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ONLINE APPENDIX
Competitive Search Equilibrium with Multidimensional Heterogeneity and Two-Sided Ex-ante Investments

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Abstract

This appendix supplements Jerez (2017) by formally describing the linear programming problems in Section 5.2.1, and by showing that these problems have optimal solutions and the same optimal value. This in turn implies that the optimal solutions to these problems can be characterized by means of the complementary slackness theorem of linear programming.

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1 The primal and dual problems

We begin with some preliminary notation. Take a metric space Z which is locally compact and separable. Let $C(Z)$ denote the space of continuous real-valued functions on Z, endowed with the topology of uniform convergence on compact sets. The topological dual of $C(Z)$ is the space $M_c(Z)$ of signed regular Borel measures on Z with compact support (see Hewitt 1959). We let $M_c(Z)$ be endowed with the weak-star topology, so $C(Z)$ is also the dual of $M_c(Z)$. The dual pair of spaces $(C(Z), M_c(Z))$ is endowed with the standard bilinear form:

$$\langle f, \gamma \rangle = \int_{z \in Z} f(z) d\gamma(z), \quad f \in C(Z),\ \gamma \in M_c(Z),$$

where the bracket notation highlights the infinite dimensional nature of the spaces in the pairing.

In the special case where Z is compact, the topological dual of $C(Z)$ is the space $M(Z)$ of signed regular Borel measures on Z.[1] (If Z is finite, both $C(Z)$ and $M(Z)$ are isomorphic to the Euclidean space). We write $C_+(Z)$, $M_{c+}(Z)$ and $M_+(Z)$ for the positive cones of the three spaces.

For any integer n, the product spaces $\prod_{j=1,\dots,n} C(Z_j)$ and $\prod_{j=1,\dots,n} M_c(Z_j)$ are endowed with the corresponding product topologies, and are also paired in duality with bilinear form:

$$\sum_{j=1}^{n} \langle f_j, \gamma_j \rangle, \quad (f_1, f_2, \dots, f_n) \in \prod_{j=1,\dots,n} C(Z_j),\ (\gamma_1, \gamma_2, \dots, \gamma_n) \in \prod_{j=1,\dots,n} M_c(Z_j).$$

The primal LP problem in Section 5.2.1 of Jerez (2017) is to find $(\mu^{\mathcal{B}}, \mu^{\mathcal{S}}) \in M_{c+}(B \times X) \times M_{c+}(S \times X)$ to solve

$$\begin{aligned}
(P_G) \qquad \sup & \int_{B \times X} [\alpha(h,a,\theta) f(h,a) - \hat{C}(b,a)] d\mu^{\mathcal{B}}(b,h,a,\theta) \\
& - \int_{S \times X} [m(h,a,\theta) v(s,h,a) + c(s,h)] d\mu^{\mathcal{S}}(s,h,a,\theta) \\
s.t. \quad & \mu^{\mathcal{B}}_B = \xi^{\mathcal{B}}, && (1.1) \\
& \mu^{\mathcal{S}}_S = \xi^{\mathcal{S}}, && (1.2) \\
& \int_{\Omega} \alpha(h,a,\theta) d\mu^{\mathcal{B}}_X(h,a,\theta) = \int_{\Omega} m(h,a,\theta) d\mu^{\mathcal{S}}_X(h,a,\theta) \quad \text{for all Borel } \Omega \subseteq X, && (1.3)
\end{aligned}$$

[1] As noted by Hewitt (1959), the topology of uniform convergence on compact sets coincides with the uniform norm topology in this case.

