



UC3M Working Papers  
Statistics and Econometrics  
17-13  
ISSN 2387-0303  
Mayo 2017

Departamento de Estadística  
Universidad Carlos III de Madrid  
Calle Madrid, 126  
28903 Getafe (Spain)  
Fax (34) 91 624-98-48

## FINDINGS ABOUT THE TWO-STATE BMMPP FOR MODELING POINT PROCESSES IN RELIABILITY AND QUEUEING SYSTEMS

Yoel G. Yera<sup>a</sup>, Rosa E. Lillo<sup>a,b</sup> Pepa Ramírez-Cobo<sup>c,d</sup>

### Abstract

The Batch Markov Modulated Poisson Process (BMMPP) is a subclass of the versatile Batch Markovian Arrival process (BMAP) which have been widely used for the modeling of dependent and correlated simultaneous events (as arrivals, failures or risk events, real-time multimedia communications). Both theoretical and applied aspects are examined in this paper. On one hand, the identifiability of the stationary BMMPP2(K) is proven, where K is the maximum batch size. This is a powerful result when inferential tasks related to real data sets are carried out. On the other hand, some findings concerning the correlation and autocorrelation structures are provided.

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**Keywords:** *Markov-modulated Poisson process (MMPP), Batch Markovian arrival process (BMAP), correlation structure, Identifiability*

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<sup>a</sup> Department of Statistics, Universidad Carlos III de Madrid.

<sup>b</sup> UC3M-BS Institute of Financial Big Data Getafe, Madrid.

<sup>c</sup> Department of Statistics and Operations Research, Universidad de Cádiz.

<sup>d</sup> IMUS, Institute of Mathematics, Univ. Sevilla, Spain.

Acknowledgements: the first and second authors acknowledge financial support from the Spanish Ministry of Economy and Competitiveness, research project ECO2015-66593-P. The Third author acknowledge financial support from the Spanish Ministry of Economy and Competitiveness, research project MTM2015-65915-R; and also from Junta de Andalucía, and BBVA Foundation, research project P11-FQM-7603 and FQM-329

# Findings about the two-state BMMPP for modeling point processes in reliability and queueing systems

Yoel G. Yera<sup>1</sup>, Rosa E. Lillo<sup>1,2</sup> Pepa Ramírez-Cobo<sup>3,4</sup>

<sup>1</sup>Department of Statistics, Universidad Carlos III de Madrid, Spain

<sup>2</sup>UC3M-BS Institute of Financial Big Data Getafe, Madrid, Spain

<sup>3</sup>Department of Statistics and Operations Research, Universidad de Cádiz, Spain

<sup>4</sup>IMUS, Institute of Mathematics, Univ. Sevilla, Spain

May 30, 2017

## Abstract

The Batch Markov Modulated Poisson Process (*BMMPP*) is a subclass of the versatile Batch Markovian Arrival process (*BMAP*) which have been widely used for the modeling of dependent and correlated simultaneous events (as arrivals, failures or risk events, real-time multimedia communications). Both theoretical and applied aspects are examined in this paper. On one hand, the identifiability of the stationary  $BMMPP_2(K)$  is proven, where  $K$  is the maximum batch size. This is a powerful result when inferential tasks related to real data sets are carried out. On the other hand, some findings concerning the correlation and autocorrelation structures are provided.

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## 1 Introduction

The Batch Markovian arrival process (noted *BMAP*, see Neuts [25]) has been suggested in the literature for modeling dependent data representing the occurrence of an arrival, failure or risk event. The *BMAPs* constitute a large class of point processes that allows for non-exponential and dependent times between (possibly correlated) consecutive batch events. It is known that the stationary (*B*)*MAPs* are capable of approximating any stationary (batch) point process (Asmussen and Koole [3]), which suggests the versatility and range of applications of such processes. Therefore, different classes of *BMAPs* have been

considered in a number of real life contexts where batch dependent occurrence times are commonly observed, as in queueing, teletraffic, reliability or insurance. See for example, Banerjee et al. [4], Sikdar and Samanta [43], Banik and Chaudhry [5], Ghosh and Banik [12], Montoro-Cazorla et al. ([22], [23]), Okamura et al. [26], Kang et al. [17], Casale et al. [7], Wu et al. [45], Ramírez-Cobo et al. ([32], [34]), Montoro-Cazorla and Pérez-Ocón [24] and Liu et al. [20]. The single arrival *BMAP* (the *MAP*) and the general *BMAP* are highly parametrized models where, in practice, only inter-event times and batch sizes are usually observed. Therefore, such processes commonly suffer from identifiability problems, which occur when different representations lead to the same likelihood function for the observed data. The study of the identifiability is crucial when estimation of the process parameters is to be considered. In particular, the non-identifiability of a process has serious negative consequences: the likelihood function may be highly multimodal, implying that standard methods (as the EM algorithm) will be strongly dependent on the starting values, running the risk of getting stuck at a poor local maximum.

Different works have dealt with the problem of identifiability in *BMAP*-related processes, especially for the *MAP* and some of its subclasses as the well-known Markov-modulated Poisson process (*MMPP*). See for example, Rydén [40], where it is proven the identifiability of the *MMPP*. On the other hand, Bodrog et al. [6] provided a canonical and unique representation for the stationary two-state *MAP*. Another example is Ramírez et. al [30], where it is shown that the  $MAP_2$  is not identifiable. Furthermore, the conditions under which two different sets of parameters induce identical stationary laws for the observable process are given. Also Ramírez-Cobo and Lillo [31] partially solved the identifiability problem for the stationary three-state *MAP*. For the non-stationary  $MAP_2$  the lack of identifiability is studied in Rodríguez et al. [39]. On the other hand, Rodríguez et al. [37] explores the identifiability of the stationary two-state *BMAP* noted  $BMAP_2(K)$ , where  $K$  is the maximum batch size and proves that for  $K \geq 2$  the process cannot be identified. For the case where events occur simultaneously, Rodríguez et al. [37] seems to be the unique paper devoted to study the identifiability issue, up to the authors knowledge.

As commented previously, the *MMPP* is an identifiable class of *MAP*, a fact that has eased its statistical estimation, see Landon et. al. [18], Özekici and Soyer ([27]-[28]), Fearnhead and Sherlock [11], Heyman and Lucatoni [14], Scott and Smyth [42] and Scott [41]. In this paper we consider the batch counterpart of the *MMPP*, the so-called Batch Markov-modulated Poisson process, noted *BMMPP*. This process has been already considered in the literature (Akar and Sohraby [1], Anastasi et. al. [2], Cordeiro and Kharoufeh [9], Takine [44], Revzina [35], and Dubin [10]) for modeling real-time multimedia communication systems and computer networks systems. However, in most of such papers, a reduced version of the *BMMPP* with batch probabilities independent from the states of the underlying Markov chain, is used. Here, we present two significant weaknesses in terms of modeling of such simplified *BMMPP* and consider the general, two-state *BMMPP*, for which two major contribu-

tions are provided. On one hand, we study the versatility of the process for modeling correlated batch events through the autocorrelation function of the batch sizes. Our findings show the suitability of the two-state *BMMPP* for fitting positively correlated batch sizes. Second, with the future aim of carrying out statistical inference of the process, we prove the identifiability of the two-state *BMMPP*.

