR moving average (ARMA) (Box and Jenkins, 1970). The QVAR model of this paper is a nonlinear and outlier-robust alternative to VARMA (Tiao and Tsay, 1989). We also refer to the following recent DCS models: Blazsek and Escribano (2016a) suggest a DCS count panel data model, which is an alternative to the dynamic count panel data models of Blundell, Griffith and Windmeijer (2002), Wooldridge (2005), and Blazsek and Escribano (2010, 2016b). Ayala, Blazsek and Escribano (2017) suggest DCS-EGARCH (exponential GARCH) models with score-driven shape parameters, which are extensions of the DCS-EGARCH models with constant shape (see, for example, Harvey, 2013). Blazsek and Ho (2017) and Blazsek, Ho and Liu (2018) suggest new Markov regime-switching DCS-EGARCH models. The works of Ayala and Blazsek (2018a, 2018b) suggest new DCS copula models for financial portfolios.

QVAR with lag-order p is a score-driven nonlinear multivariate dynamic location model, in which the conditional score vector of the log-likelihood (LL) with respect to location (hereinafter, score function) updates the dependent variables. QVAR(p) is an extension of the DCS model for the multivariate t-distribution (Harvey, 2013) that is QVAR(1) under our notation. QVAR, compared to multivariate Gaussian time series models, is robust to extreme values in the noise. For QVAR, we present the details of the econometric formulation, computation of the impulse response function (IRF), and the maximum likelihood (ML) estimation and related conditions of consistency and asymptotic normality that include the condition of invertibility.

We estimate QVAR by using quarterly macroeconomic time-series data for the period of 1987:Q1 to 2013:Q2 inclusive, from the following $I(0)$ variables: (i) quarterly percentage change in non-seasonally adjusted crude oil real price; (ii) quarterly seasonally adjusted US inflation rate; (iii) quarterly seasonally adjusted US real gross domestic product (GDP) growth. The use of these variables is motivated by several works from the body of literature, which study the question of how oil price shocks affect US real GDP growth and US inflation rate (e.g. Blanchard, 2002; Barsky and Kilian, 2004; Kilian, 2008; Kilian and Lütkepohl, 2017). The dataset of the present paper includes extreme observations (for example, those related to the 1990 oil price shock caused by the Iraqi invasion of Kuwait and also those related to the 2008 financial crisis), motivating the use of the outlier-robust QVAR(p) model. The sample size of these data is relatively small, which is frequently the case in macroeconomic data analyses. We show that the application of the nonlinear QVAR(p) model to this small dataset, is more effective in identifying stochastic seasonality effects in the data series than the application of classic linear multivariate time series models.

We compare the statistical performance of QVAR(p) with that of two linear benchmarks: (i) Gaussian QVAR(p) is a limiting special case of QVAR(p) with multivariate t distribution, when the degrees of freedom parameter goes to infinity. Gaussian QVAR(p) is a Gaussian VARMA(p,p) model with restricted vector MA (VMA) parameters. (ii) Gaussian VARMA(p,q) is a popular model in practical applications and it is also an extension of Gaussian VARMA(p,p).

In relation to Gaussian VARMA(p,q), we also consider the Gaussian VAR(p) model.

We find that the statistical performance of QVAR(p) is superior to that of Gaussian QVAR(p) and Gaussian VARMA(p,q). A relevant finding of this paper is that the nonlinear QVAR(2) with multivariate t distribution is effectively estimated by using the ML method, while for its limiting special case, the Gaussian QVAR(2) model, the ML estimator does not converge to an optimal solution. This result is due to the fact the QVAR(2) with multivariate t distribution that is updated by the score function is robust to extreme values in the irregular component, while its Gaussian benchmark is sensitive to outliers. With respect to the identification of stochastic seasonality effects, we find that QVAR(1) does not identify the aforementioned effects, motivating the extension of that model to QVAR(p) with higher lag-orders. We find that the seasonality detection mechanism is effective for QVAR(2): (i) Annual stochastic seasonality effects are identified for the non-seasonally adjusted percentage change in crude oil real price times series. (ii) Seasonality is not detected for the seasonally adjusted US inflation rate time series, thus the seasonality detection mechanism suggests that seasonality adjustment was successful at the data source. (iii) Annual stochastic seasonality effects are detected for the seasonally adjusted US real GDP growth time series, suggesting that the seasonality adjustment was not effective at the data source. With respect to the linear Gaussian alternatives, seasonality effects are not detected for any of the QVAR(p) and Gaussian VARMA(p,q) specifications of this paper.

The focus of this paper is seasonality detection, by using QVAR(p), for multivariate macroeconomic time series data. Nevertheless, QVAR(p) can also be applied to the detection of different forms of nonlinearity in time series, other than seasonality. For example, QVAR(p) can be applied to (i) the detection of regime-switching time series dynamics, (ii) the highlighting of the presence of extreme observations in a dataset, or (iii) the identification of different forms of heteroscedasticity. The QVAR(p) model of this paper identifies stochastic seasonality and it also verifies whether seasonality adjustment was successful at the data source. These results may motivate the consideration of QVAR(p) in future macroeconomic analyses, which use small datasets with extreme observations that are frequent properties of macroeconomic data.

The remainder of this paper is organized as follows. Section II presents the nonlinear QVAR(p) model. Section III presents the benchmark linear Gaussian QVAR(p) and Gaussian VARMA(p,q) models. Section IV describes the macroeconomic data. Section V summarizes the empirical results. Section VI concludes.

II. Score-driven nonlinear multivariate dynamic location model: QVAR(p)

Reduced-form and structural-form representations

The reduced-form representation of QVAR(p) for y_t ($K \times 1$) is

$$y_t = c + \mu_t + v_t, \tag{1}$$

$$\mu_t = \Phi_1 \mu_{t-1} + \ldots + \Phi_p \mu_{t-p} + \Psi_1 u_{t-1}, \tag{2}$$

where c ($K \times 1$), $\Phi_1, \ldots, \Phi_p$ (each $K \times K$) and Ψ_1 ($K \times K$) are time-constant parameters. The conditional mean of the dependent variables is given by $E(y_t | y_1, \ldots, y_{t-1}) = c + \mu_t$, because the updating term u_{t-1} ($K \times 1$) with zero unconditional mean is a function of $y_1, \ldots, y_{t-1}$ and $E(v_t) = 0_{K\times 1}$. For the first p observations, we initialize μ_t by using $\mu_t = E(\mu_t) = 0_{K\times 1}$.

With respect to the updating terms, v_t ($K \times 1$) is the reduced-form error term and u_t ($K \times 1$) is a scaled score function vector. v_t is multivariate i.i.d. with $v_t \sim t_K(0, \Sigma, \nu)$, where $\Sigma = \Omega^{-1}(\Omega^{-1})'$ is positive definite and $\nu > 2$ denotes the degrees of freedom parameter (thus, the variance of v_t is finite). The log of the conditional density of y_t is

$$\ln f(y_t | y_1, \ldots, y_{t-1}) = \ln \Gamma\left(\frac{\nu + K}{2}\right) - \ln \Gamma\left(\frac{\nu}{2}\right) - \frac{K}{2}\ln(\pi\nu) \tag{3}$$

$$-\frac{1}{2}\ln|\Sigma| - \frac{\nu + K}{2}\ln\left(1 + \frac{v_t'\Sigma^{-1}v_t}{\nu}\right).$$

The partial derivative of the log of the conditional density with respect to μ_t is

$$\frac{\partial \ln f(y_t | y_1, \ldots, y_{t-1})}{\partial \mu_t} = \frac{\nu + K}{\nu}\Sigma^{-1} \times \left(1 + \frac{v_t'\Sigma^{-1}v_t}{\nu}\right)^{-1} v_t = \frac{\nu + K}{\nu}\Sigma^{-1} \times u_t, \tag{4}$$

The latter equality defines the scaled score function u_t by using the reduced-form error term. In the definition of u_t, v_t is multiplied by $[1+(v_t'\Sigma_v^{-1}v_t)/\nu]^{-1} = \nu/(\nu+v_t'\Sigma_v^{-1}v_t) \in (0,1)$. Therefore, the scaled score function is always bounded by the reduced-form error term: $|u_t| < |v_t|$. The scaled score function u_t is multivariate i.i.d. with mean zero and covariance matrix

$$\text{Var}(u_t) = E\left[\frac{\partial \ln f(y_t|y_1,\ldots,y_{t-1})}{\partial \mu_t} \times \frac{\partial \ln f(y_t|y_1,\ldots,y_{t-1})}{\partial \mu_t'}\right] = \frac{\nu+K}{\mu+K+2}\Sigma^{-1}. \tag{5}$$

Related to the structural-form representation of QVAR(p), for the reduced-form error term $v_t \sim t_K(0,\Sigma,\nu)$ we have $E(v_t)=0$ and $\text{Var}(v_t) = \Sigma \times \nu/(\nu-2)$. We factorize $\text{Var}(v_t)$ as

$$\text{Var}(v_t) = \Sigma \times \frac{\nu}{\nu-2} = \left(\frac{\nu}{\nu-2}\right)^{1/2} \times \Omega^{-1}(\Omega^{-1})' \times \left(\frac{\nu}{\nu-2}\right)^{1/2}, \tag{6}$$

and we introduce the multivariate i.i.d. structural-form error term ϵ_t as

$$v_t = \left(\frac{\nu}{\nu-2}\right)^{1/2} \Omega^{-1} \times \epsilon_t, \tag{7}$$

where $E(\epsilon_t)=0$, $\text{Var}(\epsilon_t) = I_K$ and $\epsilon_t \sim t_K[0, I_K \times (\nu-2)/\nu, \nu]$. By substituting equation (7) into equation (1), we obtain the structural-form representation of QVAR(p):

$$\left(\frac{\nu}{\nu-2}\right)^{-1/2} \Omega y_t = \left(\frac{\nu}{\nu-2}\right)^{-1/2} \Omega c + \left(\frac{\nu}{\nu-2}\right)^{-1/2} \Omega \mu_t + \left(\frac{\nu}{\nu-2}\right)^{-1/2} \Omega v_t = \tag{8}$$

$$= \left(\frac{\nu}{\nu-2}\right)^{-1/2} \Omega c + \left(\frac{\nu}{\nu-2}\right)^{-1/2} \Omega \mu_t + \epsilon_t.$$

Furthermore, by substituting equation (7) into u_t from equation (4), we obtain

$$u_t = [(\nu-2)\nu]^{1/2}\Omega^{-1} \times \frac{\epsilon_t}{\nu-2+\epsilon_t'\epsilon_t}, \tag{9}$$

which is the representation of the scaled score function u_t according to the structural-form error term ϵ_t (we use the latter equation to obtain the IRF formulas in the next three subsections).

First-order representation

The first-order representation of the reduced-form QVAR(p) model of equations (1) and (2) is

$$Y_t = C + M_t + V_t, \tag{10}$$

$$M_t = \Phi M_{t-1} + \Psi U_{t-1}, \tag{11}$$

where

$$Y_t = \begin{bmatrix} y_t \\ y_{t-1} \\ \vdots \\ y_{t-p+1} \end{bmatrix}_{(Kp\times 1)} \qquad C = \begin{bmatrix} c \\ c \\ \vdots \\ c \end{bmatrix}_{(Kp\times 1)} \qquad M_t = \begin{bmatrix} \mu_t \\ \mu_{t-1} \\ \vdots \\ \mu_{t-p+1} \end{bmatrix}_{(Kp\times 1)} \qquad V_t = \begin{bmatrix} v_t \\ v_{t-1} \\ \vdots \\ v_{t-p+1} \end{bmatrix}_{(Kp\times 1)},$$

$$\Phi = \begin{bmatrix} \Phi_1 & \Phi_2 & \cdots & \Phi_{p-1} & \Phi_p \\ I_K & 0_{K\times K} & \cdots & \cdots & 0_{K\times K} \\ 0_{K\times K} & I_K & 0_{K\times K} & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0_{K\times K} & \cdots & 0_{K\times K} & I_K & 0_{K\times K} \end{bmatrix}_{(Kp\times Kp)}$$

$$\Psi = \begin{bmatrix} \Psi_1 & 0_{K\times K} & \cdots & 0_{K\times K} \\ 0_{K\times K} & 0_{K\times K} & \cdots & 0_{K\times K} \\ \cdots & \cdots & \cdots & \cdots \\ 0_{K\times K} & \cdots & \cdots & 0_{K\times K} \end{bmatrix}_{(Kp\times Kp)} \qquad U_t = \begin{bmatrix} u_t \\ u_{t-1} \\ \vdots \\ u_{t-p+1} \end{bmatrix}_{(Kp\times 1)}$$

Infinite vector moving average representation

From equations (10) and (11), the reduced-form nonlinear VMA(∞) representation of y_t is

$$y_t = c + \left(\sum_{j=0}^{\infty} J\Phi^j J' \Psi_1 u_{t-1-j} \right) + v_t, \tag{12}$$

$$y_t = c + \left[\sum_{j=0}^{\infty} J\Phi^j J'\Psi_1 \left(1 + \frac{v'_{t-1-j}\Sigma^{-1}v_{t-1-j}}{\nu}\right)^{-1} v_{t-1-j} \right] + v_t. \tag{13}$$

where $J = (I_K, 0_{K\times K}, \cdots, 0_{K\times K})$ $(K \times Kp)$. By using equation (7), the related structural-form nonlinear VMA(∞) representation of y_t is given by:

$$y_t = c + \left\{ \sum_{j=0}^{\infty} J\Phi^j J'\Psi_1[(\nu-2)\nu]^{1/2}\Omega^{-1} \frac{\epsilon_{t-1-j}}{\nu - 2 + \epsilon'_{t-1-j}\epsilon_{t-1-j}} \right\} + \left(\frac{\nu}{\nu-2}\right)^{1/2} \Omega^{-1}\epsilon_t. \tag{14}$$

We use C_1 to denote the maximum modulus of all eigenvalues of Φ. $C_1 < 1$ implies that the different series in equations (12) to (14) are convergent.

