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BARGAINING IN NETWORKS AND THE MYERSON VALUE *

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Abstract

We focus on a multiperson bargaining situation where the negotiation possibilities for the players are represented by a graph, that is, two players can negotiate directly with each other if and only if they are linked directly in the graph. The value of cooperation among players is given by a TU game. For the case where the graph is a tree and the TU game is strictly convex we present a noncooperative bargaining procedure, consisting of a sequence of bilateral negotiations, for which the unique subgame perfect equilibrium outcome coincides with the Myerson value of the induced graph-restricted game. In each bilateral negotiation, the corresponding pair of players bargains about the difference in payoffs to be received at the end. At the beginning of such negotiation there is a bidding stage in which both players announce prices. The player with the highest price becomes the proposer and makes a take-it-or-leave-it offer in terms of difference in payoffs to the other player. If the proposal is rejected, the proposer pays his announced price to the other player, after which this particular link is eliminated from the graph and the mechanism starts all over again for the remaining graph.

Keywords: Myerson value, implementation, cooperative games, networks.

JEL Classification: C71, C72

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1. Introduction

In this paper we study multiperson bargaining situations where the negotiation possibilities for the players are represented by a tree, that is, a connected nondirected graph without cycles. The interpretation is that two players can directly negotiate with each other if and only if there is a link between the two players in the tree. Moreover, there is a TU game which assigns to every coalition of players the surplus that can be achieved if all players in this coalition cooperate. Each pair of players linked directly in the graph bargains about their difference in payoffs to be received from cooperation. It is assumed that a coalition of players is able to extract the full surplus from cooperation if and only if each pair of coalition members linked directly in the graph has reached an agreement upon its difference in payoffs. At the end of the bargaining process the coalitions formed are exactly the maximal coalitions for which the full surplus from cooperation can be achieved. For such a coalition there is a unique allocation which distributes the full surplus among its members and respects the agreed upon bilateral differences in payoffs.

The situation described above is mathematically equivalent to a graph-restricted cooperative game, which consists of a TU game and a graph representing the communication possibilities among the players. A well-known solution for such games is the Myerson value (Myerson, 1977), which is an extension of the Shapley value to graph-restricted games. In this paper we present a noncooperative bargaining procedure which leads to the Myerson value in case the graph is a tree and the underlying TU game is strictly convex. By considering strictly convex games, we assume that the marginal contribution of a player to a coalition is increasing in the size of the coalition.

The bargaining procedure consists of a sequence of bilateral negotiations in which pairs of players bargain about their difference in payoffs to be realized at the end. Since a pair of players can only negotiate directly if there is a link connecting them, each bilateral negotiation corresponds to a link in the tree. Before the bargaining procedure starts, we choose an order of the links such that at any moment the set of remaining links is connected. For every link $\{i, j\}$, let Δ_{ij} be the difference in payoffs about which the players bargain. The interpretation of Δ_{ij} is that, if in the future all pairs of players agree upon their difference, player i's share from cooperation should exceed player j's share by the amount of Δ_{ij}. At the beginning of this bilateral negotiation both players announce a price. The player with the highest price is then allowed to propose a difference Δ_{ij}. The other player may accept or reject this difference. In case of acceptance both players commit to this difference Δ_{ij} and the mechanism turns to the following link. In case of rejection the proposer has to pay his announced price to the other player, the link $\{i, j\}$ is eliminated and the mechanism starts all over again for the remaining graph. If the remaining graph is no longer connected, the mechanism is applied separately to each of the connected components in this remaining graph.

The procedure continues until all links in the remaining graph have been considered. The final outcome of the procedure will consist of (1) a final graph, which is a forest (a disjoint collection of trees), (2) the agreed upon differences Δ_{ij} for this forest and, possibly, (3) for each player a collection of prices which he is to pay or receive as the result of rejected proposals. The final payoffs for the players are then defined as follows. For every tree in the resulting forest there is a unique allocation x which distributes the value of the tree (or, to be more precise, the value of the coalition consisting of the players in this tree) among its members, respecting the agreed upon differences Δ_{ij}. The final payoff for a player is then the sum of his payoff in x and the net sum of the prices received and the prices paid.

The main result in this paper is that there is a unique subgame perfect equilibrium outcome to this procedure, in which all proposed differences are accepted and the final payoffs for the players coincide with the Myerson value applied to the induced graph-restricted game.

At this stage we wish to compare this procedure with related bargaining mechanisms proposed in the literature. The bargaining procedures that perhaps come closest to our mechanism are the bidding-for-the-surplus procedure proposed in Pérez-Castrillo and Wettstein (2001) and Mutuswami, Pérez-Castrillo and Wettstein (2001) and the mechanism proposed by Moulin (1984). Similarly to our mechanism, the latter procedures also consist of a bidding stage, at which the role of proposer is decided by means of an auction, and a bargaining stage at which the proposer makes an offer to other players. The mechanism in Pérez-Castrillo and Wettstein (2001) implements the Shapley value for zero-monotonic TU games, whereas the mechanism in Mutuswami et al. (2001) implements the Myerson value for situations where the value from cooperation does not only depend on the coalition formed but also on the communication graph which connects the players in the coalition[1]. The bidding-for-the-surplus mechanism consists of three stages. In the first stage every player i chooses for each of the other players j a price that i is willing to pay to j. The net bid of a player is the sum of the prices he is willing to pay to the other players minus the sum of the prices the other players want to pay to him. The player with the highest net bid becomes the proposer. At the second stage, the proposer pays his prices to the other players and announces an allocation for each of them. In the third stage, the other players sequentially accept or reject this proposal. In case of a rejection by at least one player the proposer leaves the game and obtains his stand-alone payoff, and the mechanism starts again without considering this player. In case of unanimous acceptance, every player except the proposer receives his share of the allocation and the proposer gets the value of the grand coalition minus the payments made to the other players. Other noncooperative bargaining procedures which implement the Shapley value are, for instance, Dasgupta and Chiu (1998), Gul (1989), Hart and Mas-Colell (1996) and Winter (1994). Mutuswami et al. (2001) extend the bidding-for-the-surplus mechanism to the provision of public goods and network formation. The difference with respect to the former mechanism is that at the second stage the proposer announces in addition a coalition and a connected graph on such coalition. Vidal and Bergantiños (2001) apply the bidding-for-the-surplus mechanim to implement the Owen value, which is an extension of the Shapley value to cooperative situations where players are organized in a-priori unions.

Moulin (1984) implements the Kalai-Smorodinsky bargaining solution. In this mechanism there is a bidding stage and a bargaining stage. At the bidding stage players simultaneously announce a probability. The player with the highest probability, say p_i, is allowed to make an offer to the rest of players, which sequentially accept or reject such an offer starting from the player with the lowest probability. If there is a rejection, the player who rejects is allowed to make a counteroffer to the rest of players. If at least one player rejects this counteroffer,

[1] The Myerson value in this more general context is also called the Jackson and Wolinsky allocation rule (Jackson and Wolinsky (1996)).

then the status quo point is enforced. If there is unanimous approval, then this counteroffer is implemented with probability p_i, whereas the status quo point is implemented with probability $1 - p_i$. Bergantiños, Casas-Méndez and Vázquez-Brage (2000) have extended the definition of the Kalai-Smorodinsky bargaining solution to NTU games and have modified Moulin's mechanism in order to implement the latter solution.

In comparison with the mechanisms above, the procedure presented in this paper differs in various aspects. First of all, the negotiations in our procedure are always bilateral, whereas in all of the above mechanisms, except Gul's, negotiations are multilateral. Another important difference with the mechanisms above lies in the procedure which is followed after a proposal has been rejected. In the bidding-for-the-surplus mechanism, for instance, the player whose proposal has been rejected should leave the game, whereas in our mechanism only the link for which the difference has been negotiated is eliminated, but the proposer stays in the game as long as he has other links left. Finally, the role of the prices is different within our procedure in comparison with Gul (1989) and the bidding-for-the-surplus mechanim. In the latter two procedures prices are always paid in equilibrium, whereas in our mechanism the announced prices are paid only if a proposal is rejected. Hence, in equilibrium, no prices are paid.

The paper is organized as follows. Section 2 presents some basic definitions. In Section 3 we present the mechanism and show that it implements the Myerson value. Section 4 contains some concluding remarks.

2. The model

Let $N = \{1, ..., n\}$ be the set of players and let 2^N the set of all possible coalitions. A characteristic function is a function $v : 2^N \to \Re$ which defines for every coalition the value or worth obtained from cooperation among players inside this coalition. We assume that $v(\emptyset) = 0$. The pair $[N; v]$ is called a cooperative game with transferable utility (TU-game). We shall assume throughout the paper that the game $[N; v]$ is strictly convex, that is,

$$v(S \cup \{i; j\}) - v(S \cup \{i\}) > v(S \cup \{j\}) - v(S)$$

for all $i \neq j$ and all $S \subseteq N \backslash \{i; j\}$.

A TU-allocation rule is a function ψ which assigns to every TU-game $[N; v]$ a payoff vector $\psi(N; v) \in \Re^n$, where n is the number of players.