This paper is structured as follows. The two-state *BMMPP* is introduced in Section 2, where also some limitations of a reduced version of this process are described. Section 3 provides new results concerning the autocorrelation function of the batch sizes for the *BMMPP*<sub>2</sub>(*K*). In section 4 the identifiability of the two-state *BMMPP* is proven. Finally, Section 5 is devoted to summarize the conclusions and some extensions of this work.

## 2 Description of the stationary two-state *BMMPP*

In this section, the two-state Batch Markov-modulated Poisson process, noted *BMMPP*<sub>2</sub>(*K*), where *K* is the maximum batch size, is formally defined. Also, some properties that will be used throughout this paper are reviewed. The *BMMPP*<sub>2</sub>(*K*) is a Poisson process whose rate is modulated by an exogenous, irreducible Markov process,  $\{J(t) : t \geq 0\}$ , with state space  $S := \{s_1, s_2\}$ , a generator matrix *Q* and an initial distribution  $\alpha$ . Whenever  $J(t) = s_i$ , an event occur according to a Poisson process with rate  $\lambda_i$  ( $\lambda_i > 0$ ), and this status remains unchanged while the process remains in this state. As soon as *J* enters another state,  $s_j \in S$ , the arrival Poisson process alters accordingly. The process behaves as follows: at the end of an exponentially distributed sojourn time in state *i*, with mean  $1/\lambda_i$ , two possible state transitions can occur. First, with probability  $p_{ij0}$ , no event occurs and the system enters into a different state  $j \neq i$ . Second, with probability  $p_{ik}$ , a batch event of size *k* is produced, if the state of the process is  $s_i$ , and the system continues in the same state *i*.

A *BMMPP*<sub>2</sub>(*K*),  $\mathcal{B}$ , can be represented by the set of rate matrices ( $D_0, D_1, \dots, D_K$ ) such that

$$\begin{aligned} (D_0)_{ii} &= \lambda_i & 1 \leq i \leq 2 \\ (D_0)_{ij} &= \lambda_i p_{ij0} & 1 \leq i, j \leq 2 \quad i \neq j \\ (D_k)_{ik} &= \lambda_i p_{ik} & 1 \leq i \leq 2 \quad 1 \leq k \leq K, \\ (D_k)_{ijk} &= 0 & 1 \leq i, j \leq 2 \quad i \neq j, \end{aligned} \tag{1}$$

where

$$\sum_{j=1, j \neq i}^2 p_{ij0} + \sum_{k=1}^K p_{ik} = 1 \quad \text{for all } i = \{1, 2\}.$$

The definition of the rate matrices implies that

$$Q = \sum_{k=0}^K D_k$$

is the infinitesimal generator of the underlying Markov process  $J(t)$ , with stationary probability vector  $\boldsymbol{\pi} = (\pi_1, \pi_2)$ , satisfying  $\boldsymbol{\pi}Q = 0$  and  $\boldsymbol{\pi}\mathbf{e} = 1$ , where  $\mathbf{e}$  is a vector of ones.

Figure 1 illustrates a realization of the  $BMMPP_2(K)$ , where the dashed line corresponds to transitions without events and the solid lines correspond to transitions where an event of size  $b_i \in \{1, \dots, K\}$  occurs.

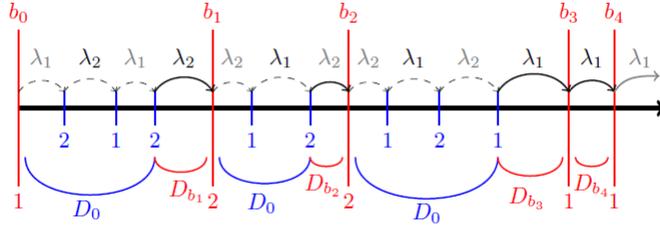


Figure 1: Transition diagram for the  $BMMPP_2(K)$ . The dashed line corresponds to transitions without events, governed by  $D_0$ , and the solid lines correspond to transitions of size  $b_k$ , governed by  $D_{b_k}$ .

## 2.1 Performance measures

A review of the performance measures regarding the  $BMMPP_2(K)$  is given next. If  $S_n$  denotes the state of the  $BMMPP$  at the time of the  $n$ -th event,  $B_n$  the batch size of that event and  $T_n$  the time between the  $(n-1)$ -th and  $n$ -th events, then the process  $\{S_{n-1}, \sum_{i=1}^n T_i, B_n\}_{n=1}^{\infty}$  is a Markov renewal process (see for example, Chakravorthy [8]). Furthermore, if

$$D = \sum_{k=1}^K D_k,$$

then  $\{S_n\}_{n=0}^{\infty}$  is a Markov chain with transition matrix

$$P^* = (-D_0)^{-1}D.$$

On the other hand, the variables  $T_n$ s are phase-type distributed with representation  $\{\boldsymbol{\phi}, D_0\}$ , such that  $\boldsymbol{\phi}$  is the stationary probability vector associated to  $P^*$  computes as  $\boldsymbol{\phi} = (\boldsymbol{\pi}D\mathbf{e})^{-1}\boldsymbol{\pi}D$  (see Chakravorthy [8] and Latouche and Ramaswami [19]). In consequence, the moments of  $T_n$  in the stationary case are given by

$$\mu_r = E(T^r) = r!\boldsymbol{\phi}(-D_0)^{-r}\mathbf{e}, \quad \text{for } r \geq 1, \quad (2)$$

and the auto-correlation function is

$$\rho_T(l) = \rho(T_1, T_{l+1}) = \frac{\boldsymbol{\mu}\boldsymbol{\pi}[(-D_0)^{-1}D]^l(-D_0)^{-1}\mathbf{e} - \mu_1^2}{\sigma^2},$$

where  $\sigma^2 = \mu_2 - \mu_1^2$ .

Also, according to Rodríguez et al. [37], the mass probability function of the stationary batch size,  $B$ , is

$$P(B = k) = \phi(-D_0)^{-1} D^k \mathbf{e}, \quad \text{for } k = 1, \dots, K,$$

from which the moments of  $B$  are obtained as

$$\beta_r = E[B^r] = \phi(-D_0)^{-1} D_r^* \mathbf{e}, \quad \text{for } r \geq 1, \quad (3)$$

where  $D_r^* = \sum_{k=1}^K k^r D_k$ . Also, the autocorrelation function in the stationary version of the process  $\rho_B(l)$  is given by

$$\rho_B(l) = \rho(B_1, B_{l+1}) = \frac{\phi(-D_0)^{-1} D_1^* [(-D_0)^{-1} D]^l (-D_0)^{-1} D_1^* \mathbf{e} - \beta_1^2}{\sigma_B^2}, \quad (4)$$

where  $\beta_1$  and  $\sigma_B^2 = \beta_2 - \beta_1^2$  are computed from (3).

Finally, in Rodríguez [37] it is proven that the Laplace-Stieltjes transform (LST) of the  $n$  first inter-event times and batch sizes of a stationary  $BMAP_2(K)$  is given by

$$f_{T,B}^*(\mathbf{s}, \mathbf{z}) = \phi(s_1 I - D_0)^{-1} \xi(z_1) \dots (s_n I - D_0)^{-1} \xi(z_n) \mathbf{e}, \quad (5)$$

where  $\mathbf{s} = (s_1, \dots, s_n)$ ,  $\mathbf{z} = (z_1, \dots, z_n)$  and  $\xi(z_i) = \sum_{k=1}^K D_k z_i^k$ , for  $i = 1, \dots, n$ .

