uc3m | Universidad Carlos III de Madrid
INDEM

WORKING PAPERS

Working Paper
Business Economic Series
WP. 18-01 February, 2018
ISSN 1989-8843

Instituto para el Desarrollo Empresarial.
Universidad Carlos III de Madrid
C/ Madrid, 126
28903 Getafe Madrid (Spain)
http://indem.uc3m.es/en/index
http://indem.uc3m.es/en/workingpapers

Relationships between the stochastic discount factor and the optimal omega ratio

Alejandro Balbás[1]
Departamento de Economía de la Empresa
Universidad Carlos III de Madrid

Beatriz Balbás[2]
Departamento de Economía y Dirección de Empresas
Universidad de Alcalá de Henares

Raquel Balbás[3]
Departamento Economía Financiera y Métodos de Decisión.
Universidad Complutense de Madrid

[1] C/ Madrid, 126. 28903 Getafe (Madrid).

[2] Pl. de la Victoria, 2. 28802 Alcalá de Henares (Madrid).

[3] Campus de Somosaguas. 28223 Pozuelo de Alarcón (Madrid).

Relationships between the stochastic discount factor and the optimal omega ratio

Alejandro Balbás*, **Beatriz Balbás**** **and Raquel Balbás*****

* University Carlos III of Madrid. C/ Madrid, 126. 28903 Getafe (Madrid, Spain). alejandro.balbas@uc3m.es.

** University of Alcalá de Henares. Pl. de la Victoria, 2. 28802 Alcalá de Henares (Madrid, Spain). beatriz.balbas@uah.es.

*** University Complutense of Madrid. Somosaguas. 28223 Pozuelo de Alarcón (Madrid, Spain). raquel.balbas@ccee.ucm.es.

Abstract The omega ratio is an interesting performance measure because it focuses on both downside losses and upside gains, and financial markets are reflecting more and more asymmetry and heavy tails. This paper focuses on the omega ratio optimization in general Banach spaces, which applies for both infinite dimensional approaches related to continuous time stochastic pricing models (Black and Scholes, stochastic volatility, etc.) and more classical problems in portfolio selection. New algorithms will be provided, as well as Fritz John-like and Karush-Kuhn-Tucker-like optimality conditions and duality results, despite the fact that omega is neither differentiable nor convex. The optimality conditions will be applied to the most important pricing models of Financial Mathematics, and it will be shown that the optimal value of omega only depends on the upper and lower bounds of the pricing model stochastic discount factor. In particular, if the stochastic discount factor is unbounded (Black and Scholes, Heston, etc.) then the optimal omega ratio becomes unbounded too (it may tend to infinity), and the introduction of several financial constraints does not overcome this caveat. The new algorithms and optimality conditions will also apply to optimize omega in static frameworks, and it will be illustrated that both infinite- and finite-dimensional approaches may be useful to this purpose.

Key words Omega Ratio, Asset Pricing Model, Stochastic Discount Factor, Representation Theorem, Optimality Conditions.

A.M.S. Classification. 91G10, 91B82, 90C46, 90C48.

J.E.L. Classification G11, G12, C61, C65.

1 Introduction

Asymmetric returns and heavy tails are provoking a growing interest in risk and performance measures beyond the classical standard deviation and Sharpe ratio. In particular, risk measures focusing on the downside risk (Value at Risk or VaR, Conditional Value at Risk or $CVaR$, etc.) are very popular amongst practitioners, regulators and researchers because they properly capture the potential capital losses of a given strategy within a planning horizon.

The omega ratio is also deserving a growing attention in financial literature for two main reasons. On the one hand, this ratio focuses on both downside potential losses and upside potential gains, since it represents the quotient between the expected payoff of a call option and the expected payoff of a put option with the same strike and on the same underlying portfolio. On the other hand, omega is becoming very tractable from a mathematical perspective. Indeed, Mausser *et al.* (2006) dealt with the Charnes and Cooper (1962) transformation and proved that the omega ratio optimization may become a simple linear programming problem if omega reaches values higher than one. Since then, linear programming linked approaches have been developed in several frameworks. Among other interesting analyses, Kapsos *et al.* (2014a) studied the omega ratio in ambiguous settings, Kapsos *et al.* (2014b) dealt with operational risk problems, Guastaroba *et al.* (2016) gave a novel relationship between omega and the Enhanced Index Tracking Problem, and Sharma *et al.* (2017) combined both omega and the $CVaR$ in portfolio choice.

All the papers above dealt with finitely many available financial assets, and, consequently, the problem to solve only involved finite-dimensional variables. Nevertheless, many classical pricing models of Financial Mathematics (Black and Scholes ($B\&S$), stochastic volatility models (SVM) such as the Heston model, etc.) often lead to problems involving infinite-dimensional spaces of random variables. This paper will focus on general optimization methods for the omega ratio applying to both finite and infinite dimensions. In particular, the optimization of omega and a general pricing model (binomial, trinomial, $B\&S$, SVM, etc.) will be integrated in a single problem.

The paper outline is as follows. Section 2 will summarize the most important background we will need. Special focus will deserve the representation of many continuous convex functions as the maximum of a *weakly*$^*-$compact (Luenberger, 1969) set of continuous linear functions, which will be called the convex function sub-gradient. Section 3 will study the omega ratio optimization in general Banach spaces. Firstly, the role of the Charnes and Cooper (1962) approach will be replaced by a transformation applying in general problems involving the optimization of ratios (Section 3.1). If both numerator and denominator of the ratio are differentiable (respectively, linear) then the resulting transformed equivalent problem will be differentiable (linear) too. This methodology will permit us to give new algorithms, optimality conditions, and duality results. In particular, Theorem 7 and Corollary 9, the most important results of Section 3.2, will provide us with Fritz John-like and Karush-Kuhn-Tucker-like optimality conditions (Craven, 1975) to optimize omega,[1] and, similarly, Theorem 10 and Remark 14 (Section 3.3) will yield duality results and complementary slackness optimality conditions if the optimization of omega can be linearized. Both the Lagrange multiplier (Fritz John) and the dual variable (linear case) will have a component belonging to the sub-gradient of both numerator and denominator of omega (notice that numerator and denominator are convex functions).

[1] It is worth remarking that the methodology of Theorem 7 also applies to potential interesting extensions of omega such that numerator and denominator do not have the same strike, though we will not address this topic for the sake of brevity.

Section 4 will apply the findings of Section 3 in problems optimizing omega under the assumptions of a general arbitrage free pricing model. Four cases will be distinguished, according to the imposed constraints: Price constraints, price and return constraints, price and risk constraints involving a general risk measure (downside risk) or a deviation measure, and price-return-risk constraints. As will be seen, risk constraints also enable us to deal with uncertainty and ambiguous settings. As will be also proved, Lagrange multipliers and dual variables will be composed of elements belonging to the sub-gradients of the involved convex measures. In particular, they will contain an element in the sub-gradient of the risk measure and another one belonging to the sub-gradient of both omega numerator and denominator. They will also have to be closely related to the Stochastic Discount Factor (SDF) of the pricing model (Duffie, 1996), in order to guarantee feasibility.

The most important results of Section 4 are Theorems 23 and 29. They show that the optimal value of omega only depends on the essential supremum and the essential infimum of the SDF. Furthermore, if the SDF essential supremum equals infinity ($B\&S$ and most of the continuous time stochastic pricing models) or its essential infimum equals zero (or both) then omega is unbounded, *i.e.*, its optimal value becomes infinity, and the sequence of investment strategies whose omega tends to infinity can be explicitly constructed (Remark 24). Hence, the investor may reach an omega ratio as large as desired with only one dollar (or one cent), an additional return and/or risk constraints do not modify this finding. The omega ratio may become as close to ∞ as desired, regardless of the amount to invest. This "surprising" property seems to be related to some "pathologies" pointed out in Balbás *et al.* (2010), and later extended in Balbás *et al.* (2016) for ambiguous investors. For many downside risk measures all the pricing models above imply the existence of a sequence of investment strategies whose couple $(risk, return)$ tends to $(-\infty, +\infty)$ or $(0, +\infty)$. Perhaps some modifications of these pricing models and their SDF could overcome this caveat, but this is obviously beyond our scope.

Section 5 will illustrate how the results of Section 3 also apply in more classical portfolio choice problems only involving buy and hold (or static) strategies. Two numerical examples will be presented. The first example will deal with with infinite-dimensional spaces in order to optimize omega in an option market, while the second one will deal with finite-dimensional spaces in order to optimize omega in a portfolio selection problem involving two international indices (SP_500 and DAX_30) and two commodities (gold and Brent). Section 6 will present some concluding remarks.

2 Background and notation

Consider the probability space $(\Xi, \mathcal{F}, \mathbb{P})$ composed of the set Ξ, the $\sigma-$algebra $\mathcal{F}$ and the probability measure $\mathbb{P}$. For $1 \leq p < \infty$ the Banach space L^p is composed of the real-valued random variables y such that $\mathbb{E}\left(|y|^p\right) < \infty$, $\mathbb{E}\left(\right)$ representing mathematical expectation. If $p = \infty$ then L^∞ is the Banach space of essentially bounded random variables. The usual norm of L^p is $\|y\|_p := (\mathbb{E}(|y|^p))^{1/p}$ for $1 \leq p < \infty$ and $\|y\|_\infty := Ess\_Sup \{|y|\}$ for $p = \infty$, Ess_Sup denoting "essential supremum". The inclusion $L^p \supset L^P$ holds for $1 \leq p \leq P \leq \infty$. If $1 \leq p < \infty$, $1 < q \leq \infty$ and $1/p + 1/q = 1$ then L^q is the dual space of L^p (Riesz Representation Theorem, Kopp, 1984). If $S \subset L^p$ is a linear subspace then its orthogonal $S^\perp = \{z \in L^q;\ \mathbb{E}(yz) = 0\ \forall y \in S\}$ is a linear and closed subspace of L^q. Moreover L^q may be endowed with the (weak*) topology $\sigma(L^q, L^p)$, which is weaker than the norm topology. Every $\sigma(L^q, L^p)-$closed and bounded subset of L^q is $\sigma(L^q, L^p)-$compact (Alaoglu's Theorem, Kopp, 1984). Consequently, the set

$$B_{(\infty,+)} := \{z \in L^\infty;\ 0 \leq z \leq 1\} \tag{1}$$

satisfies that $B_{(\infty,+)} \subset L^q$ for every $1 < q \leq \infty$, and, furthermore, $B_{(\infty,+)}$ is convex and $\sigma(L^q, L^p)-$compact. It is easy to verify that

$$\mathbb{E}\left(y^+\right) = Max\ \left\{\mathbb{E}(yz);\ z \in B_{(\infty,+)}\right\} \tag{2}$$

holds for every $y \in L^1$.[2] Further details about Banach or Hilbert spaces of random variables may be found in Kopp (1984).

If E_+ is a convex cone of a Banach space E then the relationship $x \leq y$ if $y - x \in E_+$ is an order in E. If E_+ is pointed ($E_+ \cap (-E_+) = \{0\}$) then this order is antisymmetric ($x \leq y$ and $y \leq x \Longrightarrow x = y$). If E' denotes the dual space of E, then $E'_+ = \{e' \in E';\ e'(e) \geq 0,\ \forall e \in E_+\}$ is a convex cone of E'. Further details about ordered Banach spaces may be found in Luenberger (1969).

If E_1 and E_2 are Banach spaces then the space $\mathcal{L}(E_1, E_2)$ of linear and continuous functions from E_1 to E_2 is a Banach space too. If $A \subset E_1$ is an open set, and $f : A \to E_2$ is an arbitrary function, then f is said to be Fréchet differentiable in $a \in A$ if there exist a function $o : A \to E_2$ and a linear and continuous function $f'(a, .) \in \mathcal{L}(E_1, E_2)$ such that

$$\begin{cases} f(x) = f(a) + f'(a, x - a) + \|x - a\|\, o(x), & \forall x \in A \\ \lim_{x \to a} o(x) = 0 \end{cases}$$

If so, $f'(a, .)$ is unique and is said to be the Fréchet differential of f at a. If f is Fréchet differentiable at every element of A then f is said to be Fréchet differentiable on A, and continuously Fréchet differentiable on A if the function $E_1 \supset A \ni a \to f'(a, .) \in \mathcal{L}(E_1, E_2)$ is continuous. If $A = E_1$ and $f \in \mathcal{L}(E_1, E_2)$ then $f'(a, .) = f$ holds for every $a \in A$. Further details may be found in Luenberger (1969) or Zeidler (1995).

If K is a separated compact topological space and $\mathcal{C}(K)$ denotes the space of real valued and continuous functions on K, then $\mathcal{C}(K)$ becomes a Banach space when endowed with the norm $\|f\| = Max\ \{|f(x)|;\ x \in K\}$. Its dual $\mathcal{C}'(K)$ is composed of the inner regular real valued (non necessarily signed) $\sigma-$additive measures on the Borel $\sigma-$algebra of K (Zeidler, 1995).

[2] As usual, $a^+ := Max\ \{a, 0\}$ and $a^- := Max\ \{-a, 0\}$ for every $a \in \mathbb{R}$.

If $1 \leq p < \infty$, $1 < q \leq \infty$, $1/p + 1/q = 1$, $K \subset L^q$ is convex and $\sigma(L^q, L^p)-$compact, and $\mu \in \mathcal{C}'(K)$ is a probability measure (*i.e.*, $\mu \geq 0$ and $\mu(K) = 1$), then there exists a unique $z_\mu \in K$ such that

$$\int_K \mathbb{E}(yz)\,\mu(dz) = \mathbb{E}(z_\mu y) \tag{3}$$

holds for every $y \in L^p$. A complete proof of (3) may be found, among others, in Phelps (2001).[3]

Consider $k \in \mathbb{R}$, $p \in [1, \infty)$ and $y \in L^p$. Obviously, $(k-y)^+ \in L^p$. $(k-y)^+ \geq 0$ trivially implies that $\mathbb{E}\left((k-y)^+\right) \geq 0$, and $\mathbb{E}\left((k-y)^+\right) = 0$ if and only if $(k-y)^+ = 0$ almost surely (or $\mathbb{P}(y \geq k) = 1$). If

$$\mathbb{P}(y \geq k) < 1 \tag{4}$$

or, equivalently, $\mathbb{E}\left((k-y)^+\right) > 0$, then the omega ratio $\Omega_k(y)$ of y with threshold k is defined by

$$\Omega_k(y) := \frac{\mathbb{E}\left((y-k)^+\right)}{\mathbb{E}\left((k-y)^+\right)} \geq 0, \tag{5}$$

and it has been considered a good performance measure because it depends on both downside potential capital losses (denominator) and upside potential capital gains (numerator, see Bernardo and Ledoit, 2000, or Shadwick and Keating, 2002, for further discussions about the advantages of risk/performance measures involving both losses and gains). Bearing in mind the chain of trivial equalities

$$(y-k)^+ + k = Max\ \{k, y\} = (k-y)^+ + y, \tag{6}$$

for $y \in L^1$ satisfying (4) we have that

$$\Omega_k(y) = \frac{\mathbb{E}\left((k-y)^+\right) + \mathbb{E}(y) - k}{\mathbb{E}\left((k-y)^+\right)} = \frac{\mathbb{E}(y) - k}{\mathbb{E}\left((k-y)^+\right)} + 1.$$

Asset Pricing Theory usually deals with a planning period $[0, T]$, a set $\mathcal{T}$ of trading dates satisfying $\{0, T\} \subset \mathcal{T} \subset [0, T]$, a filtration $(\mathcal{F}_\tau)_{\tau \in \mathcal{T}}$ reflecting the arrival of information and such that $\mathcal{F}_0 = \{\emptyset, \Xi\}$ and $\mathcal{F}_T = \mathcal{F}$, and stochastic processes $(y_\tau)_{\tau \in \mathcal{T}}$ adapted to $(\mathcal{F}_\tau)_{\tau \in \mathcal{T}}$ and reflecting the price evolution of self-financing investment strategies (Duffie, 1996). We will consider some $p \in [1, \infty)$, and a non-trivial closed subspace $S \subset L^p$ such that every $y \in S$ is the price at T of some self-financing portfolio. In other words, if $y \in S$ then there is a

[3] In other words, the probability measure μ may be often replaced by a Dirac Delta δ_{z_μ}, in the sense that

$$\int_K \mathbb{E}(yz)\,\mu(dz) = \int_K \mathbb{E}(yz)\,\delta_{z_\mu}(dz)$$

holds for every $y \in L^p$.

self-financing price process $(y_\tau)_{\tau\in\mathcal{T}}$ such that $y_T = y$. In general, S is said to be a subspace of marketed claims. We will only deal with closed subspaces S of marketed claims containing the riskless asset $(1 \in S)$.

The pricing rule associates the final (at T) random price $y = y_T \in S$ with the initial (at 0) numerical one $y_0 \in \mathbb{R}$. In one simplifies notations and omits subscripts, the pricing rule becomes a linear and continuous function $\Pi \in S'$, S' being the dual space of S. The Hahn Banach Theorem (Zeidler, 1995) implies the existence of an extension of this pricing rule to the whole space L^p. Since the dual of L^p is L^q, if the pricing rule extension is still denoted by Π, the Riesz Theorem implies the existence of $z_\Pi \in L^q$ such that

$$\Pi(y) = e^{-rT}\mathbb{E}(z_\Pi y) \tag{7}$$

holds for every $y \in S$, r denoting the (continuously compounded) riskless rate. Henceforth z_Π will be called stochastic discount factor (SDF) of Π. Obviously, $1 \in S$ and (7) lead to

$$\mathbb{E}(z_\Pi) = 1. \tag{8}$$

If

$$\mathbb{P}(z_\Pi = 1) = 1 \tag{9}$$

then (7) becomes $\Pi(y) = e^{-rT}\mathbb{E}(y)$, and the market is said to be risk-neutral. The price of every marketed claim equals its discounted expectation and, consequently, every expected return equals the riskless rate. Since Financial Economics usually considers risk averse agents, (9) will not hold when dealing with the standard pricing models. If the market is complete ($S = L^p$), then z_Π is unique, and

$$\mathbb{P}(z_\Pi > 0) = 1 \tag{10}$$

must hold in order to prevent the existence of arbitrage. If $p = 2$ (and therefore $q = 2$) then z_Π also becomes unique if one replaces "completeness" by the condition $z_\Pi \in S$. Further details may be found in Duffie (1996).

