A MALTHUS-SWAN-SOLOW MODEL OF ECONOMIC GROWTH

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Abstract. In this paper we introduce in the Solow-Swan growth model a labor supply based on Malthusian ideas. We show that this model may yield several steady states and that an increase in total factor productivity might decrease the capital-labor ratio in a stable equilibrium.

"Why has it taken economists so long to learn that demography influences growth?"
Jeff Williamson (1998)[13]

1. Introduction.

In this note we propose a model which combines the classical Solow (1956)[10] and Swan (1956)[11] model with ideas about population growth that are borrowed from Malthus (1798)[9]. We will refer to our model as a Malthus-Swan-Solow (MSS) model. Our model has no technical progress, no institutional change, no human capital and no land.[1] Also we do not delve into demographic variables like mortality rates, life expectancy and the like, see Galor (2005)[5] for a survey on the importance of these variables on population growth.

We assume that the rate of growth of population depends on the real wage in a continuous way. This function is a generalization of one used by Hansen and Prescott (2002)[7].

We find that, as in the classical Solow-Swan model, there exist a steady state value of capital-labor ratio, see Proposition 1. However this steady state is not necessarily unique: Proposition 2 and Example 3 show that there might be an odd number of steady state capital-labor ratios. And only the smaller and the larger values of these capital-labor ratio are locally stable, see Proposition 4. This implies that there might be two, very different values of per capita income in the steady state: one with a small and another with a large value of per capita income. Finally we find that an increase in total factor productivity may increase or decrease the capital-labor ratio in a stable steady state (Proposition 5) but it always increases per capita income (Proposition 6).

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[1] Thus the adjective Malthusian refers only to Malthus ideas on labor supply and not on his ideas on land.

Summing up, the consideration of endogenous population in the Solow-Swan model brings new insights with respect to the standard model regarding the number, stability and comparative static properties of steady states.

It goes without saying that our model is not the first blending of the Swan-Solow model with population growth: Fanti and Manfredi (2003)[4] present a model producing cycles and Guerrini (2006)[6] presents a model where population converges to a constant growth rate, as in the original Solow-Swan model. Accinelli and Brida produced two papers in 2007 [1, 2] assuming a kind of logistic population curve in the Ramsey and the Solow-Swan models. They show that, in both cases, population growth does not play any role in the steady states of these two models.

2. **The model.** There are two factors of production, labor (L) and capital (K). Capital depreciates at a constant rate d. The economy produces a unique good (Y) -which can be used as a consumption good or as investment-according with a Cobb-Douglas production function

$$Y = AK^{1-\alpha}L^{\alpha}. \tag{1}$$

where A is the total factor productivity that represents the technology, human capital, institutions or in general anything that is conductive to affect output. We assume full employment of factors.[2]

The representative firms hires labor and capital and pays a real wage of ω and a real rental rate of r. The firm maximizes profits taking prices as given. First order condition of profit maximization with respect to labor is

$$\omega = \alpha AK^{1-\alpha}L^{\alpha-1} = \alpha Ak^{1-\alpha} \tag{2}$$

where k is capital-labor ratio.

Consumers save a fixed fraction of income s. Thus, capital accumulation is

$$\dot{K} = sY - dK. \tag{3}$$

where $\dot{K}$ is the increase in K. Let g_Z be the growth rate of a generic variable Z. Taking into account this, equation (3) can be written as

$$g_K = s\frac{Y}{K} - d = sAk^{-\alpha} - d \tag{4}$$

where in the last equality we used (1).

So far this is just the Swan-Solow model. We now introduce the Malthusian component. We assume that the rate of growth of labor depends continuously on the real wage. In some papers the growth of population depends on per capita consumption (e.g. Hansen and Prescott, 2002, Irmen, 2004). However, given that the production function is Cobb-Douglas, both assumptions are equivalent.[3] In particular we assume that,

$$g_L = g(\omega) \tag{5}$$

with $g()$ continuously differentiable and such that $\exists\omega'$ such that $g(w) > -d \; \forall\omega > \omega'$. As an example consider that the rate of increase of population is a linear function of the real wage. Formally,

$$g_L = -c + b\omega, \;\; b \geq 0. \tag{6}$$

This is just a generalization of the assumption in the Solow-Swan model that the rate of population growth is given (in which case $b = 0$ and $c < 0$). According with

[2]For a Solow-Swan model with unemployment see Alonso, Echevarria and Tran (2004)[3].