Definition 2.1. The Shapley value is the allocation rule ϕ with

$$\phi_i(N; v) = \sum_{S \subseteq N \backslash \{i\}} \frac{s!(n - s - 1)!}{n!} [v(S \cup \{i\}) - v(S)]$$

for all TU-games $[N; v]$ and all players i; where $s = |S|$; $n = |N|$:

Consider a graph g consisting of a set of n nodes and a set of undirected links. The nodes represent the players and the links are denoted by $\{i; j\}$; with $i; j \in N$. The interpretation is that players i and j can negotiate directly if and only if the link $\{i; j\}$ is in g.

The triple $[N; v; g]$ is called a graph-restricted cooperative game with transferable utility. This triple is also called a network or a communication situation. An allocation rule in this context is a function which assigns to every graph-restricted cooperative game a payoff vector in $\mathbb{R}^n$, where n is the number of players.

In a graph g, a path is a sequence of adjacent links $\{i_1; i_2\}; \{i_2; i_3\}; \ldots; \{i_{K-1}; i_K\}$ with $\{i_k; i_{k+1}\} \in g$ and all $i_1; i_2; \ldots; i_K$ pairwise different. For a coalition $S \subseteq N$ let g^S be the restriction of the graph g to nodes in S. We say that S is a maximal connected coalition if (1) for every two nodes $i; j \in S$ there is a path in g^S connecting them, and (2) for every $i \in S$ and $j \notin S$ there is no path in g connecting them. Let $N|g$ be the partition of N consisting of the maximal connected coalitions induced by g: Similarly, we define $S|g$ to be the collection of maximal connected coalitions in S induced by g^S. A graph g is said to be connected if N is the unique maximal connected coalition. A graph g is called a tree if for every two nodes there is a unique path connecting them. We say that g is a forest if it is a disjoint union of trees.

Given a graph-restricted game $[N; v; g]$, we define the following auxiliary TU-game, to which we refer as the point game induced by $[N; v; g]$.

Definition 2.2. The point game $[N; v|g]$ of a graph-restricted cooperative game $[N; v; g]$ is the TU game defined by

$$v|g(S) = \sum_{T \in S|g} v(T)$$

for all $S \subseteq N$:

Hence, the value of a coalition in the point game is given by the sum of the values of its maximal connected coalitions.

Definition 2.3. The Myerson value of a graph-restricted cooperative game $[N; v; g]$ is the Shapley value applied to the induced point game $[N; v|g]$.

We denote the Myerson value of a graph-restricted cooperative game $[N; v; g]$ by $m(N; v; g)$. Myerson (1977) has provided the following axiomatic characterization.

Theorem 2.4. (Myerson, 1977) The Myerson value is the only allocation rule Ψ which satisfies the following two axioms:

1. Component efficiency
$\sum_{i \in S} \Psi_i(N; v; g) = v(S)$ for all $[N; v; g]$ and all maximal connected coalitions $S \in N|g$.
2. Fairness
$\Psi_i(N; v; g) - \Psi_i(N; v; g \backslash \{i; j\}) = \Psi_j(N; v; g) - \Psi_j(N; v; g \backslash \{i; j\})$ for all $[N; v; g]$ and all links $\{i; j\} \in g$. Here, $g \backslash \{i; j\}$ is the graph which remains after deleting the link $\{i; j\}$:

3. The mechanism

In this section we present a noncooperative bargaining procedure which for a given tree g^* and strictly convex TU-game $[N; v]$ yields the Myerson value $m(N; v; g^*)$ as the unique subgame perfect equilibrium outcome. Since N and v are fixed, we shall write $m(g^*)$ instead of $m(N; v; g^*)$.

Fix a tree g^* and an order π over the links in g^*. For every link $l \in g^*$ let $L^+(l|\pi)$ be the set of links which weakly follow link l given the order π. We say that the order π is *regular* if for every link l the graph $L^+(l|\pi) \subseteq g^*$ is connected. Note that in any regular π the link l is always an exterior link in $L^+(l|\pi)$. Let ϕ be a function which assigns to every subgraph $g \subseteq g^*$ a regular order on the links in g.

The bargaining procedure $\Gamma(v; g^*; \phi)$ is defined as follows. Let $\pi = \phi(g^*)$. Suppose that the link $\{i; j\}$ is reached by π. Then, players i and j enter the following two-step bargaining procedure.

Step 1

Players i and j simultaneously choose a non-negative price. The player with the highest bid will be the proposer in step 2. If there is a draw, the player which is not a terminal node in $L^+(\{i; j\}|\pi)$ will be the proposer in step 2. In case $\{i; j\}$ is the last link, and thus both players are terminal nodes in $L^+(\{i; j\}|\pi)$, the player with the lowest index becomes the proposer.

Step 2

The proposer offers a difference in payoffs $\Delta_{ij} \in \mathbb{R}$ and the other player can accept or reject this difference.

If Δ_{ij} is accepted, choose the next link according to the order π and return to step 1 for this link.

If Δ_{ij} is rejected, then the proposer must pay the price he bid at step 1 to the other player and the link $\{i; j\}$ is deleted from the graph. Afterwards, the procedure above is applied to the reduced graph $g^* \setminus \{i; j\}$ with respect to the order $\phi(g^* \setminus \{i; j\})$. If $g^* \setminus \{i; j\}$ is not connected, then it is understood that the procedure is applied to each of the trees in $g^* \setminus \{i; j\}$ separately. If, for instance, the difference at link $\{h; k\}$ is rejected in $g^* \setminus \{i; j\}$, the procedure starts for the remaining graph $g^* \setminus (\{i; j\} \cup \{h; k\})$, and so on, until there are no links left. In the latter case, every player i receives his stand-alone payoff $v(i)$.

The procedure stops whenever all links in the actual graph have been accepted or there are no links left. Therefore, the procedure stops after finitely many steps, since g^* has finitely many links and after every rejection one link is deleted from the actual graph. At the end, we arrive at a final subgraph $g^F \subseteq g^*$ for which all differences have been accepted. By construction, g^F is a forest. For every tree g' in g^F there is a unique component efficient payoff vector x for the players in g' which respects all the agreed upon differences for the links in g'. The final payoff for a player i in g' is given by x plus the prices received minus the prices payed as the result of rejections in the past.

We assume that the players in the bargaining procedure play a subgame perfect equilibrium with the following tie-breaking rule: (1) if a player is indifferent between accepting a difference Δ_{ij} or not, he is supposed to accept, (2) if player j is a terminal node in $L^+(\{i; j\}|\pi)$ and player i is not, then, if player i is indifferent between proposing difference Δ_{ij} and $\bar{\Delta}_{ij}$ with $\Delta_{ij} < \bar{\Delta}_{ij}$, he is supposed to propose $\bar{\Delta}_{ij}$, and (3) if player i is indifferent between choosing prices p^1 and p^2 at the bidding stage, with $p^1 < p^2$, then he is supposed to choose price p^1. In the sequel, when we write subgame perfect equilibrium, we always mean subgame perfect equilibrium satisfying this tie-breaking rule.

Theorem 3.1. *Let the TU-game $[N; v]$ be strictly convex, g^* a tree which connects all players in N and ϕ a function which assigns to every subgraph of g^* a regular order over the links.*

Then, the mechanism $\Gamma(v; g^*; \prec)$ has a unique subgame perfect equilibrium outcome. In this outcome all the differences proposed in g^* are accepted and the final payoffs for the players coincide with the Myerson value $m(N; v; g^*)$.

Proof of Theorem 3.1. We prove this result by induction on the number of links in g^*. If g^* has no link, the result is trivial. Consider now a graph g^* with K links. For the sake of convenience we assume that g^* is a tree which connects all players. If not, each tree of the forest g^* could be treated separately. Assume that the result holds for every forest with at most $K - 1$ links. We prove that the statement in the theorem holds for g^*.

We need the following notation. Let L be the set of all links in g^*. For a given link $\{i, j\}$ let $L^-(\{i, j\})$ be the set of links which preceed $\{i, j\}$ and let $L^+(\{i, j\})$ be the set of links which weakly follow $\{i, j\}$ given the order $\prec(g^*)$. For a given profile of differences $\Delta = (\Delta_l)_{l \in L}$ let $x(\Delta) \in \mathbb{R}^n$ be the unique component efficient payoff vector which respects the differences in Δ. If players i and j have agreed upon a difference Δ_{ij}, then, whenever all future differences are accepted, the payoffs x_i and x_j for players i and j are such that $x_i - x_j = \Delta_{ij}$. We use the following convention: for every link $\{i, j\}$, if we write Δ_{ij}, then j is a terminal node in $L^+(\{i, j\})$. Recall that $\prec(g^*)$ is a regular order, and hence $\{i, j\}$ is an exterior link in $L^+(\{i, j\})$. For every link $\{i, j\}$ in L let

$$\Delta^*_{ij} = m_i(N; v; g^* \setminus \{i, j\}) - m_j(N; v; g^* \setminus \{i, j\}):$$