## 2.2 A simplified version of the $BMMPP_2(K)$

Usually, it has been assumed in applications of the  $BMMPP$  that  $p_{ik} = p_{jk}$  for all  $i, j$  (in other words, there is independence between the state of the underlying Markov process and the size event), see for example, Cordeiro and Kharoufeh [9], Takine [44], Revzina [35] and Dubin [10]. This definition can be restrictive in practice as will be shown here. In this section, it will be proven that under such assumption, the autocorrelation of the random variable representing the batch size is equal to zero. Also, the correlation coefficient between the batch size and the times between the occurrence of events is zero.

**Proposition 1.** *Let  $\mathcal{B} = \{D_0, D_1, \dots, D_K\}$  a stationary  $BMMPP_2(K)$ , such that the probabilities as in (1) satisfy  $p_{ik} = p_{jk}$ ,  $i, j \in \{1, 2\}$ . Then, the autocorrelation function of the batch sizes  $\rho_B(l)$ , as in (4), satisfies*

$$\rho_B(l) = 0, \quad \text{for all } l \geq 1.$$

*Proof.* Consider the first-lag autocorrelation coefficient,  $\rho_B(1)$ , which according to (4) is written as

$$\rho_B(1) = \frac{\phi(-D_0)^{-1} D_1^* (-D_0)^{-1} D_1^* \mathbf{e} - (\phi(-D_0)^{-1} D_1^* \mathbf{e})^2}{\phi(-D_0)^{-1} D_2^* \mathbf{e} - (\phi(-D_0)^{-1} D_1^* \mathbf{e})^2}. \quad (6)$$

Using that  $p_k = p_{ik} = p_{jk}$  for  $i, j = \{1, 2\}$  and  $\Delta(\Lambda) = \text{diag}(-D_0)$ ,  $\mathcal{B}$  can be represented as  $\mathcal{B} = \{D_0, p_1\Delta(\Lambda), \dots, p_K\Delta(\Lambda)\}$  and consequently (6) can be rewritten as

$$\begin{aligned}
\rho_B(1) &= \frac{\phi(-D_0)^{-1} \left( \sum_{k=1}^K kp_k \right) \Delta(\Lambda) (-D_0)^{-1} \left( \sum_{k=1}^K kp_k \right) \Delta(\Lambda) \mathbf{e}}{\phi(-D_0)^{-1} \left( \sum_{k=1}^K k^2 p_k \right) \Delta(\Lambda) \mathbf{e} - \left[ \phi(-D_0)^{-1} \left( \sum_{k=1}^K kp_k \right) \Delta(\Lambda) \mathbf{e} \right]^2} \\
&= \frac{\left[ \phi(-D_0)^{-1} \left( \sum_{k=1}^K kp_k \right) \Delta(\Lambda) \mathbf{e} \right]^2}{\phi(-D_0)^{-1} \left( \sum_{k=1}^K k^2 p_k \right) \Delta(\Lambda) \mathbf{e} - \left[ \phi(-D_0)^{-1} \left( \sum_{k=1}^K kp_k \right) \Delta(\Lambda) \mathbf{e} \right]^2} \\
&= \frac{\left( \sum_{k=1}^K kp_k \right)^2 \left[ \phi(-D_0)^{-1} \Delta(\Lambda) (-D_0)^{-1} \Delta(\Lambda) \mathbf{e} - (\phi(-D_0)^{-1} \Delta(\Lambda) \mathbf{e})^2 \right]}{\phi(-D_0)^{-1} \left( \sum_{k=1}^K k^2 p_k \right) \Delta(\Lambda) \mathbf{e} - \left[ \phi(-D_0)^{-1} \left( \sum_{k=1}^K kp_k \right) \Delta(\Lambda) \mathbf{e} \right]^2} \\
&= \frac{\left( \sum_{k=1}^K kp_k \right)^2 \left[ \phi(-D_0)^{-1} \Delta(\Lambda) \mathbf{e} - (1)^2 \right]}{\left( \sum_{k=1}^K k^2 p_k \right) \phi(-D_0)^{-1} \Delta(\Lambda) \mathbf{e} - \left( \sum_{k=1}^K kp_k \right)^2 \left[ \phi(-D_0)^{-1} \Delta(\Lambda) \mathbf{e} \right]^2} \quad (7) \\
&= \frac{\left( \sum_{k=1}^K kp_k \right)^2 [1 - 1]}{\left( \sum_{k=1}^K k^2 p_k \right) - \left( \sum_{k=1}^K kp_k \right)^2} \\
&= 0.
\end{aligned}$$

In (7) it is used that  $(-D_0)^{-1} D \mathbf{e} = \mathbf{e}$  and  $\phi(-D_0)^{-1} D \mathbf{e} = 1$ , which can be derived since  $Q \mathbf{e} = \mathbf{0}$  as follow

$$\begin{aligned}
(-D_0)^{-1} D \mathbf{e} &= (-D_0)^{-1} (Q - D_0) \mathbf{e} \\
&= (-D_0)^{-1} Q \mathbf{e} - (-D_0)^{-1} D_0 \mathbf{e} \\
&= (-D_0)^{-1} Q \mathbf{e} + \mathbf{e} \\
&= \mathbf{e},
\end{aligned} \quad (8)$$

and consequently,

$$\phi(-D_0)^{-1} D \mathbf{e} = \phi \mathbf{e} = 1. \quad (9)$$

In Rodríguez et. al. [38], it is proven that for a general  $BMAP_2(K)$ , the auto-correlation function decreases in absolute value ( $|\rho_B(l)| \geq |\rho_B(l+1)|$  for all  $l \geq 1$ ). Therefore, if  $\rho_B(1) = 0$  implies  $\rho_B(l) = 0$  for all  $l \geq 1$ , and the proof is completed.  $\square$

**Proposition 2.** *Let  $\mathcal{B} = \{D_0, D_1, \dots, D_K\}$  be a stationary  $BMMPP_2(K)$  such that the probabilities as in (1) satisfy  $p_{ik} = p_{jk}$ ,  $i, j \in \{1, 2\}$ . Let  $T$  and  $B$  denote inter-event times and the batch size, respectively. Then, the covariance between  $T$  and  $B$  is equal to zero.*

*Proof.* First, the expression for  $E[TB]$  is deduced using the LST as given in (5)

$$\begin{aligned}
E[TB] &= E \left[ \frac{\partial(-e^{-sT} z^B)}{\partial s \partial z} \right] \Big|_{s=0, z=1} = - \frac{\partial f_{T,B}^*(s, z)}{\partial s \partial z} \Big|_{s=0, z=1} \\
&= - \frac{\partial}{\partial z} \left[ \frac{\partial}{\partial s} \left[ \phi(sI - D_0)^{-1} \left( \sum_{k=1}^K z^k D_k \right) \mathbf{e} \right] \right] \Big|_{s=0, z=1} \\
&= \frac{\partial}{\partial z} \left[ \phi(sI - D_0)^{-2} \left( \sum_{k=1}^K z^k D_k \right) \mathbf{e} \right] \Big|_{s=0, z=1} \\
&= \left[ \phi(sI - D_0)^{-2} \left( D_1 + \sum_{k=2}^K k z^{k-1} D_k \right) \mathbf{e} \right] \Big|_{s=0, z=1} \\
&= \phi(-D_0)^{-2} D_1^* \mathbf{e}.
\end{aligned}$$