[3]Our assumption is more in line with the original formulation by Malthus (1798)[9].

(6) population grows iff $\omega > c/b$. Thus c/b can be interpreted as the subsistence wage. Of course it may be not very reasonable to assume that population grows very fast when wages are high (in Example 3 below, population growth eventually decreases for sufficiently large wages). But (6) could be thought as a reasonable approximation when wages are low.

3. **The results.** Let us solve the model. Using (2), the right hand side equality of (5) becomes

$$g_L = g(\alpha A k^{1-\alpha}). \tag{7}$$

Let $g_k = g_K - g_L$ be the rate of growth of the capital-labor ratio. Using (4) and (7) we obtain that

$$g_k = sAk^{-\alpha} - d - g(\alpha A k^{1-\alpha}). \tag{8}$$

A Steady State capital-labor ratio (SS in the sequel) is a k^* such that $g_k = 0$. In the sequel all variables in the SS will be denoted by a star superscript.

Proposition 1. *The MSS model has, at least, a SS.*

Proof. It is clear that $sAk^{-\alpha} - d - g(\alpha A k^{1-\alpha})$ tends to $+\infty$ when $k \to 0$. Since it is a continuous function it takes positive values in an interval close to zero. When $k \to \infty$ this function tends to $-d - g(\alpha A k^{1-\alpha})$ which for k large enough is negative since $g(w) > -d$. $\forall \omega > \omega'$. Thus, the intermediate value theorem implies the result. □

Assume now the following condition

$$-\alpha s A(k^*)^{-\alpha-1} - \alpha A(1-\alpha)(k^*)^{-\alpha} g'(\alpha A (k^*)^{1-\alpha}) \neq 0 \tag{9}$$

where $g'()$ is the derivative of $g()$. (9) just says that the slope of the right hand side of (8) is not zero at points in which its value is zero. This assumption holds generically in the sense that if it does not hold for some functional form of the right hand side of (8), a small perturbation restores the validity of this assumption. As a consequence of Proposition 1 and this assumption we have the following:

Proposition 2. *Under (9), the MSS model has an odd number of SS.*

Proof. Recall from Proposition 1 that the right hand side of (8) tends to $+\infty$ when $k \to 0$ and when $k \to \infty$ it tends to a negative value. Since (9) says that the right hand side of (8) has a non vanishing slope at k^*, the number of intersections of this function with the k axis is odd. □

Next example shows that there might be more than one SS.

Example 3. Let $g(w) = \min\{0, \alpha A k^{1-\alpha} - (\alpha A k^{1-\alpha})^2\}$. Thus the rate of increase of population increases with the real wage up to a point. If wages keeps increasing, the rate of increase of population decrease but it is never negative. This function is not differentiable: this will be taken care of later.

Let $s = .3, A = 1, \alpha = .5, d = .1$. Disregarding for a moment the non negativity constraint (which only bites for $k > 4$), the equation determining SS is

$$\frac{0.3}{k^{0.5}} - 0.5k^{0.5} + 0.25k^{1.0} - 0.1 = 0$$

which is the solid line in Figure 1 below. The dash line is the increase in population $.5k^{.5} - (.5k^{.5})^2$.

It is clear that there are two steady states for $k \in [0.4]$ namely $k^* = .753$ and $k^* = 3.51$. For $k > 4$ the equation that determines the SS is $0.3/k^{0.5} = 0.1$ yielding

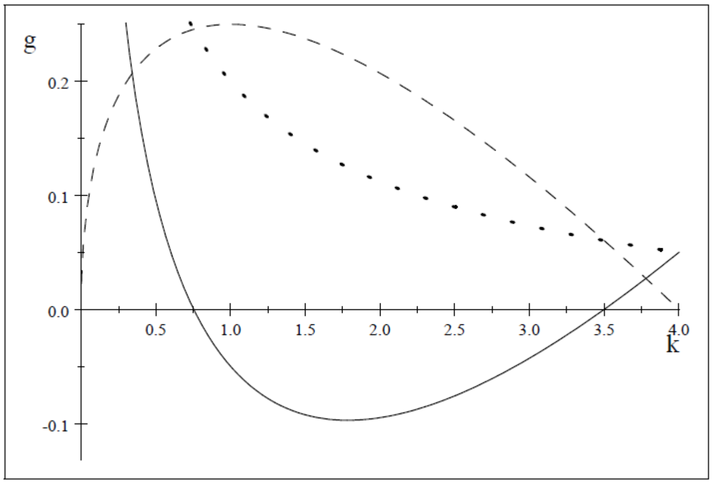

Figure 1.