We refer to Δ^*_{ij} as the fair difference at $\{i, j\}$. In the sequel, for a subgraph $g \subseteq g^*$, we simply write $m(g)$ to denote the Myerson value of the graph-restricted game $[N; v; g]$, since $[N; v]$ is fixed. By Theorem 2.4 the Myerson value $m(g^*)$ is the unique component efficient allocation which respects all fair differences in L. By $\Delta_{L^-(\{i,j\})}$ we denote a profile of past differences that has been agreed upon before reaching $\{i, j\}$. For a given $\Delta_{L^-(\{i,j\})}$, let $\Gamma\left(\Delta_{L^-(\{i,j\})}\right)$ be the subgame starting at link $\{i, j\}$ where all past differences have been accepted and coincide with $\Delta_{L^-(\{i,j\})}$. For every difference Δ_{ij} let $\Gamma\left(\Delta_{L^-(\{i,j\})}; \Delta_{ij}\right)$ be the subgame starting directly after $\{i, j\}$ in which the past differences are given by $\left(\Delta_{L^-(\{i,j\})}; \Delta_{ij}\right)$. Let $D^a_{ij}\left(\Delta_{L^-(\{i,j\})}\right)$ be the set of differences Δ_{ij} for which the induced subgame $\Gamma\left(\Delta_{L^-(\{i,j\})}; \Delta_{ij}\right)$ contains a subgame perfect equilibrium in which all proposed differences are accepted. Define $\Delta^{min}_{ij}\left(\Delta_{L^-(\{i,j\})}\right) = \inf D^a_{ij}\left(\Delta_{L^-(\{i,j\})}\right)$. Note that $\Delta^{min}_{ij}\left(\Delta_{L^-(\{i,j\})}\right)$ can be $-\infty$ or ∞. However, we will show that $\Delta^{min}_{ij}\left(\Delta_{L^-(\{i,j\})}\right)$ is finite whenever $\{i, j\}$ is not the last link. Let $X^a\left(\Delta_{L^-(\{i,j\})}; \Delta_{ij}\right)$ be the set of payoff vectors in $\mathbb{R}^n$ induced by subgame perfect equilibria σ in the subgame $\Gamma\left(\Delta_{L^-(\{i,j\})}; \Delta_{ij}\right)$ where all differences are accepted in σ. Note that $X^a\left(\Delta_{L^-(\{i,j\})}; \Delta_{ij}\right)$ is nonempty if and only if $\Delta_{ij} \in D^a_{ij}\left(\Delta_{L^-(\{i,j\})}\right)$. Let

$$X^a\left(\Delta_{L^-(\{i,j\})}\right) = \bigcup_{\Delta_{ij}} X^a\left(\Delta_{L^-(\{i,j\})}; \Delta_{ij}\right):$$

By $X^a_{ij}\left(\Delta_{L^-(\{i,j\})}\right)$ we denote the projection of the set $X^a\left(\Delta_{L^-(\{i,j\})}\right)$ on $\mathbb{R}^{\{i,j\}}$. The set $X^a_{ij}\left(\Delta_{L^-(\{i,j\})}\right)$ may be interpreted as the set of achievable payoffs for players i and j when they face a history given by $\Delta_{L^-(\{i,j\})}$ and all future differences are to be accepted. Let $\Delta^*_{L^+(\{i,j\})} = (\Delta^*_l)_{l \in L^+(\{i,j\})}$ be the profile of fair differences in the subgame $\Gamma\left(\Delta_{L^-(\{i,j\})}\right)$. We define the set

$$D_{L-\{i,j\}} = \left\{ \Delta_{L-\{i,j\}} \mid x\left(\Delta_{L-\{i,j\}}; \Delta^*_{L^+(\{i,j\})}\right) > m(g \setminus l) \text{ for all } l \in L \right\};$$

where the inequality should be read coordinatewise. We prove the following lemma.

Lemma 3.2. Consider a subgame $\Gamma\left(\Delta_{L-(\{i,j\})}\right)$. Then the following properties are satisfied.
(1.a) $D^a_{ij}\left(\Delta_{L-(\{i,j\})}\right) = \left[\Delta^{min}_{ij}\left(\Delta_{L-(\{i,j\})}\right); \infty\right)$.
(1.b) The sets $D^a_{ij}\left(\Delta_{L-(\{i,j\})}\right)$ and $X^a_{ij}\left(\Delta_{L-(\{i,j\})}\right)$ depend continously on $\Delta_{L-(\{i,j\})}$.
(1.c) The set of payoffs $X^a_{ij}\left(\Delta_{L-(\{i,j\})}\right)$ consists of a connected union of non-increasing non-horizontal[2] line segments.
(2) If $\Delta_{L-(\{i,j\})} \in D_{L-(\{i,j\})}$, then there is a unique subgame perfect equilibrium outcome in $\Gamma\left(\Delta_{L-(\{i,j\})}\right)$ where at every link the corresponding players agree on the fair difference.
(3) If $\Delta^{min}_{ij}\left(\Delta_{L-(\{i,j\})}\right) > \Delta^*_{ij}$, then in every subgame perfect equilibrium in $\Gamma\left(\Delta_{L-(\{i,j\})}\right)$ for which all differences are accepted, players i and j agree on the difference $\Delta^{min}_{ij}\left(\Delta_{L-(\{i,j\})}\right)$.
(4) If $\Delta^{min}_{ij}\left(\Delta_{L-(\{i,j\})}\right) \leq \Delta^*_{ij}$, then in every subgame perfect equilibrium in $\Gamma\left(\Delta_{L-(\{i,j\})}\right)$ for which all differences are accepted, players i and j agree on the fair difference Δ^*_{ij}.

Proof of Lemma 3.2. We prove this result by induction on the number of links that follow $\{i,j\}$. Consider a profile of differences $\Delta_{L-(\{i,j\})}$. From now on, we will omit $\Delta_{L-(\{i,j\})}$ from the variables whenever this cannot lead to confusion.

Suppose first that there is no link following $\{i,j\}$.

(1) Note that if the link $\{i,j\}$ is built, the grand coalition is formed. Given the past differences $\Delta_{L-(\{i,j\})}$ each $\Delta_{ij} \in \Re$ induces the unique payoff vector $x \in \Re^n$ satisfying

$$\sum_{r \in N} x_r = v(N);$$
$$x_h - x_k = \Delta_{hk} \text{ for all } \{h,k\} \in L. \quad (3.1)$$

Let

$$S_i = \{i\} \cup \{r \in N \mid \text{there is a path in } g \setminus \{i,j\} \text{ connecting } r \text{ and } i\}.$$

Let s_i be the number of player in S_i. Similarly we define s_j. For every link $\{h,k\} \neq \{i,j\}$ we define

$$S_h(\{h,k\}) = \{h\} \cup \{r \in N \mid \text{there is a path in } g \setminus \{h,k\} \text{ connecting } r \text{ and } h\};$$

and $s_h(\{h,k\})$ as the cardinality of $S_h(\{h,k\})$. Similarly we define $S_k(\{h,k\})$. Let

$$c(\{h,k\} \mid \{i,j\}) = \begin{cases} s_k(\{h,k\}); & \text{if path from } \{i,j\} \text{ to } k \text{ contains } h \\ -s_h(\{h,k\}); & \text{otherwise.} \end{cases}$$

[2] Our convention is to put player i's payoff on the horizontal axis and player j's payoff on the vertical axis. Recall that, by convention, player j is the exterior node in $L^+(\{i,j\})$. Non-horizontal thus means that player j's payoff cannot be constant on any of the line segments.

Then it may be verified that system (3.1) is equivalent to

$$\begin{aligned} s_i x_i + s_j x_j &= v(N) + \sum_{l \in L \setminus \{i,j\}} c(l|\{i,j\})\Delta_l; \\ x_i - x_j &= \Delta_{ij}. \end{aligned} \tag{3.2}$$

Consequently,

$$X_{ij}^a = \left\{ (x_i, x_j) \in \mathbb{R}^2 | s_i x_i + s_j x_j = v(N) + \sum_{l \in L - (\{i,j\})} c(l|\{i,j\})\Delta_l \right\}. \tag{3.3}$$

Since s_i and s_j are strictly positive and Δ_l is given for $l \neq \{i,j\}$, the set X_{ij}^a is a strictly decreasing line.

(1.a) is satisfied since $\Delta_{ij}^{min} = -1$ and $D_{ij}^a = (-1, 1)$, (1.b) and (1.c) follow inmediately from (3.3).

(2) Suppose that $\Delta_{L-(\{i,j\})} \in D_{L-(\{i,j\})}$. Let $e_i = m_i(g^* \setminus \{i,j\})$ and $e_j = m_j(g^* \setminus \{i,j\})$. Then, define the price

$$p^* = \max \left\{ p \in \mathbb{R} | \exists (x_i, x_j) \in X_{ij}^a \text{ such that } x_i \geq e_i + p \text{ and } x_j \geq e_j + p \right\}.$$

Assume that $i < j$. If a difference Δ_{ij} is rejected, the procedure for the reduced graph $g^* \setminus \{i,j\}$ starts. By the induction hypothesis at the beginning of the proof, the procedure for the reduced graph $g^* \setminus \{i,j\}$ yields the Myerson value $m(g^* \setminus \{i,j\})$. Suppose that player i is the proposer and has chosen price p. Then $e_j + p$ can be seen as the outside option for player j, since by rejecting player i's difference he receives the price p from player i and gets payoff e_j in the procedure for the reduced graph $g^* \setminus \{i,j\}$. Similarly, if player j is the proposer and has chosen price p, $e_i + p$ can be seen as the outside option for player i. Since $\Delta_{L-(\{i,j\})} \in D_{L-(\{i,j\})}$, it is easily seen that $p^* > 0$. We prove now that players i and j can guarantee a payoff $e_i + p^*$ and $e_j + p^*$, respectively, by choosing the price p^* at the bidding stage. Consider player i. If player j wins the auction, i.e., $p_j > p^*$, then player i can guarantee the payoff $x_i = e_i + p_j > e_i + p^*$ by rejecting player j's offer. If player i wins the auction by choosing p^* he may offer the difference Δ_{ij}^* which induces the payoff pair $(e_i + p^*, e_j + p^*)$. Player j will then be indifferent between accepting this difference and rejecting. By the tie-breaking rule player j will then accept. Hence, player i can guarantee payoff $e_i + p^*$ by choosing price p^*. Similarly for player j.

