Dynamic Collective Choice with Endogenous Status Quo

ONLINE APPENDIX

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Proofs for the $N-$alternative model omitted from the main Appendix

Notation 1 *Here is some useful notation that is introduced in the Appendix. For all $T \subseteq \Theta$ and $f \in \mathcal{F}^2$, denote*

$$\begin{cases} D'(f) \triangleq \{\theta \in \Theta : f_l(\theta) < 0 \text{ and } f_r(\theta) > 0\} \\ D''(f) \triangleq \{\theta \in \Theta : f_l(\theta) > 0 \text{ and } f_r(\theta) < 0\} \\ I(f) \triangleq \{\theta \in \Theta : f_l(\theta) = 0 \text{ or } f_r(\theta) = 0\} \\ D(T,f) \triangleq D'(f) \cup D''(f) \cup (I(f) \cap T) \end{cases}. \tag{1}$$

For all $T \subseteq \Theta$, we define the mapping Ω^T as follows: for all $f \in \mathcal{F}^2$, $k \in \{l,r\}$, and $\theta \in \Theta$,

$$\Omega_k^T(f,\theta) \triangleq \int_{\zeta \in D(T,f)} f_k(\zeta)\, dP_\theta(\zeta), \tag{2}$$

Claim 1 *If σ is an equilibrium of Γ_c^{en}, then $V_k^\sigma(\langle \zeta, x_{n+1}\rangle) - V_k^\sigma(\langle \zeta, x_n\rangle)$ cannot be of the same strict sign for both players*

Proof. Suppose by contradiction that $V_k^\sigma(\langle \zeta, x_{n+1}\rangle) > V_k^\sigma(\langle \zeta, x_n\rangle)$ for both players, the argument in the other case is identical. Since the proposer is indifferent between any action played with positive probability in $\langle \zeta, x_{n+1}\rangle$, it must be that (i) $V_k^\sigma(\langle \zeta, x_{n+1}, x_{n+2}\rangle) > V_k^\sigma(\langle \zeta, x_n\rangle)$ for both players, or that (ii) $V_k^\sigma(\langle \zeta, x_{n+1}, x_n\rangle) > V_k^\sigma(\langle \zeta, x_n\rangle)$ for both players. In case (i), since

$V_k^\sigma(\langle\zeta, x_n\rangle) \geq V_k^\sigma(\langle\langle\langle\zeta, x_n\rangle\rangle\rangle)$, the only stage-undominated action for both players in $\langle\zeta, x_n, x_{n+1}\rangle$ is to vote *yes*, so by playing *right* in $\langle\zeta, x_n\rangle$, the proposer can get $V_{pr}^\sigma(\langle\zeta, x_{n+1}, x_{n+2}\rangle)$, a contradiction with (i). In case (ii), since $V_{pr}^\sigma(\langle\zeta, x_n\rangle) \geq V_{pr}^\sigma(\langle\zeta, x_n, x_{n-1}\rangle)$, the proposer must vote *no* with probability 1 in $\langle\zeta, x_{n+1}, x_n\rangle$. Together with (ii), this implies that for both players, $V_k^\sigma(\langle\langle\langle\zeta, x_{n+1}\rangle\rangle\rangle) = V_k^\sigma(\langle\zeta, x_{n+1}, x_n\rangle) > V_k^\sigma(\langle\zeta, x_n\rangle) \geq V_k^\sigma(\langle\langle\langle\zeta, x_n\rangle\rangle\rangle)$, so they must vote *yes* in $\langle\zeta, x_n, x_{n+1}\rangle$. By playing *right* in $\langle\zeta, x_n\rangle$, the proposer can then get at least $V_{pr}^\sigma(\langle\langle\langle\zeta, x_{n+1}\rangle\rangle\rangle)$, which contradicts $V_{pr}^\sigma(\langle\langle\langle\zeta, x_{n+1}\rangle\rangle\rangle) > V_{pr}^\sigma(\langle\zeta, x_n\rangle)$. ■

Claim 2 *The mapping $\sigma \to \hat{\Sigma}^\sigma$ defined in the proof of Proposition 5 is an upper-hemicontinuous, contractible-valued correspondence.*

Proof. *Step 1: $\sigma \to \hat{\Sigma}^\sigma$ is upper-hemicontinuous.*

Let $(\sigma^z)_{z\in\mathbb{N}}$ and $(\tilde{\sigma}^z)_{z\in\mathbb{N}}$ be two sequences of strategy profiles that converge to some σ and $\tilde{\sigma}$, respectively, and such that for all $z \in \mathbb{N}$, $\tilde{\sigma}^z \in \hat{\Sigma}^{\sigma^z}$. Let $\theta \in \Theta$ and $k \in \{l, r\}$. For all $z \in \mathbb{N}$, by definition of $\hat{\Sigma}^{\sigma^z}$, $\tilde{\sigma}_k^z(\theta)$ prescribes stage-undominated actions given continuation play $\left(\tilde{\sigma}_k^z(\theta), \sigma_k^z(-\theta), \sigma_{k'}^z\right)$. Since continuation payoffs in Γ_c^{en} are continuous in strategies, this implies that $\tilde{\sigma}_k(\theta)$ prescribes stage-undominated actions given continuation play $(\tilde{\sigma}_k(\theta), \sigma_k(-\theta), \sigma_{k'})$, so $\tilde{\sigma}_k(\theta) \in \Sigma_k^\sigma(\theta)$. If k is not the proposer in state of nature θ, this means that $\tilde{\sigma}_k(\theta) \in \hat{\Sigma}_k^\sigma(\theta)$, as needed. If she is, then by assumption, for all $z \in \mathbb{N}$, $\left(\phi^{\tilde{\sigma}_k^z(\theta), \sigma_{k'}^z(\theta)}(\langle\theta, q\rangle)\right)_{q\in X}$ is monotonic in q, and since ϕ depends continuously on the strategy profile, so is $\phi^{\tilde{\sigma}_k(\theta), \sigma_{k'}(\theta)}(\langle\theta, q\rangle)$, and thus $\tilde{\sigma}_k(\theta) \in \hat{\Sigma}_k^\sigma(\theta)$.

For the rest of the proof, we fix a Markov strategy profile σ, $\theta \in \Theta$, $k \in \{l, r\}$. Note that for all $\sigma'_k(\theta), \sigma''_k(\theta) \in \Sigma_k^\sigma(\theta)$, $(\sigma'_k(\theta), \sigma_k(-\theta), \sigma_{k'})$ and $(\sigma''_k(\theta), \sigma_k(-\theta), \sigma_{k'})$ yield the same continuation payoff for player k starting from any state (otherwise both could not be best responses). Thus, we can define $I_v^\sigma(\theta)$ to be the set of voting states in which the state of nature is θ and both actions, *yes* and *no*, are stage-undominated for k given continuation play $(\sigma'_k(\theta), \sigma_k(-\theta), \sigma_{k'})$ for any $\sigma'_k(\theta) \in \Sigma_k^\sigma(\theta)$. Likewise, if k is the proposer in state of nature θ, we define $I_d^\sigma(\theta)$ to be the set of direction states in which the state of nature is θ and both $left$ and $right$ are stage-undominated for k given continuation play $(\sigma'_k(\theta), \sigma_k(-\theta), \sigma_{k'})$ for any $\sigma'_k(\theta) \in \Sigma_k^\sigma(\theta)$. If k is not the proposer, we set $I_d^\sigma(\theta) \doteq \emptyset$. Thereafter, $\sigma_k^*(\theta)$ refers to the stage-undominated best response to $\sigma_k(-\theta)$ and $\sigma_{k'}$ that prescribes *no* with probability 1 in $I_v^\sigma(\theta)$ and $left$ with probability 1 in $I_d^\sigma(\theta)$. From what precedes, $\sigma_k^*(\theta)$ exists and is unique.

