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Group strategy-proof stable mechanisms in priority-based resource allocation under multi-unit demand: a note*

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Abstract

In this note we prove that group strategy-proofness and strategyproofness are equivalent requirements on stable mechanisms in prioritybased resource allocation problems with multi-unit demand.

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1 Introduction

We study the compatibility of efficiency, stability and group strategy-proofness in resource allocation problems with multi-unit demand. Those properties are important for assignment mechanisms, but they are not compatible, in general (see Roth and Sotomayor, 1990).

Kojima (2013) shows that the existence of a stable and strategy-proof mechanism is equivalent to the existence of an efficient and stable mechanism. In particular, Kojima (2013) characterizes those priority structures that allow stable and strategy-proof mechanisms as the ones that satisfy a condition called essential homogeneity. If essential homogeneity is satisfied, courses can have different priorities only on top ranked students. If the courses have few seats available, essential homogeneity is extremely requiring. For instance, if one of the courses has only one seat available, essential homogeneity amounts to all courses having the same priorities. On the other hand, if the courses have a large supply of seats, essential homogeneity is more permissive.

Anecdotal evidence suggests that collusion is a common behavior among students trying to enroll in over-demanded courses. This collusion is a concern for university officers, since it makes difficult to assess the demand for courses. The possibility of group manipulation is also a relevant concern in other multi-unit assignment problems such as the assignment of landing slots (see Schummer and Vohra, 2013 and Schummer and Abizada, 2017).

In this note we explore the possibility of designing mechanisms that are stable and group strategy-proof. We exploit the characterization of essential homogeneity in terms of a serial dictatorship provided by Kojima (2013) to prove that group strategy-proofness and strategy-proofness are equivalent requirements when imposed on stable mechanisms. Thus, the essential homogeneity of a priority system characterizes both requirements. We also suggest that an alternative proof of our result could employ the equivalence between essential homogeneity and the consistency of the student-optimal stable mechanism proved by Kojima (2013).

This equivalence is a surprising, albeit simple, observation. In general, group strategy-proofness is much more requiring than strategy-proofness. In particular, in the school assignment model, the student-optimal stable mechanism always provides a stable and strategy-proof assignment. However, efficiency and group strategy-proofness require priorities to satisfy an acyclicity condition (see Ergin, 2002). In the school assignment problem, Ergin (2002) proves that the existence of a stable and group strategy-proof mechansim is equivalent to the existence of a stable and efficient mechanism and to the consistency of the student-optimal stable mechanism. Thus, our characterization extends Ergin's results to the course assignment problem and contributes in explaining the restrictiveness of imposing strategy-proofness on stable mechanisms.

Finally, we observe that restricting the demand of the students does not lead to more permissive results. Even if a student can apply to at most two courses, essential homogeneity is still a necessary condition for the existence of a stable and strategy-proof mechanism.

The paper is organized as follows. Section 2 presents the model. Section 3 presents the results. Section 4 concludes. The proofs are in the Appendix.

2 The Model

There are a finite set of courses C and a finite set of students S, with $S \cap C = \emptyset$. Each student $s \in S$ has a strict preference relation P_s over the set of subsets of C. For all $c, c' \in C$, we write cP_sc' and $cP_s\emptyset$ instead of $\{c\} P_s \{c'\}$ and $\{c\} P_s \emptyset$, respectively. We assume that the preference relation of each student is **responsive** (see Roth 1985), with **demand** q_s. Formally, we assume that for each $C' \subseteq C$ and for all $c, c' \in C \setminus C'$, the following hold:

(1) if $|C'| < q_s$, then $C' \cup \{c\} P_s C' \cup \{c'\}$ if and only if cP_sc',
(2) if $|C'| < q_s$, then $C' \cup \{c\} P_s \emptyset$ if and only if $cP_s\emptyset$,
(3) if $|C'| > q_s$, then $\emptyset P_s C'$.