So any equilibrium at this step should yield expected payoffs $x_i \geq e_i + p^*$ and $x_j \geq e_j + p^*$. But there is only one feasible pair $(x_i, x_j) \in X_{ij}$ such that $x_i \geq e_i + p^*$ and $x_j \geq e_j + p^*$, namely $(x_i^*, x_j^*) = (e_i + p^*, e_j + p^*)$. This implies that, if there is an equilibrium at this stage, it should imply payoffs $(e_i + p^*, e_j + p^*)$. It may be verified by the reader that there is a unique equilibrium behavior in which both players choose price p^* at step 1, player i offers the difference Δ_{ij}^* which induces the payoff pair $(e_i + p^*, e_j + p^*)$ and player j accepts this difference. Hence, property (2) holds.

(3) Since $\Delta_{ij}^{min} = -1$, it cannot be the case that $\Delta_{ij}^{min} > \Delta_{ij}^*$, and therefore there is nothing to show.

(4) This property is shown in the same way as (2). This completes the proof of the lemma for the last link.

Now consider some link $\{i,j\}$ which is followed by at least one other link. By induction, assume that for every link $\{h,k\}$ following $\{i,j\}$ and for every profile of differences agreed upon until $\{h,k\}$ the properties (1) to (4) hold. We prove that properties (1) to (4) hold for link $\{i,j\}$ and for every profile of differences $\Delta_{L^-(\{i,j\})}$.

(1.a) Let $\Delta^1_{ij} \in D^a_{ij}$ and $\Delta^2_{ij} \geq \Delta^1_{ij}$. We prove that $\Delta^2_{ij} \in D^a_{ij}$, which would imply (1.a). Let $\{h,k\}$ be the link which immediately follows $\{i,j\}$. By induction assumptions (3) and (4) applied to link $\{h,k\}$, we know that Δ^1_{ij} induces a unique difference Δ^1_{hk} which is agreed upon at link $\{h,k\}$ in equilibrium.

For every profile of differences $(\Delta_l)_{l \in L}$ the final payoffs for the players are given by

$$\begin{aligned} \sum_{r \in N} x_r &= v(N); \\ x_i - x_j &= \Delta_{ij}; \\ x_m - x_r &= \Delta_{mr}; \end{aligned}$$

for all $\{m,r\} \in L \setminus \{i,j\}$. This system of equations implies

$$x_i(\Delta) = \frac{1}{n}\left[v(N) + \sum_{l \in L \setminus \{i,j\}} c(l \mid \{i,j\})\, \Delta_l + s_j \Delta_{ij}\right]; \tag{3.4}$$

$$x_m(\Delta) = \frac{1}{n}\left[v(N) + s_r(\{m,r\})\, \Delta_{mr} + \sum_{l \in L \setminus (\{i,j\} \cup \{m,r\})} c(l \mid \{m,r\})\, \Delta_l + c(\{i,j\} \mid \{m,r\})\, \Delta_{ij}\right]; \tag{3.5}$$

for all links $\{m,r\} \neq \{i,j\}$; where $\Delta = (\Delta_l)_{l \in L}$.

For all links $\{m,r\} \in L^+(\{h,k\})$ we have, since the rule of order π is regular, that $c(\{i,j\} \mid \{m,r\}) = s_j$ and $c(\{h,k\} \mid \{m,r\}) = s_k$. Recall that, by convention, j is an exterior node in $L^+(\{i,j\})$ and k is an exterior node in $L^+(\{h,k\}) \subset L^+(\{i,j\})$, and hence every link $\{m,r\} \in L^+(\{h,k\})$ belongs to S_i and $S_h(\{h,k\})$. As such, $c(\{i,j\} \mid \{m,r\}) = s_j$ and $c(\{h,k\} \mid \{m,r\}) = s_k$ for all links $\{m,r\} \in L^+(\{h,k\})$. Suppose that Δ^2_{hk} is such that

$$s_j \Delta^1_{ij} + s_k(\{h,k\})\, \Delta^1_{hk} = s_j \Delta^2_{ij} + s_k(\{h,k\})\, \Delta^2_{hk}. \tag{3.6}$$

By the system of equations (3.5), it is easily verified that the subgame $\Gamma\left(\Delta_{L^-(\{i,j\})}, \Delta^1_{ij}, \Delta^1_{hk}\right)$ is equivalent, for the players remaining in this subgame, to the subgame $\Gamma\left(\Delta_{L^-(\{i,j\})}, \Delta^2_{ij}, \Delta^2_{hk}\right)$. Note that the players remaining in these subgames are exactly the players in $L^+(\{h,k\})$. The fact that both subgames are equivalent for the players in $L^+(\{h,k\})$ follows from the following observation: for every profile of differences $\Delta_{L^+(\{h,k\}) \setminus \{h,k\}}$ we have that

$$\begin{aligned} & x_m\left(\Delta_{L^-(\{i,j\})}, \Delta^1_{ij}, \Delta^1_{hk}, \Delta_{L^+(\{h,k\}) \setminus \{h,k\}}\right) \\ = \; & x_m\left(\Delta_{L^-(\{i,j\})}, \Delta^2_{ij}, \Delta^2_{hk}, \Delta_{L^+(\{h,k\}) \setminus \{h,k\}}\right) \end{aligned}$$

for every player m in $L^+(\{h,k\})$. The latter equation follows from the system of equations (3.5).

We know by induction assumptions (3) and (4) that δ^1_{ij} and δ^1_{hk} induce a unique subgame perfect equilibrium outcome in the subgame $\Gamma\left(\delta_{L-\{i,j\}}, \delta^1_{ij}, \delta^1_{hk}\right)$ in which all differences are accepted. Let δ be the profile of differences accepted in this outcome in the subgame $\Gamma\left(\delta_{L-\{i,j\}}, \delta^1_{ij}, \delta^1_{hk}\right)$. Since the subgames $\Gamma\left(\delta_{L-\{i,j\}}, \delta^1_{ij}, \delta^1_{hk}\right)$ and $\Gamma\left(\delta_{L-\{i,j\}}, \delta^2_{ij}, \delta^2_{hk}\right)$ are equivalent, we may thus conclude that the latter subgame has a unique subgame perfect equilibrium outcome in which the profile of differences δ is accepted.

Let $\left(x^1_h, x^1_k\right)$ be the payoff pair for players h and k induced by $\left(\delta_{L-\{i,j\}}, \delta^1_{ij}, \delta^1_{hk}, \delta\right)$ and let $\left(x^2_h, x^2_k\right)$ be induced by $\left(\delta_{L-\{i,j\}}, \delta^2_{ij}, \delta^2_{hk}, \delta\right)$. By equation (3.5) applied to player h we obtain that $x^1_h = x^2_h$. Recall that, by convention, player k is a terminal node in $L^+(\{h,k\})$. From equation (3.6) it follows that $\delta^2_{hk} \leq \delta^1_{hk}$. Since $x^1_k = x^1_h - \delta^1_{hk}$, $x^2_k = x^2_h - \delta^2_{hk}$ and $x^1_h = x^2_h$, it follows that $x^2_k \geq x^1_k$. Since $\left(x^1_h, x^1_k\right)$ is a subgame perfect equilibrium payoff, we know, in particular, that $\left(x^1_h, x^1_k\right)$ is not dominated, for players h and k, by any outcome in which one future difference is rejected. Since $x^1_h = x^2_h$ and $x^2_k \geq x^1_k$, the payoff pair $\left(x^2_h, x^2_k\right)$ is not dominated, for players h and k, by any outcome in which one future difference is rejected. As such, there is a subgame perfect equilibrium in $\Gamma\left(\delta_{L-\{i,j\}}, \delta^2_{ij}\right)$ where all differences are accepted. Hence, $\delta^2_{ij} \in D^a_{ij}$.

(1.b) Let $\{m,r\}$ be the link which immediately preceeds $\{i,j\}$. Fix a profile of differences $\delta_{L-\{m,r\}}$. We show that D^a_{ij} and X^a_{ij} depend continuously on δ_{mr}. By induction, this would imply eventually that D^a_{ij} and X^a_{ij} depend continuously on $\delta_{L-\{i,j\}}$. For every δ_{mr}, define $D^a_{ij}(\delta_{mr}) = D^a_{ij}\left(\delta_{L-\{m,r\}}, \delta_{mr}\right)$ and $\delta^{min}_{ij}(\delta_{mr}) = \delta^{min}_{ij}\left(\delta_{L-\{m,r\}}, \delta_{mr}\right)$. We show the following claim.