In what follows, to show that $\hat{\Sigma}_k^\sigma(\theta)$ is contractible when k is the proposer, we show that any two stage-undominated best responses $\sigma_k'(\theta)$ and $\sigma_k''(\theta)$ differ at most on $I_v^\sigma(\theta) \cup I_d^\sigma(\theta)$ (step 2), that $\sigma_k^*(\theta) \in \hat{\Sigma}_k^\sigma(\theta)$ (step 3a), and that for any other $\sigma_k'(\theta) \in \hat{\Sigma}_k^\sigma(\theta)$, one can continuously change the actions prescribed by $\sigma_k'(\theta)$ on $I_v^\sigma(\theta) \cup I_d^\sigma(\theta)$ so as to transform $\sigma_k'(\theta)$ into $\sigma_k^*(\theta)$ without violating stage undomination and monotonicity (step 3b and 3c). The simpler case in which k is the proposer is treated in step 4.

Step 2: $\sigma_k'(\theta) \in \Sigma_k^\sigma(\theta)$ *if and only if* $\sigma_k'(\theta)$ *prescribes the same (pure) action as* $\sigma_k^*(\theta)$ *in all states that are not in* $I_v^\sigma(\theta) \cup I_d^\sigma(\theta)$.

Since $\sigma_k^*(\theta) \in \Sigma_k^\sigma(\theta)$, all actions available in $I_v^\sigma(\theta) \cup I_d^\sigma(\theta)$ yield the same continuation payoff given continuation play $(\sigma_k^*(\theta), \sigma_k(-\theta), \sigma_{k'})$, so if $\sigma_k'(\theta)$ prescribes the same action as $\sigma_k^*(\theta)$ in all other states, $(\sigma_k'(\theta), \sigma_k(-\theta), \sigma_{k'})$ yields the same continuation payoff as $(\sigma_k^*(\theta), \sigma_k(-\theta), \sigma_{k'})$ starting from any state. Hence, the set of stage-undominated actions must be the same in all states for the two continuation strategies. As a result, $\sigma_k'(\theta) \in \Sigma_k^\sigma(\theta)$. Reciprocally, if $\sigma_k'(\theta) \in \Sigma_k^\sigma(\theta)$, then $(\sigma_k'(\theta), \sigma_k(-\theta), \sigma_{k'})$ must yield the same continuation payoff as $(\sigma_k^*(\theta), \sigma_k(-\theta), \sigma_{k'})$ from any state, so $\sigma_k^*(\theta)$ and $\sigma_k'(\theta)$ must prescribe the same pure action in any state in which only one action is stage-undominated, i.e., outside of $I_v^\sigma(\theta) \cup I_d^\sigma(\theta)$.

Step 3: If k *is the proposer in state* θ, $\hat{\Sigma}_k^\sigma(\theta)$ *is contractible.*

To show that $\hat{\Sigma}_k^\sigma(\theta)$ is contractible, we will construct a contraction which continuously shrinks $\hat{\Sigma}_k^\sigma(\theta)$ into $\sigma_k^*(\theta)$ by first increasing the probability that *no* is played in $I_v^\sigma(\theta)$ and then the probability that $left$ is played in $I_d^\sigma(\theta)$, while preserving stage undomination and monotonicity.

Formally, for all $\sigma_k'(\theta) \in \hat{\Sigma}_k^\sigma(\theta)$ and $a \in [0,1]$, let $f(a, \sigma_k'(\theta))$ differ from $\sigma_k'(\theta)$ only in that in any state $s \in I_v^\sigma(\theta)$, $f(a, \sigma_k'(\theta))$ prescribes *no* with probability $\max\{a, p_{no}(s)\}$, where $p_{no}(s)$ is the probability with which $\sigma_k'(\theta)$ prescribes *no* in s. For $a \in [1,2]$, $f(a, \sigma_k'(\theta))$ differs from $f(1, \sigma_k'(\theta))$ only in that in any state $s \in I_d^\sigma(\theta)$, $f(a, \sigma_k'(\theta))$ prescribes $left$ with probability $\max\{a-1, p_{left}(s)\}$, where $p_{left}(s)$ is the probability with which $\sigma_k'(\theta)$ prescribes $left$ in s.

Note that from step 2, for all $\sigma_k'(\theta) \in \Sigma_k^\sigma(\theta)$, $f(2, \sigma_k'(\theta)) = \sigma_k^*(\theta)$. To show that f is a contraction for $\hat{\Sigma}_k^\sigma(\theta)$, it remains to show that $\sigma_k^*(\theta) \in \hat{\Sigma}_k^\sigma(\theta)$, that if $\sigma_k'(\theta) \in \hat{\Sigma}_k^\sigma(\theta)$, then $f(a, \sigma_k'(\theta)) \in \hat{\Sigma}_k^\sigma(\theta)$ for all $a \in [0,2]$ (step 3b), and that $f(a, \sigma_k'(\theta))$ is continuous in $(a, \sigma_k'(\theta))$ (step 3c).

Step 3a: $\sigma_k^*(\theta) \in \hat{\Sigma}_k^\sigma(\theta)$.

By construction, $\sigma_k^*(\theta) \in \Sigma_k^\sigma(\theta)$, so it suffices to show that $\sigma_k^*(\theta)$ is monotonic. Let $x_{n-1}, x_n \in X$, and suppose by contradiction that monotonicity is violated. In that case, it must be violated between some consecutive states $\langle \theta, x_{n-1} \rangle$ and $\langle \theta, x_n \rangle$. If $\langle \theta, x_{n-1} \rangle \in I_d^\sigma(\theta)$, then by construction, $\sigma_k^*(\theta)$ prescribes k to play $left$ with probability 1 in $\langle \theta, x_{n-1} \rangle$, so monotonicity cannot be violated. If $\langle \theta, x_{n-1} \rangle \notin I_d^\sigma(\theta)$ and $\langle \theta, x_n \rangle \notin I_d^\sigma(\theta)$, monotonicity can be violated only if k strictly prefers to play $right$ in $\langle \theta, x_{n-1} \rangle$ and $left$ in $\langle \theta, x_n \rangle$, which is impossible. Finally, consider the case $\langle \theta, x_{n-1} \rangle \notin I_d^\sigma(\theta)$ and $\langle \theta, x_n \rangle \in I_d^\sigma(\theta)$. For monotonicity to be violated, $\sigma_k^*(\theta)$ must prescribe k to play $right$ with positive probability in $\langle \theta, x_{n-1} \rangle$, and since $\langle \theta, x_{n-1} \rangle \notin I_d^\sigma(\theta)$, this implies that k strictly prefer to play $right$ in $\langle \theta, x_{n-1} \rangle$. Thus, if $V_k(s)$ denote k's expected payoff from any state s given continuation play $(\sigma_k^*(\theta), \sigma_k(-\theta), \sigma_{k'})$, this means that

$$V_k(\langle \theta, x_{n-1}, x_n \rangle) > V_k(\langle \theta, x_{n-1}, x_{n-2} \rangle). \tag{3}$$