Impulse response function

From equation (14), we obtain $\mathrm{IRF}_j = \partial y_{t+j}/\partial\epsilon_t$ for $j = 0, 1, \ldots, \infty$ that is given by

$$\mathrm{IRF}_0 = \left(\frac{\nu}{\nu-2}\right)^{1/2} \Omega^{-1}, \tag{15}$$

$$\mathrm{IRF}_{jt} = J\Phi^j J'\Psi_1[(\nu-2)\nu]^{1/2}\Omega^{-1}D_{t-1-j} \quad \text{for} \quad j = 1, \ldots, \infty, \tag{16}$$

where

$$D_t = \frac{\partial \frac{\epsilon_t}{\nu-2+\epsilon'_t\epsilon_t}}{\partial\epsilon_t} = \begin{bmatrix} d_{11,t} & \cdots & d_{1K,t} \\ \cdots & \cdots & \cdots \\ d_{K1,t} & \cdots & d_{KK,t} \end{bmatrix} = \tag{17}$$

$$= \begin{bmatrix} \frac{\nu-2+\epsilon'_t\epsilon_t-2\epsilon_{1t}^2}{(\nu-2+\epsilon'_t\epsilon_t)^2} & \frac{-2\epsilon_{1t}\epsilon_{2t}}{(\nu-2+\epsilon'_t\epsilon_t)^2} & \cdots & \frac{-2\epsilon_{1t}\epsilon_{Kt}}{(\nu-2+\epsilon'_t\epsilon_t)^2} \\ \frac{-2\epsilon_{2t}\epsilon_{1t}}{(\nu-2+\epsilon'_t\epsilon_t)^2} & \frac{\nu-2+\epsilon'_t\epsilon_t-2\epsilon_{2t}^2}{(\nu-2+\epsilon'_t\epsilon_t)^2} & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \frac{-2\epsilon_{Kt}\epsilon_{1t}}{(\nu-2+\epsilon'_t\epsilon_t)^2} & \cdots & \cdots & \frac{\nu-2+\epsilon'_t\epsilon_t-2\epsilon_{Kt}^2}{(\nu-2+\epsilon'_t\epsilon_t)^2} \end{bmatrix}.$$

As IRF_{jt} for $j = 1, 2, \ldots, \infty$ depends on t, we evaluate its unconditional mean

$$\mathrm{IRF}_j = E(\mathrm{IRF}_{jt}) = J\Phi^j J'\Psi_1[(\nu-2)\nu]^{1/2}\Omega^{-1}E(D_{t-1-j}) \quad \text{for} \quad j = 1, 2, \ldots, \infty. \tag{18}$$

If all elements of D_t are covariance stationary, then $E(D_{t-1-j})$ can be estimated by using the sample average (see, for example, Hamilton, 1994). We test the covariance stationarity of D_t by using the augmented Dickey–Fuller (1979) (ADF) unit root test with constant. It is important to note, however, that an alternative to the use of the time-invariant $E(\text{IRF}_{jt})$ is the period-by-period estimation of IRF_{jt}. In those applications, IRF_{jt} is averaged, for example, for pre- and post-recession periods and the resulting different IRF estimates are compared.

Maximum likelihood estimation

We estimate the parameters of QVAR (c, $\Phi_1, \ldots, \Phi_p$, Ψ_1, Ω^{-1} and ν), by using the ML method. The ML estimator of parameters is given by

$$\hat{\Theta}_{\text{ML}} = \arg\max_{\Theta} \text{LL}(y_1, \ldots, y_T; \Theta) = \arg\max_{\Theta} \sum_{t=1}^{T} \ln f(y_t|y_1, \ldots, y_{t-1}; \Theta), \tag{19}$$

where Θ denotes the vector of parameters. We use the numerically estimated inverse information matrix for the ML standard errors (Creal, Koopman and Lucas, 2013; Harvey, 2013), and we also use results from Harvey (2013, Chapters 2.3, 2.4 and 3.3) for the conditions of consistency and asymptotic normality of the ML estimator. Related to the asymptotic properties of the ML estimator, we also study the invertibility of QVAR(p) (see, for example, Blasques, Gorgi, Koopman and Wintenberger, 2018).

First, Condition 1 is $C_1 < 1$, which ensures that μ_t is covariance stationary. Second, Condition 2 is that the scaled score function u_t ($K \times 1$) and its derivative $\partial u_t/\partial \mu_t$ ($K \times K$) have finite second moments and covariance that are dynamic and do not depend on μ_t. For this condition, we refer to the specific elements $u_{j,t}$ and $\partial u_{k,t}/\partial \mu_{l,t}$, where $j, k, l = 1, \ldots, K$. Condition 2 holds if $E[u_{j,t}^{2-i}(\partial u_{k,t}/\partial \mu_{l,t})^i] < \infty$, where $i = 0, 1, 2$ and $j, k, l = 1, \ldots, K$. We test Condition 2 by using the ADF test with constant for each $u_{j,t}^{2-i}(\partial u_{k,t}/\partial \mu_{l,t})^i$. Third, for Condition 3, we consider the representative element Ψ_{ij} from the matrix Ψ. From equation (11), we have

$$\frac{\partial M_t}{\partial \Psi_{ij}} = \Phi \frac{\partial M_{t-1}}{\partial \Psi_{ij}} + \Psi \frac{\partial U_{t-1}}{\partial \Psi_{ij}} + W_{ij} U_{t-1} \tag{20}$$

for all $t = 1, \dots, T$, where the element (i, j) of the matrix W_{ij} $(Kp \times Kp)$ is one and the rest of the elements of W_{ij} are zero. We use the chain rule to express

$$\frac{\partial U_{t-1}}{\partial \Psi_{ij}} = \frac{\partial U_{t-1}}{\partial M'_{t-1}} \frac{\partial M_{t-1}}{\partial \Psi_{ij}}, \tag{21}$$

and we substitute the latter equation into equation (20) to get the first-order AR representation

$$\frac{\partial M_t}{\partial \Psi_{ij}} = \left(\Phi + \Psi \frac{\partial U_{t-1}}{\partial M'_{t-1}} \right) \frac{\partial M_{t-1}}{\partial \Psi_{ij}} + W_{ij} U_{t-1} = X_t \frac{\partial M_{t-1}}{\partial \Psi_{ij}} + W_{ij} U_{t-1}, \tag{22}$$

where X_t $(Kp \times Kp)$ is defined by the last equality. Condition 3 is that all eigenvalues of $E(X_t)$ are within the unit circle. We denote the maximum modulus of all eigenvalues of $E(X_t)$ by using C_3. If each element of X_t is covariance stationary, then $E(X_t)$ can be estimated by using the sample average. We test covariance stationarity of X_t by using the ADF test. Condition 3 is a necessary condition of consistency and asymptotic normality of ML. Fourth, the information matrix of QVAR(p) depends on the following term, expressed by using equation (22) for the specific elements (i, j) and (k, l):

$$\frac{\partial M_t}{\partial \Psi_{ij}} \frac{\partial M'_t}{\partial \Psi_{kl}} = X_t \frac{\partial M_{t-1}}{\partial \Psi_{ij}} \frac{\partial M'_{t-1}}{\partial \Psi_{kl}} X'_t + X_t \frac{\partial M_{t-1}}{\partial \Psi_{ij}} W'_{ij} U_{t-1} + U'_{t-1} W_{kl} \frac{\partial M'_{t-1}}{\partial \Psi_{kl}} X'_t + W_{ij} U_{t-1} U'_{t-1} W'_{kl}. \tag{23}$$

We write this equation according to a first-order dynamic representation, as follows:

$$\text{vec}\left(\frac{\partial M_t}{\partial \Psi_{ij}} \frac{\partial M'_t}{\partial \Psi_{kl}} \right) = (X_t \otimes X_t) \text{vec}\left(\frac{\partial M_{t-1}}{\partial \Psi_{ij}} \frac{\partial M'_{t-1}}{\partial \Psi_{kl}} \right) + \tag{24}$$

$$+\text{vec}\left(X_t \frac{\partial M_{t-1}}{\partial \Psi_{ij}} W'_{ij} U_{t-1} \right) + \text{vec}\left(U'_{t-1} W_{kl} \frac{\partial M'_{t-1}}{\partial \Psi_{kl}} X'_t \right) + \text{vec}\left(W_{ij} U_{t-1} U'_{t-1} W'_{kl} \right),$$

where $\otimes$ is the Kronecker product and vec(x) indicates that the columns of the matrix are being stacked one upon the other. Condition 4 is that all eigenvalues of $E(X_t \otimes X_t)$ are within the unit circle. We denote the maximum modulus of all eigenvalues of $E(X_t \otimes X_t)$ by using C_4. If each element of $X_t \otimes X_t$ is covariance stationary, then $E(X_t \otimes X_t)$ can be estimated by using the sample average. We test covariance stationarity of $X_t \otimes X_t$ by using the ADF test with constant. Condition 4 is a sufficient condition of consistency and asymptotic normality of ML.

For the computation of $X_t = \Phi + \Psi(\partial U_{t-1}/\partial M'_{t-1})$, we need the formula for $\partial u_t/\partial \mu'_t$ $(K \times K)$. As aforementioned, the scaled score function is given by

$$u_t = \left(1 + \frac{v'_t \Sigma^{-1} v_t}{\nu}\right)^{-1} v_t = \frac{\nu(y_t - c - \mu_t)}{\nu + (y_t - c - \mu_t)'\Sigma^{-1}(y_t - c - \mu_t)}, \tag{25}$$

and the formula of $\partial u_t/\partial \mu'_t$ can be obtained by using standard matrix calculus.

In addition to the previous conditions, we also study the invertibility of QVAR, which is a condition of the consistency and asymptotic normality of ML. Invertibility is studied in the recent literature on DCS models (see for example: Blasques, Gorgi, Koopman and Wintenberger, 2018). From equations (10) and (11) we express:

$$Y_t = C - \Phi C + \Phi Y_{t-1} - \Phi V_{t-1} + \Psi U_{t-1} + V_t. \tag{26}$$

We substitute the scaled score function vector U_{t-1} into the previous equation and obtain:

$$Y_t = C - \Phi C + \Phi y_{t-1} + (\Psi_t - \Phi)V_{t-1} + V_t, \tag{27}$$

where

$$\Psi_t = \begin{bmatrix} \Psi_{1t} & 0_{K\times K} & \cdots & 0_{K\times K} \\ 0_{K\times K} & 0_{K\times K} & \cdots & 0_{K\times K} \\ \cdots & \cdots & \cdots & \cdots \\ 0_{K\times K} & \cdots & \cdots & 0_{K\times K} \end{bmatrix}_{(Kp\times Kp)} \tag{28}$$

and

$$\Psi_{1t} = \Psi_1 \times \left(1 + \frac{v'_{t-1}\Sigma^{-1}v_{t-1}}{\nu}\right)^{-1} = \Psi_1 \times \frac{\nu}{\nu + v'_{t-1}\Sigma^{-1}v_{t-1}}. \tag{29}$$

QVAR(p) is a VARMA(1,1) model with the VMA(1) parameter $\Psi_t - \Phi$. If the maximum modulus of eigenvalues of $\Psi_t - \Phi$ is lower than one for all t, then QVAR will be invertible.

III. Benchmark linear multivariate time series models

First benchmark: Gaussian QVAR(p)

From the nonlinear QVAR(p) model that uses the multivariate t distribution for the reduced-form error term v_t, we obtain linear multivariate time series models with multivariate normal distribution for v_t. If $\nu \to \infty$, then in the limiting case $v_t \sim t_K(0, \Sigma, \nu) \to_d N_K(0, \Sigma)$ and $u_t = v_t[1 + (v_t' \Sigma_v^{-1} v_t)/\nu]^{-1} \to_p v_t$. The multivariate model obtained for $\nu \to \infty$, named Gaussian QVAR(p), is considered as the first benchmark model. For the limiting case, QVAR(p) is

$$y_t = c - \Phi_1 c - \ldots - \Phi_p c + \Phi_1 y_{t-1} + \ldots + \Phi_p y_{t-p} + v_t + \Psi_1 v_{t-1} - \Phi_1 v_{t-1} - \ldots - \Phi_p v_{t-p}, \quad (30)$$

which is a Gaussian VARMA(p,p) specification with VMA coefficients $\Psi_1 - \Phi_1, -\Phi_2, \ldots, -\Phi_p$. For the Gaussian QVAR(1) case (Harvey, 2013), we have the reduced-form representation:

$$y_t = c - \Phi_1 c + \Phi_1 y_{t-1} + v_t + (\Psi_1 - \Phi_1) v_{t-1}, \quad (31)$$

which is a Gaussian VARMA(1,1) model with VMA coefficient $\Psi_1 - \Phi_1$. Under the restriction $\Psi_1 = \Phi_1$, we obtain the reduced-form representation of the classic Gaussian VAR(1) model:

$$y_t = c - \Phi_1 c + \Phi_1 y_{t-1} + v_t. \quad (32)$$

For the lag-orders $p > 1$, it is not possible to obtain the Gaussian VAR(p) model by using parameter restrictions from the Gaussian QVAR(p) model; see equation (30).

For Gaussian QVAR(p), which is a special case of the classic Gaussian VARMA(p,q) model, we refer to the work of Lütkepohl (2005), with respect to the structural-form and VMA(∞) representations of y_t, the IRF, and the ML estimation and related conditions of covariance stationarity and invertibility. It is noteworthy that even if the $v_t \sim N_K(0, \Sigma)$ assumption does not hold for Gaussian QVAR(p), the ML estimator still provides consistent parameter estimates due to the quasi-ML (QML) results of Gouriéroux, Monfort and Trognon (1984).