The dual problem is to find $(q^{\mathcal{B}}, q^{\mathcal{S}}, w) \in C(B) \times C(S) \times C(X)$ to solve[2]

$$
\begin{aligned}
(D_G) \qquad & \inf \int_B q^{\mathcal{B}}(b) d\xi^{\mathcal{B}}(b) + \int_S q^{\mathcal{S}}(s) d\xi^{\mathcal{S}}(s) \\
s.t. \qquad & q^{\mathcal{B}}(b) \geq \alpha(h,a,\theta)\left[f(h,a) - w(h,a,\theta)\right] - \hat{C}(b,a) \quad \forall (b,h,a,\theta) \in B \times X, \qquad (1.4) \\
& q^{\mathcal{S}}(s) \geq m(h,a,\theta)\left[w(h,a,\theta) - v(s,h,a)\right] - c(s,h) \quad \forall (s,h,a,\theta) \in S \times X. \qquad (1.5)
\end{aligned}
$$

2 Existence of optimal solutions and absence of a duality gap

Below we prove that problems (P_G) and (D_G) have optimal solutions and the same optimal value: $\nu(P_G) = \nu(D_G)$ We begin by showing that both problems are consistent (i.e. their feasible sets are not empty) and bounded (i.e. $\nu(P_G)$ and $\nu(D_G)$ are finite).

Lemma A. 1. *Problems (P_G) and (D_G) are consistent and bounded.*

Proof. An allocation where workers make no investments (so they all choose $h = h_0$), neither do firms (who choose $a = a_0$) and all agents are assigned to the fictitious market x_0 is a feasible solution for problem (P_G). Hence, problem (P_G) is consistent. Also, since total welfare is zero under this allocation, $\nu(P_G) \geq 0$.

In problem (D_G), set $w = w_o \in C(X)$ where $w_o(h,a,\theta) = 0$ for all $(h,a,\theta) \in X$. In the constraint systems (1.4) and (1.5), $\alpha(h,a,\theta)$ and $m(h,a,\theta)$ are bounded above by one and below by zero (since they are probabilities). One then can find a feasible dual solution where $w = w_o$ by choosing $q_o^{\mathcal{B}} \in C(B)$ and $q_o^{\mathcal{S}} \in C(S)$ constant so that

$$
\begin{aligned}
q_o^{\mathcal{B}}(b) &= \bar{q}_o^{\mathcal{B}} \equiv \sup_{(b,h,a) \in B \times H \times A} \left\{ f(h,a) - \hat{C}(b,a) + \epsilon \right\} > 0, \\
q_o^{\mathcal{S}}(s) &= \epsilon, \quad s \in S,
\end{aligned}
$$

where ϵ is an arbitrary positive real number. Since f and $\hat{C}$ are continuous and B, H and A are compact, the above supremum $\bar{q}_o^{\mathcal{B}}$ is attained. So problem (D_G) is consistent. (The supremum $\bar{q}_o^{\mathcal{B}}$ is clearly positive: we have assumed that for each firm type b there is a value of a, a worker type s, and a value of h such that $f(h,a) - \hat{C}(b,a) - c(s,h) - v(s,h,a) > 0$, and so $f(h,a) - \hat{C}(b,a) > 0$). Moreover,

$$
\nu(D_G) \leq \int_B q_o^{\mathcal{B}}(b) d\xi^{\mathcal{B}}(b) + \int_S q_o^{\mathcal{S}}(s) d\xi^{\mathcal{S}}(s) = \bar{q}_o^{\mathcal{B}} \xi^{\mathcal{B}}(B) + \epsilon \xi^{\mathcal{S}}(S) < \infty.
$$

Finally, by the weak duality theorem (Anderson and Nash 1987, Theorem 2.1), $\nu(P_G) \leq \nu(D_G)$, so the primal and dual problems are bounded:

$$
0 \leq \nu(P_G) \leq \nu(D_G) \leq \bar{q}_o^{\mathcal{B}} \xi^{\mathcal{B}}(B) + \epsilon \xi^{\mathcal{S}}(S) < \infty. \qquad \square
$$

Next, we show that problem (P_G) is solvable.

[2] The derivation of the dual is analogous to that in Jerez (2016), the main difference being that there α and m depend only on θ.

Theorem A. 1. *Problem (P_G) has optimal solutions.*

Proof. The feasible set of problem (P_G) is bounded, and the constraint map and objective function are weak-star continuous, so the result follows from Theorem 3.20 in Anderson and Nash (1987). $\square$

We also show that problems (P_G) and (D_G) have the same optimal value.