Next let us introduce some downside risk measures, since they are becoming more and more important in many classical problems of Financial Mathematics (see, among many others, Gilli *et al.*, 2006, Zakamouline and Koekebbaker, 2009, or Zhao and Xiao, 2016). Consider some $p \in [1, \infty)$. Throughout this paper a risk measure on L^p will be a function $\rho : L^p \longrightarrow \mathbb{R}$ satisfying the existence of a convex and $\sigma(L^q, L^p)$ $-$compact set $\Delta_\rho \subset L^q$ such that

$$\rho(y) = Max\ \{-\mathbb{E}(yz);\ z \in \Delta_\rho\} \tag{11}$$

for every $y \in L^p$, and the existence of the zero-variance random variable $z = \tilde{E} \geq 0$ such that

$$\tilde{E} \in \Delta_\rho, \tag{12}$$

and

$$\mathbb{E}(z) = \tilde{E} \tag{13}$$

for every $z \in \Delta_\rho$. If $\tilde{E} = 1$ then ρ will be said to be expectation bounded (Rockafellar *et al.*, 2006). If, moreover,

$$\mathbb{P}(z \geq 0) = 1 \tag{14}$$

also holds for every $z \in \Delta_\rho$, then ρ will be said to be coherent (Artzner *et al.*, 1999). For instance, for $p = 1$ and the very well-known Conditional Value at Risk with confidence level $1 - \alpha \in [0, 1)$, given by

$$CVaR_{1-\alpha}(y) := \frac{1}{\alpha}\int_0^\alpha VaR_{1-t}(y)\,dt, \tag{15}$$

$VaR_{1-t}(y)$ denoting the Value at Risk of y with the confidence level $1-t$,[4] one has (Rockafellar *et al.*, 2006)

$$\Delta_{CVaR_{1-\alpha}} = \left\{z \in L^\infty;\ \mathbb{E}(z) = 1 \ and\ 0 \leq z \leq 1/\alpha\right\}. \tag{17}$$

Similarly, for $p = 1$ and the Weighted Conditional Value at Risk with positive weights $W_1, W_2, ..., W_m$ such that $\sum_{j=1}^m W_j = 1$ and confidence levels $1 - \alpha_1$, $1 - \alpha_2$,...,$1 - \alpha_m \in [0, 1)$, given by

$$WCVaR_{(W_1,W_2,...,W_m,1-\alpha_1,1-\alpha_2,...,1-\alpha_m)}(y) := \sum_{j=1}^m W_j CVaR_{1-\alpha_j}(y) \tag{18}$$

one has

$$\Delta_{WCVaR} = \sum_{j=1}^m W_j \Delta_{CVaR_{1-\alpha_j}}. \tag{19}$$

As a third example let us consider $p = 2$ and the ambiguous setting of Balbás *et al.* (2016), characterized by a set of priors

$$[R, S]_1 = \{f \in L^p;\ R \leq f \leq S \ and\ \mathbb{E}(z) = 1\}.$$

In this approach the decision maker is not sure about the real probabilities of the states of nature, and he/she considers every $\mathbb{P}-$continuous probability measure $\mathbb{P}_f$ whose Radon–Nikodym derivative satisfies

$$\frac{d(\mathbb{P}_f)}{d(\mathbb{P})} \in [R, S]_1.$$

According to Balbás *et al.* (2016), the Robust (or worst-case) $CVaR$ given by

$$RCVaR_{\left([R,S]_1,1-\alpha\right)}(y) := Max\ \left\{CVaR_{1-\alpha}(y);\ f \in [R, S]_1\right\} \tag{20}$$

---

[4] Recall that the Value at Risk of a random variable y is given by

$$VaR_{1-\alpha}(y) := -Inf\ \{x \in \mathbb{R}; \mathbb{P}(y \leq x) > \alpha\}, \tag{16}$$

where $1 - \alpha \in (0, 1)$ is the level of confidence.

$\rho = VaR_{1-\alpha}$ does not satisfy any representation similar to (11), though there are alternative representation results for this risk measure (Balbás *et al.*, 2017). Throughout this paper $\rho = VaR_{1-\alpha}$ will be the only considered risk measure such that (11) may fail.

exists for every $y \in L^2$, and $RCVaR_{([R,S]_1,\alpha)}$ is a coherent and expectation bounded risk measure.[5] Furthermore,

$$\Delta_{RCVaR} = \left\{z \in L^\infty;\ \mathbb{E}(z) = 1 \ and \ \exists f \in [R,S]_1 \ with \ 0 \le z \le f/\alpha\right\}. \quad (21)$$

If $\tilde{E} = 0$ then ρ is called deviation measure (Rockafellar *et al.*, 2006). Well-known examples are the absolute deviation $\sigma_1(y) := \mathbb{E}(|y - \mathbb{E}(y)|)$ for $p = 1$, and the standard deviation $\sigma_2(y) := \sqrt{\mathbb{E}\left(|y - \mathbb{E}(y)|^2\right)}$ for $p = 2$. For them one has (Rockafellar *et al.*, 2006),[6]

$$\left\{\begin{array}{l} \Delta_{\sigma_1} = \left\{z \in L^\infty;\ z = w - \mathbb{E}(w) \ and \ 0 \le w \le 2\right\} \\ \Delta_{\sigma_2} = \left\{z \in L^2;\ \mathbb{E}(z) = 0 \ and \ \sigma_2(z) \le 1\right\}. \end{array}\right. \quad (22)$$

3 Optimizing omega

Let us focus on the optimization of the omega ratio. Fix $k \in \mathbb{R}$, $p \in [1,\infty)$, $q \in (1,\infty]$ such that $1/p + 1/q = 1$, a closed subspace $S \subset L^p$ containing the riskless asset, a non-void $Y_0 \subset S \subset L^p$, and Problem

$$\left\{\begin{array}{ll} Max & \Omega_k(y) \\ & y \in Y_0. \end{array}\right. \quad (23)$$

Assumption 1. Throughout the rest of the paper we will suppose that (4) holds for every $y \in Y_0$. □

3.1 Optimizing general ratios

This sub-section will be devoted to dealing with the optimization of general ratios. Its main results, along with (2) and (3), will provide us with new methods to optimize omega. It is remarkable that this methodology to optimize general ratios will only involve linear programming problems if one deals with ratios of two linear functions.

Throughout this sub-section we will fix is an arbitrary set U_0, and two arbitrary functions $N : U_0 \to \mathbb{R}$ and $D : U_0 \to \mathbb{R}$ such that $N(u) \ge 0$ and $D(u) > 0$ hold for every $u \in U_0$. Let us denote

$$0 \le \beta^* = Sup \left\{\frac{N(u)}{D(u)};\ u \in U_0\right\} \le \infty. \quad (24)$$

[5] Many alternative approaches about the set of priors and the robust $CVaR$ may be found in the literature about ambiguity. We have chosen the one by Balbás *et al.* (2016) because it nicely fits in the framework of this paper, since (11) holds for the representation set (21).

[6] It is easy to prove that Δ_{σ_1} may be also given by

$$\Delta_{\sigma_1} = \left\{z \in L^\infty;\ z = w - \mathbb{E}(w) \ and \ M - 1 \le w \le M + 1\right\}$$

for every $M \in \mathbb{R}$. In particular, if one takes $M = 0$ one has

$$\Delta_{\sigma_1} = \left\{z \in L^\infty;\ z = w - \mathbb{E}(w) \ and \ -1 \le w \le 1\right\},$$

and (22) arises for $M = 1$.

Lemma 1 *Suppose that* $u^* \in U_0$*.* u^* *solves*

$$Max \left\{ \frac{N(u)}{D(u)};\; u \in U_0 \right\} \tag{25}$$

if and only if u^* *solves*

$$Min \left\{ \frac{N(u^*)}{D(u^*)} D(u) - N(u);\; u \in U_0 \right\}. \tag{26}$$

Proof. Suppose that $u^* \in U_0$ solves (25). Then

$$\frac{N(u^*)}{D(u^*)} \geq \frac{N(u)}{D(u)} \tag{27}$$

holds for every $u \in U_0$, and consequently,

$$\frac{N(u^*)}{D(u^*)} D(u) - N(u) \geq 0$$

holds for every $u \in U_0$. Hence u^* solves (26) because

$$\frac{N(u^*)}{D(u^*)} D(u^*) - N(u^*) = 0. \tag{28}$$

Conversely, suppose that u^* solves (26). The equality (28) implies that $\frac{N(u^*)}{D(u^*)} D(u) - N(u) \geq 0$ for every $u \in U_0$, *i.e.*, (27) holds for every $u \in U_0$, and u^* solves (25). □

Lemma 2 $\beta \geq 0$ *is an upper bound of (25) if and only if* $\beta D(u) - N(u) \geq 0$ *holds for every* $u \in U_0$*. If so, Problem*

$$Min \;\{\beta D(u) - N(u);\; u \in U_0\} \tag{29}$$

is bounded.

Proof. Obviously, $\beta \geq \frac{N(u)}{D(u)}$ for every $u \in U_0$ if and only if $\beta D(u) - N(u) \geq 0$ for every $u \in U_0$. □

Lemma 3 *Suppose that* $\beta \geq 0$ *and* $u_\beta \in U_0$ *solves (29). Consider* β^* *given in (24).*

a) If $\beta D(u_\beta) - N(u_\beta) = 0$ *then* $\beta = \beta^*$ *and* u_β *solves (25).*

b) If $\beta D(u_\beta) - N(u_\beta) < 0$ *then* $\beta < \beta^*$*.*

c) If $\beta D(u_\beta) - N(u_\beta) > 0$ *then* $\beta \geq \beta^*$*. Moreover, if (25) is solvable (i.e.,* β^* *in (24) is attainable) then* $\beta > \beta^*$*.*

Proof. *a)* $\beta D(u_\beta) - N(u_\beta) = 0$ implies that $\beta = \dfrac{N(u_\beta)}{D(u_\beta)}$, and therefore u_β solves $Min\left\{\dfrac{N(u_\beta)}{D(u_\beta)}D(u) - N(u);\ u \in U_0\right\}$. Lemma 1 implies that u_β solves (25), and therefore (24) leads to

$$\beta^* = \frac{N(u_\beta)}{D(u_\beta)} = \beta.$$

b) $\beta D(u_\beta) - N(u_\beta) < 0$ leads to $\beta < \dfrac{N(u_\beta)}{D(u_\beta)} \leq \beta^*$.

c) If $u \in U_0$ then $\beta \leq \dfrac{N(u)}{D(u)}$ would lead to $\beta D(u) - N(u) \leq 0$, in contradiction with $\beta D(u_\beta) - N(u_\beta) > 0$. Thus, $\beta > \dfrac{N(u)}{D(u)}$ and therefore $\beta \geq \beta^*$. Moreover, if $u^* \in U_0$ and $\beta^* = \dfrac{N(u^*)}{D(u^*)}$ then $\beta = \beta^*$ would imply $\beta D(u^*) - N(u^*) = 0$, in contradiction with $\beta D(u_\beta) - N(u_\beta) > 0$. □

Next, let us show that Lemmas 1, 2 and 3 yield simple algorithms solving Problem (25).

Algorithm 4 *(Exact solution) In order to simplify the exposition, let us assume that (29) is solvable for every* $\beta \geq 0$.

Step_1. Solve (29) as a parametric problem depending on the parameter $\beta \geq 0$. *Denote by* $f(\beta) \in U_0$ *a solution of (29).*

Step_2. Solve the equation

$$\beta D(f(\beta)) - N(f(\beta)) = 0. \tag{30}$$

Lemmas 1 *and* 3 *show that the solution of (25) is also a solution of Equation (30).* □

Algorithm 5 *(Approximation of an exact solution) In order to simplify the exposition, let us assume that (29) is solvable for every* $\beta \geq 0$.

Step_1. Choose an "small enough" admissible error $\varepsilon > 0$.

Step_2. Consider the sequence $(u_n)_n \subset U_0$ *such that every* u_n *solves (29) if* $\beta = n\varepsilon$, $n = 0, 1, 2, 3, ...$

Step_3. Compute the sequence $(|n\varepsilon D(u_n) - N(u_n)|)_n$ *and choose its minimum value* $|n_0\varepsilon D(u_{n_0}) - N(u_{n_0})|$. *According to Lemma* 3, u_{n_0} *could be an approximation of the solution of (25), and* $n_0\varepsilon$ *could be an approximation for the optimal ratio.* □

Remark 6 *(Other approximations) Lemmas* 1, 2 *and* 3 *above may generate more algorithms approximating a solution of (25). In order to shorten the exposition, we will just summarize without technical detail one more general procedure.*

Step_1. Choose $\beta_0 = 0$ *and* $u_0 \in U_0$ *solving (29) for* $\beta = 0$*. Obviously,* $\beta_0 D(u_0) - N(u_0) = -N(u_0) \leq 0$*. If* $N(u_0) = 0$ *then Lemma* 3 *shows that* $u^* = u_0$ *solves (25) and the algorithm ends. Otherwise we have* $\beta_0 D(u_0) - N(u_0) < 0$*.*

Step_2. Choose $\beta_1 > 0$ *"big enough" so as to guarantee that (29) is bounded from below by zero for* $\beta = \beta_1$*. If* β_1 *cannot be found then Lemma* 2 *implies that (25) is unbounded. If* β_1 *can be found then solve (29) and let* u_1 *be its solution. Obviously,* $\beta_1 D(u_1) - N(u_1) \geq 0$*. If* $\beta_1 D(u_1) - N(u_1) = 0$ *then Lemma* 3 *shows that* $u^* = u_1$ *solves (25) and the algorithm ends. Otherwise we have* $\beta_1 D(u_1) - N(u_1) > 0$*.*

Step_3. Take $\beta_2 = (\beta_0 + \beta_1)/2$*, and* u_2 *solving (29) for* $\beta = \beta_2$*. If* $\beta_2 D(u_2) - N(u_2) = 0$ *then Lemma* 3 *shows that* $u^* = u_2$ *solves (25) and the algorithm ends. Otherwise go to Step* 4*.*

Step_4. If $\beta_2 D(u_2) - N(u_2) > 0$ *(respectively,* < 0*) then take* $\beta_3 = (\beta_0 + \beta_2)/2$ *(respectively,* $\beta_3 = (\beta_1 + \beta_2)/2$*), and* u_3 *solving (29) for* $\beta = \beta_3$*. If* $\beta_3 D(u_3) - N(u_3) = 0$ *then Lemma* 1 *shows that* $u^* = u_3$ *solves (25) and the algorithm ends. Otherwise go to Step* 5*.*

Step_5. Induction. Suppose that the solution u^* *of (25) does exist. If the algorithm does not end in finitely many steps, then construct* $(\beta_n)_{n=0}^{\infty}$ *and* $(u_n)_{n=0}^{\infty}$ *in the obvious manner. It is easy to see that* $\left|\beta_{n+1} - \beta_n\right| = \dfrac{|\beta_1 - \beta_0|}{2^n}$ *holds for every* $n \in \mathbb{N}$*, which easily implies that* $\beta = \lim_{n\to\infty} \beta_n$ *exists and equals the supremum of (25).* □

3.2 Optimizing omega: General approach

Lemmas 1, 2 and 3 show that one can optimize omega by solving

$$\begin{cases} Min\ \beta \mathbb{E}\left((k-y)^+\right) - \mathbb{E}\left((y-k)^+\right) \\ y \in Y_0 \end{cases} \tag{31}$$

According to (6),

$$\beta \mathbb{E}\left((k-y)^+\right) - \mathbb{E}\left((y-k)^+\right) = \beta\left(\mathbb{E}\left((y-k)^+\right) + k - y\right) - \mathbb{E}\left((y-k)^+\right)$$
$$= (\beta - 1)\,\mathbb{E}\left((y-k)^+\right) - \beta\mathbb{E}(y) + \beta k.$$

Thus, (31) is equivalent to

$$\begin{cases} Min\ (\beta-1)\,\mathbb{E}\left((y-k)^+\right) - \beta\mathbb{E}(y) + \beta k \\ y \in Y_0 \end{cases} \tag{32}$$

Let us deal with (31) and give necessary optimality conditions for Problem (23). Recall that (4) holds for every $y \in Y_0$ (Assumption 1).