Second benchmark: Gaussian VARMA(p,q)

The Gaussian VARMA(p,p) specification of equation (30) is a special case of the classic linear Gaussian VARMA(p,q) model that is frequently used by practitioners for the analysis of macroeconomic data. Gaussian VAR(p), which to our knowledge is even more popular for practical use, is a special case of Gaussian VARMA(p,q). Motivated by these points, we consider Gaussian VARMA(p,q) as the second benchmark model, and present estimation results for both Gaussian VARMA(p,q) and Gaussian VAR(p). The reduced-form representation of VARMA(p,q) is

$$y_t = \tilde{\mu}_t + v_t = \tilde{\mu}_t + \Omega^{-1}\epsilon_t, \tag{33}$$

where $v_t \sim N_K(0,\Sigma)$ is the multivariate i.i.d. reduced-form error term. We factorize the positive definite covariance matrix as $\text{Var}(v_t) = \Sigma = \Omega^{-1}(\Omega^{-1})'$. The multivariate i.i.d. structural-form error term $\epsilon_t = \Omega v_t \sim N_K(0, I_K)$. Furthermore, $\tilde{\mu}_t$ is the conditional mean of $y_t|(y_1,\ldots,y_{t-1})$ that is specified as

$$E(y_t|y_1,\ldots,y_{t-1}) = \tilde{\mu}_t = \tilde{c} + \Phi_1 y_{t-1} + \cdots + \Phi_p y_{t-p} + \tilde{\Psi}_1 v_{t-1} + \ldots + \tilde{\Psi}_q v_{t-q}, \tag{34}$$

where $\tilde{c}$ $(K\times 1)$, $\Phi_1,\ldots,\Phi_p$ (each $K\times K$) and $\tilde{\Psi}_1,\ldots,\tilde{\Psi}_q$ (each $K\times K$) are constant parameters. For the classic Gaussian VARMA(p,q) we use tilde notation for several parameters to indicate the difference of parameters with respect to Gaussian QVAR(p). Under the restriction $\tilde{\Psi}_j = 0_{K\times K}$ for $j = 1,\ldots,q$, we obtain the classic Gaussian VAR(p) model. For the first p observations, we initialize $\tilde{\mu}_t$ by using the unconditional mean $\tilde{\mu}_t = E(y_t) = J(I_{Kp} - \Phi)^{-1}\tilde{C}$, where

$$\Phi = \begin{bmatrix} \Phi_1 & \Phi_2 & \cdots & \Phi_{p-1} & \Phi_p \\ I_K & 0_{K\times K} & \cdots & \cdots & 0_{K\times K} \\ 0_{K\times K} & I_K & 0_{K\times K} & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0_{K\times K} & \cdots & 0_{K\times K} & I_K & 0_{K\times K} \end{bmatrix}_{(Kp\times Kp)} \qquad \tilde{C} = \begin{bmatrix} \tilde{c} \\ 0_{K\times 1} \\ \cdots \\ 0_{K\times 1} \end{bmatrix}_{(Kp\times 1)}$$

and $J = [I_K, 0_{K\times K}, \cdots, 0_{K\times K}]$ $(K \times Kp)$.

For Gaussian VARMA(p,q), we refer to the work of Lütkepohl (2005), with respect to the structural-form and VMA(∞) representations of y_t, the IRF, and the ML estimation and related conditions of covariance stationarity and invertibility. We refer to the QML asymptotic results as noted earlier, for the case when $v_t \sim N_K(0, \Sigma)$ is used for VARMA(p,q).

VI. Data

We use macroeconomic data from the book of Kilian and Lütkepohl (2017) (the data are downloaded from: http://www-personal.umich.edu/~lkilian/figure9_1_chol.zip; original data sources: Federal Reserve Economic Data, Federal Reserve Bank of St. Louis; Economagic). This dataset includes the following variables: (i) non-seasonally adjusted monthly West Texas Intermediate (WTI) spot price of crude oil for the period of December 1972 to June 2013, inclusive; (ii) seasonally adjusted quarterly US GDP deflator for the period of 1959:Q1 to 2013:Q2, inclusive; (iii) seasonally adjusted quarterly US real GDP level for the period of 1959:Q1 to 2013:Q2, inclusive. We highlight the facts that the crude oil price series is non-seasonally adjusted and the US GDP deflator and US real GDP level series are both seasonally adjusted. The stochastic seasonality detection mechanism of the present paper is able to detect the seasonality of crude oil price, and the same mechanism is also able to evaluate whether the seasonality adjustments for US GDP deflator and US real GDP level were effective at the data source.

We define: (i) percentage change in the crude oil real price y_{1t}, as the quarterly first difference of log real price of crude oil; (ii) US inflation rate y_{2t}, as the quarterly first difference of log US GDP deflator; (iii) US real GDP growth y_{3t}, as the quarterly first difference of log US real GDP level. All variables are measured in percentage points. We define $y_t = (y_{1t}, y_{2t}, y_{3t})'$, hence, $K = 3$ for all models. We use data for the period of 1987:Q1 to 2013:Q2 inclusive (see Figure 1), motivated by the work of Kilian and Lütkepohl (2017). All results of this paper are according to the variable ordering (y_{1t}, y_{2t}, y_{3t}). Nevertheless, we also perform robustness analyses by using different variable orderings (see Blazsek and Escribano, 2017). The results of this paper are supported with respect to alternative orders of variables.

In Panel A of Table 1, we present the start and end dates of the dataset, and sample size (T), minimum, maximum, mean, standard deviation, skewness and excess kurtosis for each variable. In the same table, we also present ADF test results and partial autocorrelation function (PACF) estimates (see, for example, Hamilton, 1994) up to 20 lags. The ADF test for three alternative test specifications suggests that all dependent variables are $I(0)$. The PACF estimates indicate that all variables are $I(0)$ and significant serial correlation for several lags for all variables.

In Panel B of Table 1, we report descriptive statistics of deterministic annual seasonality effects for the percentage change in crude oil real price, US inflation rate and US real GDP growth time series. For each variable y_{it}, we estimate the linear regression model $y_{it} = \delta_{i,\mathrm{Q1}} D_{\mathrm{Q1},t} + \delta_{i,\mathrm{Q2}} D_{\mathrm{Q2},t} + \delta_{i,\mathrm{Q3}} D_{\mathrm{Q3},t} + \delta_{i,\mathrm{Q4}} D_{\mathrm{Q4},t} + \eta_{it}$, where $D_{\mathrm{Q1},t}$, $D_{\mathrm{Q2},t}$, $D_{\mathrm{Q3},t}$ and $D_{\mathrm{Q4},t}$ are dummy variables indicating each quarter of the year, and η_{it} is the error term. We use heteroscedasticity and autocorrelation consistent (HAC) standard errors (Newey and West, 1987) for the ordinary least squares (OLS) estimation of this model. We test the significance of parameter differences for each linear regression. The corresponding p-values, reported in Panel B of Table 1, indicate significant deterministic seasonality effects for the variables y_{1t} and y_{3t}. These preliminary results suggest that the deterministic seasonality effects are significant for the percentage change in crude oil real price, that the seasonal adjustment of US inflation rate was successful at the data source, and that the seasonal adjustment of US real GDP growth was ineffective at the data source. We do not include seasonality dummies in QVAR, VAR and VARMA, motivated by the design of the seasonality detection mechanism of this work. The stochastic seasonality detection mechanism of this paper can detect seasonality effects in a more general way than the linear regression model with quarterly dummies and constant parameters.

The relatively small sample size of data ($T = 106$) used in this paper is a frequent property of macroeconomic time series data. In the next section, we present the consequences of the limited sample size for the effective estimation of QVAR(p) and that of the benchmark Gaussian QVAR(p) and Gaussian VARMA(p,q) models.

[APPROXIMATE LOCATION OF TABLE 1 AND FIGURE 1]

V. Empirical results

Identification of structural forms

The QVAR, VAR and VARMA models of this paper are recursively identified structural models. This identification method is supported by the argument that oil price shocks may act as domestic supply shocks for the US economy (Kilian and Lütkepohl, 2017).

In this subsection, we present the identification of the most general QVAR model of this paper: QVAR(2). The identification of the structural-form representation is identical for all other models of this paper. Let $K = 3$ and $P = 2$ in equations (1) and (2), then the reduced-form QVAR(2) is

$$\begin{bmatrix} y_{1t} \\ y_{2t} \\ y_{3t} \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} + \begin{bmatrix} \Phi_{1,11} & \Phi_{1,12} & \Phi_{1,13} \\ \Phi_{1,21} & \Phi_{1,22} & \Phi_{1,23} \\ \Phi_{1,31} & \Phi_{1,32} & \Phi_{1,33} \end{bmatrix} \begin{bmatrix} \mu_{1,t-1} \\ \mu_{2,t-1} \\ \mu_{3,t-1} \end{bmatrix} + \tag{35}$$

$$+ \begin{bmatrix} \Phi_{2,11} & \Phi_{2,12} & \Phi_{2,13} \\ \Phi_{2,21} & \Phi_{2,22} & \Phi_{2,23} \\ \Phi_{2,31} & \Phi_{2,32} & \Phi_{2,33} \end{bmatrix} \begin{bmatrix} \mu_{1,t-2} \\ \mu_{2,t-2} \\ \mu_{3,t-2} \end{bmatrix} + \begin{bmatrix} \Psi_{1,11} & \Psi_{1,12} & \Psi_{1,13} \\ \Psi_{1,21} & \Psi_{1,22} & \Psi_{1,23} \\ \Psi_{1,31} & \Psi_{1,32} & \Psi_{1,33} \end{bmatrix} \begin{bmatrix} u_{1,t-1} \\ u_{2,t-1} \\ u_{3,t-1} \end{bmatrix} + \begin{bmatrix} v_{1t} \\ v_{2t} \\ v_{3t} \end{bmatrix}$$

Let $\text{Var}(v_t) = [\nu/(\nu-2)] \times \Sigma = [\nu/(\nu-2)] \times \Omega^{-1}(\Omega^{-1})'$ where

$$\Omega^{-1} = \begin{bmatrix} \Omega^{-1}_{11} & 0 & 0 \\ \Omega^{-1}_{21} & \Omega^{-1}_{22} & 0 \\ \Omega^{-1}_{31} & \Omega^{-1}_{32} & \Omega^{-1}_{33} \end{bmatrix} \tag{36}$$

is a lower-triangular matrix. Then,

$$v_t = \left(\frac{\nu}{\nu-2}\right)^{1/2} \Omega^{-1}\epsilon_t = \left(\frac{\nu}{\nu-2}\right)^{1/2} \begin{bmatrix} \Omega^{-1}_{11}\epsilon_{1t} \\ \Omega^{-1}_{21}\epsilon_{1t} + \Omega^{-1}_{22}\epsilon_{2t} \\ \Omega^{-1}_{31}\epsilon_{1t} + \Omega^{-1}_{32}\epsilon_{2t} + \Omega^{-1}_{33}\epsilon_{3t} \end{bmatrix} \tag{37}$$

For QVAR, VARMA and VAR, the decomposition $\Sigma = \Omega^{-1}(\Omega^{-1})'$ is a Cholesky decomposition. As Σ is positive definite for all cases, the Cholesky decomposition is unique if the diagonal of Ω^{-1} includes positive elements (i.e. $\Omega_{11}^{-1} > 0$, $\Omega_{22}^{-1} > 0$ and $\Omega_{33}^{-1} > 0$).

Estimation for a small dataset with extreme observations

For QVAR(p) with multivariate t distribution, the ML procedure does not converge to an optimum for the dataset of this paper, due to the small number of observations for each variable. We use the $\Psi_1 = \Psi_{1,11} \times I_K$ restriction, where $\Psi_{1,11} \in \mathbb{R}$. This implies that Ψ_1 is diagonal with $\Psi_{1,11} = \Psi_{1,22} = \Psi_{1,33}$ (Harvey, 2013, p. 211, notes that the effective specification of matrix Ψ_1 is related to the specific application of the multivariate DCS model). Under that restriction, we identify all elements of c, Φ, Ω^{-1} and ν. Due to the small sample size, the ML procedure does not converge to an optimum for the QVAR(p) specification with $p > 2$. Therefore, for QVAR(p) with multivariate t distribution, we report results for QVAR(1) and QVAR(2) with scalar Ψ_1. In future works that use a greater sample size for the dataset, the full Ψ_1 matrix can be estimated and QVAR(p) with $p > 2$ can be used for data analysis.

For Gaussian QVAR(p) and Gaussian VARMA(p,q) the ML procedure does not converge to an optimum, due to the small number of observations. We use scalar VMA parameters for all Gaussian multivariate specifications. For Gaussian QVAR(p) we denote the scalar VMA parameter by $\Psi_{1,11}$, and for Gaussian VARMA(p,q) we denote the scalar VMA parameters by $\tilde{\Psi}_{j,11}$ with $j = 1, \ldots, q$. Under those restrictions, we identify all elements of c, Φ and Ω^{-1}. The ML estimation procedure converges for the following Gaussian models: Gaussian QVAR(1), Gaussian VARMA(1,1), Gaussian VARMA(2,1), Gaussian VAR(1) and Gaussian VAR(2).

A relevant result of the present paper is that QVAR(2) with multivariate t distribution is successfully estimated, while Gaussian QVAR(2) (i.e. the special case of QVAR(2) with multivariate t distribution) and Gaussian VARMA(2,2) are not estimated effectively. This is due to the fact that QVAR(2) is robust to extreme observations while the Gaussian multivariate models are sensitive to those observations. This result shows the advantage of QVAR(2) for practical application, with respect to the Gaussian models. In addition, the ML procedure does not converge to an optimum for the Gaussian QVAR(p) and Gaussian VARMA(p,q) specifications with $p > 2$ and $q > 2$. Therefore, we report results for the Gaussian QVAR(1), Gaussian VARMA(1,1), Gaussian VARMA(2,1), Gaussian VAR(1) and Gaussian VAR(2) models.

First-order multivariate models

We present the parameter estimates and model diagnostics for QVAR(1), Gaussian QVAR(1), Gaussian VARMA(1,1) and Gaussian VAR(1) in Table 2. We present the IRFs for those models in Figures 2 to 5, respectively. We present the invertibility of QVAR(1) in Figure 6(a).