Theorem A. 2. *There is no duality gap: $\nu(P_G) = \nu(D_G)$.*

Proof. The positive cone of $C(B \times X) \times C(S \times X)$ has a non-empty interior, denoted by Y_0. Also, $(q_o^{\mathcal{B}}, q_o^{\mathcal{S}}, w_o) \in C_+(B) \times C_+(S) \times C_+(X)$ in the proof of Lemma A.1 is a Slater point in the feasible set of problem (D_G). Since $\nu(D_G)$ is finite, Theorem 3.13 in Anderson and Nash (1987) implies that $\nu(P_G) = \nu(D_G)$. $\square$

By Theorem A.2, the Complementary Slackness Theorem (Anderson and Nash 1987, Theorem 3.2) may be applied to characterize optimal solutions for problems (P_G) and (D_G). Theorem A.3 is the general version of Theorem 1 in Jerez (2017).[3]

Theorem A. 3. *(Complementary Slackness Theorem) Feasible solutions $(\mu^{\mathcal{B}}, \mu^{\mathcal{S}})$ and $(q^{\mathcal{B}}, q^{\mathcal{S}}, w)$ for problems (P_G) and (D_G) are optimal if and only if they satisfy the complementary slackness conditions:*

$$q^{\mathcal{B}}(b) = \alpha(h,a,\theta)\,[f(h,a) - w(h,a,\theta)] - \hat{C}(b,a) \quad \text{for all } (b,h,a,\theta) \in \mathrm{supp}\mu^{\mathcal{B}}, \tag{2.1}$$

$$q^{\mathcal{S}}(s) = m(h,a,\theta)\,[w(h,a,\theta) - v(s,h,a)] - c(s,h) \quad \text{for all } (s,h,a,\theta) \in \mathrm{supp}\mu^{\mathcal{S}}. \tag{2.2}$$

We still need to show that problem (D_G) is solvable. We begin by stating two preliminary results. Lemma A.2 shows that the set of feasible dual solutions can be taken to be bounded without loss of generality. The proof uses the fact that f, v, c and $\hat{C}$ are continuous, α and m are bounded, and B, S, H, and A are compact sets. Lemma A.3 shows that the tightness level θ can be restricted without loss of generality to lie on a compact subset of $\Re_+$ (e.g. to be bounded above).

Lemma A. 2. *The set of feasible dual solutions can be taken to be bounded without loss of generality. In particular, we may take (i) $q^{\mathcal{B}}$ and $q^{\mathcal{S}}$ to be positive and bounded above, and (ii) w such that*

$$\min_{(s,h,a)\in S\times H\times A} v(s,h,a) \le w(h,a,\theta) \le \max_{(h,a)\in H\times A} f(h,a) \text{ for all } (h,a,\theta) \in X.$$

Proof. For $(h,a,\theta) = x_0$ the dual constraint systems (1.4)–(1.5) imply $q^{\mathcal{B}}, q^{\mathcal{S}} \ge 0$.

[3]The latter theorem corresponds to the model with one-sided heterogeneity, whereas Theorem A.3 corresponds to the model with two-sided heterogeneity.

Take any $(h, a, \theta) \in H \times A \times \Re_{++}$ which lies in supp$\mu_X^{\mathcal{B}}$. (There is always such a value of (h, a, θ), when one focuses on the interesting case where autarky is not an optimal allocation.) We know that $(\tilde{b}, h, a, \theta) \in$ supp$\mu^{\mathcal{B}}$ for some $\tilde{b} \in B$. Hence, $(\tilde{s}, h, a, \theta) \in$ supp$\mu^{\mathcal{S}}$ for some $\tilde{s} \in S$. This is because the restrictions of $\mu_X^{\mathcal{B}}$ and $\mu_X^{\mathcal{S}}$ to $H \times A \times \Re_{++}$ are mutually absolutely continuous measures, and so they have the same support.[4] By Theorem A.3, in this case, optimal dual solutions satisfy

$$q^{\mathcal{B}}(\tilde{b}) = \alpha(h,a,\theta)\,[f(h,a) - w(h,a,\theta)] - \hat{C}(\tilde{b},a), \tag{2.3}$$
$$q^{\mathcal{S}}(\tilde{s}) = m(h,a,\theta)\,[w(h,a,\theta) - v(\tilde{s},h,a)] - c(\tilde{s},h). \tag{2.4}$$