Hence, using (2)-(3) and (8)-(9), the nullity of  $Cov(T, B)$  is proven as follow

$$\begin{aligned}
Cov(T, B) &= \phi(-D_0)^{-2} D_1^* \mathbf{e} - \mu_1 \beta_1 \\
&= \phi(-D_0)^{-2} D_1^* \mathbf{e} - \phi(-D_0)^{-1} \mathbf{e} \phi(-D_0)^{-1} D_1^* \mathbf{e} \\
&= \phi(-D_0)^{-2} \left( \sum_{k=1}^K k p_k \right) \Delta(\Lambda) \mathbf{e} - \phi(-D_0)^{-1} \mathbf{e} \phi(-D_0)^{-1} \left( \sum_{k=1}^K k p_k \right) \Delta(\Lambda) \mathbf{e} \\
&= \left( \sum_{k=1}^K k p_k \right) [\phi(-D_0)^{-2} \Delta(\Lambda) \mathbf{e} - \phi(-D_0)^{-1} \mathbf{e} \phi(-D_0)^{-1} \Delta(\Lambda) \mathbf{e}] \\
&= \left( \sum_{k=1}^K k p_k \right) [\phi(-D_0)^{-1} \mathbf{e} - \phi(-D_0)^{-1} \mathbf{e}] \\
&= 0.
\end{aligned}$$

□

### 3 The autocorrelation function of the batch size for the $BMMPP_2(K)$

In order to avoid the limited behaviour of the simplified version of the  $BMMPP_2(K)$  presented in the previous section, we consider from now the general  $BMMPP_2(K)$  with probabilities  $p_{ik}$  dependent on state  $i \in \{1, 2\}$ . Figures 2 and 3 show the first-lag autocorrelation coefficient of the batch size and the correlation coefficient between the inter-event times and the batch sizes, respectively for a sequence of 10000 randomly simulated  $BMMPP_2(2)$ s. Since the autocorrelation function of the batch size decreases with the lag value (see, Rodríguez et al. [39]), then, it can be deduced from the figures that Propositions 1 and 2 are not true in the general case.

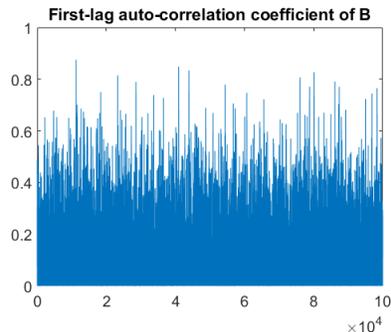


Figure 2: Values of the first-lag autocorrelation coefficient of the batch size for a total 100000 simulated  $BMMPP_2(2)$  with the general formulation.

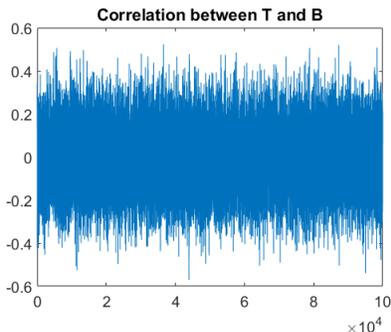


Figure 3: Values of the correlation between the batch size and the inter-event time for a total 100000 simulated  $BMMPP_2(2)$  with the general formulation.

The auto-correlation function of the event sizes is crucial when the modeling capability of the process is of interest. In Rodríguez et. al. [38] the auto-correlation functions of the inter-event times and event sizes for the  $BMAP$  are studied.

The auto-correlation function for the inter-event times,  $\rho_T$ , is the same for the  $BMAP_2(K)$  and  $MAP_2$ . Then, the results obtained for the  $MAP_2$  by Heindl et al. [13] and Ramírez-Cobo and Carrizosa [29] are also valid for the  $BMMPP_2(K)$ . In particular it is known that the lag-one auto-correlation,  $\rho_T(l)$ , is upper-bounded by 0.5 and the auto correlation function is exponentially decreasing in absolute value. On the other hand, Kang and Sung [16] prove that for any  $MMPP_2$ ,  $\rho_T(l) \geq 0$  for all  $l$ .

In the case of the event sizes, Rodríguez et. al. [38] gives a characterization of the auto-correlation functions in terms of the eigenvalues of the stochastic matrix governing  $(P^*)$ . For two states process, this representation allows to prove that the autocorrelation function decrease geometrically to zero and four different decrease patterns are provided. It is shown trough simulation how

$\rho_B(1)$  for  $BMAP_2(2)$ s may take values close to 1 or -1. In Figure 2 it can be seen how the first-lag auto-correlation coefficient of the batch sizes for the  $BMMPP_2(2)$  may also take values very close to 1, but negative values are not obtained.

In this section it is proven that the autocorrelation function of the batch sizes of the  $BMMPP_2(K)$ ,  $\rho_B(l)$  as in (4), is non-negative.

**Lemma 1.** *Consider a  $BMMPP_2(2)$  and let  $\rho_B(1)$  denote the first-lag auto-correlation coefficient of the batch sizes. Then,  $\rho_B(1) \geq 0$ .*

*Proof.* A stationary  $BMMPP_2(2)$  will be represented by  $B = \{D_0, D_1, D_2\}$  where

$$\begin{aligned} D_0 &= \begin{pmatrix} -\lambda_1 & \lambda_1 p_{120} \\ \lambda_2 p_{210} & -\lambda_2 \end{pmatrix} = \begin{pmatrix} x & y \\ r & u \end{pmatrix}, \\ D_1 &= \begin{pmatrix} p_{111} \lambda_1 & 0 \\ 0 & p_{221} \lambda_2 \end{pmatrix} = \begin{pmatrix} w & 0 \\ 0 & q \end{pmatrix}, \\ D_2 &= \begin{pmatrix} p_{112} \lambda_1 & 0 \\ 0 & p_{222} \lambda_2 \end{pmatrix} = \begin{pmatrix} -x - y - w & 0 \\ 0 & -r - u - q \end{pmatrix}. \end{aligned}$$

Consider the first-lag autocorrelation coefficient,  $\rho_B(1)$ . It is not difficult to see that after some computations, the numerator in (4), taking  $l = 1$  and  $K = 2$ , becomes

$$\phi(-D_0)^{-1} D_1^* [I - e\phi] (-D_0)^{-1} D_1^* e = \frac{ry[w(u+r) - q(x+y)]^2}{(xu - ry)(rx + ry + ry + yu)^2}.$$

Similarly, the denominator in (4) is found as

$$\phi(-D_0)^{-1} D_2^* e - (\phi(-D_0)^{-1} D_1^* e)^2 = \frac{(rw + yq)[r(-x - y - w) + y(-r - u - q)]}{(rx + ry + ry + yu)^2}.$$

Therefore,

$$\rho_B(1) = \frac{ry[w(u+r) - q(x+y)]^2}{(xu - ry)(rw + yq)[r(-x - y - w) + y(-r - u - q)]}. \quad (10)$$

Since,

$$\begin{aligned} ry &= \lambda_2 p_{210} \lambda_1 p_{120} \geq 0, \\ rw + yq &= \lambda_2 p_{210} \lambda_1 p_{111} + \lambda_1 p_{120} \lambda_2 p_{221} \geq 0, \\ r(-x - y - w) + y(-r - u - q) &= \lambda_2 p_{210} \lambda_1 p_{112} + \lambda_1 p_{120} \lambda_2 p_{222} \geq 0, \\ xu - ry &= \text{Det}(D_0) = \lambda_1 \lambda_2 (1 - p_{120} p_{210}) \geq 0, \end{aligned}$$

then, it can be concluded that  $\rho_B(1)$  as in (10) satisfies  $\rho_B(1) \geq 0$ .  $\square$

Lemma 1 is extended in the next Lemma 2 to  $BMMPP_2(K)$  with  $K \geq 3$ .