Theorem 7 *Consider Problem (23).*

a) Consider $\beta \geq 0$ and $y_\beta \in Y_0$. Then, y_β solves (31) (or (32)) if and only if there exists $(\theta_1^*, \theta_2^*, z_2^*) \in \mathbb{R} \times \mathbb{R} \times L^\infty$ such that $(\theta_1^*, \theta_2^*, y_\beta, z_2^*)$ solves

$$Min\ \beta\theta_1 + \theta_2 \begin{cases} \theta_1 - \mathbb{E}\left(z_1\left(k - y\right)\right) \geq 0, & \forall z_1 \in B_{(\infty,+)} \\ \theta_2 + \mathbb{E}\left(z_2\left(y - k\right)\right) \geq 0 \\ (\theta_1, \theta_2, y, z_2) \in \mathbb{R} \times \mathbb{R} \times Y_0 \times B_{(\infty,+)} \end{cases} \tag{33}$$

$(\theta_1, \theta_2, y, z_2) \in \mathbb{R} \times \mathbb{R} \times L^p \times L^\infty$ being the decision variable.

b) Consider $y^* \in Y_0$. y^* solves (23) if and only if there exists $(\theta_1^*, \theta_2^*, z_2^*) \in \mathbb{R} \times \mathbb{R} \times L^\infty$ such that $(\theta_1^*, \theta_2^*, y^*, z_2^*)$ solves (33) for $\beta = \Omega_k(y^*)$.

c) (Necessary Fritz John-like optimality conditions) Suppose that $S \subset L^p$ is a closed subspace containing the riskless asset, and $Y \subset S$ is a non-void open set in the relative topology of S.[7] Suppose that E is a Banach space ordered by the convex cone E_+ with non-void interior E_+°. Consider a continuously Fréchet differentiable function $g : Y \to E$. Suppose that Y_0 is given by

$$Y_0 = \{y \in Y;\ g(y) \leq g_0\} \tag{34}$$

where $g_0 \in E$. Denote E' the dual space of E. If $\beta \geq 0$ and $y_\beta \in Y_0$ solves (31) then there exist $\tau \in \mathbb{R}$, $z_1^*, z_2^* \in L^\infty$ and $e'_\beta \in E'$ such that,[8]

$$\begin{cases} e'_\beta \circ g'(y_\beta, .) - \tau\left(\beta z_1^* + z_2^*\right) \in S^\perp \\ z_2^*(\xi) = 0 \Longrightarrow \tau\left(y_\beta(\xi) - k\right) \leq 0, & \xi \in \Xi \\ z_2^*(\xi) = 1 \Longrightarrow \tau\left(y_\beta(\xi) - k\right) \geq 0, & \xi \in \Xi \\ 0 < z_2^*(\xi) < 1 \Longrightarrow \tau\left(y_\beta(\xi) - k\right) = 0, & \xi \in \Xi \\ \tau\beta\mathbb{E}\left(z_1^*\left(y_\beta - k\right)\right) \leq \tau\beta\mathbb{E}\left(z_1\left(y_\beta - k\right)\right), & \forall z_1 \in B_{(\infty,+)} \\ e'_\beta\left(g\left(y_\beta\right) - g_0\right) = 0 \\ \left(\tau, e'_\beta\right) \neq (0,0) \\ \tau \geq 0,\ z_1^*, z_2^* \in B_{(\infty,+)},\ e'_\beta \geq 0 \end{cases} \tag{35}$$

If $y^* \in Y_0$ solves (23) then there exist $\tau \in \mathbb{R}$, $z_1^*, z_2^* \in L^\infty$ and $e'_\beta \in E'$ satisfying (35) for $\beta = \Omega_k(y^*)$.[9]

Proof. *a)* (31) is obviously equivalent to

$$Min\ \beta\theta_1 + \theta_2 \begin{cases} \theta_1 \geq \mathbb{E}\left(\left(k - y\right)^+\right) \\ \theta_2 \geq -\mathbb{E}\left(\left(y - k\right)^+\right) \\ (\theta_1, \theta_2, y) \in \mathbb{R} \times \mathbb{R} \times Y_0 \end{cases}$$

[7] Henceforth we will merely say that Y is an open subset of S.

[8] Notice that the first equality in (35) is equivalent to $e'_\beta \circ g'(y_\beta, .) - \tau\left(\beta z_1^* + z_2^*\right) = 0$ if $S = L^p$ (*i.e.*, if the market is complete).

[9] It is worth remarking that the methodology of Theorem 7 also applies to extensions of omega given by

$$\frac{\sum_{i=1}^n \mathbb{E}\left(\left(y - k_i\right)^+\right)}{\sum_{j=1}^m \mathbb{E}\left(\left(k'_j - y\right)^+\right)}.$$

These extensions may be interesting because they still focus on both losses and gains, but they do not depend on a single threshold k.

$(\theta_1, \theta_2, y) \in \mathbb{R} \times \mathbb{R} \times L^p$ being the decision variable, and this problem is equivalent to (33) due to (2).

b) According to Lemma 1, y^* solves (23) if and only if y^* solves (33) for $\beta = \Omega_k(y^*)$, so the result trivially follows from *a*).

c) Suppose that there exists $(\theta_1^*, \theta_2^*, z_2^*) \in \mathbb{R} \times \mathbb{R} \times L^\infty$ such that $(\theta_1^*, \theta_2^*, y_\beta, z_2^*)$ solves (33). Problem (33) also incorporates the constraints $z \leq 0$ and $-z \leq -1$ (which are equivalent to $z \in B_{(\infty,+)}$, see (1)). Thus, the constraints of (33) are valued on $\mathcal{C}\left(B_{(\infty,+)}\right)$, $\mathbb{R}$, E, L^∞ and L^∞, respectively. Since all of these spaces have a non-negative cone with non-void interior, the Fritz John Theorem (Craven, 1975) implies the existence of $\tau \geq 0$ and

$$\left(\mu \geq 0, \lambda \geq 0, e'_\beta \geq 0, \alpha_0 \geq 0, \alpha_1 \geq 0\right) \in \mathcal{C}'\left(B_{(\infty,+)}\right) \times \mathbb{R} \times E' \times \left(L^\infty\right)' \times \left(L^\infty\right)'$$

such that $\left(\tau, \mu, \lambda, e'_\beta, \alpha_0, \alpha_1\right) \neq (0,0,0,0,0,0)$, the Lagrangian

$$\left\{\begin{array}{l} Lag\left(\tau, \theta_1, \theta_2, y, z_2, \mu, \lambda, e'_\beta, \alpha_0, \alpha_1\right) = \\ \tau\left(\beta\theta_1 + \theta_2\right) - \int_{B_{(\infty,+)}} \left(\theta_1 - \mathbb{E}\left(z_1\left(k - y\right)\right)\right)\mu\left(dz_1\right) \\ -\lambda\left(\theta_2 + \mathbb{E}\left(z_2\left(y-k\right)\right)\right) - \int_\Xi z_2(\xi)\,\alpha_0(d\xi) + \int_\Xi \left(z_2(\xi) - 1\right)\alpha_1(d\xi) \\ +e'_\beta\left(g(y) - g_0\right) \end{array}\right. \tag{36}$$

has null partial Fréchet differentials with respect to $(\theta_1, \theta_2, y, z_2)$, and the complementary slackness conditions hold. Computing derivatives in (36) with respect to (θ_1, θ_2), we get the system

$$\left\{\begin{array}{l} \tau\beta = \mu\left(B_{(\infty,+)}\right) \\ \tau = \lambda \end{array}\right. \tag{37}$$

Computing derivatives with respect to z_2, and bearing in mind that

$$\mathbb{E}\left(z_2\left(y-k\right)\right) - \int_\Xi z_2(\xi)\,\alpha_0(d\xi) + \int_\Xi z_2(\xi)\,\alpha_1(d\xi)$$

is a linear expression in the z_2-variable, we get

$$\alpha_1 - \alpha_0 = \tau\left(y_\beta - k\right) \tag{38}$$

because $\lambda = \tau$ must hold. Lastly, computing derivatives with respect to y, bearing in mind that $\tau\mathbb{E}(z_2 y) + \int_{B_{(\infty,+)}} \mathbb{E}(z_1 y)\,\mu(dz_1)$ is linear in the $y-$variable, bearing in mind (37), and applying (3), we get the existence of $z_1^*, z_2^* \in B_{(\infty,+)}$ such that

$$e'_\beta \circ g'\left(y_\beta, y\right) = \tau\mathbb{E}\left(\left(\beta z_1^* + z_2^*\right)y\right), \quad \forall y \in S.$$

The first condition in (35) was just proved. The second, third and fourth ones trivially follow from the complementary slackness conditions $\alpha_1\left(z_2^* - 1\right) =$

$\alpha_0 z_2^* = 0$ along with $\alpha_0 \geq 0$, $\alpha_1 \geq 0$ and (38). The fifth condition in (35) follows from the complementary slackness condition

$$\int_{B_{(\infty,+)}} \left(\theta_1 - \mathbb{E}\left(z_1\left(k - y_\beta\right)\right)\right) \mu\left(dz_1\right) = 0$$

along with the first constraint of (33), (37), and $\mu = \mu\left(B_{(\infty,+)}\right) \delta_{z_1^*}$ (recall (3)). The rest of conditions in (35) are clear consequences of the general Fritz John Theorem (Craven, 1975). □

Remark 8 *Under the assumptions and notation of Theorem 7c, suppose that $\tau\beta \neq 0$ in (35). The fifth condition becomes $\mathbb{E}\left(z_1^*\left(y_\beta - k\right)\right) \leq \mathbb{E}\left(z_1\left(y_\beta - k\right)\right)$ for every $z_1 \in B_{(\infty,+)}$, which trivially leads to*

$$\begin{cases} y_\beta(\xi) - k < 0 \Longrightarrow z_1^*(\xi) = 1, & \xi \in \Xi \\ y_\beta(\xi) - k > 0 \Longrightarrow z_1^*(\xi) = 0, & \xi \in \Xi \end{cases}$$

Consequently, bearing in mind the second and third conditions in (35), we have

$$\begin{cases} y_\beta(\xi) - k < 0 \Longrightarrow z_1^*(\xi) = 1 \text{ and } z_2^*(\xi) = 0, & \xi \in \Xi \\ y_\beta(\xi) - k > 0 \Longrightarrow z_1^*(\xi) = 0 \text{ and } z_2^*(\xi) = 1, & \xi \in \Xi \end{cases}$$

□

Corollary 9 *(Necessary Karush Kuhn Tucker-like optimality conditions) Consider the notations of Theorem 7c. If there exists $y_1 \in S$ such that $g'(y^*, y_1) \in E_+^\circ$, or $g'(y^*, .) : S \to E$ is onto, then one can take $\tau = 1$ in (35).*

Proof. It is sufficient to see that $\tau > 0$, since in such a case the result becomes trivial if one replaces τ with 1 and e'_β with e'_β/τ. Suppose that $\tau = 0$. The first condition in (35) implies that $e'_\beta \circ g'(y^*, .) \in S^\perp$. If $g'(y^*, .)$ is onto then $e'_\beta = 0$, contradicting Theorem 7. If the existence of $y_1 \in S$ holds and $e'_\beta \neq 0$ then $e'_\beta \circ g'(y^*, y_1) > 0$ (Luenberger, 1969), so once again we are facing a contradiction. □

3.3 Optimizing omega: Linear approach

Fix a Banach space E ordered by the convex cone E_+ with non-void interior E_+°, a linear and continuous function $g : S \to E$, and an element $g_0 \in E$. Suppose that Y_0 is given by (34) (with $Y = S$). If $\beta > 1$ then (32) becomes a convex problem, and the Lagrangian duality of Luenberger (1969) applies. Moreover, bearing in mind (2), and proceeding as in the proof of Theorem 7, Problems (31) and (32) are equivalent to the linear problem

$$Min \ (\beta - 1)\theta - \beta\mathbb{E}(y) + \beta k \begin{cases} \theta - \mathbb{E}\left(z\left(y - k\right)\right) \geq 0, & \forall z \in B_{(\infty,+)} \\ g(y) \leq g_0 \\ (\theta, y) \in \mathbb{R} \times S \end{cases} \tag{39}$$

$(\theta, y) \in \mathbb{R} \times S$ being the decision variable. The Lagrangian function becomes (Luenberger, 1969)

$$\begin{cases} Lag\,(\theta, y, \mu, e') = (\beta - 1)\theta - \beta \mathbb{E}(y) + \beta k \\ - \int_{B_{(\infty,+)}} (\theta - \mathbb{E}\,(z\,(y-k)))\,\mu\,(dz) + e'\,(g\,(y) - g_0) \end{cases}$$

for $(\theta, y, \mu, e') \in \mathbb{R} \times S \times \mathcal{C}'\left(\mathsf{B}_{(\infty,+)}\right) \times E'$, $\mu \geq 0$ and $e' \geq 0$. Manipulating,

$$\begin{cases} Lag\,(\theta, y, \mu, e') = (\beta - 1 - \mu\left(B_{(\infty,+)}\right))\theta - \beta \mathbb{E}(y) + \beta k \\ + \int_{B_{(\infty,+)}} \mathbb{E}\,(zy)\,\mu\,(dz) - k \int_{B_{(\infty,+)}} \mathbb{E}\,(z)\,\mu\,(dz) + e'\,(g\,(y) - g_0) \end{cases}$$

Bearing in mind (3), and proceeding as in the proof of Theorem 7, the Lagrangian above may simplify to

$$\begin{cases} Lag\,(\theta, y, z, e') = (\beta - 1 - \mu\left(B_{(\infty,+)}\right))\theta - \beta \mathbb{E}(y) + \beta k \\ +\mu\left(B_{(\infty,+)}\right) \mathbb{E}\,(zy) - k\mu\left(B_{(\infty,+)}\right) \mathbb{E}(z) + e'\,(g\,(y) - g_0)\,. \end{cases}$$

According to Luenberger (1969), $(z, e') \in \mathsf{B}_{(\infty,+)} \times E'$ will be dual-feasible if $e' \geq 0$ and $Lag\,(\theta, y, z, e')$ is bounded from below for $(\theta, y) \in \mathbb{R} \times Y$. Hence, $\mu\left(B_{(\infty,+)}\right) = \beta - 1$ must hold, and the Lagrangian becomes

$$\begin{cases} Lag\,(y, z, e') = \\ \mathbb{E}(y\,((\beta - 1)\,z - \beta)) + e' \circ g\,(y) + \beta k - k\,(\beta - 1)\,\mathbb{E}(z) - e'\,(g_0)\,. \end{cases} \tag{40}$$

According to Luenberger (1969), $(z, e') \in \mathsf{B}_{(\infty,+)} \times E'_+$ will be dual feasible if and only if the infimum of (40) in $y \in S$ is strictly higher than $-\infty$, which is obviously equivalent to the equality $(\beta - 1)\,z - \beta + e' \circ g \in S^{\perp}$. Moreover, the dual of (31), (32) and (39) becomes

$$Max\ \beta k - k\,(\beta - 1)\,\mathbb{E}(z) - e'\,(g_0) \begin{cases} (\beta - 1)\,z - \beta + e' \circ g \in S^{\perp} \\ (z, e') \in \mathsf{B}_{(\infty,+)} \times E'_+ \end{cases} \tag{41}$$

Theorem 10 *Suppose that $\beta > 1$. Fix a Banach space E ordered by the convex cone E_+ with non-void interior E_+°, a linear and continuous function $g : S \rightarrow E$, and an element $g_0 \in E$. Suppose Y_0 is given by (34) (with $Y = S$). Suppose lastly that there exists $\tilde{y} \in Y_0$ such that $g\,(\tilde{y}) - g_0 \in E_+^{\circ}$ (Slater qualification). Then;*

a) Problem (31) (or (32)) is bounded if and only if Problem (41) is feasible. If so, the infimum of (31) (or (32)) equals the maximum of (41) (i.e., (41) achieves its supremum and therefore it is solvable).

b) Suppose that (31) is bounded. Consider $y_\beta \in Y_0$ and $\left(z_\beta, e'_\beta\right) \in \mathsf{B}_{(\infty,+)} \times E'_+$. Then, $y_\beta \in Y_0$ solves (31) (or (32)) and $\left(z_\beta, e'_\beta\right)$ solves (41) if and only if the complementary slackness conditions

$$\begin{cases} \mathbb{E}\,(z_\beta\,(y_\beta - k)) \geq \mathbb{E}\,(z\,(y_\beta - k))\,, \quad \forall z \in \mathsf{B}_{(\infty,+)} \\ (\beta - 1)\,z_\beta - \beta + e'_\beta \circ g \in S^{\perp} \\ e'_\beta\,(g\,(y_\beta) - g_0) = 0 \end{cases} \tag{42}$$

hold.

c) Consider Problem (23) and $y^* \in Y_0$. y^* solves (23) if and only if y^* solves (31) for $\beta = \Omega_k(y^*)$.

d) Consider Problem (23) and $y^* \in Y_0$ such that $\Omega_k(y^*) > 1$. y^* solves (23) if and only there exists $(z^*, e'^*) \in \mathsf{B}_{(\infty,+)} \times E'_+$ such that (42) holds for $\left(z_\beta, e'_\beta\right) = (z^*, e'^*)$, $y_\beta = y^*$ and $\beta = \Omega_k(y^*)$.

Proof. *a*) It trivially follows from the duality theory for convex problems of Luenberger (1969).

b) Take $\theta_\beta = \mathbb{E}\left((y_\beta - k)^+\right)$. Obviously, $y_\beta \in Y_0$ solves (31) if and only if (θ_β, y_β) solves (39). According to Anderson and Nash (1987), and bearing that (39) and (41) are linear, (θ_β, y_β) solves (39) and $\left(z_\beta, e'_\beta\right)$ solves (41) if and only if they are feasible and the complementary slackness conditions

$$\begin{cases} \theta_\beta = \mathbb{E}\left(z_\beta\left(y_\beta - k\right)\right) \\ (\beta - 1) z_\beta - \beta + e'_\beta \circ g = 0 \\ e'_\beta\left(g\left(y_\beta\right) - g_0\right) = 0 \end{cases}$$

hold. Obviously, (2) implies the equivalence between $\theta_\beta = \mathbb{E}\left(z_\beta\left(y_\beta - k\right)\right)$, and the first condition in (42).

c) It is an obvious consequence of Lemma 1.

d) It is an obvious consequence of *c*) and *b*). □

Remark 11 *According to the duality theory presented in Luenberger (1969), there is a natural extension of Theorem* 10 *also applying if g is a convex function. We have decided to omit it in order to shorten the exposition.* □

Remark 12 *Bearing in mind (1), the first condition in (42) is obviously equivalent to*

$$\begin{cases} y_\beta(\xi) - k < 0 \Longrightarrow z_\beta(\xi) = 0, & \xi \in \Xi \\ y_\beta(\xi) - k > 0 \Longrightarrow z_\beta(\xi) = 1, & \xi \in \Xi \end{cases} \tag{43}$$

□

Theorems 7 and Theorem 10 provide us with optimality conditions for Problem (23). Theorem 7 applies in a much more general setting, since g does not have to be linear, the Slater qualification is not imposed, and constraint $\Omega_k(y^*) > 1$ does not have to hold. Nevertheless, when Theorem 10 applies (42) is necessary and sufficient, and much more tractable than (35), which is only necessary.[10] It is also natural to analyze the relationship between (35) and (42) when Theorem 10 applies. Let us show that they become identical.