Claim 1. For all δ^1_{mr}, δ^2_{mr} we have that

$$\delta^{min}_{ij}\left(\delta^2_{mr}\right) = \delta^{min}_{ij}\left(\delta^1_{mr}\right) + \frac{s_r(\{m,r\})}{s_j}\left[\delta^1_{mr} - \delta^2_{mr}\right].$$

Proof of Claim 1. Define

$$\delta^2_{ij} = \delta^{min}_{ij}\left(\delta^1_{mr}\right) + \frac{s_r(\{m,r\})}{s_j}\left[\delta^1_{mr} - \delta^2_{mr}\right].$$

Note that, by construction,

$$s_r(\{m,r\})\,\delta^1_{mr} + s_j\,\delta^{min}_{ij}\left(\delta^1_{mr}\right) = s_r(\{m,r\})\,\delta^2_{mr} + s_j\,\delta^2_{ij}.$$

By choosing δ^2_{ij} after δ^2_{mr} the induced subgame $\Gamma\left(\delta_{L-\{m,r\}}, \delta^2_{mr}, \delta^2_{ij}\right)$ is equivalent for the remaining players to the subgame $\Gamma\left(\delta_{L-\{m,r\}}, \delta^1_{mr}, \delta^{min}_{ij}\left(\delta^1_{mr}\right)\right)$. This follows from the system of equations (3.5). Since $\delta^{min}_{ij}\left(\delta^1_{mr}\right) \in D^a_{ij}\left(\delta^1_{mr}\right)$ we know that in the latter subgame there is a unique equilibrium outcome in which all future differences will be accepted. Hence, by choosing δ^2_{ij} after δ^2_{mr}, all future differences after $\{i,j\}$ will be accepted too. Then, by definition, $\Delta_{ij}^2 \in D_{ij}^a\left(\Delta_{mr}^2\right)$, which implies that

$$\Delta_{ij}^{min}\left(\Delta_{mr}^2\right) \le \Delta_{ij}^2 = \Delta_{ij}^{min}\left(\Delta_{mr}^1 + \frac{s_r(\{m,r\})}{s_j}\left[\Delta_{mr}^1 - \Delta_{mr}^2\right]\right). \tag{3.7}$$

By exchanging the roles of Δ_{mr}^1 and Δ_{mr}^2 we can similarly show that

$$\Delta_{ij}^{min}\left(\Delta_{mr}^1\right) \le \Delta_{ij}^{min}\left(\Delta_{mr}^2 + \frac{s_r(\{m,r\})}{s_j}\left[\Delta_{mr}^2 - \Delta_{mr}^1\right]\right). \tag{3.8}$$

It is easy to verify that inequalities (3.7) and (3.8) can only be satisfied when both are equalities. This completes the proof of Claim 1.

By (1.a) it follows that

$$D_{ij}^a\left(\Delta_{mr}^2\right) = D_{ij}^a\left(\Delta_{mr}^1 + \frac{s_r(\{m,r\})}{s_j}\left[\Delta_{mr}^1 - \Delta_{mr}^2\right]\right), \tag{3.9}$$

for all Δ_{mr}^1, Δ_{mr}^2, which implies that D_{ij}^a depends continuously on Δ_{mr}. By applying induction assumptions (1.a), (1.b), (3) and (4) to the links following $\{i,j\}$ we know that every difference $\Delta_{ij} \in D_{ij}^a$ induces a unique profile of future differences $\boldsymbol{\Delta}(\Delta_{ij})$ which depends continuously on Δ_{ij}. Since the set D_{ij}^a depends continuously on Δ_{mr}, and moreover, the payoffs x_i and x_j depend continuously on the realized differences, we may also conclude that X_{ij}^a depends continuously on Δ_{mr}. This completes the proof of property (1.b).

(1.c) Let $\{m,r\}$ be the link which immediately follows $\{i,j\}$. By induction assumptions (1.b), (3) and (4) applied to link $\{m,r\}$, it follows that every $\Delta_{ij} \in D_{ij}^a$ induces a unique subgame perfect equilibrium payoff $(x_i(\Delta_{ij}), x_j(\Delta_{ij}))$, which depends continuously on Δ_{ij}. Since by (1.a) applied to link $\{i,j\}$ we know that D_{ij}^a is connected, it follows that X_{ij}^a is connected.

We now show that X_{ij}^a consists of non-increasing non-horizontal line segments. For every $\Delta_{ij} \in D_{ij}^a$ let $\boldsymbol{\Delta}(\Delta_{ij})$ be the profile of future equilibrium differences induced by Δ_{ij}. Let $D_{ij}^1, D_{ij}^2, D_{ij}^3, \ldots$ be a partition of D_{ij}^a such that (1) $\boldsymbol{\Delta}(\Delta_{ij})$ is constant for all $\Delta_{ij} \in D_{ij}^1$ and (2) for all $k \ge 2$ there is a link l^k following $\{i,j\}$ such that l^k is the first link for which $\Delta_{l^k}(\Delta_{ij})$ changes with respect to $\Delta_{ij} \in D_{ij}^k$. We show that for every D_{ij}^k the total derivative $\frac{dx_i}{d\Delta_{ij}}$ is constant and ≥ 0, whereas $\frac{dx_j}{d\Delta_{ij}}$ is constant and < 0.

Assume first that $\Delta_{ij} \in D_{ij}^1$. Since $\Delta_l(\Delta_{ij})$ is constant on D_{ij}^1, it follows from equation (3.4) that $\frac{dx_i}{d\Delta_{ij}} = \frac{s_j}{n} > 0$ and $\frac{dx_j}{d\Delta_{ij}} = -\frac{s_i}{n} < 0$.

Assume now that Δ_{ij}^1, $\Delta_{ij}^2 \in D_{ij}^k$, where $k \ge 2$. Hence $\Delta_l\left(\Delta_{ij}^1\right) = \Delta_l\left(\Delta_{ij}^2\right)$ for all l following $\{i,j\}$ and preceding l^k, and by choosing Δ_{ij}^1 and Δ_{ij}^2 close enough we have $\Delta_{l^k}\left(\Delta_{ij}^1\right) \neq \Delta_{l^k}\left(\Delta_{ij}^2\right)$. Let $l^k = \{h,k\}$ and $\Delta_{l^k} = \Delta_{hk}$. Recall that, by our convention, player k is a terminal node in the remaining graph $L^+(\{h,k\})$. Note also that $\{h,k\}$ cannot be the last link, since in this case $\Delta_{lk}(\Delta^1_{ij}) = \Delta^*_{hk} = \Delta_{lk}(\Delta^2_{ij})$, which would be a contradiction.

Claim 2. If Δ^1_{ij} is close enough to Δ^2_{ij}, then

$$s_j \Delta^1_{ij} + s_k(\{h,k\})\,\Delta_{hk}\left(\Delta^1_{ij}\right) = s_j \Delta^2_{ij} + s_k(\{h,k\})\,\Delta_{hk}\left(\Delta^2_{ij}\right). \qquad (3.10)$$

Proof of Claim 2. Let $L_+^{\{h,k\}}$ be the set of links following $\{i,j\}$ and preceding $\{h,k\}$. Let $\Delta_{L_+^{\{h,k\}}}(\Delta^1_{ij})$ be the equilibrium differences for links in $L_+^{\{h,k\}}$ if the difference Δ^1_{ij} is agreed upon. For Δ^1_{ij} we define $D^a_{hk}(\Delta^1_{ij})$ as the set of those differences Δ_{hk} for which all differences at links following $\{h,k\}$ are accepted, given that the differences Δ^1_{ij} and $\Delta_{L_+^{\{h,k\}}}(\Delta^1_{ij})$ are already realized. Let $X^a_{hk}(\Delta^1_{ij})$ be the set of feasible payoff pairs for players h and k if the differences Δ^1_{ij} and $\Delta_{L_+^{\{h,k\}}}(\Delta^1_{ij})$ have been realized and all future differences are to be accepted. Similarly we define $D^a_{hk}(\Delta^2_{ij})$ and $X^a_{hk}(\Delta^2_{ij})$. By induction assumption we know that the sets $X^a_{hk}(\Delta^1_{ij})$ and $X^a_{hk}(\Delta^2_{ij})$ are connected unions of non-increasing non-horizontal line segments.

Let $\Delta^{min}_{hk}(\Delta^1_{ij}) = \inf D^a_{hk}(\Delta^1_{ij})$ and $\Delta^{min}_{hk}(\Delta^2_{ij}) = \inf D^a_{hk}(\Delta^2_{ij})$. We first show that $\Delta^{min}_{hk}(\Delta^1_{ij})$ and $\Delta^{min}_{hk}(\Delta^2_{ij})$ are finite numbers. Recall that player k is a terminal node in $L^+(\{h,k\})$ and that $\{h,k\}$ is not the last link. Hence, if Δ_{hk} is too small, then at every future link in $L^+(\{h,k\})$ every proposed difference will be rejected. This implies that $\Delta^{min}_{hk}(\Delta^1_{ij})$ and $\Delta^{min}_{hk}(\Delta^2_{ij})$ cannot be $-\infty$. Let $\{m,r\}$ be the link immediately following $\{h,k\}$. By choosing Δ_{hk} large enough, one can always insure that $\Delta_{L^-(\{m,r\})} \in D_{L^-(\{m,r\})}$, and hence, by induction assumption (2), all future differences are accepted. This implies that $\Delta^{min}_{hk}(\Delta^1_{ij})$ and $\Delta^{min}_{hk}(\Delta^2_{ij})$ cannot be ∞.

We now distinguish two cases.