**Lemma 2.** *Consider a  $BMMPP_2(K)$  with  $K \geq 3$  and let  $\rho_B(1)$  denote the first-lag autocorrelation coefficient of the batch sizes. Then,  $\rho_B(1) \geq 0$ .*

*Proof.* A stationary  $BMMPP_2(K)$  will be represented by  $B = \{D_0, D_1, \dots, D_K\}$  where

$$\begin{aligned} D_0 &= \begin{pmatrix} -\lambda_1 & \lambda_1 p_{120} \\ \lambda_2 p_{210} & -\lambda_2 \end{pmatrix} = \begin{pmatrix} x & y \\ r & u \end{pmatrix}, \\ D_k &= \begin{pmatrix} p_{11k} \lambda_1 & 0 \\ 0 & p_{22k} \lambda_2 \end{pmatrix} = \begin{pmatrix} w_k & 0 \\ 0 & q_k \end{pmatrix} \quad \text{for } 1 \leq k \leq K-1, \\ D_K &= \begin{pmatrix} p_{11K} \lambda_1 & 0 \\ 0 & p_{22K} \lambda_2 \end{pmatrix} = \begin{pmatrix} -x - y - \sum_{k=1}^{K-1} w_k & 0 \\ 0 & -r - u - \sum_{k=1}^{K-1} q_k \end{pmatrix}. \end{aligned}$$

In this case, the numerator in (4) with  $l = 1$  and general  $K$  turns out

$$\phi(-D_0)^{-1} D_1^* [I - \mathbf{e}\phi] (-D_0)^{-1} D_1^* \mathbf{e} = \frac{ry[W_1(u+r) - Q_1(x+y)]^2}{(xu - ry)(rx + ry + ry + yu)^2}, \quad (11)$$

where  $W_1 = \sum_{k=1}^{K-1} (K-k)w_k$  and  $Q_1 = \sum_{k=1}^{K-1} (K-k)q_k$ . Due to

$$\begin{aligned} ry &= \lambda_2 p_{210} \lambda_1 p_{120} \geq 0, \\ xu - ry &= \text{Det}(D_0) = \lambda_1 \lambda_2 (1 - p_{120} p_{210}) \geq 0, \end{aligned}$$

then Eq. (11) is positive or equal to zero. Similarly, the denominator is given by

$$\begin{aligned} \phi(-D_0)^{-1} [D_2^* - D_1^* \mathbf{e}\phi] (-D_0)^{-1} D_1^* \mathbf{e} &= \frac{[r(W_2 - KW_1) + y(Q_2 - KQ_1)](rx + 2ry + yu)}{(rx + ry + ry + yu)^2} \\ &\quad - \frac{(rW_1 + yQ_1)^2}{(rx + ry + ry + yu)^2} \end{aligned} \quad (12)$$

where  $W_2 = \sum_{k=1}^{K-1} k(K-k)w_k$  and  $Q_2 = \sum_{k=1}^{K-1} k(K-k)q_k$ . We prove next the non-negativity of (12). First, define:

$$\begin{aligned} H_K &= [r(W_2 - KW_1) + y(Q_2 - KQ_1)](rx + 2ry + yu) - (rW_1 + yQ_1)^2 \\ &= - \left( \sum_{k=1}^{K-1} (K-k)^2 [rw_k + yq_k] \right) (rx + 2ry + yu) - \left( \sum_{k=1}^{K-1} (K-k) [rw_k + yq_k] \right)^2. \end{aligned} \quad (13)$$

It can be seen that when  $K = 3$ , expression (13) reduces to:

$$\begin{aligned} H_3 &= [2(rw_1 + yq_1) + (rw_2 + yq_2)][-r(x + y + w_1 + w_2) - y(r + u + q_1 + q_2)] \\ &\quad + [rw_1 + yq_1][-r(x + y + w_1 + w_2) - r(x + y + w_1) - y(r + u + q_1 + q_2) - y(r + u + q_1)] \\ &\geq 0, \end{aligned}$$

We now proceed by induction. Assume that  $H_{K-1} \geq 0$ , then after some calculations  $H_K$  can be rewritten as

$$\begin{aligned}
H_K &= \left( \sum_{k=1}^{K-1} (K-k)[rw_k + yq_k] \right) \left[ -r(x+y + \sum_{k=1}^{K-1} w_k) - y(r+u + \sum_{k=1}^{K-1} q_k) \right] \\
&\quad + \left( \sum_{k=1}^{K-2} [(K-k-1)][rw_k + yq_k] \right) \left[ -r(x+y + \sum_{k=1}^{K-1} w_k) - y(u+r + \sum_{k=1}^{K-1} q_k) \right] \\
&\quad - \left( \sum_{k=1}^{K-2} (K-k-1)^2[rw_k + yq_k] \right) (rx + 2ry + yu) \\
&\quad - \left( \sum_{k=1}^{K-2} (K-k-1)[rw_k + yq_k] \right)^2 \\
&= \left( \sum_{k=1}^{K-1} (K-k)[rw_k + yq_k] \right) [rw_K + yq_K] + H_{K-1} \\
&\quad + \left( \sum_{k=1}^{K-2} [(K-k-1)][rw_k + yq_k] \right) [rw_K + yq_K] \\
&\geq 0,
\end{aligned}$$

since  $r, y, w_k, q_k$  are non-negative for all  $k$  and  $H_{K-1} \geq 0$  by induction hypothesis. Therefore, Eq. (12) is positive or equal to zero consequently  $\rho_B(1) \geq 0$  is proven.  $\square$

**Proposition 3.** Consider a  $BMMPP_2(K)$ , with autocorrelation function of the batch sizes given by  $\rho_B(l)$ , as in (4). Then,  $\rho_B(l) \geq 0$  for all  $l \geq 1$ .

*Proof.* In Rodríguez [38] it is proved that the autocorrelation function of the batch sizes in a  $BMAP_2(K)$  is given by

$$\rho_B(l) = \rho_B(1)q_B^{l-1}, \quad (14)$$

where  $q_B$  is the only eigenvalue of  $P^* = (-D_0)^{-1}D$  less than 1 in absolute value. Note that, in the  $BMMPP_2(K)$ ,  $D$  can be computed as

$$D = \sum_{k=1}^K D_k = \begin{pmatrix} -x-y & 0 \\ 0 & -r-u \end{pmatrix}.$$

Therefore, in this specific case,  $q_B$  is given by

$$q_B = \frac{(-x-y)(r+u)}{ry-xu}, \quad (15)$$

for all  $K$ . Since

$$\begin{aligned}
r+u &= -\lambda_2(1-p_{210}) \leq 0 \\
-x-y &= \lambda_1(1-p_{120}) \geq 0 \\
xu-ry &= \text{Det}(D_0) = \lambda_1\lambda_2(1-p_{120}p_{210}) \geq 0,
\end{aligned}$$

then, it can be concluded that  $q_B$  as in (15) satisfies  $q_B \geq 0$  and consequently, from (14) and Lemmas 1-2,  $\rho_B(l) \geq 0$  for all  $l \geq 1$ . □

## 4 Identifiability of the $BMMPP_2(K)$

Identifiability problems occur when different representations of the process lead to the same likelihood functions for the observable data. In order to develop an estimation method to fit real datasets to the model, a detailed examination of the identifiability of the process is crucial. It is well known that the *MAP* and *BMAP* processes suffer from identifiability problems, but, on the other hand, in Rydén [40], the identifiability of the *MMPP* was proven. Here, we extend such result to the  $BMMPP_2(K)$  case. First, consider the definition of identifiability.