[10] Constraint $\Omega_k(y^*) > 1$ often holds in practice (Sharma and Mehra, 2017, or Sharma *et al.*, 2017), but its fulfilment is not always guaranteed (see Theorem 29 below).

Proposition 13 *Under the notations and assumptions of Theorem* 10*, if*

$$\left(y_\beta, z_\beta, e'_\beta\right) \in Y_0 \times B_{(\infty,+)} \times E'_+$$

satisfies (42), $\tau = 1$, $z_1^* = 1 - z_\beta$, and $z_2^* = z_\beta$ then $\left(y_\beta, \tau, z_1^*, z_2^*, e'_\beta\right)$ satisfies (35).

Proof. It is obviously sufficient to see that $\beta z_1^* + z_2^* = \beta - (\beta - 1) z_\beta$, which trivially follows from $z_1^* = 1 - z_\beta$, and $z_2^* = z_\beta$. □

Remark 14 *Consider the same assumptions as in Theorem* 10*, but suppose that $\beta = 1$. (31) is still equivalent to (32), which becomes*

$$Min\ k - \mathbb{E}(y) \left\{ \begin{array}{l} g(y) \leq g_0 \\ y \in S \end{array} \right. \tag{44}$$

and is a linear problem. According to Anderson and Nash (1987), the dual of (44) is

$$Max\ k - e'(g_0) \left\{ \begin{array}{l} e' \circ g - \mathbb{E}(.) \in S^\perp \\ e' \in E'_+ \end{array} \right. \tag{45}$$

Let us assume the existence of $\tilde{y} \in Y_0$ such that $g(\tilde{y}) - g_0 \in E_+^\circ$ (Slater qualification). Problem (45) is feasible and solvable if and only if (44) is feasible and bounded, in which case there is no duality gap between both problems. The complementary slackness conditions become

$$\left\{ \begin{array}{l} e'_1 \circ g - \mathbb{E}(.) \in S^\perp \\ g(y_1) \leq g_0 \\ e'_1 \left(g(y_1) - g_0\right) = 0 \end{array} \right.$$

and they are necessary and sufficient to guarantee that $y_1 \in S$ solves (44) and $e'_1 \in E'_+$ solves (45) (Anderson and Nash, 1987). □

4 Pricing models and optimal omega

Throughout Section 4 we will fix k, C_0, $\tilde{C}_0 = C_0 e^{rT}$, E_0, $\rho_0 \in \mathbb{R}$, $p \in [1, \infty)$, $q \in (1, \infty]$ such that $1/p + 1/q = 1$, a closed subspace $S \subset L^p$ of marketed claims containing the riskless asset, a non-void $Y_0 \subset S$, a pricing rule $\Pi \in S'$, a SDF $z_\Pi \in L^q$ satisfying (7) and (10),[11] and such that (9) does not hold.[12] Constraint $y \in Y_0$ in Problem (23) admits many particular cases. Special examples are

$$Y_{0,1} := \left\{ y \in S;\ \mathbb{E}\left(z_\Pi y\right) \leq \tilde{C}_0 \right\}, \tag{46}$$

[11] Actually, (10) is equivalent to the absence of arbitrage in a complete market ($S = L^p$). In an incomplete market (10) implies the absence of arbitrage, and the converse also holds under quite general assumptions. For instance, it holds in the Heston model (Hull, 2012) and other SVM. See Balbás *et al.* (2016) for a further discussion.

[12] (9) holds if and only if the market is risk-neutral, but the standard pricing models of Mathematical Finance deal with a risk-averse framework.

$$Y_{0,2} := \left\{ y \in S;\ \mathbb{E}\left(z_\Pi y\right) \leq \tilde{C}_0,\ \mathbb{E}\left(y\right) \geq E_0 C_0 \right\}, \tag{47}$$

$$Y_{0,3} := \left\{ y \in S;\ \mathbb{E}\left(z_\Pi y\right) \leq \tilde{C}_0,\ \rho\left(y\right) \leq \rho_0 C_0 \right\}, \tag{48}$$

$$Y_{0,4} := \left\{ y \in S;\ \mathbb{E}\left(z_\Pi y\right) \leq \tilde{C}_0,\ \mathbb{E}\left(y\right) \geq E_0 C_0,\ \rho\left(y\right) \leq \rho_0 C_0 \right\}, \tag{49}$$

ρ being the value at risk (see (16)) or an arbitrary real-valued risk measure on L^p satisfying (11), (12) and (13). (46) indicates that one will not invest more than C_0. If $C_0 > 0$ then (47) also imposes a minimum expected return E_0, (48) imposes a maximum risk ρ_0 per invested dollar, and (49) imposes both a minimum expected return and a maximum relative risk.[13] The four cases can be analyzed with the Fritz John-like optimality conditions (35) and the complementary slackness conditions (42).

Proposition 15 *Consider Problem (23) and suppose that $\mathbb{E}\left(z_\Pi y\right) \leq \tilde{C}_0$ holds for every $y \in Y_0 \subset S$. If $k > \tilde{C}_0$ then (4) holds for every $y \in Y_0$.*

Proof. If $\mathbb{E}\left(z_\Pi y\right) \leq \tilde{C}_0 < k$ and $y \geq k$ then $\mathbb{E}\left(z_\Pi (y-k)\right) < k - k\mathbb{E}\left(z_\Pi\right) = 0$ (see (8)) and $y - k \geq 0$, which implies that $y - k$ is an arbitrage strategy (Duffie, 1996). □

Remark 16 *Throughout Sections* 4 *and* 5 *we will also consider that $k > \tilde{C}_0$ holds. Hence, $Y = S$ is an open set (relative to S), $Y_0 \subset Y$ under the four cases for Y_0 ($Y_{0,1}$, $Y_{0,2}$, $Y_{0,3}$ and $Y_{0,4}$), and Assumption* 1 *holds. Consequently, both Theorem* 7 *and Theorem* 10 *apply.* □

Remark 17 *Let us focus on Theorem* 7c *under the four cases for Y_0. The only difference among these cases is given by the function g, and therefore it only affects the first and sixth conditions in (35). If $Y_0 = Y_{0,1}$ then g is given by*

$$S \ni y \to g\left(y\right) = \mathbb{E}\left(z_\Pi y\right) \in \mathbb{R}, \tag{50}$$

which is linear and coincides with its Fréchet differential. Thus, the two mentioned conditions if (35) become

$$\left\{ \begin{array}{l} e'_\beta\ z_\Pi - \tau\left(\beta z_1^* + z_2^*\right) \in S^\perp \\ e'_\beta \left(\tilde{C}_0 - \mathbb{E}\left(z_\Pi y_\beta\right)\right) = 0 \end{array} \right. \tag{51}$$

where $e'_\beta \in \mathbb{R}$, $e'_\beta \geq 0$. Similarly, for $Y_0 = Y_{0,2}$ one has

$$S \ni y \to g\left(y\right) = \left(\mathbb{E}\left(z_\Pi y\right), -\mathbb{E}\left(y\right)\right) \in \mathbb{R}^2, \tag{52}$$

[13] Notice that the risk measure ρ allows us to introduce ambiguity in the analysis, in which case the decision maker will reflect uncertainty with respect to the real probability of every state of the world. Indeed, it is for instance sufficient to deal with the robust $CVaR$ (see (20) and (21)) or other robust risk measures (Balbás *et al.*, 2016).

and

$$\left\{\begin{array}{l} e'_{\beta,1}\ z_\Pi - e'_{\beta,2} - \tau\left(\beta z_1^* + z_2^*\right) \in S^\perp \\ e'_{\beta,1}\left(\tilde{C}_0 - \mathbb{E}\left(z_\Pi y_\beta\right)\right) = 0 \\ e'_{\beta,2}\left(\mathbb{E}\left(y_\beta\right) - C_0 E_0\right) = 0 \end{array}\right. , \tag{53}$$

where $e'_{\beta,1}, e'_{\beta,2} \in \mathbb{R}$, $e'_{\beta,1}, e'_{\beta,2} \geq 0$.

For $Y_0 = Y_{0,3}$ *things become more complex. Indeed,* $L^p \ni y \to \rho(y) \in \mathbb{R}$ *is not necessarily Fréchet differentiable, but one can overcome this caveat by drawing on the Representation Theorems. Indeed, if* ρ *is expectation bounded or a deviation measure then (11) implies that*

$$Y_{0,3} = \left\{y \in S;\ \mathbb{E}\left(z_\Pi y\right) \leq \tilde{C}_0 \ and\ -\mathbb{E}\left(yz\right) \leq C_0\rho_0\ \forall z \in \Delta_\rho\right\}.$$

Thus, one can consider

$$S \ni y \to g\left(y\right) = \left(\mathbb{E}\left(z_\Pi y\right), -\mathbb{E}\left(yz\right)\right) \in \mathbb{R}\times\mathcal{C}\left(\Delta_\rho\right), \tag{54}$$

which is linear again and coincides with its differential. Since the natural cone of $\mathcal{C}\left(\Delta_\rho\right)$ *has non-void interior, bearing in mind (3), the mentioned conditions in (35) imply*

$$\left\{\begin{array}{l} e'_\beta z_\Pi - \nu_\rho z_\rho - \tau\left(\beta z_1^* + z_2^*\right) \in S^\perp \\ e'_\beta\left(\tilde{C}_0 - \mathbb{E}\left(z_\Pi y_\beta\right)\right) = 0 \\ \nu_\rho\left(C_0\rho_0 + \mathbb{E}\left(y_\beta z_\rho\right)\right) = 0 \end{array}\right. \tag{55}$$

with $e'_\beta \geq 0$, $\nu_\rho \geq 0$, $z_\rho \in \Delta_\rho$ *and* $\left(\tau, e'_\beta, \nu_\rho\right) \neq (0,0,0)$. *Similarly, for* $Y_0 = Y_{0,4}$ *one gets*

$$\left\{\begin{array}{l} e'_{\beta,1}\ z_\Pi - e'_{\beta,2} - \nu_\rho z_\rho - \tau\left(\beta z_1^* + z_2^*\right) \in S^\perp \\ e'_{\beta,1}\left(\tilde{C}_0 - \mathbb{E}\left(z_\Pi y_\beta\right)\right) = 0 \\ e'_{\beta,2}\left(\mathbb{E}\left(y_\beta\right) - C_0 E_0\right) = 0 \\ \nu_\rho\left(C_0\rho_0 + \mathbb{E}\left(y_\beta z_\rho\right)\right) = 0 \end{array}\right. \tag{56}$$

with $e'_{\beta,1}, e'_{\beta,2} \geq 0$, $\nu_\rho \geq 0$, $z_\rho \in \Delta_\rho$ *and* $\left(\tau, e'_{\beta,1}, e'_{\beta,2}, \nu_\rho\right) \neq (0,0,0,0)$. □

Remark 18 *Let us focus on Theorem* 10b *under the four cases for* Y_0. *As above, the only difference affects the second and third conditions in (42). If* $Y_0 = Y_{0,1}$ *then (50) implies that* $e'_\beta \in \mathbb{R}$, $e'_\beta \geq 0$ *and*

$$\left\{\begin{array}{l} \left(\beta - 1\right) z_\beta - \beta + e'_\beta z_\Pi \in S^\perp \\ e'_\beta\left(\tilde{C}_0 - \mathbb{E}\left(z_\Pi y_\beta\right)\right) = 0 \end{array}\right. . \tag{57}$$

For $Y_0 = Y_{0,2}$ *Expressions (42) and (52) lead to*

$$\left\{\begin{array}{l} \left(\beta - 1\right) z_\beta - \beta + e'_{\beta,1} z_\Pi - e'_{\beta,2} \in S^\perp \\ e'_{\beta,1}\left(\tilde{C}_0 - \mathbb{E}\left(z_\Pi y_\beta\right)\right) = 0 \\ e'_{\beta,2}\left(\mathbb{E}\left(y_\beta\right) - C_0 E_0\right) = 0 \end{array}\right. \tag{58}$$

with $e'_{\beta,1}, e'_{\beta,2} \in \mathbb{R}, e'_{\beta,1}, e'_{\beta,2} \geq 0$. If $Y_0 = Y_{0,3}$ and ρ is expectation bounded or a deviation measure then (11), (42) and (54) lead to

$$\left\{ \begin{array}{l} (\beta - 1)\, z_\beta - \beta + e'_\beta z_\Pi - \nu_\rho z_\rho \in S^\perp \\ e'_\beta \left(\tilde{C}_0 - \mathbb{E}\left(z_\Pi y_\beta\right) \right) = 0 \\ \nu_\rho \left(C_0 \rho_0 + \mathbb{E}\left(y_\beta z_\rho\right)\right) = 0 \end{array} \right. \tag{59}$$

Similarly, for $Y_0 = Y_{0,4}$ one gets

$$\left\{ \begin{array}{l} (\beta - 1)\, z_\beta - \beta + e'_{\beta,1}\; z_\Pi - e'_{\beta,2} - \nu_\rho z_\rho \in S^\perp \\ e'_{\beta,1} \left(\tilde{C}_0 - \mathbb{E}\left(z_\Pi y_\beta\right) \right) = 0 \\ e'_{\beta,2} \left(\mathbb{E}\left(y_\beta\right) - C_0 E_0\right) = 0 \\ \nu_\rho \left(C_0 \rho_0 + \mathbb{E}\left(y_\beta z_\rho\right)\right) = 0 \end{array} \right. \tag{60}$$

with $e'_{\beta,1}, e'_{\beta,2} \geq 0$, $\nu_\rho \geq 0$, $z_\rho \in \Delta_\rho$. □

Remark 19 *Obviously, $\|z_\Pi\|_\infty = Ess\_Sup\{z_\Pi\} \leq \infty$, and (10) clearly implies that $Ess\_Inf\{z_\Pi\} \geq 0$. Henceforth we will denote $\|z_\Pi\|_0 = Ess\_Inf\{z_\Pi\}$. Moreover, (8) trivially shows that $\|z_\Pi\|_\infty \geq 1$ and $\|z_\Pi\|_0 \leq 1$, and it also shows that $\|z_\Pi\|_0 = \|z_\Pi\|_\infty$ if and only if $z_\Pi = 1$ (i.e., $\|z_\Pi\|_0 = \|z_\Pi\|_\infty$ if and only if the market is risk-neutral, which is a excluded case). To sum up*

$$0 \leq \|z_\Pi\|_0 < 1 < \|z_\Pi\|_\infty \leq \infty. \tag{61}$$

If $\|z_\Pi\|_\infty = \infty$ or $\|z_\Pi\|_0 = 0$ (or both) we will accept the usual convention $\|z_\Pi\|_\infty / \|z_\Pi\|_0 = \infty$. More generally, we will take $\infty/0 = \infty$ and $\infty/a = a/0 = \infty$ for $a > 0$. □

4.1 Price and return constraints

Let us focus on the particular cases $Y_0 = Y_{0,1}$ and $Y_0 = Y_{0,2}$. If the market is complete ($S = L^p$), we will show that the solution of (23) only depends on the properties of z_Π. Needless to say, the most important pricing models of Financial Mathematics (binomial model, trinomial models, $B\&S$, Heston, other SVM, etc.) are complete.[14]

Lemma 20 *Suppose that $S = L^p$ and $Y_0 = Y_{0,1}$. Consider $\beta > 1$. The dual problem (41) is feasible if and only if $0 < \|z_\Pi\|_0 < \|z_\Pi\|_\infty < \infty$ and $\beta \geq \|z_\Pi\|_\infty / \|z_\Pi\|_0$. If so, the dual problem (41) becomes*

$$Max \;\; \left(k - \tilde{C}_0\right) e'_\beta \left\{ \begin{array}{l} z = (\beta - e'_\beta z_\Pi)/\,(\beta - 1) \\ 1/\,\|z_\Pi\|_0 \leq e'_\beta \leq \beta/\,\|z_\Pi\|_\infty \end{array} \right. \tag{62}$$

[14] Formally, SVM may be incomplete, but in practice it is assumed the existence of volatility dependent assets making them complete. Otherwise it would be impossible to use these models to give a unique price of the usual derivatives.

its solution is

$$\left(z_\beta, e'_\beta\right) = \left(\frac{\beta\left(\|z_\Pi\|_\infty - z_\Pi\right)}{(\beta-1)\|z_\Pi\|_\infty}, \frac{\beta}{\|z_\Pi\|_\infty}\right) \tag{63}$$

and its optimal value equals $\left(\left(k-\tilde{C}_0\right)\beta\right)/\|z_\Pi\|_\infty$.