Now, since $q^{\mathcal{B}}(\tilde{b}), q^{\mathcal{S}}(\tilde{s}) \geq 0$ and α, m, $\hat{C}$ and c are all positive functions, it follows that $v(\tilde{s}, h, a) \leq w(h, a, \theta) \leq f(h, a)$. Hence,

$$0 \leq \inf_{s \in S} v(s,h,a) \leq w(h,a,\theta) \leq f(h,a). \tag{2.5}$$

The continuity of v and the compactness of S imply that the infimum in (2.5) is attained. Since w is continuous, (2.5) also holds for $(h, a, 0) \in \mu_X^{\mathcal{B}}$. Equation (2.5) in turn implies that the terms on right-hand side of (2.1) and (2.2) are bounded above (since matching probabilities are bounded, f, v, c and $\hat{C}$ are continuous functions, and H, A, B and S are compact). So there is no loss of generality in assuming that $q^{\mathcal{B}}$ and $q^{\mathcal{S}}$ are bounded above.

On the other hand, if $(h, a, \theta) \notin$ supp$\mu_X^{\mathcal{B}}$ then $(h, a, \theta) \notin$ supp$\mu_X^{\mathcal{S}}$. In this case, market (h, a, θ) is inactive, and its shadow price $w(h, a, \theta)$ can be chosen arbitrarily among all the values that satisfy

$$q^{\mathcal{B}}(b) \geq \alpha(h,a,\theta)\,[f(h,a) - w(h,a,\theta)] - \hat{C}(b,a), \quad \forall b \in B, \tag{2.6}$$
$$q^{\mathcal{S}}(s) \geq m(h,a,\theta)\,[w(h,a,\theta) - v(s,h,a)] - c(s,h). \quad \forall s \in S. \tag{2.7}$$

In particular, we may restrict without loss of generality to values of $w(h, a, \theta)$ satisfying (2.5). Indeed, if (2.6) and (2.7) hold for $w(h, a, \theta) > f(h, a)$ then they must hold for $w(h, a, \theta) = f(h, a)$ since $q^{\mathcal{B}} \geq 0$. Likewise, if these equations hold for $w(h, a, \theta) < v(\underline{s}, h, a) = \min_{s \in S} v(s, h, a)$ then they must also hold for $w(h, a, \theta) = v(\underline{s}, h, a)$ since $q^{\mathcal{S}} \geq 0$. Finally, since (2.5) holds for all $(h, a, \theta) \in X$ and (again H, A and S are compact and f and v are continuous functions),[5]

$$\min_{(s,h,a) \in S \times H \times A} v(s,h,a) \leq w(h,a,\theta) \leq \max_{(h,a) \in H \times A} f(h,a), \quad \forall (h,a,\theta) \in X. \tag{2.8}$$

$\square$

Lemma A. 3. *There exists a sufficiently large $\tilde{\theta} \in \Re_+$ such that, if all the constraints which are associated with elements $\theta > \tilde{\theta}$ are eliminated from problem (D_G), the set of optimal dual solutions does not change.*

Proof. Suppose the statement in Lemma A.3 were not true. Let $(\mu^{\mathcal{B}}, \mu^{\mathcal{S}})$ be an optimal primal solution. Take an increasing sequence $\{\theta_j\} \subset \Re_+$ with $\theta_j \to \infty$. For each j there then exists $(b_j, h_j, a_j, \hat{\theta}_j) \in$ supp$\mu^{\mathcal{B}}$ with $\hat{\theta}_j > \theta_j$. Equivalently, $(s_j, h_j, a_j, \hat{\theta}_j) \in$ supp$\mu^{\mathcal{S}}$ for some $s_j \in S$, since

[4] The corresponding Radon-Nikodym derivatives are f and $1/f$ where $f(h, a, \theta) = \theta$. This follows from equation (1.3) since $\alpha(h, a, \theta) = m(h, a, \theta)\theta^{-1}$ and $\alpha(h, a, \theta) > 0$ for all $\theta > 0$.