**Definition 1.** *Let  $\mathcal{B}$  be a representation of a  $BMMPP_2(K)$  and let  $T_n$  and  $B_n$  denote the time between the  $(n-1)$ -th and  $n$ -th event occurrences, and the batch size of the  $n$ -th event occurrence, respectively. Then  $\mathcal{B}$  is said to be identifiable if there does not exist a different parametrization  $\tilde{\mathcal{B}}$ , such that*

$$(T_1, \dots, T_n, B_1, \dots, B_n) \stackrel{d}{=} (\tilde{T}_1, \dots, \tilde{T}_n, \tilde{B}_1, \dots, \tilde{B}_n), \quad \text{for all } n \geq 0,$$

where  $\tilde{T}_i$  and  $\tilde{B}_i$  are defined in analogous way as  $T_i$  and  $B_i$ , and where  $\stackrel{d}{=}$  denotes equality in distribution.

In what follows we will concentrate on the LST, given in (5), in order to prove the identifiability of  $BMMPP_2(K)$ . Note that the equality in distribution is equivalent to the equality of the LSTs,  $f_{T,B}^*(\mathbf{s}, \mathbf{z}) = f_{\tilde{T},\tilde{B}}^*(\mathbf{s}, \mathbf{z})$ , for all  $\mathbf{s}, \mathbf{z}$ . Next, we review the concept of permutation matrix and some of its properties that are useful to obtain the main result (for more details, see for example Horn and Johnson [15]).

**Definition 2.** *A square matrix  $P$  is a permutation matrix if exactly one entry in each row and column is equal to 1 and all other entries are 0.*

Some properties concerning permutations matrices are:

- P1.  $PA$  where  $A \in M_{m,n}$  implies a permutations of the rows of  $A$ .
- P2.  $AP$  permutes the columns of  $A$ .
- P3. A permutation matrix is orthogonal ( $P^{-1} = P^T$ )
- P4. The permutation matrices are closed under product.

Next result establishes how to obtain equivalent representations to a given one, using permutation matrices.

**Lemma 3.** Let  $\mathcal{B} = \{D_0, \dots, D_K\}$  be a representation of a  $BMMPP_2(K)$  and let  $\mathcal{B}_P = \{PD_0P, PD_1P, \dots, PD_KP\}$  be a different representation where  $P$  is a permutation matrix. Then,  $\mathcal{B}$  and  $\mathcal{B}_P$  are equivalent representations of the same process.

*Proof.* We demonstrate the equivalency between  $\mathcal{B}$  and  $\mathcal{B}_P$  by proving the equality of their respective LTs as in (5). Consider first  $\pi_P$ , the stationary probability vector related to representation  $\mathcal{B}_P$ , which satisfies  $\pi_P Q_P = 0$ . It is not difficult to see

$$\begin{aligned}\pi_P Q_P &= \pi_P \left( \sum_{k=0}^K PD_kP \right) \\ &= \pi_P P \left( \sum_{k=0}^K D_k \right) P \\ &= \pi_P P Q P\end{aligned}$$

and

$$\pi_P P Q P = 0 \iff \pi_P P Q = 0.$$

But  $\pi_Q = 0$  and  $\pi P P Q = \pi Q$ , therefore  $\pi_P = \pi P$ . On the other hand, let  $\phi_P$  denote the stationary probability vector with transitions events related to representation  $\mathcal{B}_P$ . Then

$$\begin{aligned}\phi_P &= (\pi_P P D P e)^{-1} \pi_P P D P \\ &= \pi_P (P D_1 P + \dots + P D_k P) e]^{-1} \pi_P (P D_1 P + \dots + P D_k P) \\ &= [\pi P P D P e]^{-1} \pi P P D P \\ &= [\pi D e]^{-1} \pi D P \\ &= \phi P.\end{aligned}\tag{16}$$

From, the fact that  $P e = e$ , the property P3 of permutation matrices and (16), we have

$$\begin{aligned}f_{T_P, B_P}^*(s, z) &= \phi_P \left[ \prod_{i=1}^n (s_i I - P D_0 P)^{-1} \left( \sum_{l=1}^k P D_l P z_i^l \right) \right] e \\ &= \phi P \left[ \prod_{i=1}^n P^T (s_i I - D_0)^{-1} P^T P \left( \sum_{l=1}^k D_l z_i^l \right) P \right] e \\ &= \phi P P^T \left[ \prod_{i=1}^n (s_i I - D_0)^{-1} \left( \sum_{l=1}^k D_l z_i^l \right) \right] P e \\ &= \phi \left[ \prod_{i=1}^n (s_i I - D_0)^{-1} \left( \sum_{l=1}^k D_l z_i^l \right) \right] e = f_{T, B}^*(s, z),\end{aligned}$$

and the lemma is proven.  $\square$

**Remark 1.** Lemma 3 holds true for general  $BMMPP_m(K)$  with  $m \geq 2$ .

The next result is a direct consequence of the identifiability of the  $MMPP$ s.

**Lemma 4.** Let  $\mathcal{B} = \{D_0, D_1, \dots, D_k\}$  and  $\tilde{\mathcal{B}} = \{\tilde{D}_0, \tilde{D}_1, \dots, \tilde{D}_k\}$  be two different but equivalent representations of a  $BMMPP_2(K)$ . Then,  $D_0 = \tilde{D}_0$  and  $\Lambda = \tilde{\Lambda}$ , except by permutation, where  $\Lambda$  ( $\tilde{\Lambda}$ ) is the vector of exponential rates of  $\mathcal{B}$  ( $\tilde{\mathcal{B}}$ ).