Proof. $Y_0 = Y_{0,1}$ and (50) imply that (41) becomes

$$Max\ \beta k - (\beta-1)\, k\mathbb{E}(z) - \tilde{C}_0 e'_\beta \begin{cases} (\beta-1)\, z - \beta + e'_\beta z_\Pi = 0 \\ (z, e') \in \mathsf{B}_{(\infty,+)} \times \mathbb{R},\ e' \geq 0 \end{cases}$$

Its constraint leads to

$$z = \frac{\beta - e'_\beta z_\Pi}{\beta - 1}, \tag{64}$$

and therefore the dual objective equals (see (8))

$$\beta k - k\left(\beta - e'_\beta\right) - \tilde{C}_0 e'_\beta = \left(k - \tilde{C}_0\right) e'_\beta. \tag{65}$$

(64) and $z \in \mathsf{B}_{(\infty,+)}$ lead to $0 \leq \beta - e'_\beta z_\Pi \leq \beta - 1$, *i.e.*, $1 \leq e'_\beta z_\Pi \leq \beta$. (10) leads to $1/z_\Pi \leq e'_\beta \leq \beta/z_\Pi$, which is equivalent to

$$\frac{1}{\|z_\Pi\|_0} \leq e'_\beta \leq \frac{\beta}{\|z_\Pi\|_\infty}. \tag{66}$$

(61), (64), (65) and (66) imply that (41) becomes feasible if and only if $0 < \|z_\Pi\|_0 < \|z_\Pi\|_\infty < \infty$ and $\beta \geq \|z_\Pi\|_\infty / \|z_\Pi\|_0$, in which case its solution is (63) and its optimal value becomes $\left(\left(k-\tilde{C}_0\right)\beta\right)/\|z_\Pi\|_\infty$. □

Lemma 21 *Suppose that* $S = L^p$. $\Omega_k(y) \leq \|z_\Pi\|_\infty / \|z_\Pi\|_0$ *for every* $y \in Y_{0,1}$.

Proof. Let us assume that $\|z_\Pi\|_\infty / \|z_\Pi\|_0 < \infty$ (otherwise the result is obvious). Theorem 10*a* and Lemma 20 show that (41) is feasible and (31) is bounded if $\beta = \|z_\Pi\|_\infty / \|z_\Pi\|_0 > 1$, and

$$\left(\|z_\Pi\|_\infty / \|z_\Pi\|_0\right) \mathbb{E}\left((k-y)^+\right) - \mathbb{E}\left((y-k)^+\right) \geq \frac{\left(k-\tilde{C}_0\right)\left(\|z_\Pi\|_\infty / \|z_\Pi\|_0\right)}{\|z_\Pi\|_\infty} > 0,$$

for every $y \in Y_{0,1}$. Therefore $\|z_\Pi\|_\infty / \|z_\Pi\|_0 \geq \Omega_k(y)$ for every $y \in Y_{0,1}$. □

Lemma 22 *Suppose that* $S = L^p$, $Y_0 = Y_{0,2}$, *and* $\beta > 1$. *Then, the dual problem (41) becomes*

$$Max\ \left(k-\tilde{C}_0\right) e'_{\beta,1} + \left(\tilde{C}_0 E_0 - k\right) e'_{\beta,2} \begin{cases} z = (\beta - e'_{\beta,1} z_\Pi + e'_{\beta,2}) / (\beta-1) \\ 1 \leq \|z_\Pi\|_0\, e'_{\beta,1} - e'_{\beta,2} \\ \|z_\Pi\|_\infty\, e'_{\beta,1} - e'_{\beta,2} \leq \beta \\ e'_{\beta,1}, e'_{\beta,2} \geq 0 \end{cases} \tag{67}$$

This problem is feasible if and only if $\beta \geq \|z_\Pi\|_\infty / \|z_\Pi\|_0$, *in which case its optimal value is strictly positive.*

Proof. $Y_0 = Y_{0,2}$ and (52) imply that (41) becomes

$$Max\ \beta k - (\beta-1)\, k\mathbb{E}(z) - \tilde{C}_0 e'_{\beta,1} + \tilde{C}_0 E_0 e'_{\beta,2} \left\{ \begin{array}{l} (\beta-1)\, z - \beta + e'_{\beta,1} z_\Pi - e'_{\beta,2} = 0 \\ (z, e') \in \mathsf{B}_{(\infty,+)} \times \mathbb{R},\ e' \geq 0 \end{array} \right.$$

The constraints lead to

$$0 \leq z = \frac{\beta - e'_{\beta,1} z_\Pi + e'_{\beta,2}}{\beta - 1} \leq 1, \tag{68}$$

i.e., $0 \leq \beta - e'_{\beta,1} z_\Pi + e'_{\beta,2} \leq \beta - 1$, or $1 \leq \|z_\Pi\|_0\, e'_{\beta,1} - e'_{\beta,2} \leq \beta$, which is obviously equivalent to

$$\left\{ \begin{array}{l} 1 \leq \|z_\Pi\|_0\, e'_{\beta,1} - e'_{\beta,2} \\ \|z_\Pi\|_\infty\, e'_{\beta,1} - e'_{\beta,2} \leq \beta \end{array} \right. \tag{69}$$

Moreover, the objective function becomes (see (8) and (68))

$$\beta k - k(\beta-1)\, \mathbb{E}(z) - \tilde{C}_0 e'_{\beta,1} + e'_{\beta,2} = k e'_{\beta,1} - k e'_{\beta,2} - \tilde{C}_0 e'_{\beta,1} + \tilde{C}_0 E_0 e'_{\beta,2}.$$

Replace "$\leq$" with "$=$" in (69) to get a system of equations. It is easy to see that the intersection of (68) and $e'_{\beta,1}, e'_{\beta,2} \geq 0$ is non void if and only if the system solution

$$\left(e'_{\beta,1}, e'_{\beta,2}\right) = \left(\frac{\beta - 1}{\|z_\Pi\|_\infty - \|z_\Pi\|_0}, \frac{\beta \tilde{z}_\Pi - \|z_\Pi\|_\infty}{\|z_\Pi\|_\infty - \|z_\Pi\|_0} \right),$$

satisfies $e'_{\beta,2} \geq 0$, *i.e.*, $\beta \geq \|z_\Pi\|_\infty / \|z_\Pi\|_0$. If so, $\left(e'_{\beta,1}, e'_{\beta,2}\right) = \left(\frac{1}{\|z_\Pi\|_0}, 0 \right)$ is also feasible, and the problem optimal value will have to equal, at least, $\left(k - \tilde{C}_0\right) e'_{\beta,1} = \left(k - \tilde{C}_0\right) / \|z_\Pi\|_0 > 0$. □

Theorem 23 *Suppose that $S = L^p$.*

$$Sup\left\{\Omega_k(y);\ y \in Y_{0,2}\right\} = Sup\left\{\Omega_k(y);\ y \in Y_{0,1}\right\} = \|z_\Pi\|_\infty / \|z_\Pi\|_0 \leq \infty.$$

Proof. According to Lemma 21 and the inclusion $Y_{0,2} \subset Y_{0,1}$, it is sufficient to see that $\|z_\Pi\|_\infty / \|z_\Pi\|_0 \leq Sup\left\{\Omega_k(y);\ y \in Y_{0,2}\right\}$. Let us consider $1 < \beta < \|z_\Pi\|_\infty / \|z_\Pi\|_0$, and let us show that $\beta < Sup\left\{\Omega_k(y);\ y \in Y_{0,2}\right\}$. Lemma 22 and $1 < \beta < \|z_\Pi\|_\infty / \|z_\Pi\|_0$ imply that (41) is unfeasible, and Theorem 10a shows that (31) is unbounded. Hence, Lemma 2 implies that β is not an upper bound of $Sup\left\{\Omega_k(y);\ y \in Y_{0,2}\right\}$. □

Remark 24 *The most important continuous time pricing models of Financial Mathematics (B&S, Heston, more general SVM, etc.) have a non-essentially bounded SDF. In fact, both $\|z_\Pi\|_\infty = \infty$ and $\|z_\Pi\|_0 = 0$ often hold. Consequently, Theorem* 23 *implies that (23) is unbounded, and the investor may reach an omega ratio as large as desired with only one dollar (or one cent), and an additional return constraint $\mathbb{E}(y) \geq C_0 E_0$ does no provoke any essential change. The omega ratio may become as close to ∞ as desired, regardless of the amount* *to invest and the expected return to reach. This "surprising" finding is related with some "pathologies" pointed out in Balbás et al. (2010), and later extended in Balbás et al. (2016). If ρ is a coherent and expectation bounded risk measure then all the pricing models above imply the existence of a sequence of investment strategies whose couple $(risk, return)$ tends to $(-\infty, +\infty)$ or $(0, +\infty)$, where the risk is measured by ρ.*

Let us highlight the important role plaid by the SDF. Actually, the optimal value of (23) only depends on it, and the threshold k is not at all relevant. Moreover, if $\Omega_k(y)$ is replaced by the ratio

$$\frac{\mathbb{E}\left((y-k_1)^+\right)}{\mathbb{E}\left((k-y)^+\right)}$$

with $k_1 \geq k$, then

$$\Omega_{k_1}(y) \leq \frac{\mathbb{E}\left((y-k_1)^+\right)}{\mathbb{E}\left((k-y)^+\right)} \leq \Omega_k(y),$$

and therefore, for $Y_0 = Y_{0,1}$ or $Y_0 = Y_{0,2}$

$$Sup\left\{\frac{\mathbb{E}\left((y-k_1)^+\right)}{\mathbb{E}\left((k-y)^+\right)};\ y \in Y_0\right\} = \frac{\|z_\Pi\|_\infty}{\|z_\Pi\|_0} \leq \infty.$$

Similarly,

$$Sup\left\{\frac{\mathbb{E}\left((y-k)^+\right)}{\mathbb{E}\left((k_2-y)^+\right)};\ y \in Y_0\right\} = \frac{\|z_\Pi\|_\infty}{\|z_\Pi\|_0} \leq \infty$$

if $\tilde{C}_0 < k_2 \leq k$.

As said above, the threshold k is not relevant to compute the optimal value of (23), but k affects the optimal solution y^*, if it exists. Actually, if (23) is solved by some $y^* \in Y_{0,1}$ (or $y^* \in Y_{0,2}$) then one can take $\beta = \|z_\Pi\|_\infty / \|z_\Pi\|_0$ and deal with conditions (42), (43) and (57) (or (58)) in order to find y^*. If this system is complex, one can previously solve the dual problem of Lemma 20 *(Lemma* 22*), since it is quite simple and only involves one (two) real variable(s).*

However, the absence of solution y^* of (23) may hold, as can be easily seen by dealing with the procedure above and the simple binomial model (we do not include this example in order to shorten the exposition). Moreover, the lack of a solution is obvious if $\|z_\Pi\|_\infty / \|z_\Pi\|_0 = \infty$.

If a solution y^* of (23) does not exist, it may be interesting to construct a sequence of strategies $\{y_n\}_{n=1}^\infty \subset Y_{0,1}$ such that $\lim_{n\to\infty} \Omega_k(y_n) = \|z_\Pi\|_\infty / \|z_\Pi\|_0$. Let us do that, and let us point out that straightforward modifications of this construction will imply the inclusion $\{y_n\}_{n=1}^\infty \subset Y_{0,2}$.

Consider a strictly decreasing sequence $\left\{\delta_n^{(1}\right\}_{n=1}^{\infty} \subset \mathbb{R}$ *and a strictly increasing one* $\left\{\delta_n^{(2}\right\}_{n=1}^{\infty} \subset \mathbb{R}$ *such that* $\lim_{n\to\infty} \delta_n^{(1} = \|z_\Pi\|_0$*,* $\delta_n^{(1} < \delta_n^{(2}$ *and* $\lim_{n\to\infty} \delta_n^{(2} = \|z_\Pi\|_\infty$*. Consider the disjoint intervals*

$$I_{n,1} = \left[\|z_\Pi\|_0, \delta_n^{(1}\right), I_{n,2} = \left[\delta_n^{(1}, \delta_n^{(2}\right), I_{n,3} = \left[\delta_n^{(2}, \|z_\Pi\|_\infty\right]$$

and $\Xi_{n,j} = z_\Pi^{-1}(I_{n,j})$*,* $j = 1, 2, 3$*. Consider*

$$h_{n,j} = \left(\int_{\Xi_{n,j}} z_\Pi(u)\,\mathbb{P}(du)\right) / \mathbb{P}(\Xi_{n,j})$$

(take $h_{n,2} = 0$ *if* $\mathbb{P}(\Xi_{n,2}) = 0$*, and notice that both* $\mathbb{P}(\Xi_{n,1}) > 0$ *and* $\mathbb{P}(\Xi_{n,3}) > 0$ *hold), and take,* $m, M \in \mathbb{R}$ *and*

$$y_n(\xi) = \begin{cases} M, & \xi \in \Xi_{n,1} \\ k, & \xi \in \Xi_{n,2} \\ -m, & \xi \in \Xi_{n,3} \end{cases}$$

Obviously,

$$\mathbb{E}(z_\Pi y_n) = M h_{n,1}\mathbb{P}(\Xi_{n,1}) + k h_{n2}\mathbb{P}(\Xi_{n,2}) - m h_{n,3}\mathbb{P}(\Xi_{n,3}),$$

so $\mathbb{E}(z_\Pi y_n) = \tilde{C}_0$ *if*

$$M = \frac{\tilde{C}_0 + m h_{n,3}\mathbb{P}(\Xi_{n,3}) - k h_{n2}\mathbb{P}(\Xi_{n,2})}{h_{n,1}\mathbb{P}(\Xi_{n,1})}.$$

Since this expression tends to ∞ *as so does* m*, fix* m *large enough so as to guarantee that* $-m < k$ *and* $M > k$*. Obviously,*

$$\mathbb{E}\left((y_n - k)^+\right) = (M - k)\,\mathbb{P}(\Xi_{n,1}) = \frac{\tilde{C}_0 + m h_{n,3}\mathbb{P}(\Xi_{n,3}) - k\left(\sum_{j=2}^{3} h_{n,j}\mathbb{P}(\Xi_{n,j})\right)}{h_{n,1}}$$

and $\mathbb{E}\left((k - y_n)^+\right) = (m + k)\,\mathbb{P}(\Xi_{n,3})$*, so*

$$\Omega_k(y_n) = \frac{\tilde{C}_0 + m h_{n,3}\mathbb{P}(\Xi_{n,3}) - k\left(\sum_{j=2}^{3} h_{n,j}\mathbb{P}(\Xi_{n,j})\right)}{(m + k)\,\mathbb{P}(\Xi_{n,3})\,h_{n,1}}.$$

Thus, $\Omega_k(y_n)$ *tends to* $h_{n,3}/h_{n,1}$ *as* m *tends to* ∞*, and* m *may be selected in such a manner that*

$$\Omega_k(y_n) \geq \frac{h_{n,3}}{h_{n,1}} - \frac{1}{n}$$

holds for every $n \in \mathbb{N}$ large enough. If

$$\lim_{n\to\infty} \frac{h_{n,3}}{h_{n,1}} = \frac{\|z_\Pi\|_\infty}{\|z_\Pi\|_0}, \tag{70}$$

then Lemma 20 will lead to $\lim_{n\to\infty} \Omega_k(y_n) = \|z_\Pi\|_\infty / \tilde{z}_\Pi$. (70) will trivially follow from

$$\begin{cases} \lim_{n\to\infty} h_{n,1} = \|z_\Pi\|_0 \\ \lim_{n\to\infty} h_{n,3} = \|z_\Pi\|_\infty \end{cases} \tag{71}$$

Clearly, (71) follows from the construction of the sequences $\{h_{n,1}\}_{n=1}^\infty$ and $\{h_{n,3}\}_{n=1}^\infty$, since $\|z_\Pi\|_0 \le h_{n,1} \le \delta_n^{(1}$ and $\delta_n^{(2} \le h_{n,3} \le \|z_\Pi\|_\infty$ hold. □

4.2 Risk constraints

Throughout this sub-section we will deal with a complete market and a risk measure ρ such that (11), (12) and (13) hold. We will also fix $C_0 > 0$, $E_0 > 0$, $\rho_0 \in \mathbb{R}$ and $\tilde{C}_0 = C_0 e^{rT}$. Furthermore, we will suppose that (23) is feasible and satisfies the Slater qualification, *i.e.*, there exists $y \in Y_{0,4}$ such that $\mathbb{E}(z_\Pi y) < \tilde{C}_0$, $\mathbb{E}(y) > E_0 C_0$ and $\rho(y) < \rho_0 C_0$.[15] Notice that for $Y_0 = Y_{0,3}$ and $\rho_0 > 0$ the Slater qualification is guaranteed since one can take $y = 0$.