For all models, we find that conditions of consistency and asymptotic normality of ML are supported by the C_1, C_2, C_3 and C_4 metrics (Table 2). With respect to QVAR(1), the null hypothesis of the ADF test with constant is always rejected for conditions C_2, C_3, C_4 and matrix D_t (Table 2). In Figure 6(a), we present the evolution of the maximum modulus of eigenvalues of $\Psi_t - \Phi$ for QVAR(1), with the related $\pm 2\sigma$ interval, for the period of 1987:Q1 to 2013:Q2, inclusive. Those results suggest that QVAR(1) is invertible. We compare the statistical performance of the first-order models, by using the following likelihood-based metrics: LL and Akaike information criterion (AIC). Both metrics suggest that the statistical performance of QVAR(1) is superior to the other first-order alternatives of this paper (Table 2).

We find that some elements of Φ_1 and Ω^{-1} are significantly different from zero for all models (Table 2). To check the robustness of our results, we also estimate a restricted version of all models, for which each non-significant parameter in Φ_1 and Ω^{-1} is restricted to the value zero (see Blazsek and Escribano, 2017). All results reported in this work are robust to those restrictions. This suggests significant dynamic and contemporaneous interaction effects, respectively, among the percentage change in crude oil real price, US inflation rate and US real GDP growth. We study the dynamic interaction effects by using the IRFs in Figures 2 to 5, for which we report the mean IRF estimates up to 20 leads. We do not report the IRF confidence intervals in those figures, due to the small number of observations in the dataset and due to the fact that the main focus of the present paper is seasonality detection based on the mean IRF estimates. A common finding for all first-order models is that the stochastic seasonality effects are not identified in any of the IRF figures. These results show that the score-driven first-order multivariate t distribution model, also named QVAR(1) model, does not identify the seasonality effects for the small macroeconomic dataset of this paper. This motivates the application of higher-order QVAR(p) specifications, which are presented in the following subsection.

[APPROXIMATE LOCATION OF TABLE 2 AND FIGURES 2 TO 6]

Second-order multivariate models

We present the parameter estimates and model diagnostics for QVAR(2), Gaussian VARMA(2,1) and Gaussian VAR(2) in Table 3. We present the IRFs for those models in Figures 7 to 9, respectively. We present the invertibility of QVAR(2) in Figure 6(b).

All conditions of consistency and asymptotic normality of ML are supported by the C_1, C_2, C_3 and C_4 metrics (Table 3). With respect to QVAR(2), the null hypothesis of the ADF test with constant is always rejected for conditions C_2, C_3, C_4 and matrix D_t (Table 3). In Figure 6(b), we present the evolution of the maximum modulus of eigenvalues of $\Psi_t - \Phi$ for QVAR(2), with the related $\pm 2\sigma$ interval, for the period of 1987:Q1 to 2013:Q2, inclusive. Those results suggest that QVAR(2) is invertible. The LL and AIC metrics suggest that the statistical performance of QVAR(2) is superior to the other second-order alternatives of this paper (Table 3).

We find that some elements of Φ_1, Φ_2 and Ω^{-1} are significantly different from zero for all models (Table 3), suggesting significant dynamic and contemporaneous interaction effects among the percentage change in crude oil real price, US inflation rate and US real GDP growth. To check the robustness of our results, we also estimate a restricted version of all models, for which each non-significant parameter in Φ_1, Φ_2 and Ω^{-1} is restricted to the value zero (see Blazsek and Escribano, 2017). All results reported in this work are robust to those restrictions. We study the dynamic interaction effects by using the IRFs in Figures 7 to 9. In those figures, we report the mean IRF estimates up to 20 leads (as aforementioned, we do not report IRF confidence intervals due to the small number of observations). According to the IRF estimates, clear annual stochastic seasonality effects are observed for QVAR(2) for several years (Figure 7), while the same effects are not found for the Gaussian VARMA(2,1) and Gaussian VAR(2) specifications

(Figures 8 and 9, respectively).

We obtain the following conclusions from the diagonal panels of Figure 7. For the non-seasonally adjusted percentage change in crude oil real price time series, the seasonality detection mechanism for QVAR(2) suggests that the percentage change in crude oil real price time series includes a significant annual stochastic seasonality component. For the seasonally adjusted US inflation rate time series, the seasonality detection mechanism of QVAR(2) suggests that seasonality correction was effective at the data source. For the seasonally adjusted US real GDP time series, the seasonality detection mechanism for QVAR(2) suggests that seasonality correction was not effective at the data source, because the seasonally adjusted US real GDP growth time series includes a significant annual stochastic seasonality component.

Our results for QVAR(2) with multivariate t distribution may motivate the use of the new score-driven nonlinear multivariate QVAR(p) model, in order to identify stochastic seasonality effects in small macroeconomic datasets with extreme observations and to verify the effectiveness of different deseasonalization methods for macroeconomic variables.

[APPROXIMATE LOCATION OF TABLE 3 AND FIGURES 7 TO 9]

VI. Conclusions

In this paper, we have introduced a new mechanism of seasonality detection for multivariate macroeconomic time series, by extending the DCS model for the multivariate t distribution. We have named the extended model QVAR(p). We have used a relatively small macroeconomic dataset that includes extreme observations for the period of 1987:Q1 to 2013:Q2, inclusive. The variables considered have been quarterly percentage change in non-seasonally adjusted crude oil real price, quarterly seasonally adjusted US inflation rate, and quarterly seasonally adjusted US real GDP growth. We have found that the statistical performance of QVAR(p) is superior to that of the linear Gaussian multivariate alternatives. Stochastic seasonality effects have not been detected by the linear Gaussian QVAR(p), Gaussian VARMA(p,q) and Gaussian VAR(p), whereas those effects have been detected by the nonlinear QVAR(p) with multivariate t distribution. Our results have indicated that, for those cases when the seasonality adjustment is not correct or the data series includes a significant seasonality component, the VAR and VARMA specifications of this paper are unable to detect seasonality effects, while QVAR is able to detect seasonality effects, even for small macroeconomic datasets that are frequent in practical applications.

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TABLE 1

Descriptive statistics of percentage change in crude oil real price y_{1t}, US inflation rate y_{2t}, and US real GDP growth y_{3t}

A. Descriptive statistics	Crude oil y_{1t}	Inflation y_{2t}	GDP growth y_{3t}	B. Deterministic seasonality effects	Estimates
Start date	1987Q1	1987Q1	1987Q1	Crude oil y_{1t}:	
End date	2013Q2	2013Q2	2013Q2	$\delta_{1,\mathrm{Q1}}$	4.9367*(2.5909)
Sample size T	106	106	106	$\delta_{1,\mathrm{Q2}}$	2.6584(2.6120)
Minimum	−93.1315	−0.1560	−2.1746	$\delta_{1,\mathrm{Q3}}$	4.0292(3.4734)
Maximum	68.2694	1.2005	1.8712	$\delta_{1,\mathrm{Q4}}$	−7.3139*(4.2138)
Mean	1.1289	0.5548	0.6430	p-value $\mathrm{H}_0 : \delta_{1,\mathrm{Q1}} - \delta_{1,\mathrm{Q2}} = 0$	0.6267
Standard deviation	17.6076	0.2423	0.6212	p-value $\mathrm{H}_0 : \delta_{1,\mathrm{Q1}} - \delta_{1,\mathrm{Q3}} = 0$	0.8477
Skewness	−1.0045	0.2055	−1.3129	p-value $\mathrm{H}_0 : \delta_{1,\mathrm{Q1}} - \delta_{1,\mathrm{Q4}} = 0$	**0.0107**
Excess kurtosis	8.3888	0.4469	4.0935	Inflation y_{2t}:	
ADF with constant	−9.4521***	−2.8647**	−6.3930***	$\delta_{2,\mathrm{Q1}}$	0.6081***(0.0494)
ADF with constant and linear trend	−9.4633***	−3.4689**	−6.7048***	$\delta_{2,\mathrm{Q2}}$	0.5507***(0.0499)
ADF with constant and quadratic trend	−9.4560***	−3.5659*	−6.7431***	$\delta_{2,\mathrm{Q3}}$	0.5593***(0.0465)
PACF(1)	−0.0238	0.6536***	0.4319***	$\delta_{2,\mathrm{Q4}}$	0.4991***(0.0374)
PACF(2)	−0.263***	0.2862***	0.24**	p-value $\mathrm{H}_0 : \delta_{2,\mathrm{Q1}} - \delta_{2,\mathrm{Q2}} = 0$	0.3867
PACF(3)	0.0209	0.1336	−0.0769	p-value $\mathrm{H}_0 : \delta_{2,\mathrm{Q1}} - \delta_{2,\mathrm{Q3}} = 0$	0.4664
PACF(4)	−0.0407	0.0507	0.0806	p-value $\mathrm{H}_0 : \delta_{2,\mathrm{Q1}} - \delta_{2,\mathrm{Q4}} = 0$	0.1053
PACF(5)	−0.2431**	−0.0418	−0.0463	GDP growth y_{3t}:	
PACF(6)	−0.0473	0.0859	0.0067	$\delta_{3,\mathrm{Q1}}$	0.4619***(0.1209)
PACF(7)	0.0178	−0.1441	0.0306	$\delta_{3,\mathrm{Q2}}$	0.8190***(0.0860)
PACF(8)	−0.1096	−0.0356	−0.0325	$\delta_{3,\mathrm{Q3}}$	0.6279***(0.0949)
PACF(9)	−0.1454	−0.0469	0.1543	$\delta_{3,\mathrm{Q4}}$	0.6634***(0.1569)
PACF(10)	−0.0401	−0.0035	−0.0603	p-value $\mathrm{H}_0 : \delta_{3,\mathrm{Q1}} - \delta_{3,\mathrm{Q2}} = 0$	**0.0358**
PACF(11)	−0.076	0.0794	−0.2349**	p-value $\mathrm{H}_0 : \delta_{3,\mathrm{Q1}} - \delta_{3,\mathrm{Q3}} = 0$	0.3296
PACF(12)	0.0923	−0.0351	−0.0327	p-value $\mathrm{H}_0 : \delta_{3,\mathrm{Q1}} - \delta_{3,\mathrm{Q4}} = 0$	0.2371
PACF(13)	−0.169*	−0.1464	−0.0068		
PACF(14)	0.1392	0.0772	0.1337		
PACF(15)	−0.0532	−0.0354	0.0196		
PACF(16)	0.0546	0.1065	0.0285		
PACF(17)	−0.0422	−0.0276	−0.0064		
PACF(18)	−0.0357	−0.1058	−0.1212		
PACF(19)	0.0062	−0.1567	0.1495		
PACF(20)	0.0228	0.1108	0.0017		

Note: United States (US); gross domestic product (GDP); augmented Dickey–Fuller test statistic (ADF); partial autocorrelation function (PACF). *, ** and *** indicate significance at the 10%, 5% and 1% levels, respectively. We highlight the p-values that indicate significant parameter differences by using bold numbers.

Data are downloaded from: http://www-personal.umich.edu/∼lkilian/figure9_1_chol.zip.

TABLE 2

Parameter estimates and model diagnostics for first-order multivariate time series models

A. Parameters	QVAR(1)	Gaussian QVAR(1)	A. Parameters	Gaussian VARMA(1,1)	Gaussian VAR(1)
c_1	2.7357(1.7330)	1.5209(2.0607)	$\tilde{c}_1$	0.4544(3.2069)	2.2474(7.1518)
c_2	0.5628***(0.0688)	0.5592***(0.0600)	$\tilde{c}_2$	0.0037(0.0241)	0.1359**(0.0640)
c_3	0.6944***(0.0831)	0.6647***(0.0984)	$\tilde{c}_3$	0.1913**(0.0879)	0.4377**(0.2137)
$\Phi_{1,11}$	0.4059(0.2914)	−0.2759(0.2245)	$\Phi_{1,11}$	0.4148***(0.1526)	−0.0483(0.1323)
$\Phi_{1,12}$	−17.4749(11.5521)	−10.7715(15.4631)	$\Phi_{1,12}$	−1.5651(5.0321)	−5.0643(11.3223)
$\Phi_{1,13}$	−10.9163(6.8459)	−8.0130(7.9166)	$\Phi_{1,13}$	1.7358(1.8527)	2.7481(3.5702)
$\Phi_{1,21}$	0.0083*(0.0045)	0.0147***(0.0050)	$\Phi_{1,21}$	0.0017(0.0012)	−0.0001(0.0018)
$\Phi_{1,22}$	1.1893***(0.1546)	1.0510***(0.1675)	$\Phi_{1,22}$	0.9241***(0.0387)	0.6615***(0.0923)
$\Phi_{1,23}$	0.2019**(0.0866)	0.1268(0.0841)	$\Phi_{1,23}$	0.0535***(0.0185)	0.0776**(0.0367)
$\Phi_{1,31}$	−0.0038(0.0092)	−0.0069(0.0135)	$\Phi_{1,31}$	−0.0030(0.0028)	−0.0012(0.0031)
$\Phi_{1,32}$	−0.3238(0.3283)	−0.3548*(0.2150)	$\Phi_{1,32}$	−0.1899(0.1417)	−0.1380(0.3169)
$\Phi_{1,33}$	0.6674***(0.1809)	0.7500***(0.1080)	$\Phi_{1,33}$	0.8847***(0.0702)	0.4428***(0.1133)
$\Psi_{1,11}$	0.3871***(0.1018)	0.2486***(0.0522)	$\tilde{\Psi}_{1,11}$	−0.5969***(0.0752)	NA
Ω_{11}^{-1}	12.8909***(1.3657)	17.8343***(1.2332)	Ω_{11}^{-1}	17.1544***(1.1348)	17.4057***(1.2077)
Ω_{21}^{-1}	0.0087(0.0167)	0.0199(0.0163)	Ω_{21}^{-1}	0.0121(0.0247)	0.0078(0.0229)
Ω_{22}^{-1}	0.1354***(0.0130)	0.1595***(0.0107)	Ω_{22}^{-1}	0.1568***(0.0113)	0.1743***(0.0102)
Ω_{31}^{-1}	0.0828(0.0528)	0.1354***(0.0520)	Ω_{31}^{-1}	0.1342**(0.0568)	0.1359**(0.0671)
Ω_{32}^{-1}	0.0099(0.0597)	0.0030(0.0472)	Ω_{32}^{-1}	−0.0125(0.0638)	−0.0218(0.0585)
Ω_{33}^{-1}	0.4347***(0.0394)	0.5217***(0.0352)	Ω_{33}^{-1}	0.5242***(0.0431)	0.5390***(0.0433)
ν	5.2255***(1.3534)	NA	ν	NA	NA
B. Diagnostics	QVAR(1)	Gaussian QVAR(1)	B. Diagnostics	Gaussian VARMA(1,1)	Gaussian VAR(1)
C_1	0.8660	0.8468	C_1	0.9029	0.6017
C_2	NA	0.2486	C_2	0.5969	NA
C_2 ADF	All stationary	NA	C_2 ADF	NA	NA
C_3	0.8230	NA	C_3	NA	NA
C_3 ADF	All stationary	NA	C_3 ADF	NA	NA
C_4	0.6791	NA	C_4	NA	NA
C_4 ADF	All stationary	NA	C_4 ADF	NA	NA
D_t ADF	All stationary	NA	D_t ADF	NA	NA
LL	**−4.5353**	−4.6514	LL	−4.6000	−4.7484
AIC	**9.4479**	9.6612	AIC	9.5586	9.8365