Note that $V_k(\langle \theta, x_{n-1}, x_{n-2} \rangle) \geq V_k(\langle\langle \theta, x_{n-1} \rangle\rangle)$ and that $V_k(\langle \theta, x_{n-1}, x_n \rangle)$ is a convex combination of $V_k(\langle\langle \theta, x_{n-1} \rangle\rangle)$ and $V_k(\langle \theta, x_n, x_{n+1} \rangle)$, so (3) implies that $V_k(\langle \theta, x_n, x_{n+1} \rangle) > V_k(\langle \theta, x_{n-1}, x_{n-2} \rangle)$. Since $\langle \theta, x_n \rangle \in I_d^\sigma(\theta)$, $V_k(\langle \theta, x_n, x_{n+1} \rangle) = V_k(\langle \theta, x_n, x_{n-1} \rangle)$, which with the previous inequality delivers $V_k(\langle \theta, x_n, x_{n-1} \rangle) > V_k(\langle \theta, x_{n-1}, x_{n-2} \rangle)$. This implies that $\sigma_k^*(\theta)$ prescribes k to vote no with probability 1 in $\langle \theta, x_n, x_{n-1} \rangle$, and so $V_k(\langle \theta, x_n \rangle) = V_k(\langle\langle \theta, x_n \rangle\rangle)$. This, in turn, means that voting no in $\langle \theta, x_n, x_{n+1} \rangle$ is stage-undominated. By construction, $\sigma_k^*(\theta)$ prescribes no whenever voting no is stage-undominated, so k votes no in $\langle \theta, x_n, x_{n+1} \rangle$. So $\phi(\langle \theta, x_n \rangle) = x_n$ with probability 1 and $\phi(\langle \theta, x_{n-1} \rangle) \leq x_n$ with probability 1, so monotonicity is not violated.

Step 3b: If $\sigma'_k(\theta) \in \hat{\Sigma}_k^\sigma(\theta)$, *then* $f(a, \sigma'_k(\theta)) \in \hat{\Sigma}_k^\sigma(\theta)$ *for all* $a \in [0, 2]$.

Let $\sigma'_k(\theta) \in \hat{\Sigma}_k^\sigma(\theta)$. By construction, $f(a, \sigma'_k(\theta))$ differs from $\sigma'_k(\theta)$ only in $I_v^\sigma(\theta) \cup I_d^\sigma(\theta)$, so from step 2, $f(a, \sigma'_k(\theta)) \in \Sigma_k^\sigma(\theta)$. Thus, we only need to show that $f(a, \sigma'_k(\theta))$ is monotonic when $\sigma'_k(\theta)$ is monotonic.

In the case $a \in [0, 1]$, the monotonicity of $f(a, \sigma'_k(\theta))$ follows from the fact that one cannot violate monotonicity by increasing the probability that a player votes no in some voting state. To see this, note that a violation of monotonicity implies that for some $x_n, x_{n+1} \in X$, the path of play from $\langle \theta, x_n \rangle$ reaches $\langle \theta, x_{n+1}, x_{n+2} \rangle$ with a greater probability than from $\langle \theta, x_{n+1} \rangle$, or that the path of play from $\langle \theta, x_{n+1} \rangle$ reaches $\langle \theta, x_n, x_{n-1} \rangle$ with a greater probability than from $\langle \theta, x_n \rangle$. Consider the former kind of monotonicity violation, the proof for the latter kind is analogous. If $p_a(s)$ denotes the probability that the proposer plays some action a in some state s, then the probability that the path of play reaches $\langle \theta, x_{n+1}, x_{n+2} \rangle$ from $\langle \theta, x_n \rangle$ and from $\langle \theta, x_{n+1} \rangle$ is $p_{right}(\langle \theta, x_n \rangle) \times (1 - p_{no}(\langle \theta, x_n, x_{n+1} \rangle))$ and $p_{right}(\langle \theta, x_{n+1} \rangle)$, respectively. Thus, the considered violation of monotonicity occurs if and only if $p_{right}(\langle \theta, x_n \rangle) \times (1 - p_{no}(\langle \theta, x_n, x_{n+1} \rangle)) > p_{right}(\langle \theta, x_{n+1} \rangle)$. The conclusion follows from the fact that one cannot revert this inequality by increasing $p_{no}(\langle \theta, x_n, x_{n+1} \rangle)$.

To conclude the proof of step 3b, it remains to show that $f(a, \sigma'_k(\theta))$ is monotonic for $a \in [1, 2]$. Suppose this is not true. As shown above, $f(1, \sigma'_k(\theta))$ is monotonic, and since $f(a, \sigma'_k(\theta))$ differs from $f(1, \sigma'_k(\theta))$ only in that it prescribes $left$ with greater probability in states $I^{\sigma}_d(\theta)$, monotonicity must be violated between some consecutive states $\langle \theta, x_{n-1} \rangle$ and $\langle \theta, x_n \rangle$ such that $\langle \theta, x_n \rangle \in I^{\sigma}_d(\theta)$. Moreover, $f(a, \sigma'_k(\theta))$ must differ from $f(1, \sigma'_k(\theta))$ in state $\langle \theta, x_n \rangle$, so $f(1, \sigma'_k(\theta))$ must prescribe $left$ with probability at most $a - 1$ in $\langle \theta, x_n \rangle$, and therefore $f(a, \sigma'_k(\theta))$ prescribe $left$ with probability $a - 1$ in $\langle \theta, x_n \rangle$. Suppose first that $\langle \theta, x_{n-1} \rangle \in I^{\sigma}_d(\theta)$. In that case, by construction of f, $f(a, \sigma'_k(\theta))$ prescribes left with probability at least $a - 1$ in $\langle \theta, x_{n-1} \rangle$, and thus with a greater probability than in $\langle \theta, x_n \rangle$, so monotonicity cannot be violated between $\langle \theta, x_{n-1} \rangle$ and $\langle \theta, x_n \rangle$.

Suppose now that $\langle \theta, x_{n-1} \rangle \notin I^{\sigma}_d(\theta)$. Then k is not indifferent between $right$ and $left$ in $\langle \theta, x_{n-1} \rangle$, and for monotonicity to be violated, she must strictly prefer to play $right$. Let $V_k(s)$ denote k's expected payoff from any state s given continuation play $(\sigma'_k(\theta), \sigma_k(-\theta), \sigma_{k'})$. The preceding implies

$$V_k(\langle \theta, x_{n-1}, x_n \rangle) > V_k(\langle \theta, x_{n-1}, x_{n-2} \rangle). \tag{4}$$

Note that $V_k(\langle \theta, x_{n-1}, x_{n-2} \rangle) \geq V_k(\langle\langle \theta, x_{n-1} \rangle\rangle)$ and that $V_k(\langle \theta, x_{n-1}, x_n \rangle)$ is a convex combination of $V_k(\langle\langle \theta, x_{n-1} \rangle\rangle)$ and $V_k(\langle \theta, x_n, x_{n+1} \rangle)$, so (4) implies that $V_k(\langle \theta, x_n, x_{n+1} \rangle) > V_k(\langle \theta, x_{n-1}, x_{n-2} \rangle)$. Since $\langle \theta, x_n \rangle \in I^{\sigma}_d(\theta)$, $V_k(\langle \theta, x_n, x_{n+1} \rangle) = V_k(\langle \theta, x_n, x_{n-1} \rangle)$, which with the previous inequality delivers $V_k(\langle \theta, x_n, x_{n-1} \rangle) > V_k(\langle \theta, x_{n-1}, x_{n-2} \rangle)$. This implies that k votes *no* with probability 1 in $\langle \theta, x_n, x_{n-1} \rangle$, and so $V_k(\langle \theta, x_n \rangle) = V_k(\langle\langle \theta, x_n \rangle\rangle)$. This, in turn, means that voting *no* in $\langle \theta, x_n, x_{n+1} \rangle$ is stage-undominated. By construction of f, for $a \geq 1$, $f(a, \sigma'_k(\theta))$ prescribes *no* whenever voting *no* is stage-undominated, so k votes *no* in $\langle \theta, x_n, x_{n+1} \rangle$. So $\phi(\langle \theta, x_n \rangle) = x_n$ with probability 1 and $\phi(\langle \theta, x_{n-1} \rangle) \leq x_n$ with probability 1, so monotonicity is not violated.