Lemma 25 *Suppose that $S = L^p$ and $Y_0 = Y_{0,3}$. Consider $\beta > 1$. The dual problem (41) becomes*

$$\begin{cases} Max\ \left(k - \tilde{C}_0\right) e'_\beta - \left(\rho_0 C_0 + k\tilde{E}\right) \nu_\rho \\ e'_\beta z_\Pi + (\beta - 1) z - \nu_\rho z_\rho = \beta \\ e'_\beta, \nu_\rho \ge 0,\ z \in B_{(\infty,+)},\ z_\rho \in \Delta_\rho \end{cases} \tag{72}$$

$\left(e'_\beta, z, \nu_\rho, z_\rho\right)$ being the decision variable. If $\left(e'_\beta, z, \nu_\rho, z_\rho\right)$ is feasible then

$$\begin{cases} 1 + \tilde{E}\nu_\rho \le e'_\beta \le \beta + \tilde{E}\nu_\rho \\ \dfrac{1 + \nu_\rho z_\rho}{e'_\beta} \le z_\Pi \le \dfrac{\beta + \nu_\rho z_\rho}{e'_\beta} \end{cases} \tag{73}$$

Proof. Bearing in mind (8) and (54), and according to (11) and (41), the dual problem becomes

$$\begin{cases} Max\ \beta k - k(\beta - 1)\mathbb{E}(z) - \tilde{C}_0 e'_\beta - \rho_0 C_0 \int_{\Delta_\rho} \nu(dz) \\ (\beta - 1)\mathbb{E}(yz) - \beta\mathbb{E}(y) + e'_\beta \mathbb{E}(y z_\Pi) - \int_{\Delta_\rho \beta} \mathbb{E}(y\tilde{z})\nu(d\tilde{z}) = 0,\ \forall y \in L^p \\ \left(z, e'_\beta, \nu\right) \in \mathsf{B}_{(\infty,+)} \times \mathbb{R} \times \mathcal{C}'(\Delta_\rho) \\ e'_\beta \ge 0,\ \nu \ge 0 \end{cases} \tag{74}$$

[15] Notice that $\rho_0 > 0$ must be imposed if ρ is the absolute or the standard deviation. Indeed, if $\rho_0 \le 0$ and $C_0 > 0$ then $\rho(y) < \rho_0 C_0$ can never hold. More generally, (11) and (12) imply that $\rho_0 > 0$ must hold if $\tilde{E} = 0$.

$\left(e'_\beta, z, \nu\right)$ being the decision variable. According to (3), $\nu \in \mathcal{C}'(\Delta_\rho)$ may be replaced by $\nu(\Delta_\rho)\,\delta_{z_\rho}$ for some $z_\rho \in \Delta_\rho$. Thus, denoting $\nu_\rho = \nu(\Delta_\rho)$, (74) trivially leads to

$$\left\{\begin{array}{l} Max\ \beta k - (\beta-1)\, k\mathbb{E}(z) - \tilde{C}_0 e'_\beta - \rho_0 C_0 \nu_\rho \\ e'_\beta z_\Pi + (\beta-1)\, z - \nu_\rho z_\rho = \beta \\ e'_\beta, \nu_\rho \geq 0,\ z \in B_{(\infty,+)},\ z_\rho \in \Delta_\rho \end{array}\right. ,$$

which is equivalent to (72) if one notices that its first constraint, (8) and (13) imply that $(\beta-1)\,\mathbb{E}(z) = \beta - e'_\beta + \tilde{E}\nu_\rho$.

Besides, taking expectations in the first constraint of (72), and bearing in mind (8) and (13), one has $e'_\beta + (\beta-1)\,\mathbb{E}(z) - \nu_\rho \tilde{E} = \beta$, *i.e.*, $e'_\beta = \beta - (\beta-1)\,\mathbb{E}(z) + \nu_\rho \tilde{E}$. Hence, $0 \leq \mathbb{E}(z) \leq 1$ trivially leads to the first condition in (73). Similarly, $e'_\beta \geq 1 + \tilde{E}\nu_\rho \geq 1 > 0$ and the first constraint of (72) imply that

$$z_\Pi = \frac{\beta - (\beta-1)\, z + \nu_\rho z_\rho}{e'_\beta},$$

and the second condition in (73) trivially follows from $0 \leq z \leq 1$. □

Next let us give the dual problem for $Y_0 = Y_{0,4}$. The proof is similar and therefore omitted.

Lemma 26 *Suppose that $S = L^p$ and $Y_0 = Y_{0,4}$. Consider $\beta > 1$. The dual problem (41) becomes*

$$\left\{\begin{array}{l} Max\ \left(k - \tilde{C}_0\right) e'_{\beta,1} + (E_0 C_0 - k)\, e'_{\beta,2} - \left(\rho_0 C_0 + k\tilde{E}\right) \nu_\rho \\ e'_{\beta,1} z_\Pi + (\beta-1)\, z - e'_{\beta,2} - \nu_\rho z_\rho = \beta \\ e'_{\beta,1}, e'_{\beta,2}, \nu_\rho \geq 0,\ z \in B_{(\infty,+)},\ z_\rho \in \Delta_\rho \end{array}\right. \tag{75}$$

$\left(e'_{\beta,1}, e'_{\beta,2}, z, \nu_\rho, z_\rho\right)$ *being the decision variable. If* $\left(e'_{\beta,1}, e'_{\beta,2}, z, \nu_\rho, z_\rho\right)$ *is feasible then*

$$\left\{\begin{array}{l} 1 + e'_{\beta,2} + \tilde{E}\nu_\rho \leq e'_{\beta,1} \leq \beta + e'_{\beta,2} + \tilde{E}\nu_\rho \\ \dfrac{1 + e'_{\beta,2} + \nu_\rho z_\rho}{e'_{\beta,1}} \leq z_\Pi \leq \dfrac{\beta + e'_{\beta,2} + \nu_\rho z_\rho}{e'_{\beta,1}} \end{array}\right.$$

hold. □

Lemma 27 *Suppose that $S = L^p$ and $\beta = 1$.*

a) If $Y_0 = Y_{0,3}$ then the dual problem (45) becomes

$$Max\ k - \tilde{C}_0 e' - \rho_0 C_0 \nu_\rho \left\{\begin{array}{l} e' z_\Pi - \nu_\rho z_\rho = 1 \\ e', \nu_\rho \geq 0,\ z_\rho \in \Delta_\rho \end{array}\right. \tag{76}$$

(e', ν_ρ, z_ρ) *being the decision variable.*

b) If $Y_0 = Y_{0,4}$ then the dual problem (45) becomes

$$Max\ k - \tilde{C}_0 e_1' + E_0 C_0 e_2' - \rho_0 C_0 \nu_\rho \begin{cases} e_1' z_\Pi - e_2' - \nu_\rho z_\rho = 1 \\ e', e_2', \nu_\rho \geq 0,\ z_\rho \in \Delta_\rho \end{cases} \tag{77}$$

$(e_1', e_1', \nu_\rho, z_\rho)$ being the decision variable.

Proof. Suppose that $Y_0 = Y_{0,3}$. According to (6), (11) and (54), Problem (31) becomes

$$Min\ k - \mathbb{E}(y) \begin{cases} \mathbb{E}(z_\Pi y) \leq \tilde{C}_0 \\ -\mathbb{E}(zy) \leq \rho_0 C_0,\ \forall z \in \Delta_\rho \end{cases}$$

which is linear and whose dual becomes (Anderson and Nash, 1987, and (45))

$$Max\ k - \tilde{C}_0 e' - \rho_0 C_0 \int_{\Delta_\rho} \mu(dz) \begin{cases} e' \mathbb{E}(z_\Pi y) - \int_{\Delta_\rho} \mathbb{E}(zy)\mu(dz) = \mathbb{E}(y),\ \forall y \in L^p \\ e' \geq 0,\ \mu \in \mathcal{C}'(\Delta_\rho),\ \mu \geq 0 \end{cases}$$

Bearing in mind (3), and denoting $\nu_\rho = \int_{\Delta_\rho} \mu(dz)$, we have

$$Max\ k - \tilde{C}_0 e' - \rho_0 C_0 \nu_\rho \begin{cases} \mathbb{E}(y(e' z_\Pi - \nu_\rho z_\rho)) = \mathbb{E}(y),\ \forall y \in L^p \\ e', \nu_\rho \geq 0,\ z_\rho \in \Delta_\rho \end{cases}$$

which trivially leads to Statement *a*). The proof of *b*) is similar and therefore omitted. □

Proposition 28 *Suppose that $p = 2$ and $S = L^2$. If $Y_0 = Y_{0,4}$, ρ is the standard deviation and $\rho_0 > 0$ then $E_0 \leq e^{rT} + \rho_0 \rho(z_\Pi)$.*[16]

Proof. Consider Problem (31) with $\beta = 1$. If its dual (77) (Lemma 27*b*) is feasible then it must be bounded since otherwise Remark 14 will show that (31) is not feasible ($Y_{0,4} = \varnothing$), against the imposed Slater qualification. Taking expectations in the constraint of (77), and bearing in mind (8), (13) and $\tilde{E} = 0$, we have that $e_1' = 1 + e_2'$. If $\nu_\rho = 0$ the same constraints lead to $z_\Pi = (1 + e_2')/e_1'$, and (8) leads to $z_\Pi = 1$, against (61), so $\nu_\rho > 0$ and the constraint becomes

$$z_\rho = \frac{(1 + e_2') z_\Pi - e_2' - 1}{\nu_\rho}.$$

Hence, $\rho(z_\rho) = ((1 + e_2')\rho(z_\Pi))/\nu_\rho$ and, according to (22), $\nu_\rho \geq (1 + e_2')\rho(z_\Pi)$ must hold. Problem (77) becomes

$$Max\ k - \tilde{C}_0 + e_2' \left(-\tilde{C}_0 + E_0 C_0\right) - \rho_0 C_0 \nu_\rho \begin{cases} z_\rho = \dfrac{(1 + e_2') z_\Pi - e_2' - 1}{\nu_\rho} \\ e_2' \geq 0,\ \nu_\rho \geq (1 + e_2')\rho(z_\Pi) \end{cases},$$

which is feasible and whose solution must obviously satisfy $\nu_\rho = (1 + e_2')\rho(z_\Pi)$. Then, the objective becomes

$$k - \tilde{C}_0 + e_2' \left(-\tilde{C}_0 + E_0 C_0 - \rho_0 C_0 \rho(z_\Pi)\right) - \rho_0 C_0 \rho(z_\Pi)$$

[16] In oder words, if $E_0 > e^{rT} + \rho_0 \rho(z_\Pi)$ then the Slater qualification above does not hold.

Thus, $-\tilde{C}_0 + E_0 C_0 - \rho_0 C_0 \rho(z_\Pi) \leq 0$ must hold because otherwise (77) becomes unbounded. □

Theorem 29 *Suppose that* $S = L^p$.

a) If $\|z_\Pi\|_0 = 0$ *and the risk measure* ρ *is coherent and expectation bounded then*

$$Sup\{\Omega_k(y);\ y \in Y_{0,4}\} = Sup\{\Omega_k(y);\ y \in Y_{0,3}\} = \infty.$$

b) If $\|z_\Pi\|_\infty = \infty$, ρ *is expectation bounded and* $\Delta_\rho \subset L^\infty$ *then*

$$Sup\{\Omega_k(y);\ y \in Y_{0,4}\} = Sup\{\Omega_k(y);\ y \in Y_{0,3}\} = \infty.$$

c) If $\|z_\Pi\|_\infty = \infty$, ρ *is the absolute deviation and* $\rho_0 > 0$ *then*

$$Sup\{\Omega_k(y);\ y \in Y_{0,4}\} = Sup\{\Omega_k(y);\ y \in Y_{0,3}\} = \infty.$$

d) Suppose that $p = 2$. *If* ρ *is the standard deviation,* $\rho_0 > 0$ *and* $k \geq \tilde{C}_0 + \rho_0 C_0 \rho(z_\Pi)$ *then*

$$0 \leq Sup\{\Omega_k(y);\ y \in Y_{0,4}\} \leq Sup\{\Omega_k(y);\ y \in Y_{0,3}\} \leq 1.$$

e) Suppose that $p = 2$. *If* ρ *is the standard deviation,* $\rho_0 > 0$ *and* $\tilde{C}_0 < k < \tilde{C}_0 + \rho_0 C_0 \rho(z_\Pi)$ *then*

$$1 < Sup\{\Omega_k(y);\ y \in Y_{0,4}\} \leq Sup\{\Omega_k(y);\ y \in Y_{0,3}\} \leq \|z_\Pi\|_\infty / \|z_\Pi\|_0.$$

Proof. *a*) Suppose that $Y_0 = Y_{0,3}$. Consider $\beta > 1$. If $\left(e'_\beta, z, \nu_\rho, z_\rho\right)$ is (72)-feasible then (14) and (73) imply that $e'_\beta \geq 1$ and

$$\frac{1}{e'_\beta} \leq \frac{1 + \nu_\rho z_\rho}{e'_\beta} \leq z_\Pi,$$

and therefore $0 < 1/e'_\beta \leq \tilde{z}_\Pi$, contradicting the assumptions. Thus, (72) is not feasible and Theorem 10*a* implies that (31) is not bounded. Consequently, Lemma 2 implies that β is not an upper bound of (23).

If $Y_0 = Y_{0,4}$ the proof is similar, but Lemma 26 must play the role of Lemma 25.

b) Suppose that $Y_0 = Y_{0,3}$. As in the proof of *a*) we have that

$$z_\Pi \leq \frac{\beta + \nu_\rho z_\rho}{e'_\beta} \leq \frac{\beta + \nu_\rho \|z_\rho\|_\infty}{e'_\beta} < \infty, \tag{78}$$

so $\|z_\Pi\|_\infty \leq \left(\beta + \nu_\rho \|z_\rho\|_\infty\right)/e'_\beta$ and we have a contradiction again. Thus, (72) is not feasible and Theorem 10*a* and Lemma 2 apply.

If $Y_0 = Y_{0,4}$ the proof is similar, but Lemma 26 must play the role of Lemma 25.

c) Suppose that $Y_0 = Y_{0,3}$. Consider $\beta > 1$. If $\left(e'_\beta, z, \nu_\rho, z_\rho\right)$ is (72)-feasible then $\tilde{E} = 0$, (22) and (73) imply that $e'_\beta \geq 1$ and (78) again holds. Hence,

$\|z_\Pi\|_\infty \leq \left(\beta + \nu_\rho \|z_\rho\|_\infty\right)/e'_\beta$ and we have a contradiction again. Thus, (72) is not feasible and Theorem 10a and Lemma 2 apply.

Once again, the proof is similar if $Y_0 = Y_{0,4}$.

d) $0 \leq Sup\left\{\Omega_k(y);\ y \in Y_{0,3}\right\}$ trivially follows from (5), so let us see that $Sup\left\{\Omega_k(y);\ y \in Y_{0,3}\right\} \leq 1$. Consider Problem (31) with $\beta = 1$ and its dual Problem (76) (Remark 14 and Lemma 27). Taking expectations in the constraint of (76), and bearing in mind (8), (13) and $\tilde{E} = 0$, we have that $e' = 1$. If $\nu_\rho = 0$ the same constraint leads to $z_\Pi = 1$, against (61), so $\nu_\rho > 0$ and the constraint becomes

$$z_\rho = \frac{z_\Pi - 1}{\nu_\rho}.$$

Hence, $\rho(z_\rho) = (\rho(z_\Pi))/\nu_\rho$ and, according to (22), Problem (76) becomes

$$Max\ k - \tilde{C}_0 e' - \rho_0 C_0 \nu_\rho \left\{ \begin{array}{l} z_\rho = \dfrac{z_\Pi - 1}{\nu_\rho} \\ e' = 1,\ \nu_\rho \geq \rho(z_\Pi) \end{array} \right. ,$$

whose obvious solution is $e' = 1$, $\nu_\rho = \rho(z_\Pi)$ and $z_\rho = (z_\Pi - 1)/\rho(z_\Pi)$. Its optimal value becomes $k - \tilde{C}_0 - \rho_0 C_0 \rho(z_\Pi)$, which is non negative. Consequently, Remark 14 implies that the objective function of (31) remains non negative if $\beta = 1$, and Lemma 2 implies that $\beta = 1$ is an upper bound of (23).