*Proof.* It is clear that if representations  $\mathcal{B}$  and  $\tilde{\mathcal{B}}$  are equivalent, then, the  $MMPP$ s representations  $\mathcal{B}' = (D_0, D)$  and  $\tilde{\mathcal{B}}' = \{\tilde{D}_0, \tilde{D}\}$  will be also equivalent, where  $D = D_1 + \dots + D_K$  and  $\tilde{D} = \tilde{D}_1 + \dots + \tilde{D}_K$ . From the identifiability of the  $MMPP$ ,  $D_0 = \tilde{D}_0$  and  $D = \tilde{D}$  except by permutation. Hence, from the first equality, either  $(\lambda_1, \lambda_2) = (\tilde{\lambda}_1, \tilde{\lambda}_2)$  or  $(\lambda_1, \lambda_2) = (\tilde{\lambda}_2, \tilde{\lambda}_1)$  is obtained.  $\square$

**Remark 2.** Lemma 4 holds true for general  $BMMPP_m(K)$  for  $m \geq 2$ .

**Proposition 4.** The  $BMMPP_2(2)$  is identifiable except by permutation.

*Proof.* Consider two stationary  $BMMPP_2(2)$ s represented by  $\mathcal{B} = \{D_0, D_1, D_2\}$  and  $\tilde{\mathcal{B}} = \{\tilde{D}_0, \tilde{D}_1, \tilde{D}_2\}$ . The LST of  $\mathcal{B}$  when  $n = 1$  is, according to (5) is equal to

$$f_{T,\mathcal{B}}^*(s, z) = \phi(sI - D_0)^{-1} \xi(z) e = \frac{z(s\alpha + \beta) + z^2(s\gamma - \beta + \eta)}{s^2 + s\nu + \eta} \quad (17)$$

where

$$\begin{aligned} \alpha &= \phi(w - q) + q \\ \beta &= \phi(-uw - rw + yq + xq) + (rw - xq) \\ \gamma &= \phi(r + u + q - x - y - w) - (r + u + q) \\ \eta &= xu - ry \\ \nu &= -x - u, \end{aligned}$$

and similarly for  $\tilde{\mathcal{B}}$ . If  $\mathcal{B}$  and  $\tilde{\mathcal{B}}$  are equivalent, then from Lemma 4,  $D_0 = \tilde{D}_0$ , and therefore  $x = \tilde{x}$ ,  $y = \tilde{y}$ ,  $u = \tilde{u}$ ,  $r = \tilde{r}$ , and consequently,  $\nu = \tilde{\nu}$  and  $\eta = \tilde{\eta}$ . This implies that the equality of LSTs,

$$f_{T,\mathcal{B}}^*(s, z) = f_{T,\tilde{\mathcal{B}}}^*(s, z), \quad \text{for all } s, z.$$

becomes in

$$z(s\alpha + \beta) + z^2(s\gamma - \beta) = z(s\tilde{\alpha} + \tilde{\beta}) + z^2(s\tilde{\gamma} - \tilde{\beta}), \quad \text{for all } s, z. \quad (18)$$

By substituting first  $s = 0$  and  $z = 2$  in (18), and later from  $s = 1$  and  $z = -1$ , one leads to  $\beta = \tilde{\beta}$  and  $\alpha = \tilde{\alpha}$ :

$$\begin{aligned} \beta &= \phi(-uw - rw + yq + xq) + (rw - xq) \\ &= \phi(-u\tilde{w} - r\tilde{w} + y\tilde{q} + x\tilde{q}) + (r\tilde{w} - x\tilde{q}) \\ &= \tilde{\beta} \end{aligned} \quad (19)$$

Similarly,

$$\alpha = \phi(w - q) + q = \phi(\tilde{w} - \tilde{q}) + \tilde{q} = \tilde{\alpha}$$

from which

$$\tilde{w} = \frac{\phi(w - q + \tilde{q}) + q - \tilde{q}}{\phi}. \quad (20)$$

If (20) is substituted in (19), then

$$\begin{aligned} \phi(-uw - rw + yq + xq) + (rw - xq) &= -[\phi(w - q + \tilde{q}) + q - \tilde{q}](u + r) \\ &\quad + \phi(y\tilde{q} + x\tilde{q}) + \left( r \frac{\phi(w - q + \tilde{q}) + q - \tilde{q}}{\phi} - x\tilde{q} \right), \end{aligned}$$

hence

$$\begin{aligned} \phi(yq + xq) - xq &= -[\phi(-q + \tilde{q}) + q - \tilde{q}](u + r) \\ &\quad + \phi(y\tilde{q} + x\tilde{q}) + \left( r \frac{\phi(-q + \tilde{q}) + q - \tilde{q}}{\phi} - x\tilde{q} \right), \\ &= \left( u + r - \frac{r}{\phi} \right) (1 - \phi)(\tilde{q} - q) + \tilde{q}(\phi(y + x) - x). \quad (21) \end{aligned}$$

From (21) it can be concluded that  $q = \tilde{q}$ . Therefore, from (20)  $\tilde{w} = w$  and consequently,  $D_1 = \tilde{D}_1$  and  $D_2 = \tilde{D}_2$ .  $\square$

The Proposition 4 proves that the stationary  $BMMPP_2(2)$  is an identifiable process. The next Theorem goes further and ensures the identifiability of the stationary  $BMMPP_2(K)$ , for all  $K \geq 3$ . But before we need to prove the follow lemma.

**Lemma 5.** *Let  $\mathcal{B}_K = \{D_0, D_1, \dots, D_K\}$  and  $\tilde{\mathcal{B}}_K = \{\tilde{D}_0, \tilde{D}_1, \dots, \tilde{D}_K\}$  be two different but equivalent representations of a  $BMMPP_2(K)$ . Then,  $\mathcal{B}_{K-1}^1$  is equivalent to  $\tilde{\mathcal{B}}_{K-1}^1$ , and  $\mathcal{B}_{K-1}^2$  is equivalent to  $\tilde{\mathcal{B}}_{K-1}^2$ , where  $\mathcal{B}_{K-1}^1$  and  $\mathcal{B}_{K-1}^2$  are representations of a  $BMMPP_2(K-1)$  given by*

$$\begin{aligned} \mathcal{B}_{K-1}^1 &= \{D_0, D_1 + D_2, \dots, D_K\}, \\ \mathcal{B}_{K-1}^2 &= \{D_0, D_1, \dots, D_{K-1} + D_K\}, \end{aligned}$$

(similarly for  $\tilde{\mathcal{B}}_{K-1}^1$  and  $\tilde{\mathcal{B}}_{K-1}^2$ ).

See Appendix 1 for the proof.

**Theorem 1.** *The  $BMMPP_2(K)$  is identifiable except by permutation.*

*Proof.* We proceed by induction in  $K$ . The initial case ( $K = 2$ ) was proven in Proposition 4. Assume that the equivalence of two arbitrary  $BMMPP_2(K-1)$ s represented by  $\mathcal{B}_{K-1} = \{D_0, D_1, \dots, D_{K-1}\}$  and  $\tilde{\mathcal{B}}_{K-1} = \{\tilde{D}_0, \tilde{D}_1, \dots, \tilde{D}_{K-1}\}$ , implies the equality of the corresponding rate matrices,  $D_0 = \tilde{D}_0$ ,  $D_1 = \tilde{D}_1$ , ...  $D_{K-1} = \tilde{D}_{K-1}$ . Consider two equivalent  $BMMPP_2(K)$ s given by  $\mathcal{B}_K =$

$\{D_0, D_1, \dots, D_k\}$  and  $\tilde{\mathcal{B}}_K = \{\tilde{D}_0, \tilde{D}_1, \dots, \tilde{D}_k\}$  and obtain from them, as in Lemma 5, the  $BMMPP_2(K-1)$  representations  $\mathcal{B}_{K-1}^1$  and  $\tilde{\mathcal{B}}_{K-1}^1$ . Then, Lemma 5 guarantees the equivalence of  $\mathcal{B}_{K-1}^1$  and  $\tilde{\mathcal{B}}_{K-1}^1$  and from the hypothesis induction,

$$D_1 + D_2 = \tilde{D}_1 + \tilde{D}_2 \quad \text{and} \quad D_k = \tilde{D}_k \quad \text{for } 3 \leq k \leq K. \quad (22)$$