Step 3c: f is continuous.

By construction of f, for all $a, a' \in [0,2]$ and all $\sigma'_k(\theta), \sigma''_k(\theta) \in \Sigma^\sigma_k(\theta)$, the distance between $f(a, \sigma'_k(\theta))$ and $f(a', \sigma'_k(\theta))$ in the sup norm is smaller than $|a' - a|$, and from step 2, $f(a, \sigma'_k(\theta))$ and $f(a, \sigma''_k(\theta))$ coincide outside $I^\sigma_v(\theta) \cup I^\sigma_d(\theta)$ and f can only decrease the difference between $\sigma'_k(\theta)$ and $\sigma''_k(\theta)$ on $I^\sigma_v(\theta) \cup I^\sigma_d(\theta)$, so the distance between $f(a, \sigma'_k(\theta))$ and $f(a, \sigma''_k(\theta))$ is smaller than the distance between $\sigma'_k(\theta)$ and $\sigma''_k(\theta)$.

Step 4: If k is not the proposer in state θ, $\hat{\Sigma}^\sigma_k(\theta)$ is contractible.

Since k is not the proposer, then by definition $\hat{\Sigma}^\sigma_k(\theta) = \Sigma^\sigma_k(\theta)$. From step 2, $\Sigma^\sigma_k(\theta)$ is the set of strategies which prescribes the same pure action as $\sigma^*_k(\theta)$ outside $I^\sigma_v(\theta)$, and any action on $I^\sigma_v(\theta)$. So $\Sigma^\sigma_k(\theta)$ is homeomorphic to the nonempty, convex set $[0,1]^{I^\sigma_v(\theta)}$. As such, it is contractible (see, e.g., Reny 2011, p. 503). ■

Proof of equilibrium existence for Proposition 6. Suppose that Θ_1 is finite. Let $W \in \left[-\frac{m}{1-\delta}, \frac{m}{1-\delta}\right]^{\{l,r\}\times\Theta_1\times X}$, where m is the uniform bound on U. Let $\sigma(W)$ be a strategy profile such that for all $\theta \in \Theta$, the restriction of $\sigma(W)$ to the Markov states in which the state of nature is θ is an equilibrium of a one-period version of Γ^{en}_c in which the final payoff at any payoff state $\langle\langle\theta, x\rangle\rangle$ is $V^W(\langle\langle\theta, x\rangle\rangle) \circeq U(\theta, x) + \delta W(\theta_1, x)$. Let $W^{\sigma(W)}$ be the corresponding continuation value in Γ^{en}_c. Below, we show that irrespective of how $\sigma(W)$ is picked for each W, the map $W \to W^{\sigma(W)}$ is continuous. Since it maps $\left[-\frac{m}{1-\delta}, \frac{m}{1-\delta}\right]^{\{l,r\}\times\Theta_1\times X}$ into itself, Brower's fixed point theorem implies that it has a fixed point W^*. By construction, the corresponding strategy profile $\sigma(W^*)$ is an equilibrium of Γ^{en}_c.

Let $(W^z)_{z\in\mathbb{N}}$ be a sequence that converges to some W and let $q \in X$. For all $z \in \mathbb{N}$, let

$$\Phi^z \circeq \left\{\zeta \in \Theta : \phi^{\sigma(W^z)}(\langle\langle\zeta, q\rangle\rangle) = \phi^{\sigma(W)}(\langle\langle\zeta, q\rangle\rangle)\right\}$$

and let 1_{Φ^z} denote its indicator function. Note that for all $\zeta \in \Phi^z$,

$$V^{W^z}_k\left(\zeta, \phi^{\sigma(W^z)}(\langle\langle\zeta, q\rangle\rangle)\right) - V^W_k\left(\zeta, \phi^{\sigma(W)}(\langle\langle\zeta, q\rangle\rangle)\right) = \delta\left(W^z_k\left(\zeta_1, \phi^{\sigma(W)}(\langle\langle\zeta, q\rangle\rangle)\right) - W_k\left(\zeta_1, \phi^{\sigma(W)}(\langle\langle\zeta, q\rangle\rangle)\right)\right).$$

So for all $\theta_1 \in \Theta_1$,

$$W_k^{\sigma(W^z)}(\theta_1, q) - W_k^{\sigma(W)}(\theta_1, q) \tag{5}$$
$$= \delta \int \left[W_k^z \left(\zeta_1, \phi^{\sigma(W)}(\langle \zeta, q \rangle) \right) - W_k \left(\zeta_1, \phi^{\sigma(W)}(\langle \zeta, q \rangle) \right) \right] 1_{\Phi^z}(\zeta)\, dP_{\theta_1}(\zeta)$$
$$+ \int \left[V_k^{W^z} \left(\zeta, \phi^{\sigma(W^z)}(\langle \zeta, q \rangle) \right) - V_k^W \left(\zeta, \phi^{\sigma(W)}(\langle \zeta, q \rangle) \right) \right] 1_{(\Phi^z)^C}(\zeta)\, dP_{\theta_1}(\zeta).$$

Since $W^z \to W$, the dominated convergence theorem implies that the first integral on the right-hand side of (5) tends to 0. As for the second integral, observe that for all $\zeta_1 \in \Theta_1$, for almost all $U \in \mathbb{R}^{\{l,r\} \times X}$, the profile of payoff $(U_k(x) + \delta W_k(\zeta_1, x))_{k \in \{l,r\}, x \in X}$ induces strict ordinal preferences over X. Therefore, for any such U, for z sufficiently large, $(U_k(x) + \delta W_k^z(\zeta_1, x))_{k \in \{l,r\}, x \in X}$ induces the same strict ordinal preferences, and thus the same outcome. Together with Assumption 1 (i), this implies that for $P_{\theta_1}-$almost all ζ, for z sufficiently large, $\phi^{\sigma(W^z)}(\langle \zeta, q \rangle) = \phi^{\sigma(W)}(\langle \zeta, q \rangle)$, so $1_{(\Phi^z)^C}(\zeta) = 0$. The dominated convergence theorem implies then that the second integral on the right hand-side of (5) also tends to 0. ■

Proof of Proposition 7. For all $b \in \mathbb{R}^2$, let U^b denote the profile of quadratic current preferences with biases b, and for all finite subsets X of $\mathbb{R}$, let $\Gamma_c^{ex}(X, U^b)$ and $\Gamma_c^{en}(X, U^b)$ denote the game with an exogenous and endogenous status quo, respectively, in which players' current preferences are U^b and the set of alternatives is X. For all $p_l \leq 0$ and $p_r \geq 0$, let $\sigma^{ex}(b+p)$ denote the equilibrium of $\Gamma_c^{ex}(X, U^{b+p})$, as described in Section 4.2.1, and let $W^{\sigma^{ex}(b+p)}$ denote players' continuation value when $\sigma^{ex}(b+p)$ is played in $\Gamma_c^{en}(X, U^b)$. For all $x, y \in X$, $k \in \{l, r\}$, and $\theta \in \mathbb{R}$, denote

$$U_k^{x,y,b}(\theta) \doteq -(y - \theta - b_k)^2 + (x - \theta - b_k)^2 = 2(y - x)\left(\theta - \left(\frac{x+y}{2} - b_k\right)\right). \tag{6}$$