In order to prove that $Sup\left\{\Omega_k(y);\ y \in Y_{0,4}\right\} \leq 1$, notice that Proposition 28 leads to $-\tilde{C}_0 + E_0 C_0 - \rho_0 C_0 \leq 0$. As in the proof of this proposition, for $\beta = 1$ the dual (77) of (31) becomes

$$\left\{ \begin{array}{ll} Max & k - \tilde{C}_0 + e'_2\left(-\tilde{C}_0 + E_0 C_0 - \rho_0 C_0\right) - \rho_0 C_0 \rho(z_\Pi) \\ & z_\rho = \dfrac{(1 + e'_2) z_\Pi - e'_2 - 1}{\nu_\rho} \\ & e'_2 \geq 0,\ \nu_\rho = (1 + e'_2)\rho(z_\Pi) \end{array} \right.$$

whose optimal value is attained at $e'_2 = 0$ and becomes $k - \tilde{C}_0 - \rho_0 C_0 \rho(z_\Pi)$. The rest of the proof is similar to that already given for $Y_0 = Y_{0,3}$.

e) $Sup\left\{\Omega_k(y);\ y \in Y_{0,3}\right\} \leq \|z_\Pi\|_\infty / \|z_\Pi\|_0$ trivially follows from $Y_{0,3} \subset Y_{0,1}$ and Lemma 21, so let us see that $1 < Sup\left\{\Omega_k(y);\ y \in Y_{0,3}\right\}$. Consider Problem (31) with $\beta = 1$ and its dual Problem (76) (Remark 14 and Lemma 27). As above, its optimal value equals $k - \tilde{C}_0 - \rho_0 C_0 \rho(z_\Pi)$, which is negative. Consequently, Remark 14 implies that the objective function of (31) takes negative values if $\beta = 1$, and Lemma 2 implies that $\beta = 1$ is not an upper bound of (23). If $Y_{0,3}$ is replaced by $Y_{0,4}$ then proceed as in the proof of *d*). □

Remark 30 *Suppose that $S = L^p$ and $Y_0 = Y_{0,3}$ or $Y_0 = Y_{0,4}$. Theorem* 29 *shows that most of the caveats pointed out in Remark* 24 *still apply if risk constraints are imposed. Pricing models such as B&S and most of the SVM generate a SDF such that $\|z_\Pi\|_0 = 0$ and $\|z_\Pi\|_\infty = \infty$. Hence, the existence of risk constraints does not prevent a supremum of (23) equaling ∞*

for every expectation bounded and coherent risk measure, or for every risk measure whose representation set Δ_ρ is composed of essentially bounded random variables. Very important examples are $CVaR$ and $WCVaR$ (see (15), (17), (18) and (19)), among others. Furthermore, if the expectation bounded measure is replaced by the absolute deviation then (23) is still unbounded, as pointed out in Statement e). The standard deviation is the only example of Section 2 that prevents this caveat, but, under the presence of asymmetries, this is also the unique example reflecting lack of compatibility with the second order stochastic dominance and the standard utility functions (Ogryczak and Ruszczynski, 1999 and 2002). Needless to say, all of these pricing models are used to price options and other derivatives, which are often very asymmetric. To sum up, and bearing in mind both Theorem 29 and the results of Balbás et al. (2010), the most important continuous time pricing models of Financial Mathematics lead to the existence of sequences of strategies whose price is lower than one dollar (or one cent), whose expected return and omega ratio are as large as desired, and whose risk (or absolute deviation) is as close as desired to zero. Moreover, if the selected risk measure is VaR, and bearing in mind that $CVaR_{1-\alpha}(y) \leq \rho_0 C_0 \implies VaR_{1-\alpha}(y) \leq \rho_0 C_0$ because $VaR_{1-\alpha}(y) \leq CVaR_{1-\alpha}(y)$, the caveat above still holds, despite the fact that VaR is not expectation bounded. □

Remark 31 *Suppose that $S = L^p$ and $Y_0 = Y_{0,3}$ or $Y_0 = Y_{0,4}$. If the inequality $\|z_\Pi\|_\infty / \|z_\Pi\|_0 < \infty$ holds then (23) becomes bounded (Lemma 21). Approximations to the optimal value of (23) may be given by means of minor modifications of Algorithms 4 and 5 or Remark 6. In particular, if this optimal value is higher than one (for instance, under the conditions of Theorems 23 or 29e), then one can choose a "small enough" error $\varepsilon > 0$ and solve the dual problem (41) (which may become (62), (67), (72) or (75)) for $\beta = 1 + n\varepsilon$, $n = 1, 2, \ldots$ The optimal value $f(n)$ of (41) will equal the optimal value of (31) (Theorem 10), and Lemmas 1 and 2 justify the choice of n_0 minimizing $|f(n)|$. Once we know an approximations for the dual solution and the optimal value $\beta = 1 + n_0\varepsilon$, the primal solution of (31) will give approximations to the solution of (23). Solutions of (31) for $\beta = 1 + n_0\varepsilon$ may be estimated by means of (42) (which may become (57), (58), (59) or (60)).*

If the optimal value of (23) is not higher than one (for instance, under the conditions of Theorem 29d), the sequence $\beta = 1 + n\varepsilon$, $n = 1, 2, ..$above must be replaced by $\beta = n\varepsilon$, $n = 1, 2, ..$, and the role of the duality theory must be replaced by Conditions (35) (which may become (51), (53), (55) or (56)). Obviously, in this case things may become more complex since both the solution of (31) and its multipliers must be simultaneously estimated. Lastly, if in advance one cannot anticipate whether the optimal value of (23) is higher than one,, then one will take $\beta = n\varepsilon$, $n = 1, 2, ..$and combine both procedures, since, depending on n, both $n\varepsilon \leq 1$ and $n\varepsilon \geq 1$ may hold. □

5 Buy and hold approaches

This section will be devoted to optimizing omega in static frameworks. Consequently, there are only two trading dates, namely, $t = 0$ (agents make the investment decision) and $t = T$ (agents recover the invested amount plus random earnings/losses). The strategy is selected at $t = 0$ and it will not be rebalanced before $t = T$.

Consider a finite set of available assets

$$S_0 = \{1, y_1, y_2, ..., y_n\} \subset L^2, \tag{79}$$

1 representing the pay-off at T of the riskless asset and $y_j \in L^2$, $j = 1, 2, ..., n$, representing the pay-off of the $j-th$ risky asset. Obviously, the space S of marketed claims equals the linear subspace generated by S_0, while the orthogonal space $S^{\perp}$ will be composed of those $y \in L^2$ such that

$$\begin{cases} \mathbb{E}(y) = 0 \\ \mathbb{E}(y_j y) = 0, \quad j = 1, 2, ..., n \end{cases} \tag{80}$$

The set of current prices will be denoted by

$$\left\{e^{-rT} = \Pi(1), \Pi_1 = \Pi(y_1), \Pi_2 = \Pi(y_2), ..., \Pi_n = \Pi(y_n)\right\}.$$

As indicated in the second section, there is a unique SDF belonging to S, so it will be given by

$$z_\Pi = z_{\Pi,0} + \sum_{j=1}^{n} z_{\Pi,j} y_j, \tag{81}$$

where $(z_{\Pi,j})_{j=0}^{n} \in \mathbb{R}^{n+1}$ solves the linear system

$$(z_{\Pi,0}, z_{\Pi,1}, ..., z_{\Pi,n}) M = \left(1, \tilde{\Pi}_1, \tilde{\Pi}_2, ..., \tilde{\Pi}_n\right)$$

with $\tilde{\Pi}_j = e^{rT}\Pi_j$, $j = 1, 2, ..., n$, and M being the $(n+1) \times (n+1)$ symmetric matrix

$$\begin{cases} M_{0,0} = 1 \\ M_{0,j} = M_{j,0} = \mathbb{E}(y_j), & j = 1, 2, ..., n \\ M_{i,j} = M_{j,i} = \mathbb{E}(y_i y_j), & i, j = 1, 2, ..., n \end{cases} \tag{82}$$

Without lost of generality one can assume that M is regular,[17] so the SDF will be given by (81), with

$$(z_{\Pi,0}, z_{\Pi,1}, ..., z_{\Pi,n}) = \left(1, \tilde{\Pi}_1, \tilde{\Pi}_2, ..., \tilde{\Pi}_n\right) M^{-1} \tag{83}$$

[17] M is regular if and only if the $n+1$ available assets are linearly independent, *i.e.*, their unique vanishing linear combination is the trivial one. If M were singular then there would exist redundant securities which could be removed, the problem dimension would decrease and "the new $M-$matrix" would become regular.

and M being given by (82). As in Section 4, let us suppose that the market is not risk neutral, *i.e.*, $z_\Pi \neq 1$ (or $(z_{\Pi,0}, z_{\Pi,1}, ..., z_{\Pi,n}) \neq (1, 0, ..., 0)$).

Let us deal with Problem (23) for $Y_0 = Y_{0,4}$ (see(49)).[18] Take $C_0 > 0$, $\tilde{C}_0 = C_0 e^{rT}$, $E_0 > e^{rT} > 0$,[19] $\rho_0 \in \mathbb{R}$,[20] and $k > \tilde{C}_0$, and let us assume that $Y_{0,4} \neq \oslash$ (*i.e.*, (23) is feasible) and the Slater qualification holds (*i.e.*, the three constraints of (49) can simultaneously hold as strict inequalities). The fulfillment of (4) and Assumption 1 are guaranteed by Proposition 15. Denote

$$0 \leq \beta^* = Sup \ \{\Omega_k(y);\ y \in Y_{0,4}\} \leq \infty \tag{84}$$

(see (24)) the optimal value of (23). It is important to analyze whether the inequality $\beta^* > 1$ holds, since its fulfillment (failure) will allow (impose) us to deal with Theorem 10 (7).

Proposition 32 *Consider β^* given by (84), and the set*

$$A = \left\{(\alpha, z_\rho) \in (0, \infty) \times \Delta_\rho;\ \mathbb{E}(y_j z_\rho) = \tilde{E}\tilde{\Pi}_j + \alpha\left(\tilde{\Pi}_j - \mathbb{E}(y_j)\right),\ j = 1, ..., n\right\}.$$

If $A \neq \varnothing$ then take $\alpha^* = Sup\ \{\alpha;\ \ (\alpha, z_\rho) \in A\} \in (0, +\infty]$.

a) If $A = \varnothing$ then $\beta^* > 1$.

b) If $A \neq \varnothing$ then $\alpha^* < +\infty$ and there exists $z^*_\rho \in \Delta_\rho$ with $\left(\alpha^*, z^*_\rho\right) \in A$ and $E_0 \leq e^{rT} + \left(\rho_0 + e^{rT}\right)/\alpha^*$.

c) If $A \neq \varnothing$ and $k/C_0 < e^{rT} + \left(\rho_0 + e^{rT}\right)/\alpha^*$ then $\beta^* > 1$.

d) If $A \neq \varnothing$ and $k/C_0 < E_0$ then $\beta^* > 1$.

e) If $A \neq \varnothing$ and $k/C_0 \geq e^{rT} + \left(\rho_0 + e^{rT}\right)/\alpha^*$ then $\beta^* \leq 1$.

Proof. Consider $\beta = 1$. According to Remark 14, (31), (32) and (44) are the same problem. Moreover, its dual becomes (see (3), (45) and (80))

$$\begin{cases} Max\ k - \tilde{C}_0 e'_{1,1} + E_0 C_0 e'_{1,2} - \left(\rho_0 C_0 + \tilde{C}_0\right)\nu_\rho \\ \mathbb{E}\left(e'_{1,1} z_\Pi - e'_{1,2} - \nu_\rho z_\rho\right) = 1 \\ \mathbb{E}(y_j\left(e'_{1,1} z_\Pi - e'_{1,2} - \nu_\rho z_\rho - 1\right)) = 0, & j = 1, 2, ..., n \\ e'_{1,1}, e'_{1,2}, \nu_\rho \geq 0,\ z_\rho \in \Delta_\rho \end{cases}$$

The first constraint leads to (see (8) and (13)) $e'_{1,1} = 1 + e'_{1,2} + \tilde{E}\nu_\rho$ so straightforward manipulations imply the equivalence between the dual above and

$$\begin{cases} Max\ k - \tilde{C}_0 + \left(E_0 C_0 - \tilde{C}_0\right) e'_{1,2} - \left(\rho_0 C_0 + \tilde{C}_0\right)\nu_\rho \\ \left(\tilde{\Pi}_j - \mathbb{E}(y_j)\right) e'_{1,2} + \left(\tilde{E}\tilde{\Pi}_j - \mathbb{E}(y_j z_\rho)\right)\nu_\rho = \mathbb{E}(y_j) - \tilde{\Pi}_j,\ j = 1, 2, ..., n \\ e'_{1,2}, \nu_\rho \geq 0,\ z_\rho \in \Delta_\rho \end{cases} \tag{85}$$

[18] Similar analyses could be implemented if $Y_{0,4}$ were replaced by $Y_{0,1}$, $Y_{0,2}$ or $Y_{0,3}$ (see (46), (47) or (48)), but let us shorten the exposition.

[19] *i.e.*, the investor expected return is higher than the risk free rate.

[20] Footnote 15 applies.

$\left(e'_{1,2}, \nu_\rho,\ z_\rho\right)$ being the decision variable. Furthermore, if $\left(e'_{1,2}, \nu_\rho,\ z_\rho\right)$ is (85)-feasible then $\nu_\rho > 0$, because $\nu_\rho = 0$ implies $\left(\tilde{\Pi}_j - \mathbb{E}\left(y_j\right)\right) e'_{1,2} = \mathbb{E}\left(y_j\right) - \tilde{\Pi}_j$, and $e'_{1,2} \neq -1$ (or $e'_{1,2} \geq 0$) implies $\tilde{\Pi}_j - \mathbb{E}\left(y_j\right) = 0$, against the assumptions (the market is not risk neutral).

a) If $A = \varnothing$ then (85) has no feasible solutions, since otherwise

$$\left(\alpha = \frac{1 + e'_{1,2}}{\nu_\rho}, z_\rho\right) \in A.$$

Remark 14 implies that (31) (or (32), or (44)) is unbounded for $\beta = 1$, so there exists $y_0 \in Y_{0,4}$ such that

$$\mathbb{E}\left(\left(k - y_0\right)^+\right) - \mathbb{E}\left(\left(y_0 - k\right)^+\right) < 0, \tag{86}$$

i.e., $\beta^* \geq \Omega_k\left(y_0\right) = \mathbb{E}\left(\left(y_0 - k\right)^+\right) / \mathbb{E}\left(\left(k - y_0\right)^+\right) > 1$.

b) If $(\alpha, z_\rho) \in A$ then take $e'_{1,2} \geq 0$ and $\nu_\rho = \left(1 + e'_{1,2}\right)/\alpha$. $\left(e'_{1,2}, \nu_\rho,\ z_\rho\right)$ trivially becomes (85)-feasible. Furthermore, the objective of (85) equals

$$\left(k - \tilde{C}_0\right) - \frac{\rho_0 C_0 + \tilde{C}_0}{\alpha} + \left(E_0 C_0 - \tilde{C}_0 - \frac{\rho_0 C_0 + \tilde{C}_0}{\alpha}\right) e'_{1,2}. \tag{87}$$

If the coefficient of $e'_{1,2}$ were positive then the expression above would tend to $+\infty$ as $e'_{1,2} \to +\infty$, so (Remark 14) (31) (or (32), or (44)) would not be feasible (or $Y_{0,4} = \varnothing$), against the assumptions. Thus,

$$E_0 C_0 - \tilde{C}_0 - \frac{\rho_0 C_0 + \tilde{C}_0}{\alpha} \leq 0, \tag{88}$$

the supremum of (87) is achieved for $e'_{1,2} = 0$ when α has been fixed, and the maximum in (85) equals

$$\left(k - \tilde{C}_0\right) - \frac{\rho_0 C_0 + \tilde{C}_0}{\alpha^*} = C_0 \left(\frac{k}{C_0} - e^{rT} - \frac{\rho_0 + e^{rT}}{\alpha^*}\right). \tag{89}$$

In particular, $\alpha^* < \infty$ because otherwise the optimal value of (85) would not be attainable.

c) As in *a*), it is sufficient to show the existence of $y_0 \in Y_{0,4}$ satisfying (86), which becomes obvious because the optimal value (89) of (85) is negative.

d) (88) for $\alpha = \alpha^*$ implies that $E_0 \leq e^{rT} + \dfrac{\rho_0 + e^{rT}}{\alpha^*}$.

e) The optimal value (89) of (85) is not negative, and it is a lower bound for (31) (or (44)) if $\beta = 1$. Thus,

$$\mathbb{E}\left(\left(k - y\right)^+\right) - \mathbb{E}\left(\left(y - k\right)^+\right) \geq C_0 \left(\frac{k}{C_0} - e^{rT} - \frac{\rho_0 + e^{rT}}{\alpha^*}\right) \geq 0$$

holds for every $y \in Y_{0,4}$, *i.e.*, $\Omega_k\left(y\right) \leq 1$ for every $y \in Y_{0,4}$. □

Remark 33 *Proposition* 32 *shows that the linear problem*

$$Max\ \alpha \left\{ \begin{array}{l} \mathbb{E}\left(y_j z_\rho\right) = \tilde{E}\tilde{\Pi}_j + \alpha\left(\tilde{\Pi}_j - \mathbb{E}\left(y_j\right)\right),\ j = 1, 2, ..., n \\ \alpha \geq 0,\ z_\rho \in \Delta_\rho \end{array} \right. \tag{90}$$

is solvable if it is feasible, and it permits us to know whether $\beta^* > 1$ holds. In particular, if (90) is feasible, then $\beta^* \leq 1$ will hold for k "large enough". Once we can compare β^* with 1*, we will apply Algorithms* 4 *or* 5 *(or extensions, see Remark* 6*) in order to estimate β^* and solve (23). We will deal with Theorem* 7 *(*10*) if $\beta^* \leq 1$ ($\beta^* > 1$). Tow numerical examples will be given in Sections* 5.1 *and* 5.2*.* □

5.1 Derivative markets: Numerical experiment

Consider an underlying asset (bond, stock, or likely an international stock index) and denote by $y_1 \in L^2$, $y_1 \geq 0$, its random price at T. Suppose that there exists a derivative market where European calls on this asset can be traded.[21] Suppose that T is the maturity of them all,[22] and denote by

$$0 = a_1 < a_2 < ... < a_n < \infty$$

the available strikes. Obviously, $(y_1 - a_j)^+$ is the random pay-off of the $j - th$ option, $j = 1, 2, ..., n$, and the first option ($j = 1$) is the underlying security. Furthermore, the set (79) of available assets is

$$S_0 = \left\{1, (y_1 - a_1)^+ = y_1, (y_1 - a_2)^+, ..., (y_1 - a_n)^+\right\},$$

and generates the matrix M above leading to the SDF (see (81), (82) and (83)).

Remark 34 *It is known that every "smooth enough" European style derivative of $y_1 \in L^2$ may be replicated by a static portfolio containing "infinitely many" European options (Haugh and Lo, 2001). Since the space of continuous functions is often dense in L^2 (Anderson and Nash, 1987), every European style derivative of y_1 has a "good enough" approximation containing "many" European calls. This intuitive argument, along with Theorem* 29*, may be used in order to prove that, under the assumptions of the classical pricing models of Financial Mathematics (B&S, SVM, etc.), the optimal omega ratio will tend to infinity if $n \to \infty$, $\lim_{n\to\infty}\left(Max\ \left\{a_2 - a_1, a_3 - a_2, ..., a_n - a_{n-1}\right\}\right) = 0$ and one deals with a coherent and expectation bounded risk measure or with the absolute deviation. Nevertheless, in practice there are only finitely many available options, and the non available options cannot be replicated in a buy and hold approach, so the optimal omega may remain bounded.* □

[21] Put options may be accepted as well, but we will not deal with them because the put/call parity (Hull, 2012) makes them theoretically redundant.

[22] This assumption may be relaxed, but it simplifies the mathematical framework.