Case 1. If $\Delta^{min}_{hk}(\Delta^1_{ij}) > \Delta^*_{hk}$. Then, if Δ^2_{ij} is close enough to Δ^1_{ij}, we have that $\Delta^{min}_{hk}(\Delta^2_{ij}) > \Delta^*_{hk}$. The latter follows from the fact that $\Delta^{min}_{hk}(\Delta_{ij})$ depends continuously on Δ_{ij}. Then, by induction assumption (3) of our lemma, it holds that $\Delta_{hk}(\Delta^1_{ij}) = \Delta^{min}_{hk}(\Delta^1_{ij})$ and $\Delta_{hk}(\Delta^2_{ij}) = \Delta^{min}_{hk}(\Delta^2_{ij})$.

From above, we know that

$$x_i(\Delta_{ij}) = \frac{1}{n}\left[v(N) + \sum_{l \in L \setminus \{i,j\}} c(l \mid \{i,j\})\,\Delta_l + s_j \Delta_{ij}\right]; \qquad (3.11)$$

$$x_m(\Delta_{ij}) = \frac{1}{n}\left[v(N) + s_r(\{m,r\})\,\Delta_{mr} + \sum_{l \in L \setminus (\{i,j\} \cup \{m,r\})} c(l \mid \{m,r\})\,\Delta_l + c(\{i,j\} \mid \{m,r\})\,\Delta_{ij}\right]; \qquad (3.12)$$

for all links $\{m,r\} \neq \{i,j\}$.

For all links $\{m,r\} \in L^+(\{h,k\})$ we have, since the rule of order $\Pi(g^*)$ is regular, that $c(\{i,j\}|\{m,r\}) = s_j$ and $c(\{h,k\}|\{m,r\}) = s_k(\{h,k\})$. Suppose that $\tilde{\Delta}_{hk}$ is such that

$$s_j\Delta^1_{ij} + s_k(\{h,k\})\Delta_{hk}\left[\Delta^1_{ij}\right] = s_j\Delta^2_{ij} + s_k(\{h,k\})\tilde{\Delta}_{hk}.$$

Recall that, by assumption, $\Delta_{L_+^{\{h,k\}}}\left(\Delta^1_{ij}\right) = \Delta_{L_+^{\{h,k\}}}\left(\Delta^2_{ij}\right)$, where $L_+^{\{h,k\}}$ is the set of links following $\{i,j\}$ and preceding $\{h,k\}$. Then, by the system of equations (3.12), it is easily verified that the subgame $\Gamma\left(\Delta_{L\setminus(\{i,j\})};\Delta^1_{ij};\Delta_{L_+^{\{h,k\}}}\left(\Delta^1_{ij}\right);\Delta_{hk}\left(\Delta^1_{ij}\right)\right)$ is equivalent, for the players remaining in this subgame, to the subgame $\Gamma\left(\Delta_{L\setminus(\{i,j\})};\Delta^2_{ij};\Delta_{L_+^{\{h,k\}}}\left(\Delta^2_{ij}\right);\tilde{\Delta}_{hk}\right)$.

We show that

$$\Delta_{hk}\left[\Delta^2_{ij}\right] = \tilde{\Delta}_{hk} = \Delta_{hk}\left[\Delta^1_{ij}\right] + \frac{s_j}{s_k(\{h,k\})}\left[\Delta^1_{ij} - \Delta^2_{ij}\right];$$

which would complete the proof of the claim for case 1. By choosing $\tilde{\Delta}_{hk}$ after Δ^2_{ij} the induced subgame $\Gamma\left(\Delta_{L\setminus(\{i,j\})};\Delta^2_{ij};\Delta_{L_+^{\{h,k\}}}\left(\Delta^2_{ij}\right);\tilde{\Delta}_{hk}\right)$ is equivalent for the remaining players to the subgame $\Gamma\left(\Delta_{L\setminus(\{i,j\})};\Delta^1_{ij};\Delta_{L_+^{\{h,k\}}}\left(\Delta^1_{ij}\right);\Delta_{hk}\left(\Delta^1_{ij}\right)\right)$. Since we know that in the latter subgame there is a unique equilibrium outcome in which all future differences will be accepted, we know that by choosing $\tilde{\Delta}_{hk}$ after Δ^2_{ij}, all future differences after $\{h,k\}$ will be accepted too. Hence, by definition, $\tilde{\Delta}_{hk} \in D^a_{hk}\left(\Delta^2_{ij}\right)$. Since $\Delta_{hk}\left(\Delta^2_{ij}\right) = \Delta^{min}_{hk}\left(\Delta^2_{ij}\right)$ it follows that

$$\Delta_{hk}\left[\Delta^2_{ij}\right] \leq \tilde{\Delta}_{hk} = \Delta_{hk}\left[\Delta^1_{ij}\right] + \frac{s_j}{s_k(\{h,k\})}\left[\Delta^1_{ij} - \Delta^2_{ij}\right]. \tag{3.13}$$

By exchanging the roles of Δ^1_{ij} and Δ^2_{ij} we can similarly show that

$$\Delta_{hk}\left[\Delta^1_{ij}\right] \leq \Delta_{hk}\left[\Delta^2_{ij}\right] + \frac{s_j}{s_k(\{h,k\})}\left[\Delta^2_{ij} - \Delta^1_{ij}\right]. \tag{3.14}$$

It is easy to verify that inequalities (3.13) and (3.14) can only be satisfied when both are equalities. This completes the proof of the claim for case 1.

Case 2. If $\Delta^{min}_{hk}\left(\Delta^1_{ij}\right) < \Delta^*_{hk}$. Then, if Δ^2_{ij} is close enough to Δ^1_{ij}, we have that $\Delta^{min}_{hk}\left(\Delta^2_{ij}\right) < \Delta^*_{hk}$. By induction assumption (4) applied to link $\{h,k\}$ we know that $\Delta_{hk}\left(\Delta^1_{ij}\right) = \Delta_{hk}\left(\Delta^2_{ij}\right) = \Delta^*_{hk}$, which is a contradiction to the assumption that $\Delta_{hk}\left(\Delta^1_{ij}\right) \neq \Delta_{hk}\left(\Delta^2_{ij}\right)$. This completes the proof of the claim.

By the claim we have that $s_j\Delta_{ij} + s_k(\{h,k\})\Delta_{hk}(\Delta_{ij})$ is constant on D^k_{ij}. From the above we know that all the subgames $\Gamma\left(\Delta_{L\setminus(\{i,j\})};\Delta_{ij};\Delta_{L_+^{\{h,k\}}}(\Delta_{ij});\Delta_{hk}(\Delta_{ij})\right)$ for $\Delta_{ij} \in D^k_{ij}$ are equivalent for the players remaining after $\{h,k\}$. From induction assumptions (3) and (4) in the lemma it follows that each of these subgames has a unique profile of equilibrium differences. Therefore, all these subgames induce the same profile of equilibrium differences. From equation (3.11) we have therefore that

$$\frac{dx_i(\Delta_{ij})}{d\Delta_{ij}} = \frac{1}{n}\left[s_j + c(\{h,k\}|\{i,j\})\frac{\partial \Delta_{hk}(\Delta_{ij})}{\partial \Delta_{ij}}\right].$$

Recall that

$$c(\{h,k\}|\{i,j\}) = \begin{cases} s_k(\{h,k\}), & \text{if path from } \{i,j\} \text{ to } k \text{ contains } h \\ -s_h(\{h,k\}), & \text{otherwise.} \end{cases}$$

Since $s_j\Delta_{ij} + s_k(\{h,k\})\Delta_{hk}(\Delta_{ij})$ is constant, we know that

$$\frac{\partial \Delta_{hk}(\Delta_{ij})}{\partial \Delta_{ij}} = -\frac{s_j}{s_k(\{h,k\})},$$

which implies that

$$\frac{dx_i(\Delta_{ij})}{d\Delta_{ij}} = \begin{cases} 0, & \text{if path from } \{i,j\} \text{ to } k \text{ contains } h \\ \frac{s_j}{s_k(\{h,k\})}, & \text{otherwise.} \end{cases}$$

Suppose that the path from $\{i,j\}$ to k does not contain h. Then, the path from $\{i,j\}$ to h contains k. We therefore know that $S_k(\{h,k\}) \supseteq S_j \cup \{j\}$. Recall that j is a terminal node in $L^+(\{i,j\})$ and $\{h,k\} \in L^+(\{i,j\})$. This implies that $s_j \le s_k(\{h,k\}) - 1$.

Thus, for every $\Delta_{ij} \in D^k_{ij}$ it holds that

$$0 \le \frac{dx_i(\Delta_{ij})}{d\Delta_{ij}} < 1.$$

Since $x_j(\Delta_{ij}) = x_i(\Delta_{ij}) - \Delta_{ij}$ it follows that

$$-1 \le \frac{dx_j(\Delta_{ij})}{d\Delta_{ij}} < 0$$

for every $\Delta_{ij} \in D^k_{ij}$. Given that both $\frac{dx_i(\Delta_{ij})}{d\Delta_{ij}}$ and $\frac{dx_j(\Delta_{ij})}{d\Delta_{ij}}$ remain constant in D^k_{ij}, we may conclude that the set of feasible payoff pairs $(x_i(\Delta_{ij}), x_j(\Delta_{ij}))$ for $\Delta_{ij} \in D^k_{ij}$ constitutes a non-increasing non-horizontal line segment in $\Re^{\{i,j\}}$.

We may thus conclude that the set of feasible payoff pairs X^a_{ij} is a connected union of non-increasing non-horizontal line segments. We have thus shown property (1.c).