Step 1: For all $p_l \leq 0$ and $p_r \geq 0$, $\sigma^{ex}(b+p)$ is an equilibrium of $\Gamma_c^{en}(X, U^b)$ if for all $\theta \in \Theta$, $k \in \{l, r\}$, and for any two consecutive alternatives $x, y \in X$ such that $x < y$,

$$U_k^{x,y,b}(\theta) + \delta\left(W_k^{\sigma^{ex}(b+p)}(\theta, y) - W_k^{\sigma^{ex}(b+p)}(\theta, x)\right) \begin{cases} \geq 0 \text{ if } \theta > \frac{x+y}{2} - b_k - p_k \\ \leq 0 \text{ if } \theta < \frac{x+y}{2} - b_k - p_k \end{cases}. \tag{7}$$

By construction of $\sigma^{ex}(b+p)$ (see Section 4.2.1), it should be clear that $\sigma^{ex}(b+p)$ is also an equilibrium of $\Gamma_c^{ex}(X,U)$ for any payoff profile U such that for all $\theta \in \theta$ and $k \in \{l,r\}$, $(U_k(\theta,x))_{x\in X}$ is weakly increasing to the left of the peak of $\left(-(x-\theta-b_k-p_k)^2\right)_{x\in X}$ and weakly decreasing to its right. From (6), this is satisfied if for any two *consecutive* alternatives $x, y \in X$, $U_k(\theta,y) - U_k(\theta,x)$ is weakly of the same sign as $U_k^{x,y,b+p}(\theta)$. Moreover, in any period of $\Gamma_c^{en}\left(X,U^b\right)$, players play the same game form as in $\Gamma_c^{ex}\left(X,U^{b+p}\right)$ with players' continuation payoff in $\Gamma_c^{en}\left(X,U^b\right)$ of implementing an alternative x being $-(x-\theta-b_k)^2 + \delta W_k^{\sigma^{ex}(b+p)}(\theta,x)$. Therefore, using (6), $\sigma^{ex}(b+p)$ is an equilibrium of $\Gamma_c^{en}\left(X,U^b\right)$ if for any two consecutive alternatives $x,y \in X$, for all $\theta \in \theta$ and $k \in \{l,r\}$, $U_k^{x,y,b}(\theta) + \delta\left(W_k^{\sigma^{ex}(b+p)}(\theta,y) - W_k^{\sigma^{ex}(b+p)}(\theta,x)\right)$ is weakly of the same sign as $U_k^{x,y,b+p}(\theta)$. Step 1 follows then from (6).

Step 2: For all $p_l \leq 0$, $p_r \geq 0$, and $k \in \{l,r\}$, for any two consecutive alternatives $x,y \in X$ such that $x<y$, the function $\theta \to W_k^{\sigma^{ex}(b+p)}(\theta,y) - W_k^{\sigma^{ex}(b+p)}(\theta,x)$ is uniquely characterized by the following condition: for all $\theta \in \Theta$,

$$W_k^{\sigma^{ex}(b+p)}(\theta,y) - W_k^{\sigma^{ex}(b+p)}(\theta,x) \tag{8}$$
$$= \int_{\frac{x+y}{2}-b_r-p_r}^{\frac{x+y}{2}-b_l-p_l} \left[U_k^{x,y,b}(\zeta) + \delta\left(W_k^{\sigma^{ex}(b+p)}(\zeta,y) - W_k^{\sigma^{ex}(b+p)}(\zeta,x)\right)\right] f(\zeta-\theta)\, d\zeta.$$

By construction of $\sigma^{ex}(b+p)$ (see Section 4.2.1), if x and y are consecutive alternatives, status quo x leads to a different outcome than status quo y under $\sigma^{ex}(b+p)$ in the game $\Gamma_c^{ex}\left(X,U^{b+p}\right)$ in a period with state ζ only if players disagree on how to rank x and y, that is, if $U_l^{x,y,b+p}(\zeta) < 0$ and $U_r^{x,y,b+p}(\zeta) > 0$.[1] From (6), this happens when $\zeta \in \left(\frac{x+y}{2} - b_r - p_r, \frac{x+y}{2} - b_l - p_l\right)$. In such states, either status quos $q \in \{x,y\}$ stays in place. So if $\sigma^{ex}(b+p)$ is played in $\Gamma_c^{en}(b)$, in such states, players' continuation payoff from status quo $q \in \{x,y\}$ is $-(q-\zeta-b_k)^2 + \delta W_k^{\sigma^{ex}(b+p)}(\zeta,q)$. Using (6), this shows that (8) holds. Uniqueness follows from the fact that the right-hand side of (8) is a contraction in $\theta \to W_k^{\sigma^{ex}(b+p)}(\theta,y) - W_k^{\sigma^{ex}(b+p)}(\theta,x)$ for the sup norm.

Claim 3 *For all $x,y \in \mathbb{R}$ with $x<y$, let $\Gamma^{en}\left(\{x,y\},U^b\right)$ denote the $2-$alternative game defined in Section 3 in which players have quadratic preferences with bias b and $X=\{x,y\}$. This game has a least polarized equilibrium $\sigma^{x,y}(b)$ (in the sense of Proposition 2) and the corresponding*

[1] We ignore indifferences as they happen with probability 0.

continuation value $W^{\sigma^{x,y}(b)}$ *satisfies the same condition as* $\theta \to W^{\sigma^{ex}(b+p)}(\theta,y) - W^{\sigma^{ex}(b+p)}(\theta,x)$ *in (7) and in (8) for some* p^* *that does not depend on* $\{x,y\}$.

Before we prove this claim, let us argue that together with step 1 and 2 above, it completes the proof. For any two consecutive alternatives $x, y \in X$, $W^{\sigma^{x,y}(b)}$ as constructed in Claim 3 satisfies (8) for $p = p^*$, so from Step 2, $\sigma^{ex}(b+p^*)$ is such that $W^{\sigma^{ex}(b+p^*)}(\theta,y) - W^{\sigma^{ex}(b+p^*)}(\theta,x)$ is equal to $W^{\sigma^{x,y}(b)}$. Claim 3 implies then that $W^{\sigma^{ex}(b+p)}(\theta,y) - W^{\sigma^{ex}(b+p)}(\theta,x)$ satisfies (7) for $p = p^*$, and since p^* does not depend on $\{x,y\}$, step 1 implies in turn that $\sigma^{ex}(b+p^*)$ is an equilibrium of $\Gamma_c^{en}\left(X, U^b\right)$ for any X. ■

Proof of Claim 3. Let $(W^z)_{z\in\mathbb{N}}$ be the sequence defined recursively by $W^0 = \left(0^{\mathcal{F}}, 0^{\mathcal{F}}\right)$ and $W^{z+1} = \left(\Omega^{\emptyset} \circ V\right)(W^z)$ as in the proof of Proposition 2 with current preferences $U^{x,y,b}$ as in (6).

Step A: $\sigma^{x,y}(b)$ *exists,* $(W^z)_{z\in\mathbb{N}}$ *is increasing in the order* $(\leq,\geq)$*, and it converges to* $W^{\sigma^{x,y}(b)}$. These results are established in the proof of Proposition 2. Note that this proof assumes that U is bounded over Θ, which is not true in $\Gamma^{en}(\{x,y\}, b)$ since $U = U_k^{x,y,b}$ and $\Theta = \mathbb{R}$. However, the boundedness of U is used only to establish the existence of a $\bar{W}$ such that (i) $\bar{W}(\leq,\geq)\left(\Omega^{\emptyset}\circ V\right)\left(\bar{W}\right)$ and (ii) $\bar{W}(\leq,\geq)\left(0^{\mathcal{F}}, 0^{\mathcal{F}}\right)$. Consider the following candidate: for all $\theta \in \Theta$ and $k \in \{l, r\}$, $\bar{W}_k(\theta) \mathrel{\hat{=}} E_{\theta(0)=\theta}\left[a_k \sum_{t=1}^{\infty} \delta^t \left|U_k^{x,y,b}(\theta(t))\right|\right]$, with $a_r = 1$ and $a_l = -1$.[2] Property (ii) is clearly satisfied. To prove (i), note that $\bar{W}_r(\theta) = \int_{\zeta\in\Theta}\left(\left|U_r^{x,y,b}(\zeta)\right| + \delta\bar{W}_r(\zeta)\right) f(\theta-\zeta)\, d\zeta$, so from formula (2),

$$\left(\Omega^{\emptyset}\circ V\right)_r\left(\bar{W}\right)(\theta) = \int_{\zeta\in D\left(\emptyset, V\left(\bar{W}\right)\right)}\left(\left|U_r^{x,y,b}(\zeta)\right| + \delta\bar{W}_r(\zeta)\right) f(\theta-\zeta)\, d\zeta \leq \bar{W}_r(\theta).$$

The proof for $\bar{W}_l$ is analogous.