Note: Quasi-vector autoregressive (QVAR); VAR moving average (VARMA); not available (NA); augmented Dickey–Fuller (ADF); log-likelihood (LL); Akaike information criterion (AIC). For all models, $|C_1| < 1$ indicates covariance stationarity. For QVAR(1), C_2 ADF indicates that u_t and its derivative $\partial u_t/\partial \mu_t$ have finite second moments and covariance that are dynamic and do not depend on μ_t. For all Gaussian models, $|C_2| < 1$ indicates invertibility. For QVAR(1), $|C_3| < 1$ and $|C_4| < 1$ indicate necessary and sufficient conditions for the asymptotic properties of the ML estimator, respectively. For QVAR(1), D_t ADF indicates that all time series formed by the elements of D_t are covariance stationary. Bold likelihood-based metrics indicate superior model performance. Standard errors are reported in parentheses. *, ** and *** indicate significance at the 10%, 5% and 1% levels, respectively.

TABLE 3

Parameter estimates and model diagnostics for second-order multivariate time series models

A. Parameters	QVAR(2)	A. Parameters	Gaussian VARMA(2,1)	Gaussian VAR(2)
c_1	3.6927(2.9517)	$\tilde{c}_1$	−0.3198(3.6659)	−0.6156(6.3289)
c_2	0.5982***(0.0836)	$\tilde{c}_2$	0.0086(0.0321)	0.0719(0.0626)
c_3	0.6997***(0.0635)	$\tilde{c}_3$	0.2031*(0.1137)	0.4321**(0.2091)
$\Phi_{1,11}$	0.0434(0.4088)	$\Phi_{1,11}$	0.4535**(0.1885)	−0.0551(0.1381)
$\Phi_{1,12}$	−14.0767(17.4553)	$\Phi_{1,12}$	−2.4832(16.8823)	−11.6530(14.6404)
$\Phi_{1,13}$	−14.9229(10.5581)	$\Phi_{1,13}$	2.8715(4.9101)	2.8974(4.1635)
$\Phi_{1,21}$	0.0093*(0.0057)	$\Phi_{1,21}$	0.0006(0.0018)	0.0003(0.0018)
$\Phi_{1,22}$	1.6014***(0.1649)	$\Phi_{1,22}$	0.8223***(0.2101)	0.4208***(0.1187)
$\Phi_{1,23}$	0.2874***(0.1111)	$\Phi_{1,23}$	0.0776*(0.0453)	0.0698(0.0447)
$\Phi_{1,31}$	−0.0058(0.0156)	$\Phi_{1,31}$	−0.0018(0.0029)	−0.0012(0.0028)
$\Phi_{1,32}$	0.6641(0.8137)	$\Phi_{1,32}$	−0.1016(0.4782)	−0.2377(0.4349)
$\Phi_{1,33}$	0.8487***(0.2772)	$\Phi_{1,33}$	0.8758***(0.1683)	0.3270***(0.1212)
$\Phi_{2,11}$	0.5330(0.3849)	$\Phi_{2,11}$	−0.2073*(0.1205)	−0.2819**(0.1287)
$\Phi_{2,12}$	18.0614(14.6695)	$\Phi_{2,12}$	2.2500(16.3773)	10.0400(12.6349)
$\Phi_{2,13}$	10.0051(10.7389)	$\Phi_{2,13}$	−0.7693(5.6546)	1.8566(5.0159)
$\Phi_{2,21}$	0.0003(0.0063)	$\Phi_{2,21}$	0.0025(0.0016)	0.0023*(0.0014)
$\Phi_{2,22}$	−0.6958***(0.1458)	$\Phi_{2,22}$	0.0909(0.1742)	0.3371***(0.0998)
$\Phi_{2,23}$	−0.0595(0.1270)	$\Phi_{2,23}$	−0.0242(0.0411)	0.0195(0.0399)
$\Phi_{2,31}$	−0.0492**(0.0224)	$\Phi_{2,31}$	−0.0036(0.0041)	−0.0029(0.0052)
$\Phi_{2,32}$	−0.3714(1.0000)	$\Phi_{2,32}$	−0.0704(0.4839)	−0.0640(0.4330)
$\Phi_{2,33}$	−1.2614***(0.4601)	$\Phi_{2,33}$	−0.0134(0.1463)	0.2882**(0.1291)
$\Psi_{1,11}$	0.4251***(0.1046)	$\tilde{\Psi}_{1,11}$	−0.5495***(0.1424)	NA
Ω_{11}^{-1}	12.2978***(1.3441)	Ω_{11}^{-1}	16.7792***(1.3796)	16.6400***(1.5254)
Ω_{21}^{-1}	0.0034(0.0163)	Ω_{21}^{-1}	0.0184(0.0301)	0.0134(0.0314)
Ω_{22}^{-1}	0.1157***(0.0109)	Ω_{22}^{-1}	0.1506***(0.0114)	0.1582***(0.0096)
Ω_{31}^{-1}	0.1033**(0.0501)	Ω_{31}^{-1}	0.1237*(0.0644)	0.1326**(0.0617)
Ω_{32}^{-1}	0.0084(0.0549)	Ω_{32}^{-1}	−0.0065(0.0818)	−0.0236(0.0738)
Ω_{33}^{-1}	0.3825***(0.0378)	Ω_{33}^{-1}	0.5222***(0.0450)	0.5176***(0.0470)
ν	3.6079***(0.9566)	ν	NA	NA
B. Diagnostics	**QVAR(2)**	**B. Diagnostics**	**Gaussian VARMA(2,1)**	**Gaussian VAR(2)**
C_1	0.8932	C_1	0.8874	0.7779
C_2	NA	C_2	0.5495	NA
C_2 ADF	All stationary	C_2 ADF	NA	NA
C_3	0.9010	C_3	NA	NA
C_3 ADF	All stationary	C_3 ADF	NA	NA
C_4	0.8137	C_4	NA	NA
C_4 ADF	All stationary	C_4 ADF	NA	NA
D_t ADF	All stationary	D_t ADF	NA	NA
LL	**−4.4445**	LL	−4.5342	−4.5664
AIC	**9.4362**	AIC	9.5968	9.6422

Note: Quasi-vector autoregressive (QVAR); VAR moving average (VARMA); not available (NA); augmented Dickey–Fuller (ADF); log-likelihood (LL); Akaike information criterion (AIC). For all models, $|C_1| < 1$ indicates covariance stationarity. For QVAR(1), C_2 ADF indicates that u_t and its derivative $\partial u_t/\partial \mu_t$ have finite second moments and covariance that are dynamic and do not depend on μ_t. For all Gaussian models, $|C_2| < 1$ indicates invertibility. For QVAR(1), $|C_3| < 1$ and $|C_4| < 1$ indicate necessary and sufficient conditions for the asymptotic properties of the ML estimator, respectively. For QVAR(1), D_t ADF indicates that all time series formed by the elements of D_t are covariance stationary. Bold likelihood-based metrics indicate superior model performance. Standard errors are reported in parentheses. *, ** and *** indicate significance at the 10%, 5% and 1% levels, respectively.