We have adopted the $B\&S$ model and have supposed that y_1 is the value at $T = 1$ of a Geometric Brownian Motion with drift equal to 0.03 and volatility equal to 0.15. The interest rate vanishes, the current price of y_1 equals 1, and there are three available European calls whose strikes equal 0.9, 0.95 and 1, respectively. The $B\&S$ formula leads to

$$\left(1, \tilde{\Pi}_1, \tilde{\Pi}_2, \tilde{\Pi}_3, \tilde{\Pi}_4\right) =$$
$$(1,\ 1,\ 0.120217274,\ 0.086658648,\ 0.059785288)\,,$$

it is easy to see that Matrix M becomes

$$\begin{pmatrix} 1 & 1.030454534, & 0.144873428, & 0.107893997, & 0.077130424 \\ 1.030454534, & 1.085998673, & 0.169908609, & 0.129431978, & 0.094866155 \\ 0.144873428, & 0.169908609, & 0.039522523, & 0.032327381, & 0.025448773 \\ 0.107893997 & 0.129431978, & 0.032327381, & 0.026932681, & 0.021592251 \\ 0.077130424 & 0.094866155, & 0.025448773, & 0.021592251, & 0.01773573 \end{pmatrix}$$

and (83) leads to

$$\begin{pmatrix} z_{\Pi,0} \\ z_{\Pi,1} \\ z_{\Pi,2} \\ z_{\Pi,3} \\ z_{\Pi,4} \end{pmatrix} = \begin{pmatrix} 3.18150435 \\ -2.266672452 \\ 1.118091119 \\ -0.858038333 \\ 1.099359264 \end{pmatrix}$$

Consider that $\rho = \sigma_1$ is the absolute deviation.[23] Bearing in mind the equality $\tilde{E} = 0$ (Section 2) and Expression (22), Problem (90) becomes

$$Max\ \alpha \left\{ \begin{array}{l} \mathbb{E}\left(y_j\left(w - \mathbb{E}\left(w\right)\right)\right) = \alpha\left(\tilde{\Pi}_j - \mathbb{E}\left(y_j\right)\right), \qquad j = 1,2,3,4 \\ \alpha \in \mathbb{R},\ \alpha \geq 0,\ w \in L^{\infty},\ 0 \leq w \leq 2 \end{array} \right.$$

The value of every $\tilde{\Pi}_j$ is given above, and the value of every $\mathbb{E}\left(y_j\right)$ is given by the first column of M. Thus, every parameter in the infinite-dimensional linear problem above is known. The problem can be solved with the algorithms presented in Anderson and Nash (1987),[24] and its optimal value is $\alpha^* = 0$. Accordingly, the set A of Proposition 32 is void, and $\beta^* > 1$ (Statement a) for every $k > \tilde{C}_0$ and every (E_0, ρ_0) making (23) feasible and satisfying the Slater condition for $Y_0 = Y_{0,4}$.

In order to solve (23) with $Y_0 = Y_{0,4}$ let us deal with Algorithm 5. We will have to solve (29), which becomes (32) because $\beta > 1$. According to Theorem

[23] Recall that the absolute deviation is consistent with the second order stochastic dominance and the usual utility functions, while this property fails for the standard deviation if there are asymmetric returns (Ogryczak and Ruszczynski, 1999). Needless to say, asymmetries are usual in derivative markets.

Many authors have illustrated the importance of preserving the second order stochastic dominance in portfolio analysis (see, for instance, Singh and Dharmaraja, 2018, for a recent discussion).

[24] See also Balbás and Heras (1993).

10, it is sufficient to solve the dual problem (41). As in Lemmas 25 and 26, and bearing in mind (80) and $\tilde{E} = 0$, it is easy to see that (41) becomes

$$\begin{cases} Max \ \left(k - \tilde{C}_0\right) e'_{\beta,1} + \left(E_0 C_0 - k\right) e'_{\beta,2} - \rho_0 C_0 \nu_\rho \\ (\beta - 1)\, \mathbb{E}\,(z) + e'_{\beta,1} - e'_{\beta,2} = \beta \\ (\beta - 1)\, \mathbb{E}\,(y_j z) + \tilde{\Pi}_j e'_{\beta,1} - \mathbb{E}\,(y_j)\, e'_{\beta,2} - \mathbb{E}\,(y_j z_\rho)\, \nu_\rho = \beta \mathbb{E}\,(y_j)\,, \quad j = 1,2,3,4 \\ e'_{\beta,1}, e'_{\beta,2}, \nu_\rho \geq 0, \ z \in B_{(\infty,+)}, \ z_\rho \in \Delta_\rho \end{cases} \tag{91}$$

Let us choose $C_0 = 1$, $E_0 = 1.01$, $\rho_0 = 0.2$ and $k = 1.2$. This infinite-dimensional problem becomes linear if ν_ρ is fixed, and consequently it can be solved with the methods of Anderson and Nash (1987). Algorithm 5 leads to the optimal omega ratio $\beta^* = 1.06$, in which case the optimal value of (91) equals -0.000856258 (quite close to zero, see Lemma 3a and Algorithm 5). Once the solution of (91) is known, Problem (23) can be easily solved by means of (43) and (60). Let us omit this step in order to shorten the exposition.

5.2 Portfolio choice: Numerical experiment

Let us illustrate some potential applications of Section 3 in Portfolio Choice Problems involving equity and commodity markets. In particular, let us deal with biweekly quotations related to four future contracts. The database was already used in Balbás *et al.* (2016) and (2017), and it is available in both references, where further details are given. It contains the quotations in the future market of two international stock indices ($DAX-30$ and $S\&P-500$) and two commodities (Gold and Brent) during 2010. Since there are 26 quotations we have 25 biweekly returns. Therefore, the set of estimated states of nature is finite and contains 25 elements, and we will assume that the probability of every scenario equals 1/25. Moreover, $T = 1/26$ years (two weeks) and, according to the information provided by the European Central Bank, we have taken $r = 0.004 = 0.4\%$.[25] As in Section 5.1, it is easy to see that

$$(1, \Pi_1, \Pi_2, \Pi_3, \Pi_4) = (1, 1, 1, 1, 1,)\,,$$

Matrix M equals

$$\begin{pmatrix} 1, & 1.006961159, & 1.00507663, & 1.009494879, & 1,00783441 \\ 1.006961159, & 1.015132529, & 1.013122039, & 1.016534871, & 1.01586752 \\ 1.00507663, & 1.013122039, & 1.011533078, & 1.014879184, & 1.014559008 \\ 1.009494879, & 1.016534871, & 1.014879184, & 1.019918871, & 1.0181489 \\ 1,00783441 & 1.01586752 & 1.014559008 & 1.0181489 & 1.019340047 \end{pmatrix}$$

[25] See http://www.ecb.int/stats/money/yc/html/index.en.html

and the coordinates of the SDF are

$$\begin{pmatrix} z_{\Pi,0} \\ z_{\Pi,1} \\ z_{\Pi,2} \\ z_{\Pi,3} \\ z_{\Pi,4} \end{pmatrix} = \begin{pmatrix} 20.82062209 \\ -16.33843094 \\ 11.10470761 \\ -14.98314601 \\ 0.591238321 \end{pmatrix} \tag{92}$$

Let us consider $\rho = CVaR_{90\%}$.[26] Since $\tilde{E} = 1$ (Section 2), Expression (17) implies that Problem (90) becomes

$$Max\ \alpha \left\{ \begin{array}{ll} \mathbb{E}\left(y_j z_\rho\right) = \tilde{\Pi}_j + \alpha \left(\tilde{\Pi}_j - \mathbb{E}\left(y_j\right)\right), & j = 1,2,3,4 \\ \mathbb{E}\left(z_\rho\right) = 1 & \\ \alpha \in \mathbb{R},\ \alpha \geq 0,\ z_\rho \in \mathbb{R}^{25},\ 0 \leq z_\rho \leq 10 & \end{array} \right.$$

The parameters of this finite-dimensional linear problem (there are only 25 states of nature) are known, and the usual algorithms of Linear Programming lead to the optimal value $\alpha^* = 2.788695457$. Propositions 32*b*) and 32*c*) imply that

$$E_0 \leq e^{rT} + \left(\rho_0 + e^{rT}\right)/\alpha^* = 1.000038462 + \left(\rho_0 + 1,000038462\right)/2.788695457$$

must hold to prevent the equality $Y_{0,4} = \varnothing$, and

$$k/C_0 < 1.000038462 + \left(\rho_0 + 1,000038462\right)/2.788695457$$

guarantees the inequality $\beta^* > 1$. If one takes $\rho_0 = -0.4$,[27] then one obtains $k/C_0 < 1.215363395$. This upper bound is quite realistic since it is 21% higher than the invested amount. Moreover $E_0 \leq 1.215363395$ is more than natural too, since biweekly returns higher than 21% hardly make sense. Thus, we have taken $\rho_0 = -0.4$, $E_0 = 1.000039$ (a little bit higher than $e^{rT} = 1.000038462$), $C_0 = 1$ and $k = 1.1$.

Proceeding as in Section 5.1, and bearing in mind that $\tilde{E} = 1$, it is easy to see that (41) becomes

$$\left\{ \begin{array}{l} Max\ \left(k - \tilde{C}_0\right) e'_{\beta,1} + \left(E_0 C_0 - k\right) e'_{\beta,2} - \left(\rho_0 C_0 + k\right) \nu_\rho \\ \left(\beta - 1\right) \mathbb{E}\left(z\right) + e'_{\beta,1} - e'_{\beta,2} - \nu_\rho = \beta \\ \left(\beta - 1\right) \mathbb{E}\left(y_j z\right) + \tilde{\Pi}_j e'_{\beta,1} - \mathbb{E}\left(y_j\right) e'_{\beta,2} - \mathbb{E}\left(y_j z_\rho\right) \nu_\rho = \beta \mathbb{E}\left(y_j\right), \quad j = 1,2,3,4 \\ e'_{\beta,1}, e'_{\beta,2}, \nu_\rho \geq 0,\ z \in B_{(\infty,+)},\ z_\rho \in \Delta_\rho \end{array} \right. \tag{93}$$

As in Section 5.1, this finite-dimensional problem becomes linear if ν_ρ is fixed, and consequently it can be solved by standard methods. Algorithm 5 leads to

[26] Recall that this risk measure is consistent with the second order stochastic dominance (Ogryczak and Ruszczynski, 2002).

[27] *i.e.*, $CVaR_{90\%}\left(y\right) \leq -0.4C_0$, or $CVaR_{90\%}\left(y - C_0\right) \leq -0.4C_0 + C_0 = 0.6C_0$, which means that the $CVaR_{90\%}$ of the capital earnings cannot be higher than 60 cents per invested dollar.

the optimal omega ratio $\beta^* = 1.4$, in which case the optimal value of (93) equals 0.001012978 (again close to zero) and its solution becomes

$$\left\{ \begin{array}{ll} z_8 = 0.647356271 & z_{\rho,2} = 9.498564473 \\ z_9 = 0 & z_{\rho,8} = 0.480696573 \\ z_j = 1, \ otherwise & z_{\rho,9} = 10 \\ e'_{\beta,1} = 4.971018485 & z_{\rho,14} = 5.020738954 \\ e'_{\beta,2} = 3.496670898 & z_{\rho,j} = 0, \ otherwise \\ \nu_\rho = 0.2 & \end{array} \right. \tag{94}$$

The combination $(x_0, x_1, x_2, x_3, x_4)$ of the available assets leading to the optimal omega 1.4 above may be trivially computed from (43), (60), and (94).

6 Conclusion

Asymmetric returns and heavy tails are provoking a growing interest in risk and performance measures beyond the classical standard deviation and Sharpe ratio. Thus, the omega ratio is deserving more and more attention in financial literature, since it focuses on both downside potential losses and upside potential gains.

This paper has provided new algorithms, optimality conditions and duality results to optimize omega in general Banach spaces. Both finite- and infinite-dimensional frameworks are accepted. This seems to be an interesting topic since many classical arbitrage free pricing models of Financial Mathematics ($B\&S$, Heston, SVM, etc.) often lead to problems involving infinite-dimensional spaces of random variables.

Representation theorems have plaid a critical role in both optimality conditions and duality results. As a consequence, for many important pricing models the optimal value of omega is often given by the quotient between the essential supremum and the essential infimum of the SDF. In particular, most of the continuous time stochastic pricing models make omega unbounded, and the sequence of investment strategies whose omega tends to infinity can be explicitly built. Hence, the investor may reach an omega ratio as large as desired with only one dollar (or one cent), an additional return and/or risk constraints do not modify this finding. The omega ratio may become as close to ∞ as desired, regardless of the amount to invest. This "surprising" property seems to be related with some "pathologies" pointed out in Balbás *et al.* (2010). For many coherent risk measures and pricing models there exist sequences of investment strategies whose couple $(risk, return)$ tends to $(-\infty, +\infty)$ or $(0, +\infty)$.

Section 5 has illustrated how the optimality conditions also apply in more classical portfolio choice problems only dealing with buy and hold investment strategies. Two numerical examples have been presented. The first example dealt with with infinite-dimensional spaces in order to optimize omega in an option market, while the second one dealt with finite-dimensional spaces in order to optimize omega in a portfolio selection problem involving two international indices and two commodities.

References

Anderson, E.J. and P. Nash, 1987. *Linear programming in infinite-dimensional spaces*. John Wiley & Sons, New York.

Artzner, P., F. Delbaen, J.M. Eber and D. Heath, 1999. Coherent measures of risk. *Mathematical Finance*, 9, 203–228.

Balbás, A., B. Balbás and R. Balbás, 2010. $CAPM$ and APT like models with risk measures. *Journal of Banking & Finance*, 34, 1166 - 1174.

Balbás, A., B. Balbás and R. Balbás 2016. Good deals and benchmarks in robust portfolio selection. *European Journal of Operational Research*, 250, 666 - 678.

Balbás, A., B. Balbás and R. Balbás, 2017. Differential equations connecting VaR and $CVaR$. *Journal of Computational and Applied Mathematics*, 326, 247 - 267.

Balbás, A. and A. Heras, 1993. Duality theory for infinite-dimensional multiobjective linear programming. *European Journal of Operational Research*, 68, 3, 379-388.

Bernardo, A.E. and O. Ledoit, 2000. Gain, loss, and asset pricing. *Journal of Political Economy*, 108, 144–172.

Charnes, A. and W. Cooper, 1962. Programming with linear fractional functionals. *Naval Research Logistics Quarterly*, 9, 181–186.

Craven, B.D., 1975. Sufficient Fritz John optimality conditions. *Bulletin of Australian Mathematical Society*, 13, 411 - 419.

Duffie, D., 1996. *Security Markets: Stochastic Models*. Addison-Wesley.

Gilli, M., E. Kellezi and H. Hysi, 2006. A data-driven optimization heuristic for downside risk minimization. *Journal of Risk*, 8, 3, 1–19.

Guastaroba, G., R. Mansini, W. Ogryczak and M.G. Speranza, 2016. Linear programming models based on omega ratio for the enhanced index tracking problem. *European Journal of Operational Research*, 251, 938–956.

Haugh, M.B. and A. W. Lo, 2001. Asset allocation and derivatives. *Quantitative Finance*, 1, 45-72.

Hull, J.C., 2012. *Options, futures, and other derivatives*. Prentice Hall (eight edition).

Kapsos, M., N. Christofides and B. Rustem, 2014a. Optimizing the omega ratio using linear programming. *European Journal of Operational Research*, 234, 499–507.

Kapsos, M., S. Zymler, N. Christofides and B. Rustem, 2014b. Optimizing the omega ratio using linear programming. *Journal of Operational Risk*, 9, 4, 49–57.

Kopp, P.E., 1984. *Martingales and stochastic integrals*. Cambridge University Press.

Luenberger, D.G., 1969. *Optimization by vector spaces methods*. John Wiley & Sons.

Mausser, H., D. Saunders and L. Seco, 2006. Optimizing omega. *Risk Magazine*, 11, 88–92.

Ogryczak, W. and A. Ruszczynski, 1999. From stochastic dominance to mean risk models: Semideviations and risk measures. *European Journal of Operational Research*, 116, 33–50.

Ogryczak, W. and A. Ruszczynski, 2002. Dual stochastic dominance and related mean risk models. *SIAM Journal on Optimization*, 13, 60–78.

Phelps, R.R., 2001. *Lectures on Choquet's theorem, 2nd edn.* In: Lecture Notes in Mathematics 1757. Springer.

Rockafellar, R.T., S. Uryasev and M. Zabarankin, 2006. Generalized deviations in risk analysis. *Finance & Stochastics*, 10, 51-74.

Shadwick, W. and C. Keating, 2002. A universal performance measure. *Journal of Performance Measurement*, 6, 3, 59–84.

Sharma A. and A Mehra, 2017. Extended omega ratio optimization for risk-averse investors. *International Transactions in Operational Research*, 24, 485–506.

Sharma A., S. Utz and A Mehra, 2017. Omega$-CVaR$ portfolio optimization and its worst case analysis. *OR Spectrum*, 39, 505–539.

Singh, A. and S. Dharmaraja, 2018. Optimal portfolio trading subject to stochastic dominance constraints under second-order autoregressive price dynamics. *International Transactions in Operational Research*, forthcoming, DOI: 10.1111/itor.12435.

Zakamouline, V. and S. Koekebbaker, 2009. Portfolio performance evaluation with generalized Sharpe ratios: Beyond the mean and variance. *Journal of Banking & Finance*, 33, 1242-1254.

Zeidler, E., 1995. *Applied functional analysis: Main principles and their applications.* Springer.

Zhao, P. and Q. Xiao, 2016. Portfolio selection problem with Value-at-Risk constraints under non-extensive statistical mechanics. *Journal of Computational and Applied Mathematics*, 298, 74-91.