Similarly, consider  $\mathcal{B}_{K-1}^2$  and  $\tilde{\mathcal{B}}_{K-1}^2$  as defined in Lemma 5. Again, from the hypothesis of induction and Lemma 5,  $\mathcal{B}_{K-1}^2$  and  $\tilde{\mathcal{B}}_{K-1}^2$  are equivalent and thus,

$$D_{K-1} + D_K = \tilde{D}_{K-1} + \tilde{D}_K \quad \text{and} \quad D_k = \tilde{D}_k \quad \text{for } 1 \leq k \leq K-2. \quad (23)$$

From (22) and (23),  $D_k = \tilde{D}_k$ , for all  $k$ , which complete the proof.  $\square$

## 5 Conclusions

This paper considers the two-state batch counterpart of the well-known  $MMPP$ , the  $BMMPP_2(K)$ , a point process of interest in real-life contexts as reliability or queueing, since it allows for the modeling of dependent inter-event times and dependent batch sizes. Two main problems concerning the  $BMMPP_2(K)$  have been addressed. On one hand, the non-negativity of the autocorrelation function of the batch sizes is proven. This property makes the process suitable when positively correlated batch sizes are observed. On the other hand, we prove the identifiability of the process, a property inherited from that of the  $MMPP$ . The identifiability of the process is of crucial importance when inference is to be considered, as we plan to do as future work. Either from a Bayesian viewpoint as in Scott (1999) [41] or Ramírez-Cobo et al. (2017) [33], or from a moments matching method as in Rodríguez et al. [36], a statistical inference approach may be defined for fitting real datasets. Other prospects regarding this work is the study of the identifiability of higher order  $BMMPP_m(K)$  for  $m \geq 3$ , which is a challenging task that may be solved using similar arguments as in this work but by applying matrix analysis. Finally, an important extension to this work would be to derive theoretical properties concerning correlation bounds for both the inter-event times and batch size autocorrelation functions when  $m \geq 3$ . Work on this issues is underway.

## Acknowledgments

Research partially supported by research grants and projects MTM2015-65915-R and ECO2015-66593-P (Ministerio de Economía y Competitividad, Spain) and P11-FQM-7603, FQM-329 (Junta de Andalucía, Spain) and Fundación BBVA

## A Appendix: Proof of Lemma 5

Since  $\mathcal{B}_K = \{D_0, D_1, \dots, D_K\}$  and  $\tilde{\mathcal{B}}_K = \{\tilde{D}_0, \tilde{D}_1, \dots, \tilde{D}_K\}$  are equivalent, then  $D_0 = \tilde{D}_0$  and

$$f_{T,B}^*(s, z) = f_{\tilde{T}, \tilde{B}}^*(s, z), \quad \text{for all } s, z,$$

or equivalently,

$$\phi S(s_1, \dots, s_n, z_1, \dots, z_n) \mathbf{e} = \phi \tilde{S}(s_1, \dots, s_n, z_1, \dots, z_n) \mathbf{e} \quad \text{for all } s, z, \quad (24)$$

where

$$S(s_1, \dots, s_n, z_1, \dots, z_n) = (s_1 I - D_0)^{-1} \sum_{k_1=1}^K D_{k_1} z_1^{k_1} \dots (s_n I - D_0)^{-1} \sum_{k_n=1}^K D_{k_n} z_n^{k_n}$$

and similarly for  $\tilde{S}$ . Equality (24) can be rewritten as

$$\begin{aligned} \phi S(s_1, \dots, s_{n-1}, z_1, \dots, z_{n-1}) \times \\ \times (s_n I - D_0)^{-1} \sum_{k_n=1}^K D_{k_n} z_n^{k_n} \mathbf{e} &= \phi \tilde{S}(s_1, \dots, s_{n-1}, z_1, \dots, z_{n-1}) \times \\ &\times (s_n I - D_0)^{-1} \sum_{k_n=1}^K \tilde{D}_{k_n} z_n^{k_n} \mathbf{e}. \end{aligned} \quad (25)$$

Next, consider the following three block of calculations related to value  $z_n$ :

**Step 1:** Compute  $K$  times in both sides of (25) the partial derivative with respect to  $z_n$ :

$$\begin{aligned} \phi S(s_1, \dots, s_{n-1}, z_1, \dots, z_{n-1}) \times \\ \times (s_n I - D_0)^{-1} D_K K! \mathbf{e} &= \phi \tilde{S}(s_1, \dots, s_{n-1}, z_1, \dots, z_{n-1}) \times \\ &\times (s_n I - D_0)^{-1} \tilde{D}_K K! \mathbf{e}. \end{aligned} \quad (26)$$

**Step 2:** Multiply (26) by  $(z_n^{K-1} - z_n^K)/K!$ :

$$\begin{aligned} \phi S(s_1, \dots, s_{n-1}, z_1, \dots, z_{n-1}) \times \\ \times (s_n I - D_0)^{-1} D_K (z_n^{K-1} - z_n^K)! \mathbf{e} &= \phi \tilde{S}(s_1, \dots, s_{n-1}, z_1, \dots, z_{n-1}) \times \\ &\times (s_n I - D_0)^{-1} \tilde{D}_K (z_n^{K-1} - z_n^K) \mathbf{e}. \end{aligned} \quad (27)$$

**Step 3:** Summing (25) + (27), we arrive to

$$\phi S^*(s_1, \dots, s_n, z_1, \dots, z_n) \mathbf{e} = \phi \tilde{S}^*(s_1, \dots, s_n, z_1, \dots, z_n) \mathbf{e}, \quad (28)$$

where

$$S^*(s_1, \dots, s_n, z_1, \dots, z_n) = \phi S(s_1, \dots, s_{n-1}, z_1, \dots, z_{n-1}) (s_n I - D_0)^{-1} \left[ \sum_{k_n=1}^{K-1} D_{k_n} z_n^{k_n} + D_K z_n^{K-1} \right] \mathbf{e}$$

and equivalently

$$\tilde{S}^*(s_1, \dots, s_n, z_1, \dots, z_n) = \phi \tilde{S}(s_1, \dots, s_{n-1}, z_1, \dots, z_{n-1})(s_n I - D_0)^{-1} \left[ \sum_{k_n=1}^{K-1} \tilde{D}_{k_n} z_n^{k_n} + \tilde{D}_K z_n^{K-1} \right] \mathbf{e}.$$

If the previous Steps 1-3 are reproduced for  $z_{n-1}, z_{n-2}, \dots, z_1$ , the equivalency between  $\mathcal{B}_{K-1}^2$  and  $\tilde{\mathcal{B}}_{K-1}^2$  is obtained. Since the LTS for  $\mathcal{B}_{K-1}^2$  is

$$\begin{aligned} f_{T,B}^*(s, z) &= \phi(s_1 I - D_0)^{-1} \left[ \sum_{k_1=1}^{K-1} D_{k_1} z_1^{k_1} + D_K z_1^{K-1} \right] \times \\ &\quad \dots \times (s_n I - D_0)^{-1} \left[ \sum_{k_n=1}^{K-1} D_{k_n} z_n^{k_n} + D_K z_n^{K-1} \right] \mathbf{e} \end{aligned}$$

and similarly for  $\tilde{\mathcal{B}}_{K-1}^2$ .

Finally, by a parallel procedure, the equivalence between  $\mathcal{B}_{K-1}^1$  and  $\tilde{\mathcal{B}}_{K-1}^1$  is also derived.

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