Step B: There exists a negative and decreasing sequence $(p_l^z)_{z\in\mathbb{N}}$ *and a positive and increasing*

[2]Note that under our assumptions, $\bar{W}$ is well defined, that is, conditional on $\theta(0)$, $\sum_{t=1}^{\infty}\delta^t\left|U_k^{x,y,b}(\theta(t))\right|$ is integrable. To see this, note that from (6), $U_k^{x,y,b}(\theta)$ is linear in θ, so it suffices to prove that conditional on $\theta(0)$, $\sum_{t=1}^{\infty}\delta^t|\theta(t)-\theta(0)|$ is integrable. Note that $\theta(t)-\theta(0) = \sum_{t'=1}^{t} v(t')$, so for all $T\in\mathbb{N}$,

$$\sum_{t=1}^{T}\delta^t|\theta(t)-\theta(0)| \leq \sum_{t=1}^{T}\delta^t\sum_{t'=1}^{t}|v(t')| = \sum_{t'=1}^{T}|v(t')|\sum_{t=t'}^{T}\delta^t \leq \sum_{t'=1}^{T}|v(t')|\sum_{t=t'}^{\infty}\delta^t = \sum_{t'=1}^{T}|v(t')|\frac{\delta^{t'}}{1-\delta}.$$

Note that $\sum_{t'=0}^{T}E[|v(t')|]\frac{\delta^{t'}}{1-\delta}$ converges to a finite limit as $T\to\infty$ since by assumption, $E[|v(t')|]$ is constant and finite. Therefore, the monotone convergence theorem implies that $\sum_{t'=1}^{\infty}|v(t')|\frac{\delta^{t'}}{1-\delta}$ is integrable, and from the above inequality, so is $\sum_{t=1}^{\infty}\delta^t|\theta(t)-\theta(0)|$.

sequence $(p_r^z)_{z\in\mathbb{N}}$ *such that for all* $z\in\mathbb{N}$, $k\in\{l,r\}$ *and* $\theta\in\mathbb{R}$,

$$U_k^{x,y,b}(\theta)+\delta W_k^z(\theta)\begin{cases}>0 \text{ if } \theta>\frac{x+y}{2}-b_k-p_k^z\\ <0 \text{ if } \theta<\frac{x+y}{2}-b_k-p_k^z\end{cases}, \tag{9}$$

and the sequence $(W^z)_{z\in\mathbb{N}}$ *satisfies*

$$W_k^{z+1}(\theta)=\int_{\frac{x+y}{2}-b_r-p_r^z}^{\frac{x+y}{2}-b_l-p_l^z}\left[U_k^{x,y,b}(\zeta)+\delta W_k^z(\zeta)\right]f(\zeta-\theta)\,d\zeta. \tag{10}$$

We prove step B by induction on z. Since $W^0=\left(0^{\mathcal{F}},0^{\mathcal{F}}\right)$, from (6), (9) is satisfied for $z=0$ and $p^0=(0,0)$. Suppose now that (9) holds for some $z\in\mathbb{N}$ and some $p_l^z\leq 0$, and $p_r^z\geq 0$. From (1) and (9), $D(\emptyset,V(W^z))=\left(\frac{x+y}{2}-b_l-p_l^z,\frac{x+y}{2}-b_r-p_r^z\right)$. Substituting this expression for $D(\emptyset,V(W^z))$ and (2) in $W^{z+1}=\left(\Omega^{\emptyset}\circ V\right)(W^z)$, we obtain (10).

To complete the induction argument, we need to show that if (9) and (10) holds for some $z\in\mathbb{N}$ and some $p_l^z\leq 0$, and $p_r^z\geq 0$, then (9) holds for $z+1$ and some $p_l^{z+1}\leq p_l^z$, and $p_r^{z+1}\geq p_r^z$. Since f is single-peaked, for all $\zeta,\theta\in\mathbb{R}$ such that $\theta\leq\frac{x+y}{2}-b_r-p_r^z\leq\zeta$, $f(\zeta-\theta)$ is increasing in θ. Hence, (9) and (10) imply that for all $\theta\leq\frac{x+y}{2}-b_r-p_r^z$, $W_r^{z+1}(\theta)$ is increasing in θ, and from (6), $U_r^{x,y,b}(\theta)+\delta W_r^{z+1}(\theta)$ is strictly increasing in θ. Since f is single-peaked, as $\theta\to-\infty$, $f(\zeta-\theta)\to 0$, so from (10), $W_r^{z+1}(\theta)\to 0$, whereas from (6), $U_r^{x,y,b}(\theta)\to-\infty$, so $U_r^{x,y,b}(\theta)+\delta W_r^{z+1}(\theta)\to-\infty$. Moreover, from step A, $W_r^{z+1}\geq W_r^z$. Together with the induction hypothesis (9), this implies that for all $\theta>\frac{x+y}{2}-b_r-p_r^z$,

$$U_r^{x,y,b}(\theta)+\delta W_r^{z+1}(\theta)\geq U_r^{x,y,b}(\theta)+\delta W_r(\theta)>0.$$

Given the above properties of $\theta\to U_r^{x,y,b}(\theta)+\delta W_r^{z+1}(\theta)$, there exist $\theta^*\leq\frac{x+y}{2}-b_r-p_r^z$ such that $U_r^{x,y,b}(\theta)+\delta W_r^{z+1}(\theta)\gtrless 0$ when $\theta\gtrless\theta^*$. Setting $p_r^{z+1}=\frac{x+y}{2}-b_r-\theta^*$, we obtain $p_r^{z+1}\geq p_r^z$ and (9) for $z+1$. The proof for l is analogous.

Step C: $(p^z)_{z\in\mathbb{N}}$ *converges to some finite* p^* *such that* $p_l^*<0<p_r^*$ *and* $W_k^{\sigma^{x,y}(b)}$ *satisfies the same condition as* $\theta\to W_k^{\sigma^{ex}(b+p)}(\theta,y)-W_k^{\sigma^{ex}(b+p)}(\theta,x)$ *in (8) and (7) for* $p=p^*$.

By step B, $(p^z)_{z\in\mathbb{N}}$ tends to some $p^*\in[-\infty,0]\times[0,+\infty]$, and $\left(\frac{x+y}{2}-b_r-p_r^z,\frac{x+y}{2}-b_l-p_l^z\right)$ is increasing in z in the inclusion sense. Therefore, if we let $z\to\infty$ in (10), the monotone convergence theorem implies that $W_k^{\sigma^{x,y}(b)}(\theta)$ satisfies the same condition as $\theta \to W_k^{\sigma^{ex}(b+p)}(\theta, y) - W_k^{\sigma^{ex}(b+p)}(\theta, x)$ in (8) for $p = p^*$.