The set of all responsive preferences is denoted by $\mathcal{P}$. For each $s \in S$, R_s denotes weak preferences. For the preferences of students on individual courses we use the notation $P_s : c_1, c_2, ..., c_h$ meaning that $c_i P_s c_j$ for $i < j$ and $c_h P_s \emptyset$. For each $S' \subseteq S$, set $P_{S'} = (P_s)_{s \in S'} \in \mathcal{P}^{|S'|}$. For each $s \in S$ set $P_{-s} = P_{S \setminus \{s\}}$. Let $s \in S$, and let $C' \subseteq C$. The choice set from C', $Ch_s(C', P_s)$ is s's favorite subset of C'; formally, $Ch_s(C', P_s) = A$ if and only if $A \subseteq C'$, AR_sA' for each $A' \subseteq C'$. When there is no ambiguity about preferences, we write $Ch_s(C')$ for $Ch_s(C', P_s)$.

Each course $c \in C$ is characterized by **priority** $\succ_c$, which is a strict, complete, and transitive binary relation over S. Each $c \in C$ has a **supply** of q_c, which is the maximum number of students who can enroll in c. A **priority structure** is a pair $(\succ, q_C)$, where $\succ = (\succ_c)_{c \in C}$ and $q_C = (q_c)_{c \in C}$.

A **matching** is a function $\mu : S \cup C \to 2^C \cup 2^S$ such that, for each $s \in S$ and each $c \in C$, $\mu(s) \in 2^C$, $\mu(c) \in 2^S$, $|\mu(c)| \leq q_c$ and $c \in \mu(s)$ if and only if $s \in \mu(c)$. The set of all matchings is denoted by $\mathcal{M}$. Matching μ is **Pareto efficient** if there is no matching μ' such that $\mu'(s) R_s \mu(s)$ for each $s \in S$ and $\mu'(s) P_s \mu(s)$ for at least one $s \in S$.

Matching μ is **blocked** by a pair $(s, c) \in S \times C$ if $s \notin \mu(c)$, $c \in Ch_s(\mu(s) \cup \{c\})$ and either $|\mu(c)| < q_c$ or there exists $s' \in \mu(c)$ such that $s \succ_c s'$. Matching μ is **individually rational** if, for each $s \in S$, $Ch_s(\mu(s)) = \mu(s)$. Finally, a matching μ is **stable** for $(S, C, P, \succ, q_C)$ if it is individually rational and there exists no pair blocking it.

A **mechanism** is a function $\varphi : \mathcal{P}^{|S|} \to \mathcal{M}$. It is **efficient** if $\varphi(P)$ is a Pareto efficient matching for each $P \in \mathcal{P}^{|S|}$. It is **stable** if $\varphi(P)$ is a stable matching for each $P \in \mathcal{P}^{|S|}$. It is **strategy-proof** if $\varphi(P) R_s \varphi(P'_s, P_{-s})$ for each $P \in \mathcal{P}^{|S|}$, $s \in S$ and $P'_s \in \mathcal{P}$. It is **group strategy-proof** if there do not exist $S' \subseteq S$ and $P'_{S'} \in \mathcal{P}^{|S'|}$ such that $\varphi(P'_{S'}, P_{-S'}) R_s \varphi(P)$ for each $s \in S'$ and $\varphi(P'_{S'}, P_{-S'}) P_s \varphi(P)$ for at least one $s \in S'$.

If each agent has responsive preferences the set of stable matchings is a nonempty complete lattice (see Blair, 1988) and there exists a mechanism which is Pareto superior to all other stable mechanisms, which is called **student-optimal stable mechanism** and is denoted by $\mu^S(P)$. Formally, for each $P \in \mathcal{P}^{|S|}$ and for each matching μ, stable for $(S, C, P, \succ, q_C)$, $\mu^S(P)(s)\, R_s \mu(s)$ for each $s \in S$.

The priority structure $(\succ, q_C)$ satisfies satisfies **essential homogeneity** if there exist no $a, b \in C$ and $t, u \in S$ such that:

(1) $t \succ_a u$ and $u \succ_b t$, and

(2) there exist $S_a, S_b \subseteq S \setminus \{a, b\}$ such that $|S_a| = q_a - 1$, $|S_b| = q_b - 1$ and $s \succ_a u$ for each $s \in S_a$, $s \succ t$ for each $s \in S_b$.