(2) Let $\Delta_{L-\{i,j\}} \in D_{L-\{i,j\}}$. Let $\{h,k\}$ be the link which directly follows $\{i,j\}$. Then, by definition of $D_{L-\{i,j\}}$ and $D_{L-\{h,k\}}$, there is an open interval $\left(\Delta^*_{ij} - \epsilon, \Delta^*_{ij} + \epsilon\right)$ such that for every $\Delta_{ij} \in \left(\Delta^*_{ij} - \epsilon, \Delta^*_{ij} + \epsilon\right)$ we have that $\left(\Delta_{L-\{i,j\}}, \Delta_{ij}\right) \in D_{L-\{h,k\}}$. By applying

induction assumption (2) to link $\{h,k\}$ we know that for every $\Delta_{ij} \in \left(\Delta^*_{ij} - \epsilon, \Delta^*_{ij} + \epsilon\right)$ it holds that $\Delta_{L^+(\{h,k\})} = \Delta^*_{L^+(\{h,k\})}$ in equilibrium. Let

$$p^* = \max\left\{p \in \mathbb{R} \mid \exists (x_i, x_j) \in X^a_{ij} \text{ such that } x_i \geq e_i + p \text{ and } x_j \geq e_j + p\right\}.$$

This implies that $(e_i + p^*, e_j + p^*) \in X^a_{ij}$ and, moreover, belongs to the relative interior of a strictly decreasing line segment in X^a_{ij}. By (1.c) we know that X^a_{ij} is a connected union of non-increasing non-horizontal line segments. Hence,

$$\{(x_i, x_j) \in X_{ij} \mid x_i \geq e_i + p^* \text{ and } x_j \geq e_j + p^*\} = \{(e_i + p^*, e_j + p^*)\}.$$

Note that players i and j can guarantee $e_i + p^*$ and $e_j + p^*$. Moreover, we know that $(e_i + p^*, e_j + p^*)$ dominates every payoff pair (x_i, x_j) corresponding to an equilibrium in which some future difference is rejected. Thus it follows, similarly to the proof of property (2) for the last link, that there is a unique equilibrium behavior where players i and j agree on the fair difference Δ^*_{ij}.

(3) Let $\Delta^{min}_{ij}\left(\Delta_{L-\{i,j\}}\right) > \Delta^*_{ij}$. We define the price p^* by

$$p^* = \max\left\{p \in \mathbb{R} \mid \exists (x_i, x_j) \in X^a_{ij} \text{ such that } x_i \geq e_i + p \text{ and } x_j \geq e_j + p\right\}.$$

Since X^a_{ij} is a connected union of non-increasing non-horizontal line segments and $\Delta^{min}_{ij}\left(\Delta_{L-\{i,j\}}\right) > \Delta^*_{ij}$, it may be verified easily that

$$\left\{(x_i, x_j) \in X^a_{ij} \mid x_i \geq e_i + p^* \text{ and } x_j \geq e_j + p^*\right\} = \{(e_i + \tilde{p}, e_j + p^*)\},$$

for some $\tilde{p} \geq p^*$. Players i and j can guarantee payoffs $e_i + p^*$ and $e_j + p^*$ by choosing price p^* at step 1 of the mechanism. Hence, if there is a subgame perfect equilibrium in which all differences are accepted, the equilibrium payoffs for players i and j should be $(e_i + \tilde{p}, e_j + p^*)$. Since $\Delta^{min}_{ij}\left(\Delta_{L-\{i,j\}}\right)$ is the unique difference which induces the payoffs $e_i + \tilde{p}$ and $e_j + p^*$ we may conclude that in every subgame perfect equilibrium in $\Gamma\left(\Delta_{L-\{i,j\}}\right)$ for which all differences are accepted, players i and j agree on the difference $\Delta^{min}_{ij}\left(\Delta_{L-\{i,j\}}\right)$.

(4) Let $\Delta^{min}_{ij}\left(\Delta_{L-\{i,j\}}\right) \leq \Delta^*_{ij}$. We distinguish two cases.

Case 1. If $\left(x_i\left(\Delta^*_{ij}\right), x_j\left(\Delta^*_{ij}\right)\right)$ belongs to the relative interior of a strictly decreasing line segment in X^a_{ij}. Define the price p^* by

$$p^* = \max\left\{p \in \mathbb{R} \mid \exists (x_i, x_j) \in X^a_{ij} \text{ such that } x_i \geq e_i + p \text{ and } x_j \geq e_j + p\right\}.$$

Since by property (1.c), the set X^a_{ij} consists of a connected union of non-increasing non-horizontal line segments, it may be verified that

$$\left\{(x_i, x_j) \in X^a_{ij} \mid x_i \geq e_i + p^* \text{ and } x_j \geq e_j + p^*\right\} = \{(e_i + p^*, e_j + p^*)\}.$$

We know that players i and j can guarantee payoffs $e_i + p^*$ and $e_j + p^*$ by choosing price p^* at step 1 of the mechanism. Hence, if there is a subgame perfect equilibrium in which all differences are accepted, the equilibrium payoffs for players i and j should be $(e_i + p^*, e_j + p^*)$, and hence both players should agree on $\left(\Delta^*_{ij}\right)$.

Case 2. If $\left(x_i\left(\Delta^*_{ij}\right), x_j\left(\Delta^*_{ij}\right)\right)$ belongs to a vertical line segment in X^a_{ij}. Define the price p^* by

$$p^* = \max\left\{p \in \mathbb{R} \mid \exists (x_i, x_j) \in X^a_{ij} \text{ such that } x_i \geq e_i + p \text{ and } x_j \geq e_j + p\right\}.$$

We first show that in every equilibrium in which the difference at $\{i, j\}$ is accepted, player j chooses a price $p_j \leq p^*$. Suppose that player j chooses a price $p_j > p^*$ in equilibrium. We distinguish two cases. If player j becomes the proposer, then player i only accepts the difference if he receives at least $e_i + p_j > e_i + p^*$. This implies that player j's payoff is strictly less than $e_j + p^*$, which is a contradiction since player j can always guarantee a payoff equal to $e_j + p^*$. If player i becomes the proposer, that is, $p_i \geq p_j > p^*$, then player j should get at least $e_j + p_i > e_j + p^*$ and player i receives at most $e_i + p^*$. However player i can get more than $e_i + p^*$ by choosing some p_i^0 with $p^* < p_i^0 < p_j$ and rejecting player j's offer, which is a contradiction. Hence, we may conclude that in every equilibrium in which the difference Δ_{ij} is accepted, player j chooses a price $p_j \leq p^*$.

Define

$$\bar{p} = \max\left\{p \mid (e_i + p^*, e_j + p) \in X^a_{ij}\right\}.$$

Since, by assumption, $(e_i + p^*, e_j + p^*) \in X^a_{ij}$ we have that $\bar{p} \geq p^*$. We show that in every equilibrium in which the difference Δ_{ij} is accepted, player i chooses a price $p_i \in [p^*, \bar{p}]$. Suppose first that $p_i > \bar{p}$. Since we know that player j chooses $p_j \leq p^*$, player i becomes the proposer and should give at least $e_j + p_i > e_j + \bar{p}$ to player j. However this implies that player i gets strictly less than $e_i + p^*$, which is a contradiction since player i can guarantee $e_i + p^*$. Suppose now that $p_i < p^*$. We distinguish two cases. If player i becomes the proposer, player j would obtain $e_j + p_i < e_j + p^*$, which is a contradiction since player j can always guarantee a payoff $e_j + p^*$. If player j becomes the proposer, that is, $p_j > p_i$, then it would be strictly better for player j to choose some p_j^0 with $p_i < p_j^0 < p_j$, because in the latter case he only has to give $e_i + p_j^0 < e_i + p_j$ to player i. The reason that this is strictly better for player j follows from the fact that there are no horizontal parts in X^a_{ij}. This is a contradiction. Hence, we may conclude that in every equilibrium in which Δ_{ij} is accepted, player i chooses a price $p_i \in [p^*, \bar{p}]$.

Since player j chooses a price $p_j \leq p^*$, then, if player i chooses a price $p_i \in [p^*, \bar{p}]$ his final payoff is always $e_i + p^*$. Hence, player i is indifferent among all prices in $[p^*, \bar{p}]$. By the tie-breaking rule player i is supposed to choose the price p^*.

Let $\bar{\Delta}_{ij}$ be the difference which induces the payoff pair $(e_i + p^*, e_j + \bar{p})$. Hence, $\bar{\Delta}_{ij} \leq \Delta^*_{ij}$. Given that player i chooses the price p^*, player i is indifferent among all differences in $\left[\bar{\Delta}_{ij}, \Delta^*_{ij}\right]$. By the tie-breaking rule, player i is supposed to choose Δ^*_{ij}. Hence, we may conclude that in every subgame perfect equilibrium in $\Gamma\left(\Delta_{L_i(\{i,j\})}\right)$ for which all differences are accepted, players i and j agree on the fair difference Δ^*_{ij}. This completes the proof of Lemma 3.2.

In order to prove the statement in Theorem 3.1 we need the following lemma.

Lemma 3.3. If the TU-game $[N, v]$ is strictly convex and g is a tree which connects all players in N, then $m_i(N, v, g) > m_i(N, v, g \backslash a)$ for all links $a \in g$ and for all players $i \in N$.

The proof of this lemma is given in the appendix.