$k^* = 9$ (the dotted line in Figure 1 is $0.3/k^{0.5} - 0.1$). Note that for $k = 4$ the values of the dotted and the solid lines are the same). Thus there are three SS. The two depicted in Figure 1 and $k^* = 9$. The last is a kind of Swan-Solow equilibrium in which population is exogenous and the SS at $k^* = .753$ is a kind of a Malthusian population trap. To end the example we have to get rid of the non differentiable point of the population equation. By the Weierstrass approximation theorem, there is a polynomial which approximates any continuous function with a required degree of approximation. Thus a polynomial approximation of the population equation yields SS arbitrarily closed to those found before and does not create any new intersection. Since polynomials are differentiable as many times as we wish, this completes the example.

We now study the stability of SS. We say that a SS value of k, say k^*, is locally stable if a sufficiently small perturbation of k^* generates a dynamics in (8) such that k tends to k^*. For further reference we remark that in a stable SS

$$\alpha s A(k^*)^{-\alpha-1} + \alpha A(1-\alpha)(k^*)^{-\alpha} g'(\alpha A(k^*)^{1-\alpha}) > 0 \tag{10}$$

Our next result is a consequence of Proposition 2.

Proposition 4. *Under (9), the largest and the smaller values of the steady states are locally stable. Stable and unstable SS alternate.*

Note that (9) generalizes (6). In the latter, the capital-labor ratio in the SS, k^*, is unique and globally stable because (8) now looks like

$$g_k = sAk^{-\alpha} - d + c - b\alpha A k^{1-\alpha} \tag{11}$$

and the right hand side of (11) is strictly decreasing in k.

Proposition 4 implies that an economy may get trapped in a SS in which income per capita is low but there is another SS in which per capita income is much larger.[4] We remark that the difficulty of achieving the high level of per capita income does not depend on profit maximization, i.e. on the existence of a private property

[4] A similar point was raised in a more complicated model by Voigtl?nder and Voth (2007)[12].

society. The same problem arises if all income is given to labor. In this case all our analysis holds with per capita income replacing real wages.

Let us now study comparative statics of the SS. Our next result shows that an increase in the technology parameter A has consequences in the MSS model that may be different from those in the Swan-Solow model.

Proposition 5. *Suppose the SS is stable. An small increase in A increases k^* iff $s > \alpha k^* g'(\alpha A(k^*)^{1-\alpha})$ and under (6), iff $d > c$*

Proof. Totally differentiating the right hand side of (8) when it is equal to zero we obtain that

$$\frac{dk^*}{dA} = \frac{s - \alpha k^* g'(\alpha A(k^*)^{1-\alpha})}{\alpha A s(k^*)^{-1} + \alpha(1-\alpha)A g'(\alpha A(k^*)^{1-\alpha})} \tag{12}$$

The denominator of (12) is positive in the SS, see (10), so the sign is determined by the numerator. Under (6) the numerator is $d - c$ and the result follows. □

Note that in the Solow-Swan model $g'() = 0$ so an increase in A always increases k^*. But in the MSS it is possible that an increase in the technology decreases the capital-labor ratio because it increases g_L.

Our last result studies the effect of the technological parameter A on per capita income in the SS (y^*).

Proposition 6. *In a stable SS a small increase of A increases per capita income.*

Proof. We readily see that $y^* = A(k^*)^{1-\alpha}$. Thus,

$$\frac{dy^*}{dA} = (k^*)^{1-\alpha} + A(1-\alpha)(k^*)^{-\alpha}\frac{dk^*}{dA} \tag{13}$$

$$= \frac{As(k^*)^{-\alpha}}{\alpha A s(k^*)^{-1} + \alpha(1-\alpha)A g' \alpha A(k^*)^{1-\alpha}}. \tag{14}$$

where (14) comes from plugging (12) into (13). The denominator is the stability condition (10) which is positive as it is the numerator so the result follows. □

We end this note by noting that in any stable SS, k^* is increasing in the savings rate s and the decreasing in the depreciation rate d. Both results are obtained by the same reasonings done in Propositions 5 and 6.

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