[5] Equation (2.8) is intuitive: it says that we can assume that the wage in a given market is lower than the output generated by the most productive job, and higher than the lower disutility of labor a worker may experience at a job.

the restrictions of $\mu_X^{\mathcal{B}}$ and $\mu_X^{\mathcal{S}}$ to $H \times A \times \Re_{++}$ are mutually exclusive measures which have the same support. (If not, the complementary slackness theorem would imply that the dual constraints associated to all $\theta > \theta_j$ can be ignored without loss of generality, since they do not bind). But then the support of $\mu^{\mathcal{B}}$ contains the sequence $\{(b_j, h_j, a_j, \hat{\theta}_j)\}$ where $\lim \hat{\theta}_j \to \infty$, leading to a contradiction (since this support is compact by definition). □

The solvability of problem (D_G) cannot be settled using an argument similar to that in Theorem A.1 because the space of continuous functions on a compact set is not the dual of any normed space. We follow the approach used in Anderson and Nash (1987) for the continuous transportation problem (see their Theorem 5.2) and repose problem (D_G) in an enlarged space which does have this property. Then we appeal to the continuity of the f, v, C and c and the matching function, and the compactness of B, H and A to show that an optimal solution in the enlarged space lies in the original space.

Theorem A. 4. *Problem (D_G) has optimal solutions.*

Proof. Let us repose problem (D_G) with $(q^{\mathcal{B}}, q^{\mathcal{S}}, w)$ in $L^\infty(\xi^{\mathcal{B}}) \times L^\infty(\xi^{\mathcal{S}}) \times L^\infty(\mu_X^{\mathcal{B}})$, where $\mu^{\mathcal{B}}$ corresponds to an optimal solution for problem (P). (This space is the dual of $L^1(\xi^{\mathcal{B}}) \times L^1(\xi^{\mathcal{S}}) \times L^1(\mu_X^{\mathcal{B}})$). The new dual problem is solvable by Theorem 3.20 in Anderson and Nash (1987) since Lemma A.2 implies that its feasible set can be taken to be bounded without loss of generality.

We now show that there exists an optimal solution of this new problem where the functions $q^{\mathcal{B}}$, $q^{\mathcal{S}}$ and w are continuous. Suppose $(q^{\mathcal{B}}, q^{\mathcal{S}}, w)$ is optimal for the new dual problem. Feasibility requires that

$$\alpha(h,a,\theta)w(h,a,\theta) \geq \alpha(h,a,\theta)f(h,a) - \hat{C}(b,a) - q^{\mathcal{B}}(b), \quad \forall (b,h,a,\theta) \in B \times H \times A \times \Re_+, \tag{2.9}$$

and that

$$q^{\mathcal{S}}(s) + m(h,a,\theta)v(s,h,a) + c(s,h) \geq m(h,a,\theta)w(h,a,\theta), \quad \forall (s,h,a,\theta) \in S \times H \times A \times \Re_+. \tag{2.10}$$

Recall that $w(x_0) = 0$. Define w_1 so

$$\alpha(h,a,\theta)w_1(h,a,\theta) = \max\{0, \sup_{b \in B} \left\{\alpha(h,a,\theta)f(h,a) - \hat{C}(b,a) - q^{\mathcal{B}}(b)\right\}\}, \tag{2.11}$$

for $(h,a,\theta) \in H \times A \times \Re_+$, and $w_1(x_0) = 0$. Then, $(q^{\mathcal{B}}, q^{\mathcal{S}}, w_1)$ is another optimal solution.