Since by assumption the game $[N; v]$ is strictly convex and the Myerson value is the unique component efficient allocation rule which respects all fair differences, Lemma 3.3 implies for every link $\{i; j\} \in L$: if $\phi_{L-\{i;j\}} = (\phi^*_l)_{l \in L-\{i;j\}}$ then $\phi_{L-\{i;j\}} \in D_{L-\{i;j\}}$. But then, by applying property (2) of Lemma 3.2 recursively, starting at the first link, we have that there is a unique subgame perfect equilibrium outcome in which at every link the fair difference is proposed and accepted. Consequently, the unique subgame perfect equilibrium outcome of the mechanism is the Myerson value. $\blacksquare$

4. An extension

In this paper we have restricted ourselves to cooperative games in which the surplus from cooperation depends only on the coalition and not on the graph connecting that coalition. There is a more general model in which the surplus from cooperation depends on the particular network fromed. Thus, two networks connecting the same group of players can have different values. Let g^N be the complete graph on N and let $C(g)$ be the set of connected components in some graph g. The value or worth of a graph is represented by a function $w : \{g \mid g \subseteq g^N\} \to \Re$. The function w is called component additive if for every graph g

$$w(g) = \sum_{h \in C(g)} w(h):$$

The function w is called strictly convex if

$$w(g) - w(g \backslash a) > w(g \backslash l) - w(g \backslash \{l; a\})$$

for every tree g and pair of links $l; a \in g$; $l \neq a$. For a given graph g and value function w we may define the TU-game $[N; U_g]$ by

$$U_g(S) = \sum_{h \in C(g^S)} w(h):$$

The Myerson value is defined as the Shapley value of the game $[N; U_g]$, i. e.,

$$m(N; w; g) = \Phi(N; U_g):$$

We can prove the following lemma.

Lemma 4.1. Let w be a function from $\{g \mid g \subseteq g^N\}$ to $\Re$ and let g be a tree which connects all players in N. If w is strictly convex and component additive, then $m_i(N; w; g) > m_i(N; w; g \backslash a)$ for all links $a \in g$ and all players $i \in N$.

The proof of Lemma 4.1 is similar to the proof of Lemma 3.3 and is therefore omitted.

Our mechanism $\Gamma(v; g^*; \sigma)$ may be defined for this more general context, too. By making use of Lemma 4.1 and using the fact that the Myerson value in this context is the unique allocation rule satisfying component efficiency and fairness, we may prove that this mechanism has a unique subgame perfect equilibrium outcome, which coincides with the Myerson value.

5. Appendix

Proof of Lemma 3.3. Consider a reduced graph $g \setminus a$, and the corresponding point game $[N; v|(g \setminus a)]$. Define the TU-game $[N; w]$, where

$$w(S) = v|g(S) - v|(g \setminus a)(S) = \sum_{T \in S|g} v(T) - \sum_{T \in S|(g \setminus a)} v(T).$$

We show that for all $i \in N$ it holds that $w(S) - w(S \setminus \{i\}) \geq 0$ for all $S \subseteq N$ and $i \in S$, and $w(S) - w(S \setminus \{i\}) > 0$ for some $S \subseteq N$ and $i \in S$.

Case 1. Assume first that player i is one of the two nodes in a, namely $a = \{i; j\}$. Recall that $g^S = \{\{i; j\} \in g \mid i \in S \text{ and } j \in S\}$ is the graph g restricted to S. This graph g^S is a forest, given that $g^S \subseteq g$, and g is a tree. We know, by strict convexity of the game, that $w(S) > 0$ whenever $S|g \neq S|(g \setminus a)$. If $S|g = S|g \setminus a$, it holds that $w(S) = 0$. If players i and j belong to S we have that $w(S) > 0$ and $w(S \setminus \{i\}) = 0$, and hence $w(S) - w(S \setminus \{i\}) > 0$. If $i \in S$ but $j \notin S$, we have that $w(S) = w(S \setminus \{i\}) = 0$, hence $w(S) - w(S \setminus \{i\}) = 0$.

Case 2. Now assume that player i is not a node in a. We use the following notation. Let $a = \{j; k\}$. Given g is a tree, once this link a is deleted, player i will be (directly or indirectly) connected with just one of these two players, say player j. If coalition S does not contain j or k or both, then we have that $S|g = S|(g \setminus a)$ and therefore $w(S) = w(S \setminus \{i\}) = 0$. Let S be such that players i, j and k belong to S: Consider the set

$$S_j(\{j; k\}) = \{j\} \cup \left\{ r \in S \mid \text{there is a path in } g^S \setminus \{j; k\} \text{ connecting } r \text{ and } j \right\}$$

and

$$S_k(\{j; k\}) = \{k\} \cup \left\{ r \in S \mid \text{there is a path in } g^S \setminus \{j; k\} \text{ connecting } r \text{ and } k \right\}:$$

It is easy to see that $S = S_j(\{j; k\}) \cup S_k(\{j; k\})$ and $S_j(\{j; k\}) \cap S_k(\{j; k\}) = \emptyset$. Furthermore, the sets $S_j(\{j; k\})$ and $S_k(\{j; k\})$ are disconnected in $g \setminus a$. By assumption, $i \in S_j(\{j; k\})$. Hence,

$$w(S) = v|g(S) - v|(g \setminus a)(S) = v|g(S) - v|g(S_j(\{j; k\})) - v|g(S_k(\{j; k\}))$$

and

$$\begin{aligned} w(S \setminus \{i\}) &= v|g(S \setminus \{i\}) - v|(g \setminus a)(S \setminus \{i\}) = \\ &= v|g(S \setminus \{i\}) - v|g(S_j(\{j; k\}) \setminus \{i\}) - v|g(S_k(\{j; k\})); \end{aligned}$$

since $i \notin S_k(\{j; k\})$. Hence,

$$w(S) - w(S \setminus \{i\}) = [v|g(S) - v|g(S \setminus \{i\})] - [v|g(S_j(\{j; k\})) - v|g(S_j(\{j; k\}) \setminus \{i\})]:$$

Given that $S_j(\{j; k\}) \subseteq S$ and the game $[N; v|g]$ is convex[3], we know that $w(S) - w(S \setminus \{i\}) \geq 0$. It remains to check that there exists at least one S such that $w(S) - w(S \setminus \{i\}) > 0$.

[3] Van den Nouweland (1993) proves in Theorem 2.4.2 that if the underlying TU-game is convex and the graph has no cycles, then the point game is also convex.

Since g is a tree, there exists a unique path going from player i to player j. Let P be the set of players on the path from i to j. Note that the minimal number of players in P is two, which is the case when players i and j are directly connected. Take $S = P \cup \{k\}$. Note that S, $S \setminus \{j,k\}$, $S \setminus \{i\}$ and $S \setminus \{j,k\} \setminus \{i\}$ are connected in g. Thus,

$$w(S) - w(S \setminus \{i\}) = v(S) - v(S \setminus \{i\}) - [v(S \setminus \{j,k\}) - v(S \setminus \{j,k\} \setminus \{i\})] > 0,$$

by strict convexity of the game $[N; v]$.

Applying the Shapley value to $[N; w]$ yields

$$\phi_i(N; w) = \sum_{\substack{S \subseteq N \\ i \in S}} \frac{(s-1)!(n-s)!}{n!} [w(S) - w(S \setminus \{i\})] > 0,$$

since there always exist at least one $S \subseteq N$ such that $w(S) - w(S \setminus \{i\}) > 0$, while in general for any other S we know that $w(S) - w(S \setminus \{i\}) \geq 0$. But, by additivity of the Shapley value, this implies

$$\phi_i(N; w) = \phi_i(N; v|g) - \phi_i(N; v|(g \setminus a)) = m_i(N; v; g) - m_i(N; v; g \setminus a) > 0.$$

This completes the proof of Lemma 3.3. $\blacksquare$

References

[1] Bergantiños, G., Casas-Méndez, B. and Vázquez-Brage, M. (2000), A Non-Cooperative Bargaining Procedure generalising the Kalai-Smorodinsky Bargaining Solution to NTU games. International Game Theory Review 2(4), 273-286.

[2] Dasgupta, A. & Chiu, Y. S. (1998), On implementation via demand commitment games. International Journal of Game Theory 27 (2): 161-189.

[3] Gul, F. (1989), Bargaining Foundations of the Shapley Value. Econometrica 57: 81-95.

[4] Hart, S. & Mas-Colell, A. (1996), Bargaining and Value. Econometrica 64 (2): 357-380.

[5] Jackson, M. O. & Wolinsky, A. (1996), A Strategic Model of Social and Economic Networks. Journal of Economic Theory 71: 44-74.

[6] Moulin, H. (1984), Implementing the Kalai-Smorodinsky Bargaining Solution. Journal of Economic Theory 33,32-45.

[7] Mutuswami, S., Pérez-Castrillo, D. & Wettstein, D. (2001), Bidding for the surplus: Realizing efficient outcomes in general economic environments. Mimeo.

[8] Myerson, R. B. (1977), Graphs and cooperation in games. Mathematics of Operations Research, 2: 225-229.

[9] Pérez-Castrillo, D. & Wettstein, D. (2001), Bidding for the Surplus: A non-cooperative approach to the Shapley value. Journal of Economic Theory, 100 (2): 274-294.

[10] Vidal, J. And Bergantiños, G. (2001), An implementation of the coalitional value. Mimeo. Universidad de Vigo.

[11] Winter, E. (1994), The demand commitment bargaining and snowballing cooperation. Economic Theory 4: 255-273.