To show that $p_r^* \in (0, +\infty)$, suppose first that $p_r^* = 0$. Then since $p_r^* \geq p_r^1 \geq 0$, $p_r^1 = 0$. Since $W^0 = \left(0^{\mathcal{F}}, 0^{\mathcal{F}}\right)$, from (10), W^1 is continuous, so (9) implies that $U_r^{x,y,b}\left(\frac{x+y}{2} - b_r\right) + \delta W_r^1\left(\frac{x+y}{2} - b_r\right) = 0$, which, from (10) and (6), implies that

$$0 = \delta 2\left(y - x\right) \int_{\frac{x+y}{2} - b_r}^{\frac{x+y}{2} - b_l} \left(\zeta - \left(\frac{x+y}{2} - b_r\right)\right) f\left(\zeta - \left(\frac{x+y}{2} - b_r\right)\right) d\zeta,$$

which is impossible since the above integral is strictly positive, so $p_r^* > 0$. Suppose now that $p_r^* = +\infty$. As argued above, if we let $z \to \infty$ in (10), we get

$$W_r^{\sigma^{ex}(b+p)}(\theta) = \int_{-\infty}^{\frac{x+y}{2} - b_l - p_l^*} \left[U_k^{x,y,b}(\zeta) + \delta W_r^{\sigma^{ex}(b+p)}(\zeta)\right] f(\zeta - \theta)\, d\zeta.$$

The above equation means that $W_r^{\sigma^{ex}(b+p)}(\theta) = E_{\theta(0)=\theta}\left[\sum_{t=1}^{\hat{t}-1} U_k^{x,y,b}(\theta(t))\right]$ where $\hat{t}$ is the first period $t \geq 1$ such that $\theta(t) > \frac{x+y}{2} - b_l - p_l^*$. Since $\{\theta(t) : t \geq 0\}$ is a random walk and $U_r^{x,y,b}(\theta) \to -\infty$ as $\theta \to -\infty$, then clearly, $W_r^{\sigma^{ex}(b+p)}(\theta) < 0$ as $\theta \to -\infty$, which is impossible since from step A, $W_r^{\sigma^{ex}(b+p)}(\theta) \geq \bar{W}^0 = 0^{\mathcal{F}}$. The proof that $p_l^* \in (-\infty, 0)$ is analogous. Finally, if we let $z \to \infty$ in (9), we obtain that $W_k^{\sigma^{x,y}(b)}(\theta)$ satisfies the same condition as $W_k^{\sigma^{ex}(b+p)}(\theta, y) - W_k^{\sigma^{ex}(b+p)}(\theta, x)$ in (7) for $p = p^*$.

Step D: p^* does not depend on x, y.

Let $x', y' \in \mathbb{R}$, and let $(W'^z)_{z \in \mathbb{N}}$ and $(p'^z)_{z \in \mathbb{N}}$ be the sequences defined in step B for the game $\Gamma^{en}\left(\{x', y'\}, U^b\right)$. We show by induction on z that for all $z \in \mathbb{N}$, $p'^z = p^z$ and for all $\theta \in \Theta$, $W'^z(\theta) = \frac{y'-x'}{y-x} W^z\left(\theta + \frac{x+y}{2} - \frac{x'+y'}{2}\right)$. Since $W'^0 = W^0 = \left(0^{\mathcal{F}}, 0^{\mathcal{F}}\right)$, from (9), $p'^0 = p^0 = (0,0)$, and the property is true for $z = 0$. Suppose it is true for some $z \in \mathbb{N}$. From (6), $U_k^{x',y',b}(\theta) = \frac{y'-x'}{y-x} U_k^{x,y,b}\left(\theta + \frac{x+y}{2} - \frac{x'+y'}{2}\right)$. Using successively (10) for $(W'^z)_{z \in \mathbb{N}}$, the latter equality, the induction hypothesis, and the change of variable $\zeta' = \zeta + \frac{x+y}{2} - \frac{x'+y'}{2}$, we obtain

$$\begin{aligned} W_k'^{z+1}(\theta) &= \int_{\frac{x'+y'}{2}-b_r-p_r'^z}^{\frac{x'+y'}{2}-b_l-p_l'^z} \left[U_k^{x',y',b}(\zeta) + \delta W_k'^z(\zeta)\right] f(\zeta-\theta)\, d\zeta \\ &= \frac{y'-x'}{y-x} \int_{\frac{x+y}{2}-b_r-p_r^z}^{\frac{x+y}{2}-b_l-p_l^z} \left[U_k^{x,y,b}(\zeta') + \delta W_k^z(\zeta')\right] f\left(\zeta' - \left(\theta + \frac{x+y}{2} - \frac{x'+y'}{2}\right)\right) d\zeta' \\ &= \frac{y'-x'}{y-x} W_k^{z+1}\left(\theta + \frac{x+y}{2} - \frac{x'+y'}{2}\right). \end{aligned}$$

Thus, $U^{x',y',b}(\theta) + \delta W'^{z+1}(\theta) = \frac{y'-x'}{y-x}\left(U^{x,y,b}\left(\theta + \frac{x+y}{2} - \frac{x'+y'}{2}\right) + \delta W^{z+1}\left(\theta + \frac{x+y}{2} - \frac{x'+y'}{2}\right)\right)$, so one can easily see from (9) that necessarily, $p'^{z+1} = p^{z+1}$. ■

Proof of Proposition 8. Let $V_k^{\Delta,\sigma}(\theta, x)$ be the continuation payoff in $\Gamma_\wp^{en}(\Delta)$ from implementing x at a bargaining time t in which $\theta(t) = \theta$ given continuation play σ. Note that under our assumption on $\wp$, $V_k^{\Delta,\sigma}(\theta(t), x) \geq V_k^{\Delta,\sigma}(\theta(t), q(t))$ must hold for $k \in \{l, r\}$ and any policy x implemented with positive probability in t; hence, if given an opportunity to revert to $q(t)$ at the end of bargaining in t, no player would have a strict incentive to exercise this option. Hence, any equilibrium σ of $\Gamma_\wp^{en}(\Delta)$ can be mapped into an outcome equivalent equilibrium of an auxiliary game $\hat{\Gamma}_\wp^{en}(\Delta)$ which differs from $\Gamma_\wp^{en}(\Delta)$ only in that in each bargaining time t, at the end of the path of play of the within-period bargaining game $\wp(\theta(t), q(t))$ but before $x(t)$ is implemented, each player can unilaterally revert to $q(t)$.

Let $(\Delta^z)_{z\in\mathbb{N}}$ be a sequence such that $\lim_{z\to\infty} \Delta^z = 0$ and let $(\sigma^z)_{z\in\mathbb{N}}$ be a sequence of equilibria of $\Gamma_\wp^{en}(\Delta^z)$ such that there exists $Z \in \mathbb{N}$, a neighborhood B' of θ and $\varepsilon > 0$ such that for all $\zeta \in B'$ and $z \geq Z$, $\phi^{\sigma^z}(\zeta, q, x) \geq \varepsilon$ and $\phi^{\sigma^z}(\zeta, x, x) = 1$. Let $(\hat{\sigma}^z)_{z\in\mathbb{N}}$ be the corresponding sequence of equilibria of $\left(\hat{\Gamma}_\wp^{en}(\Delta^z)\right)_{z\in\mathbb{N}}$. Suppose by contradiction with Proposition 8 that $u_k(\theta, q) > u_k(\theta, x)$ for some $k \in \{l, r\}$. Since u_k is continuous in θ, there exists a neighborhood B'' of θ such that for all $\zeta \in B''$, $u_k(\zeta, q) \geq u_k(\zeta, x) + \epsilon$ for some $\epsilon > 0$. Let $B = B' \cap B''$ and let $\hat{t}_B$ be the first exit time of $\{\theta(t) : t \geq 0\}$ from B.