Fig. 1(a) Percentage change in crude oil real price y_{1t}

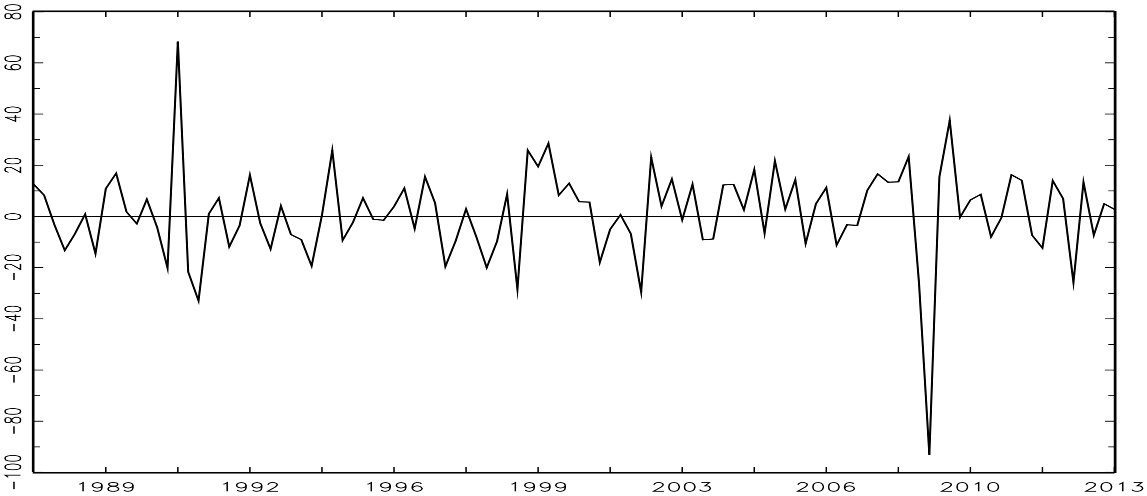

Fig. 1(b) US inflation rate y_{2t}

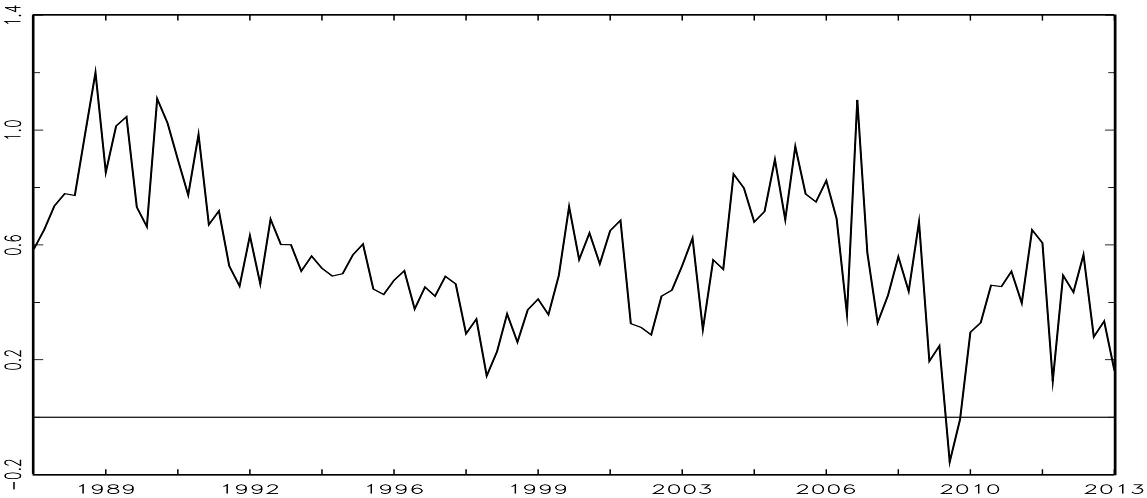

Fig. 1(c) US real GDP growth y_{3t}

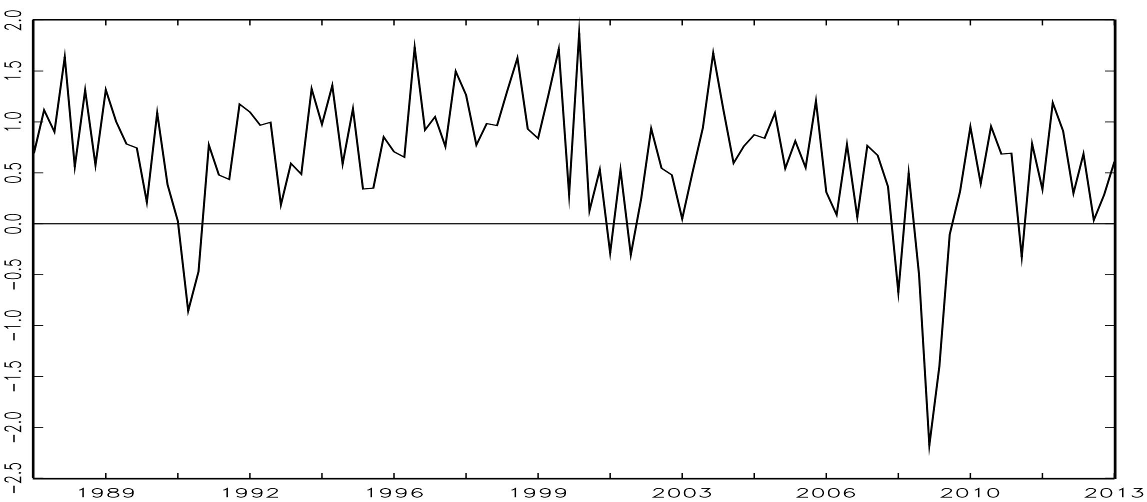

Figure 1. Dataset for the period of 1987:Q1 to 2013:Q2, inclusive

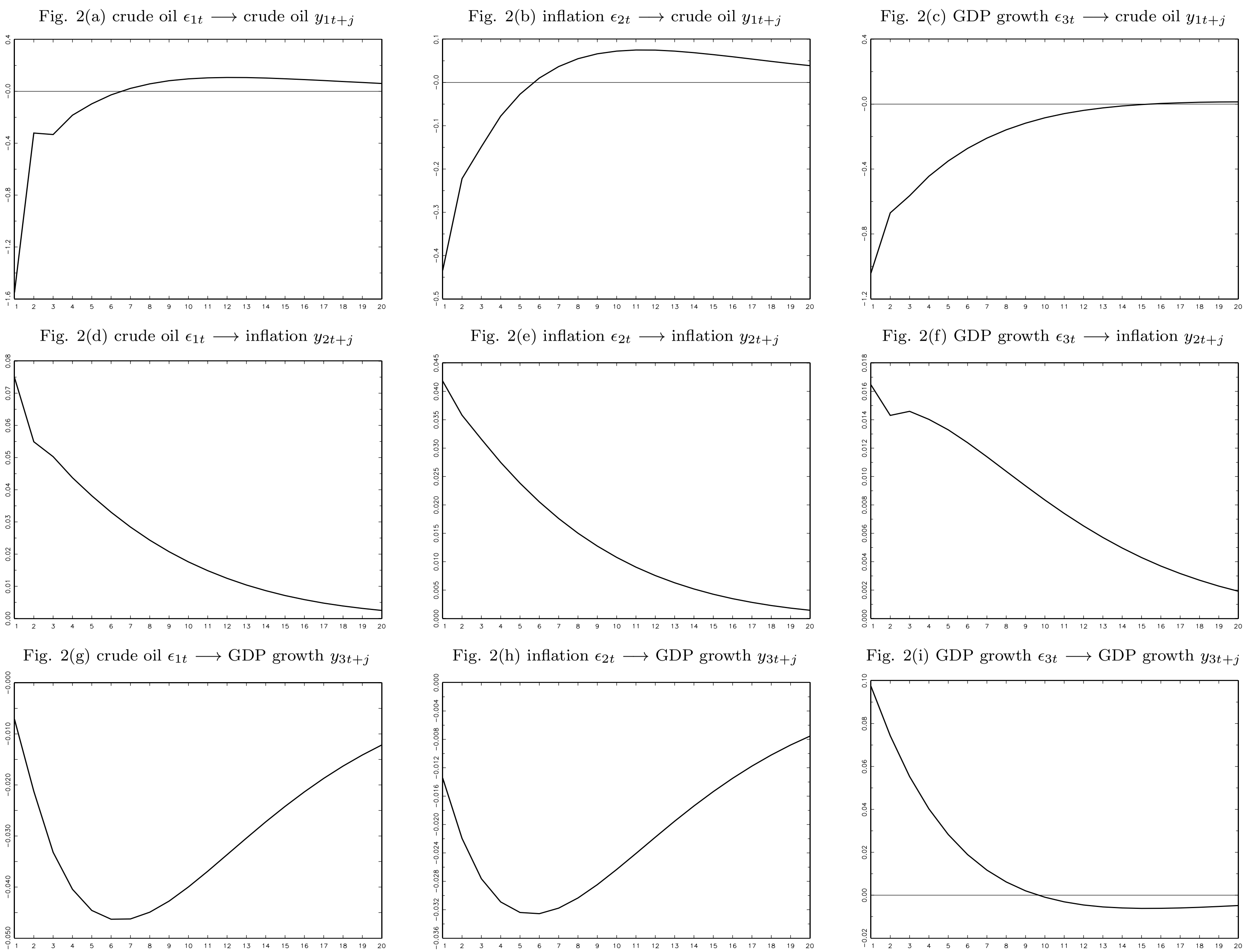

Figure 2. Impulse response function of QVAR(1)

Fig. 3(a) crude oil $\epsilon_{1t} \longrightarrow$ crude oil y_{1t+j}

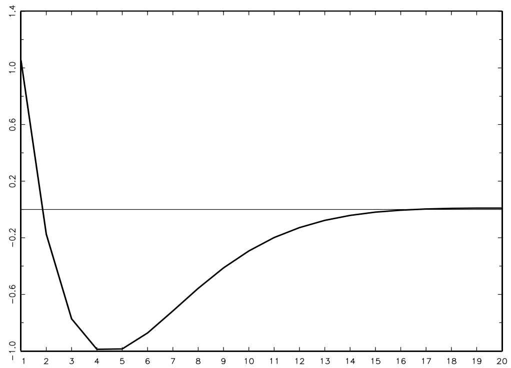

Fig. 3(b) inflation $\epsilon_{2t} \longrightarrow$ crude oil y_{1t+j}

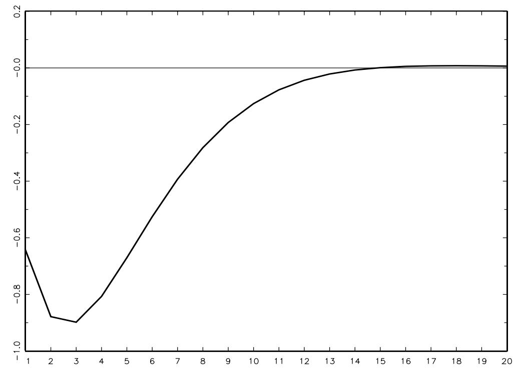

Fig. 3(c) GDP growth $\epsilon_{3t} \longrightarrow$ crude oil y_{1t+j}

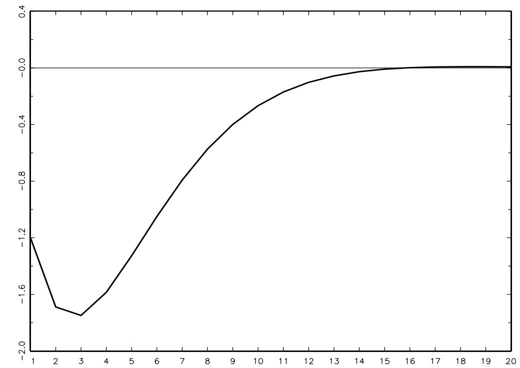

Fig. 3(d) crude oil $\epsilon_{1t} \longrightarrow$ inflation y_{2t+j}

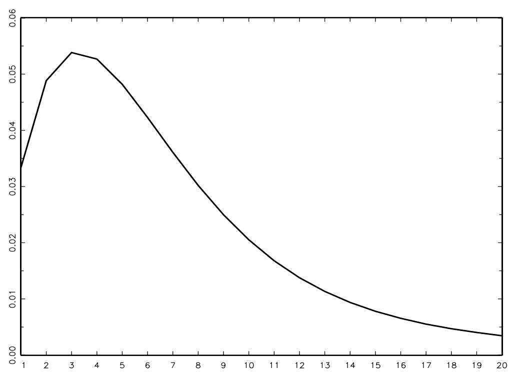

Fig. 3(e) inflation $\epsilon_{2t} \longrightarrow$ inflation y_{2t+j}

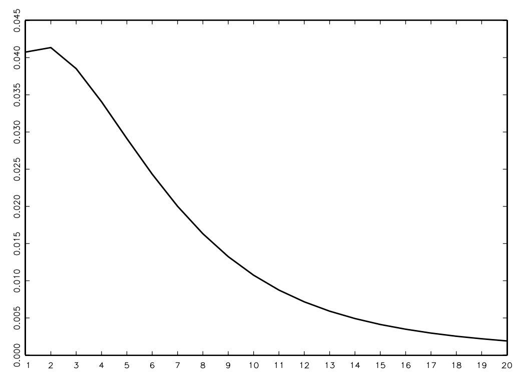

Fig. 3(f) GDP growth $\epsilon_{3t} \longrightarrow$ inflation y_{2t+j}

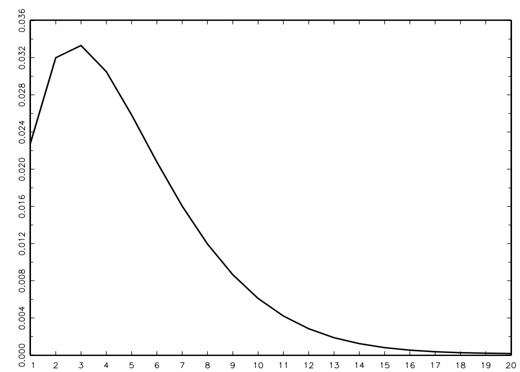

Fig. 3(g) crude oil $\epsilon_{1t} \longrightarrow$ GDP growth y_{3t+j}

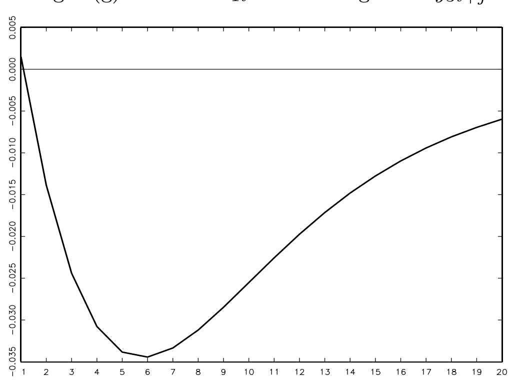

Fig. 3(h) inflation $\epsilon_{2t} \longrightarrow$ GDP growth y_{3t+j}

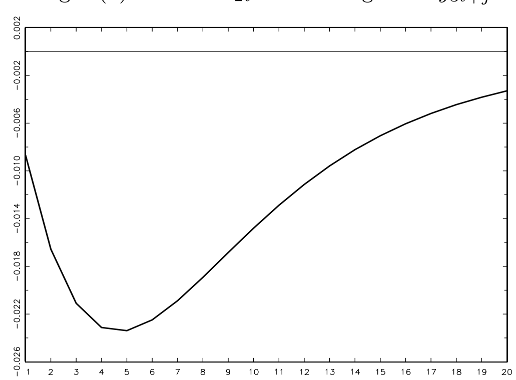

Fig. 3(i) GDP growth $\epsilon_{3t} \longrightarrow$ GDP growth y_{3t+j}

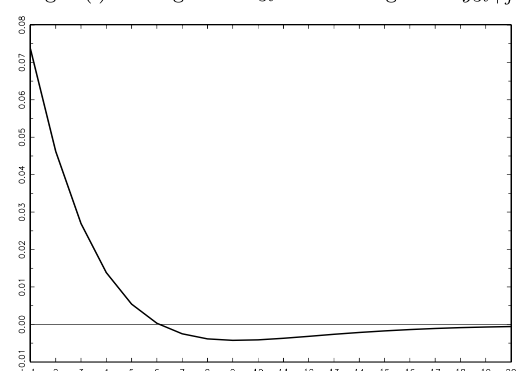

Figure 3. Impulse response function of Gaussian QVAR(1)

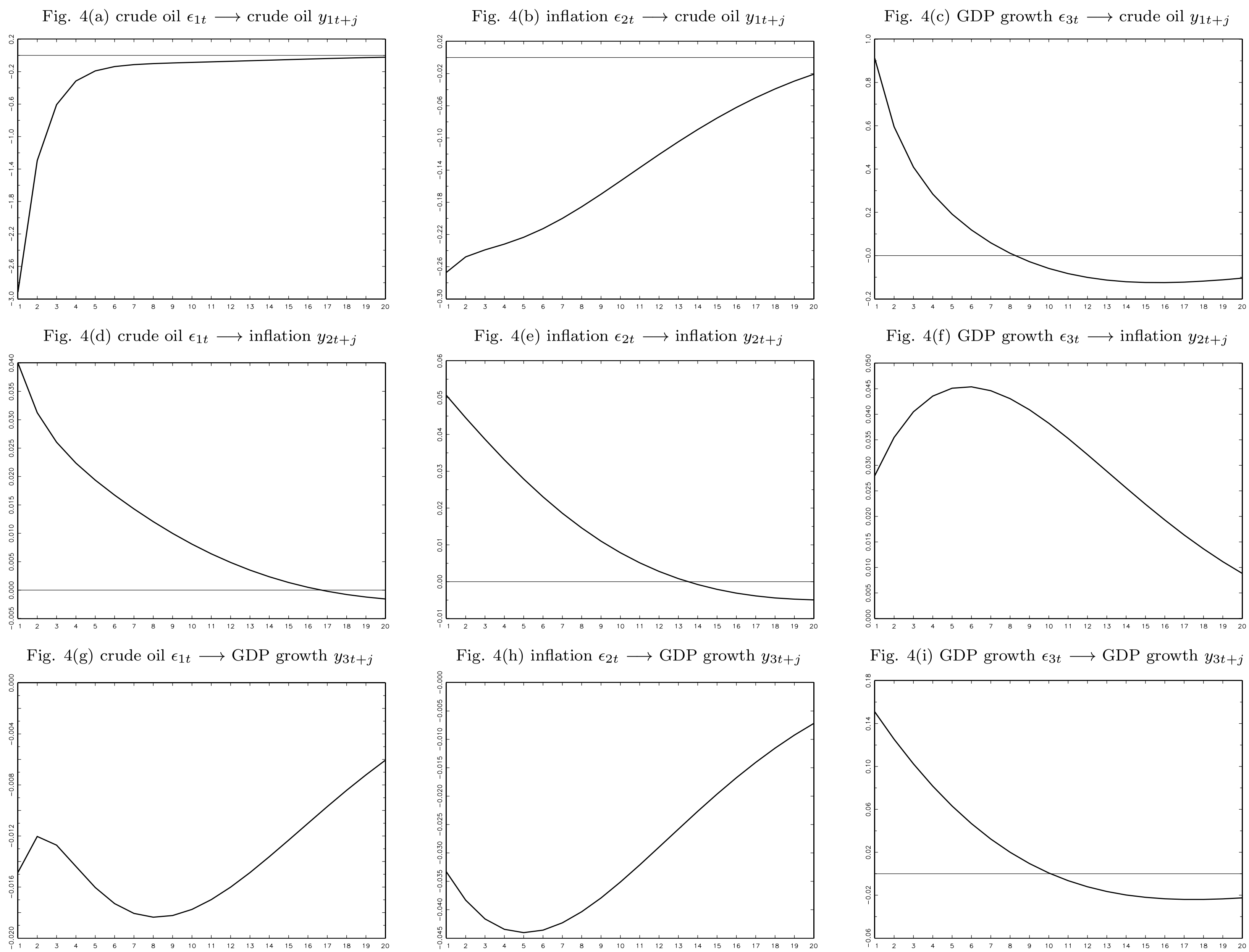

Figure 4. Impulse response function of Gaussian VARMA(1,1)