Theorem 1, in Kojima (2013) proves that the essential homogeneity of a priority structure is equivalent to the existence of a stable and efficient mechanism and to the existence of a stable and strategy-proof mechanism.

3 Results

First, we prove that the student-optimal stable mechanism is group strategy-proof when the essential homogeneity condition holds. The proof is straightforward. Theorem 3 in Kojima (2013) shows that, if a priority structure satisfies essential homogeneity, the student-optimal stable mechanism can be obtained as a serial dictatorship, where the students choose in the order determined by the priorities of any course of minimal capacity. Thus, the proof consists in observing that a serial dictatorship is also group strategy-proof. For completeness, we provide a proof in the appendix.

Lemma 1 *Assume that the priority structure $(\succ, q_C)$ satisfies essential homogeneity. Then the student-optimal stable mechanism is group strategy-proof.*

Integrating Lemma 1 and Theorem 1 in Kojima (2013) we obtain.

Theorem 1 *Given $(\succ, q_C)$, the following conditions are equivalent.*

(1) There exists a stable and efficient mechanism.

(2) There exists a stable and strategy-proof mechanism.

(3) There exists a stable and group strategy-proof mechanism.

(4) The priority structure $(\succ, q_C)$ is essentially homogeneous.

A mechanism is consistent if, whenever the assignment of a course to a student is removed from the problem, the assignment of the remaining seats (for all courses) does not change. Theorem 4 in Kojima (2013) proves the equivalence between the existence of a stable and efficient mechanism and the consistency of the student-optimal stable mechanism. An alternative, but less transparent proof of Theorem 1 could employ this result following the lines of the proof of Theorem 1 in Ergin (2002).

Allowing for multi-unit demand makes strategy-proofness a very requiring condition in assignment models with priorities. The reader might wonder whether this is a consequence of the fact that the designer must consider any possible demand of the students. This assumption may appear too restrictive since in real world applications students can enroll in a limited number of courses. We prove that this is not the case: if students can enroll in at most two courses, essential homogeneity is still a necessary requirement for the existence of a strategy-proof mechanism.

Proposition 1 *Assume that* $(\succ, q_C)$ *is not essentially homogeneous. Then there exists* $P \in \mathcal{P}^{|S|}$ *such that* $q_s \leq 2$ *for each* $s \in S$ *and* $P'_t \in \mathcal{P}$ *with* $q'_t \leq 2$*, for some* $t \in S$ *such that, for any stable mechanism,* φ, $\varphi\left(P'_t, P_{-t}\right)(t)\, P_t \varphi\left(P\right)(t)$.

The intuition behind Proposition 1 can be explained through a simple example. Let us assume that there are are only two students, two courses and that each course has one vacant seat. Let $(\succ, q_C)$ be not essentially homogeneous. Let $a, b \in C$ and $t, u \in S$ as in the definition of essential homogeneity and assume that $P_t : b, a$, $q_t = 2$, $P_u : a, b$, $q_u = 1$. There is a unique stable matching μ, where $\mu(t) = a$ and $\mu(u) = b$. In this situation, student t competes with student u for both courses, losing her favorite course b. However, if she reveals preferences $P'_t : b$, she does no longer compete for course b since student u's favorite course is a. Indeed, this deviation is profitable for t when any stable mechanism is employed because she obtains course b.

4 Conclusions

In this paper, we show the equivalence of imposing group strategy-proofness, strategy-proofness or efficiency on stable mechanisms when studying the allocation of indivisible goods to a set of agents with multi-unit demand. The essential homogeneity of the priority structure is a necessary and sufficient condition for the existence such mechanisms. We also find that it is not possible to relax the characterization by imposing caps on agents' demands. Future research looking for positive results on a larger set of priority structures should move in a different direction. For instance, it could explore non-revelation mechanisms, relaxing the equilibrium concept. An alternative to this approach is to reduce the stability requirement on the mechanism, when looking for strategy-proof mechanisms.