Let $z > Z$ and suppose that in $\hat{\Gamma}_\wp^{en}(\Delta^z)$, in some bargaining time t, which without loss of generality we normalize to 0, the state is $\theta(0) = \theta$, the status quo is q, and the realization of $\hat{\sigma}^z$ is such that x is implemented. Consider the following deviation of player k : veto x in $t = 0$, at the end of each subsequent bargaining period, impose the status quo q until there is a bargaining period $p_x^z > 0$ in which the outcome of the bargaining is x, and play $\hat{\sigma}^z$ from p_x^z onwards. Let $\Pi_k^z(\theta, q)$

denote the expected payoff from that deviation. If $\hat{t}_B > p_x^z \Delta^z$, then for all $t < p_x^z \Delta^z$ the flow payoff of k on the deviation path is $u_k(\theta(t), q)$ while on the equilibrium path, it is $u_k(\theta(t), x)$, and for all $t \geq p_x^z \Delta^z$, the deviation and the equilibrium yield the same outcome, and thus the same flow payoff. Since $u(.,.)$ is bounded by m, if $\hat{t}_B \leq p_x^z \Delta^z$, the expected payoff on the deviation path is no smaller than the expected payoff on the equilibrium path minus $2m/\rho$. Therefore,

$$\begin{aligned}
&\Pi_k^z(\theta, q) - V_k^{\Delta^z, \sigma^z}(\theta, q) \qquad (11)\\
&\geq \Pr_{\theta(0)=0}\left(\hat{t}_B > p_x^z \Delta^z\right) E_{\theta(0)=0}\left[\int_0^{p_x^z \Delta^z} \left(u_k(\theta(t), q) - u_k(\theta(t), x)\right) e^{-\rho t} dt | \hat{t}_B > p_x^z \Delta^z\right] \\
&- \Pr_{\theta(0)=0}\left(\hat{t}_B \leq p_x^z \Delta^z\right) \frac{2m}{\rho} \\
&\geq \left(1 - \Pr_{\theta(0)=0}\left(\hat{t}_B \leq p_x^z \Delta^z\right)\right) \frac{\epsilon}{\rho}\left(1 - e^{-\rho \Delta^z}\right) - \Pr_{\theta(0)=0}\left(\hat{t}_B \leq p_x^z \Delta^z\right) \frac{2m}{\rho}.
\end{aligned}$$

Note that $1 - e^{-\rho \Delta^z} \sim \rho \Delta^z$ as $\Delta^z \to 0$, so if we can show that $\Pr_{\theta(0)=0}\left(\hat{t}_B \leq p_x^z \Delta^z\right) = o(\Delta^z)$, the above inequality implies that $\Pi_k^z(\theta, q) > V_k^{\Delta^z, \sigma^z}(\theta, q)$ for Δ^z sufficiently small. Thus, the considered deviation is profitable for player k, a contradiction. To conclude, let us now show that $\Pr_{\theta(0)=0}\left(\hat{t}_B \leq p_x^z \Delta^z\right) = o(\Delta^z)$. Let $F(t) \mathrel{\hat=} \Pr_{\theta(0)=0}\left(\hat{t}_B \leq t\right)$. Using Bayes rule, we can write

$$\begin{aligned}
\frac{\Pr_{\theta(0)=0}\left(\hat{t}_B \leq p_x^z \Delta^z\right)}{\Delta^z} &= \int_0^\infty \Pr\left(p_x^z \geq \frac{t}{\Delta^z} | \hat{t}_B = t\right) \frac{1}{\Delta^z} dF(t) \\
&\leq \frac{1}{1-\varepsilon} \int_0^\infty (1-\varepsilon)^{\frac{t}{\Delta^z}} \frac{1}{\Delta^z} dF(t) = \frac{|\ln(1-\varepsilon)|}{1-\varepsilon} \int_0^\infty (1-\varepsilon)^\tau \tau \frac{F(\tau \Delta^z)}{\tau \Delta^z} d\tau,
\end{aligned}$$

where the last equality comes from integrating by parts and then using the change of variable $\tau = t/\Delta^z$. The term $F(t)/t$ is bounded for all $t > 0$ by some finite M (because by Assumption 2 its limit is 0 as $t \to 0$, and it is bounded by $1/t$ elsewhere), so $(1-\varepsilon)^\tau \tau \frac{F(\tau\Delta^z)}{\tau\Delta^z} d\tau$ is bounded by $(1-\varepsilon)^\tau \tau M$ for all $\tau > 0$. Since $(1-\varepsilon)^\tau \tau M$ is integrable over $(0, +\infty)$, and since Assumption 2 implies $\lim_{t\to 0} F(t)/t = 0$, the dominated convergence theorem implies

$$\begin{aligned}
\lim_{\Delta^z \to 0} \frac{\Pr_{\theta(0)=0}\left(\hat{t}_B \leq p_x^z \Delta^z\right)}{\Delta^z} &\leq \frac{|\ln(1-\varepsilon)|}{1-\varepsilon} \lim_{\Delta^z \to 0} \int_0^\infty (1-\varepsilon)^\tau \tau \frac{F(\tau \Delta^z)}{\tau \Delta^z} d\tau \\
&= \frac{|\ln(1-\varepsilon)|}{1-\varepsilon} \int_0^\infty (1-\varepsilon)^\tau \tau \left(\lim_{\Delta^z \to 0} \frac{F(\tau \Delta^z)}{\tau \Delta^z}\right) d\tau = 0.
\end{aligned}$$

■

Remark 1 *Assumption 2 is satisfied for Brownian processes.*

Proof. Let $\{\theta(t) : t \geq 0\}$ is a standard Brownian motion, and let $M(t) = \max_{s\in[0,t]}(\theta(s) - \theta(0))$ denote its running maximum. As is well known, $M(t)/\sqrt{t}$ has a normal distribution truncated at 0. By symmetry, $-m(t)$ has the same distribution as $M(t)$, where $m(t) = \min_{s\in[0,t]}(\theta(s) - \theta(0))$. If $D(t) = \max_{s\in[0,t]}|\theta(s) - \theta(0)|$ denotes the running maximal deviation, $D(t)/\sqrt{t}$ is equal to $\max\left\{M(t)/\sqrt{t}, -m(t)/\sqrt{t}\right\}$, so $D(t)$ is F.O.S. dominated by twice a truncated normal. If f denotes the p.d.f. of the normal distribution, and if $\hat{t}$ denotes the first exit time from $[\theta - b, \theta + b]$,

$$\Pr_{\theta(0)=\theta}\left(\hat{t} \leq t\right) = \Pr_{\theta(0)=\theta}\left(\frac{D(t)}{\sqrt{t}} \geq \frac{b}{\sqrt{t}}\right) \leq \int_{\frac{b}{\sqrt{t}}}^{\infty} 2f(\mu)\,d\mu = 2\frac{t}{b^2}\int_{\frac{b}{\sqrt{t}}}^{\infty}\frac{b^2}{t}f(\mu)\,d\mu \leq 2\frac{t}{b^2}\int_{\frac{b}{\sqrt{t}}}^{\infty}\mu^2 f(\mu)\,d\mu.$$

Since the normal distribution has a finite second moment, $\int_{\frac{b}{\sqrt{t}}}^{\infty}\mu^2 f(\mu)\,d\mu \to 0$ as $t \to 0$. So the above inequality shows that $\Pr\left(\hat{t} \leq t\right) = o(t)$ as $t \to 0$. ■

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