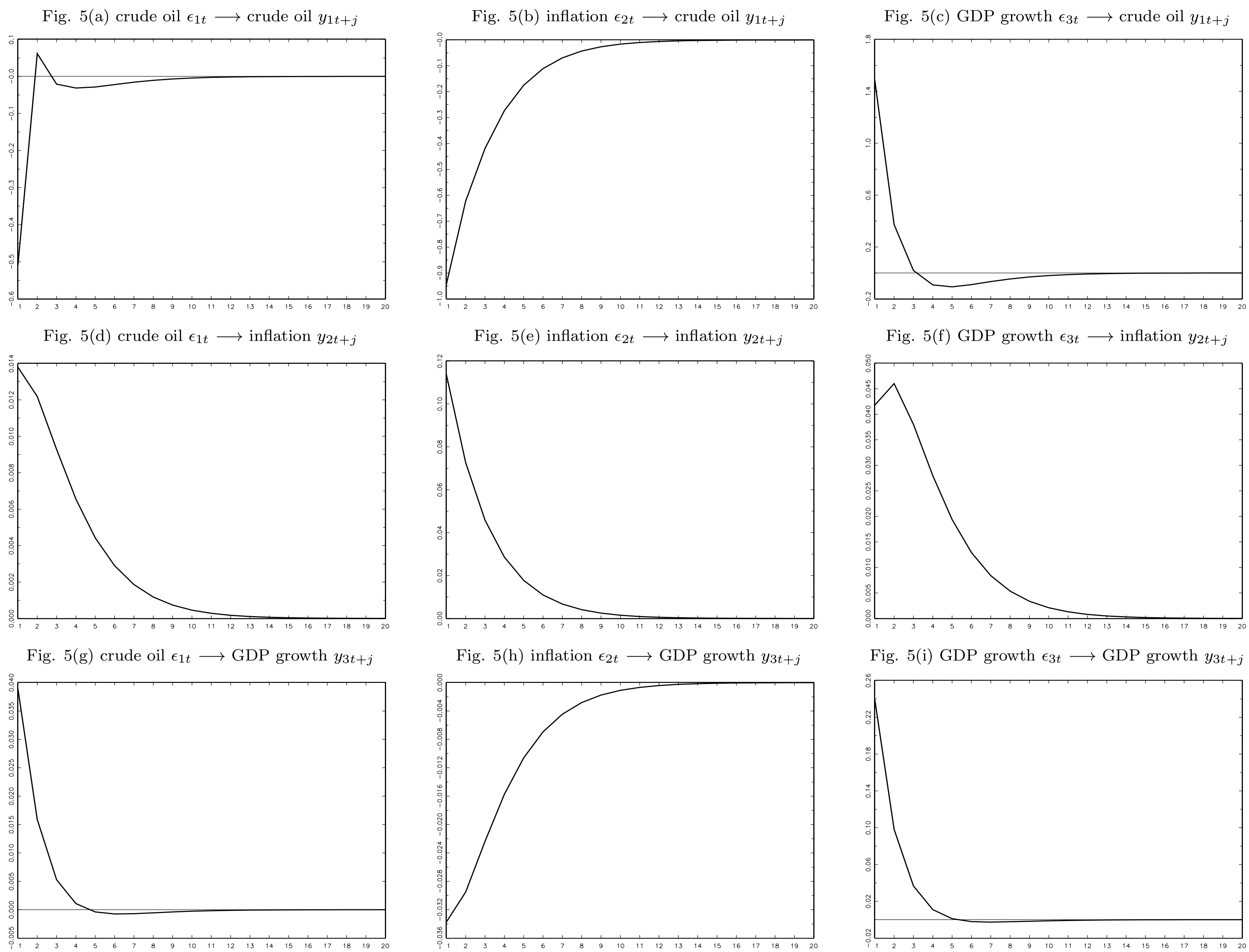

Fig. 5(a) crude oil $\epsilon_{1t} \longrightarrow$ crude oil y_{1t+j}

Fig. 5(b) inflation $\epsilon_{2t} \longrightarrow$ crude oil y_{1t+j}

Fig. 5(c) GDP growth $\epsilon_{3t} \longrightarrow$ crude oil y_{1t+j}

Fig. 5(d) crude oil $\epsilon_{1t} \longrightarrow$ inflation y_{2t+j}

Fig. 5(e) inflation $\epsilon_{2t} \longrightarrow$ inflation y_{2t+j}

Fig. 5(f) GDP growth $\epsilon_{3t} \longrightarrow$ inflation y_{2t+j}

Fig. 5(g) crude oil $\epsilon_{1t} \longrightarrow$ GDP growth y_{3t+j}

Fig. 5(h) inflation $\epsilon_{2t} \longrightarrow$ GDP growth y_{3t+j}

Fig. 5(i) GDP growth $\epsilon_{3t} \longrightarrow$ GDP growth y_{3t+j}

Figure 5. Impulse response function of Gaussian VAR(1)

Fig. 6(a) QVAR(1): Maximum modulus of eigenvalues of $\Psi_t - \Phi$ with $\pm 2\sigma$ estimates

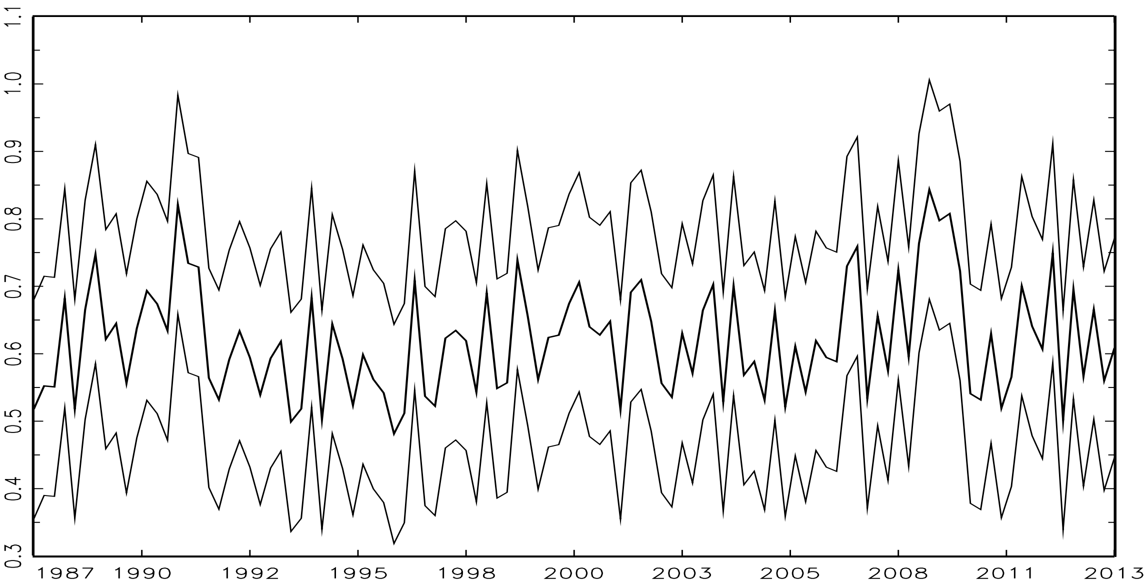

Fig. 6(a) QVAR(2): Maximum modulus of eigenvalues of $\Psi_t - \Phi$ with $\pm 2\sigma$ estimates

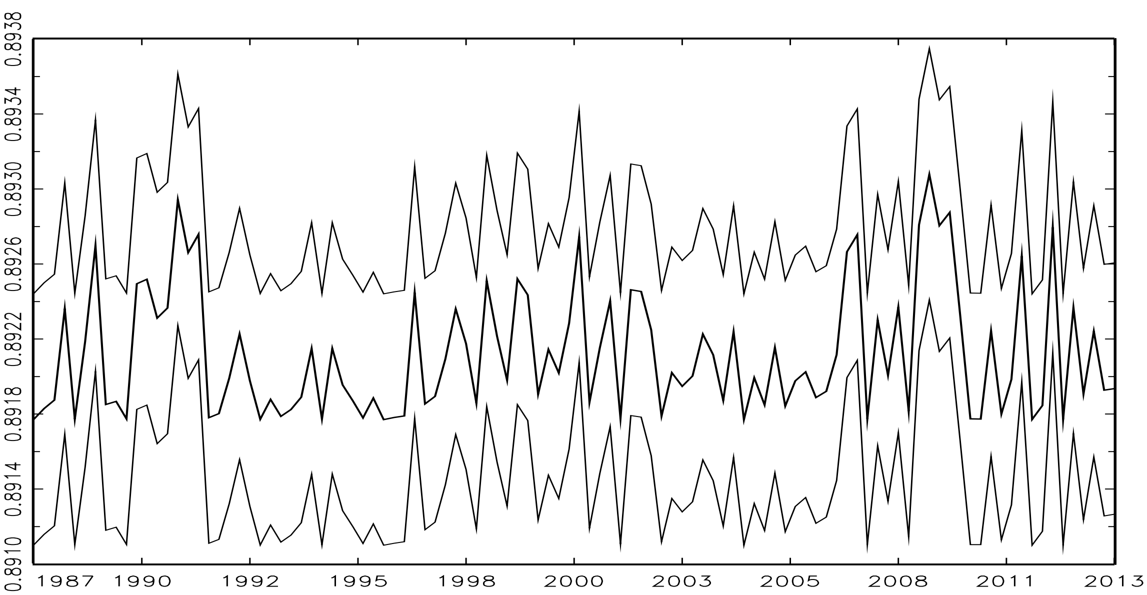

Figure 6. Invertibility of QVAR(1) and QVAR(2) for the period of 1987:Q1 to 2013:Q2, inclusive