Appendix

Proof of Lemma 1. Let c be a course of minimal capacity, which is let c such that $q_c = \min\{q_{c'} \mid c' \in C\}$. For each $l = 1, ..., |S|$, let $s_l \in S$ be the l-th ranked student according to $\succ_c$, formally $s = s_l$ if and only if $|\{s' \in S \mid s' \succ_c s\}| = l - 1$. Define $\mu(P)(s_1) = Ch_{s_1}(C)$. For each t, $2 \leq l \leq |S|$, let $A_l(P) = \left\{c' \in C \mid \bigcup_{l' < l, c' \in \mu(P,q)(s_{l'})} \{s_{l'}\} < q_{c'}\right\}$, be the set of courses that have vacant seats when is s_l's turn. Define $\mu(P)(s_l) = Ch_{s_l}(A_l(P))$. For each $c \in C$, set $\mu(P)(c) = \bigcup_{c \in \mu(P)(s_l)} \{s_l\}$. Since $(\succ, q_C)$ satisfies essential homogeneity, Theorem 3 in Kojima (2013) implies $\mu(P) = \mu^S(P)$ for each $P \in$

$\mathcal{P}^{|S|}$. Thus, in order to complete the proof of the claim, it suffices to show that mechanism $\mu(P)$ is group strategy-proof. The proof is by contradiction. Assume that there exists a nonempty set of agents $S' \subset S$, P and $P'_{S'} = (P'_s)_{s \in S'}$ such that $\mu(P'_{S'}, P_{S \setminus S'})(s) \, R_s \mu(P)(s)$ for each $s \in S'$ and $\mu^W(P'_{S'}, P_{S \setminus S'})(s') \, P_{s'} \mu^W(P)(s')$ for some $s' \in S'$.
Let $l = \min\left\{i \mid \mu(P'_{S'}, P_{S \setminus S'})(s_i) \neq \mu(P)(s_i)\right\}$. For each $i < l$, $\mu(P'_{S'}, P_{S \setminus S'})(s_i) = \mu(P)(s_i)$, then $A_l\left(P'_{S'}, P_{S \setminus S'}\right) = A_l(P)$. First assume $s_l \notin S'$. In this case $\mu\left(P'_{S'}, P_{S \setminus S'}\right)(s_l) = \mu(P)(s_l)$, which yields a contradiction. Next assume $s_l \in S'$, then $\mu(P'_{S'}, P_{S \setminus S'})(s_l) \, P_{s_l} \mu(P)(s_l)$. Since $A_l\left(P'_{S'}, P_{S \setminus S'}\right) = A_l(P)$, $\mu(P)(s_l) \, R_{s_l} \mu(P'_{S'}, P_{S \setminus S'})(s_l)$, which yields a contradiction. □

Proof of Theorem 1. From Theorem 1 in Kojima (2013) (1), (2) and (4) are equivalent. Any group strategy-proof mechanism is, in particular strategy proof, thus we have (3) $\implies$ (2). Then, from Lemma 1 we derive the implication (4) $\implies$ (3), which completes the proof of the claim. □

Proof of Proposition 1. We adapt the argument from the proof of Theorem 1 in Kojima (2013). Let $a, b \in C$ and $t, u \in S$ as in the definition of essential homogeneity. Let P_t and P_u such that $P_t : b, a$ and $P_u : a, b$. Let $q_t = 2$ and $q_u = 1$. Let P'_t be such that $P'_t : b$. For each $s \in S_a \setminus S_b$, let $P_s : a, b$ and $q_s = 1$. For each $s \in S_b \setminus S_a$, let $P_s : b, a$ and $q_s = 1$. For each $s \in S_b \cap S_a$, set $P_s : a, b$ and $q_s = 2$. For each $s \in S \setminus (S_a \cup S_c \cup \{t, u\})$, let P_s be such that $\emptyset P_s a$, $\emptyset P_s b$. Then, in any stable mechanism φ, $\varphi(P)(t) = \{a\}$ and $\varphi(P'_t, P_{-t})(t) = \{b\}$, which completes the proof of the claim. □

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