We now show that the restriction of w_1 to $H \times A \times (0, \bar{\theta}]$ is continuous (for an arbitrary value $\bar{\theta}$ of the market tightness). Take a sequence $\{b_i\}$ in B such that $(\alpha(h,a,\theta)f(h,a) - \hat{C}(b_i,a) - q^{\mathcal{B}}(b_i))$ converges to $\alpha(h,a,\theta)w_1(h,a,\theta)$. Since $B \times H \times A \times [0,\bar{\theta}]$ is compact, $\alpha(h,a,\theta)f(h,a) - \hat{C}(b_i,a)$ is uniformly continuous on $B \times H \times A \times [0,\bar{\theta}]$. For any $\epsilon > 0$ there then exists δ such that

$$|\alpha(h,a,\theta)f(h,a) - \hat{C}(b_i,a) - \alpha(h',a',\theta')f(h',a') + \hat{C}(b_i,a')| < \epsilon, \quad i = 1,2,\ldots \tag{2.12}$$

whenever (h',a',θ') lies in a δ-neighborhood of (h,a,θ), and $(h',a',\theta'), (h,a,\theta) \in H \times A \times [0,\bar{\theta}]$.

Equations (2.9) and (2.12) then imply that

$$\begin{aligned}\alpha(h',a',\theta')w_1(h',a',\theta') &\geq \alpha(h',a',\theta')f(h',a') - \hat{C}(b_i,a') - q^{\mathcal{B}}(b_i) && (2.13)\\ &> \alpha(h,a,\theta)f(h,a) - \hat{C}(b_i,a) - q^{\mathcal{B}}(b_i) - \epsilon, \quad i=1,2,\ldots && (2.14)\end{aligned}$$

Taking the limit yields

$$\alpha(h',a',\theta')w_1(h',a',\theta') + \epsilon \geq \alpha(h,a,\theta)w_1(h,a,\theta). \tag{2.15}$$

A symmetric argument implies that

$$\alpha(h,a,\theta)w_1(h,a,\theta) + \epsilon \geq \alpha(h',a',\theta')w_1(h',a',\theta'). \tag{2.16}$$

for any such (h',a',θ') and (h,a,θ). Hence, the restriction of $\alpha(h,a,\theta)w_1(h,a,\theta)$ to $H \times A \times [0,\bar{\theta}]$ is continuous. Since α is continuous and strictly positive on $H \times A \times [0,\bar{\theta}]$, the restriction of $w_1(h,a,\theta)$ to $H \times A \times [0,\bar{\theta}]$ (the quotient of two continuous functions) is continuous.

To see that w_1 is continuous, take an increasing sequence of compact sets $\{\Theta_j\}$ converging to Θ; e.g. $\Theta_j = [0,\theta_j] \cup \{\theta_0\}$ with $\theta_j \uparrow \infty$. Consider the sequence of functions $\{f_j\}$ where $f_j = \chi_{\Theta_j} w_1$, where χ_{Θ_j} denotes the characteristic function on Θ_j (so w_1 and f_j coincide on Θ_j). Since f_j is continuous on Θ_j and $w_1 = \lim_{j\to\infty} f_j$, it follows that w_1 is continuous.

Finally, defining

$$\begin{aligned} q_1^{\mathcal{B}}(b) &= \max_{(h,a,\theta)\in H\times A\times([0,\tilde{\theta}]\cup\{\theta_0\})} \alpha(h,a,\theta)\left[f(h,a) - w_1(h,a,\theta)\right] - \hat{C}(b,a), && (2.17)\\ q_1^{\mathcal{S}}(s) &= \max_{(b,h,a,\theta)\in B\times H\times A\times([0,\tilde{\theta}]\cup\{\theta_0\})} m(h,a,\theta)\left[w_1(h,a,\theta) - v(s,h,a)\right] - c(s,h), && (2.18)\end{aligned}$$

yields yet another optimal solution $(q_1^{\mathcal{B}}, q_1^{\mathcal{S}}, w_1)$ since, by Lemma A.3, the constraints associated with elements $\theta > \tilde{\theta}$ can be ignored without loss of generality in (1.4)–(1.5). By Berge's Maximum Theorem, $q_1^{\mathcal{B}}$ and $q_1^{\mathcal{S}}$ are continuous. □

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