Fig. 7(a) crude oil $\epsilon_{1t} \longrightarrow$ crude oil y_{1t+j}

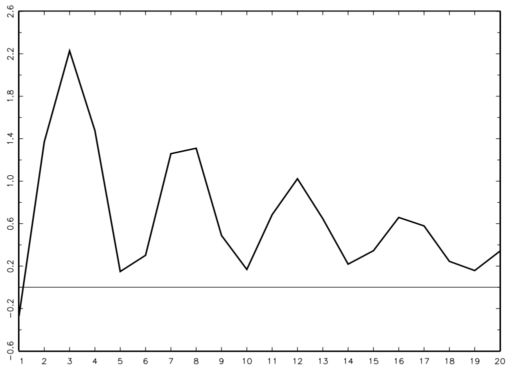

Fig. 7(b) inflation $\epsilon_{2t} \longrightarrow$ crude oil y_{1t+j}

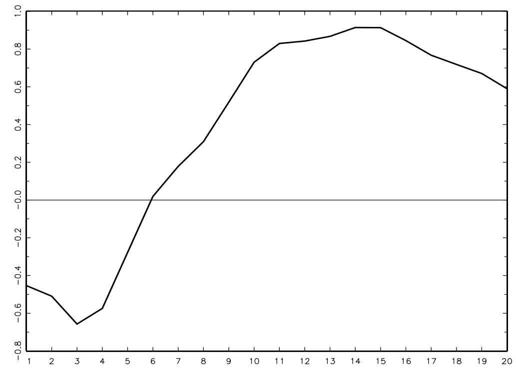

Fig. 7(c) GDP growth $\epsilon_{3t} \longrightarrow$ crude oil y_{1t+j}

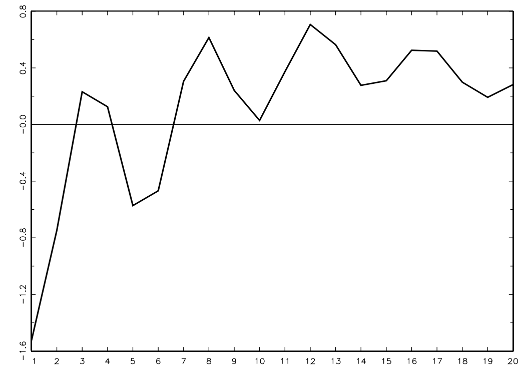

Fig. 7(d) crude oil $\epsilon_{1t} \longrightarrow$ inflation y_{2t+j}

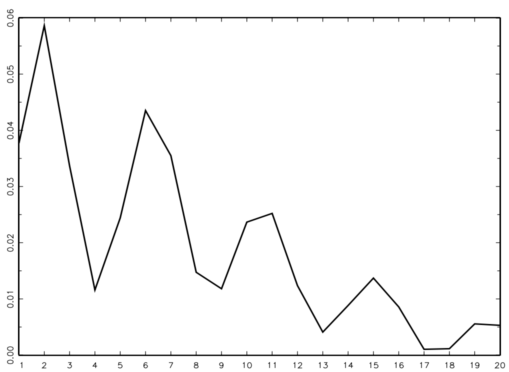

Fig. 7(e) inflation $\epsilon_{2t} \longrightarrow$ inflation y_{2t+j}

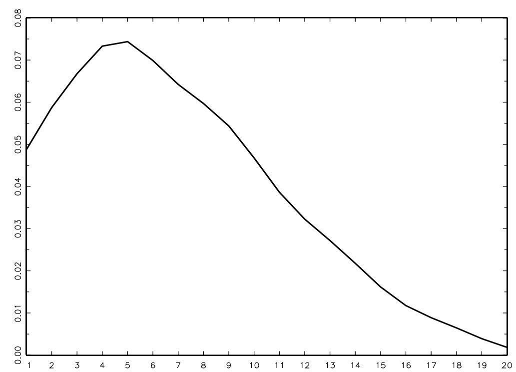

Fig. 7(f) GDP growth $\epsilon_{3t} \longrightarrow$ inflation y_{2t+j}

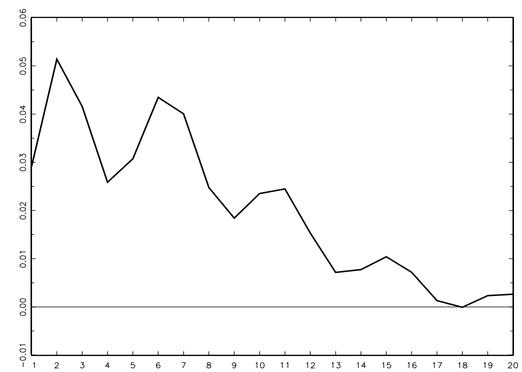

Fig. 7(g) crude oil $\epsilon_{1t} \longrightarrow$ GDP growth y_{3t+j}

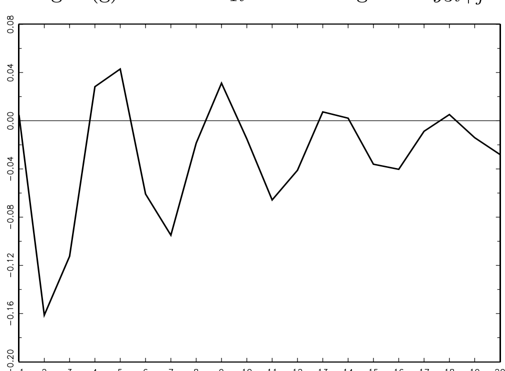

Fig. 7(h) inflation $\epsilon_{2t} \longrightarrow$ GDP growth y_{3t+j}

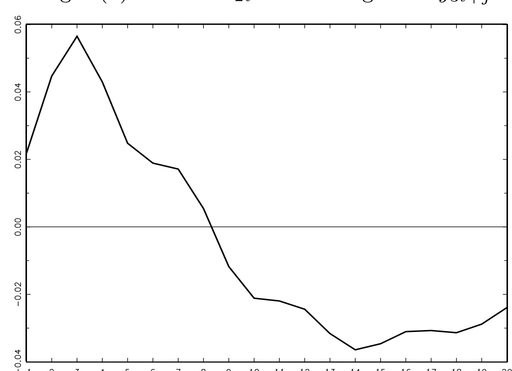

Fig. 7(i) GDP growth $\epsilon_{3t} \longrightarrow$ GDP growth y_{3t+j}

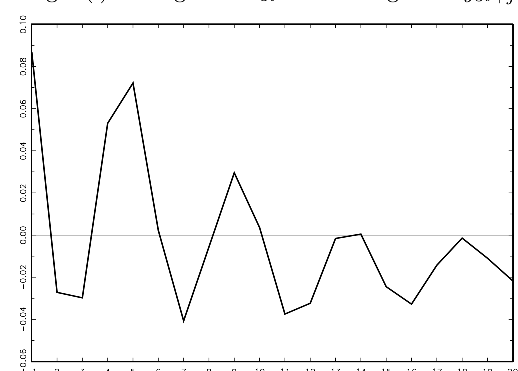

Figure 7. Impulse response function of QVAR(2)

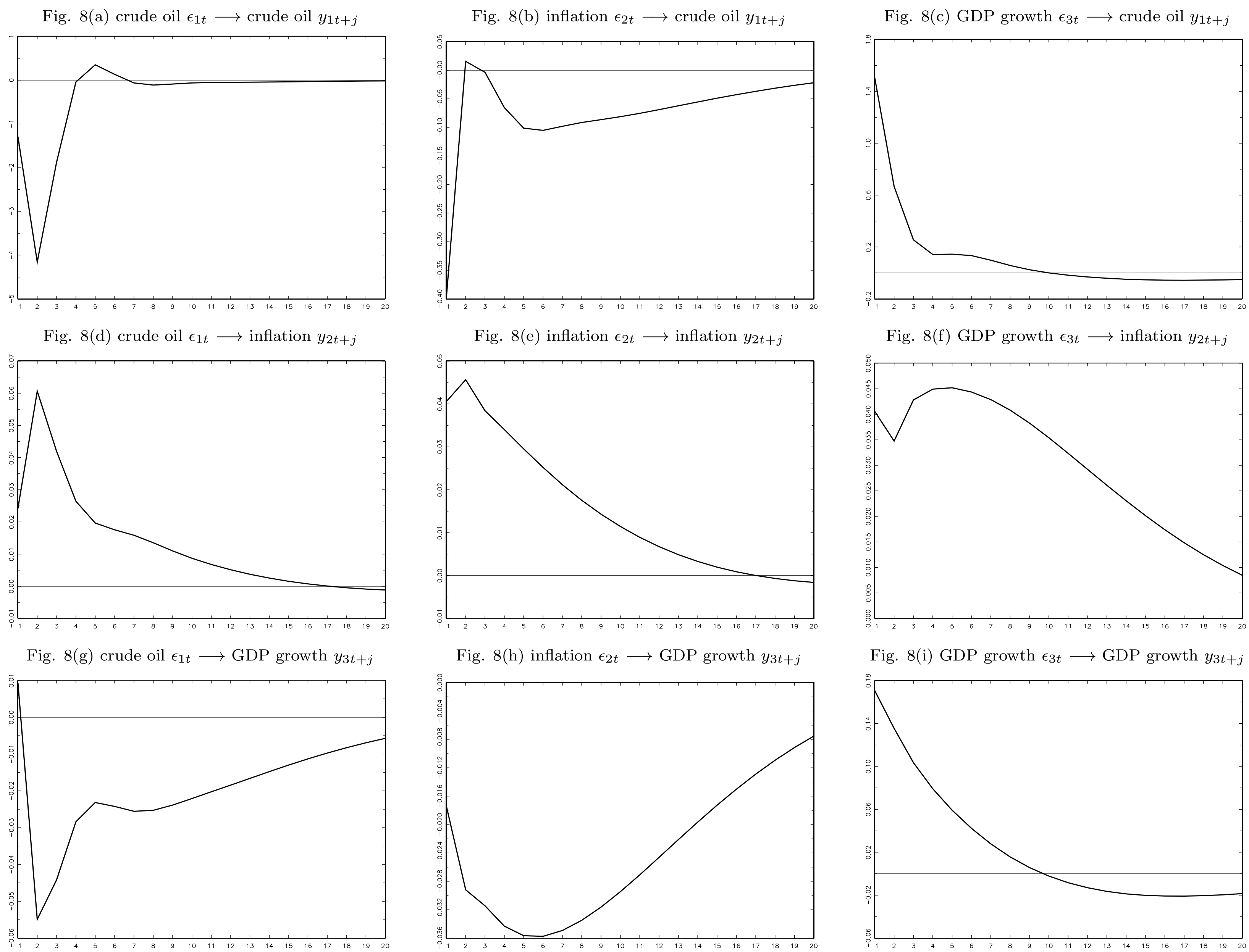

Figure 8. Impulse response function of Gaussian VARMA(2,1)

Fig. 9(a) crude oil $\epsilon_{1t} \longrightarrow$ crude oil y_{1t+j}

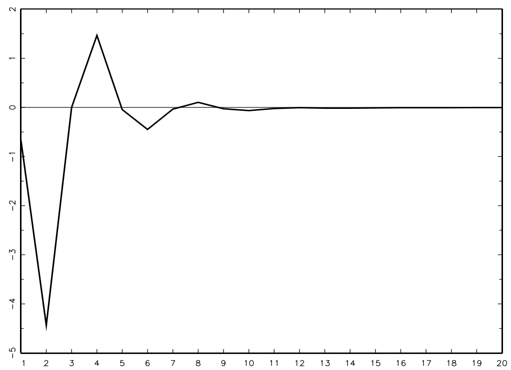

Fig. 9(b) inflation $\epsilon_{2t} \longrightarrow$ crude oil y_{1t+j}

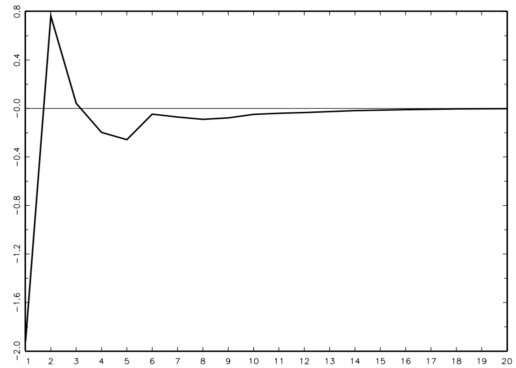

Fig. 9(c) GDP growth $\epsilon_{3t} \longrightarrow$ crude oil y_{1t+j}

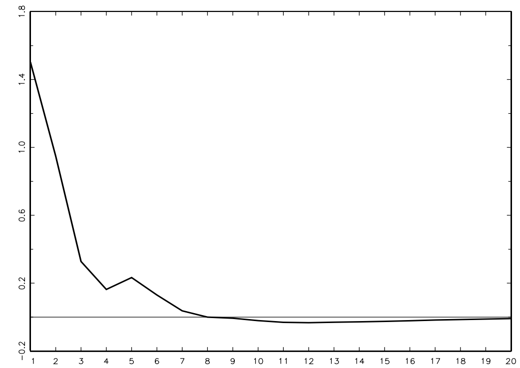

Fig. 9(d) crude oil $\epsilon_{1t} \longrightarrow$ inflation y_{2t+j}

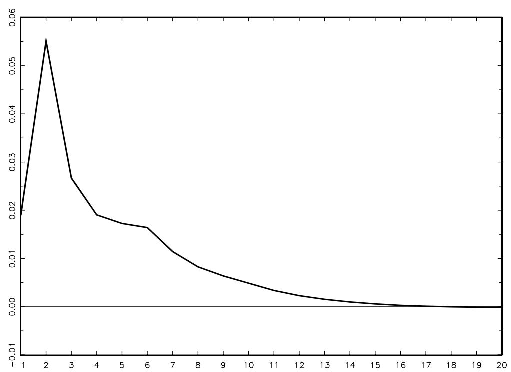

Fig. 9(e) inflation $\epsilon_{2t} \longrightarrow$ inflation y_{2t+j}

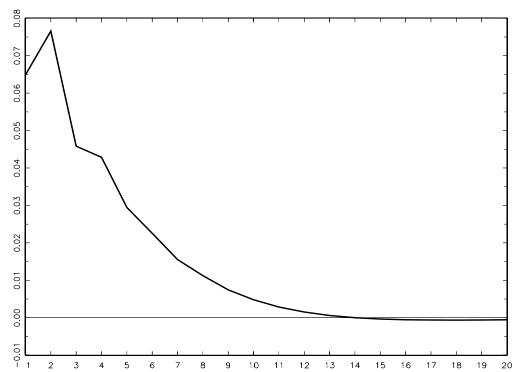

Fig. 9(f) GDP growth $\epsilon_{3t} \longrightarrow$ inflation y_{2t+j}

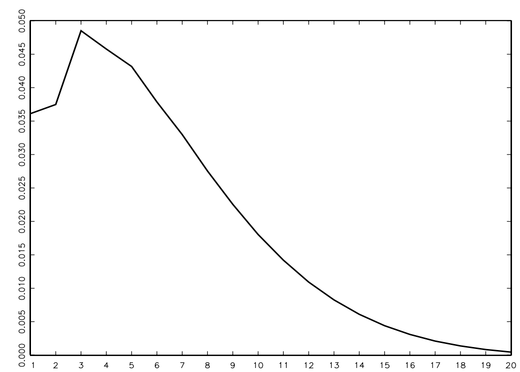

Fig. 9(g) crude oil $\epsilon_{1t} \longrightarrow$ GDP growth y_{3t+j}

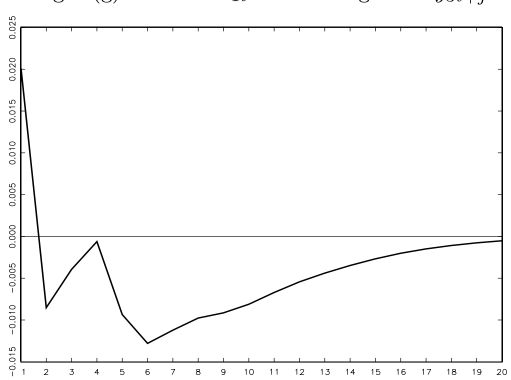

Fig. 9(h) inflation $\epsilon_{2t} \longrightarrow$ GDP growth y_{3t+j}

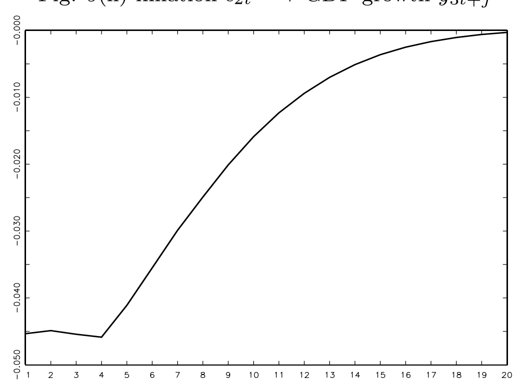

Fig. 9(i) GDP growth $\epsilon_{3t} \longrightarrow$ GDP growth y_{3t+j}

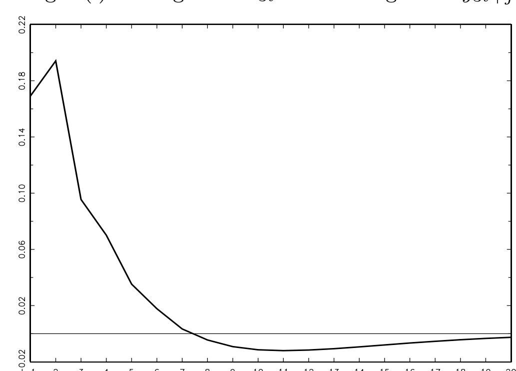

Figure 9. Impulse response function of Gaussian VAR(2)