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Keywords: risk process; ruin probability; Markov decision processes; proportional reinsurance.

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OPTIMAL POLICIES FOR DISCRETE TIME RISK PROCESSES WITH A MARKOV CHAIN INVESTMENT MODEL
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Abstract
We consider a discrete risk process modelled by a Markov Decision Process. The surplus could be invested in stock market assets. We adopt a realistic point of view and we let the investment return process to be statistically dependent over time. We assume that follows a Markov Chain model. To minimize the risk there is a possibility to reinsure a part or the whole reserve. We consider proportional reinsurance. Recursive and integral equations for the ruin probability are given. Generalized Lundberg inequalities for the ruin probabilities are derived. Stochastic optimal control theory is used to determine the optimal stationary policy which minimizes the ruin probability. To illustrate these results numerical examples are included.

1 Introduction
Recently, a large body of work has been devoted to extend risk models to allow for decisor’s actions thus continuing the classical works from the seventies by e.g., H.U. Gerber, H. Bühlan and A. Martin-Löf. Controlled risk processes are considered by Hipp and Taksar (2000), Hipp and Plum (2000), Schmidli (2001, 2002), Hipp and Schmidli (2004), among others. Most of the papers consider continuous-time models and some specific intervention, which allow them to explicitly obtain the optimal policy. Schäl (2004) discusses discrete-time risk processes controlled by reinsurance and non-statistically-dependent over time investment. The effect of stochastic investment returns minimizing the ruin probability in continuous-time models is considered by Browne (1995), Hipp & Plum (2000) and Gaier et al. (2003). Several models
of interest rate on ruin probabilities has been discussed by Sundt and Teugels (1995, 1997), Yang (1999), Norbert (1997) and more recently for discrete time risk processes see Cai (2002) and Cai and Dickson (2004) and references therein.

At first view, the ruin probability is not a classical performance criterion for control problems. As is pointed out by Schäli (2004) one can write the ruin probability as some total cost without discounting where one has to pay one unit of cost when entering a ruin state. After this simple observation, the results from discret-time dynamic programming apply. Nevertheless obtaining explicit optimal solutions is a difficult task in a general setting. An analytic method commonly used in ruin theory is to derive inequalities for ruin probabilities (see Grandell (1991), Willmot, et al. (2000) and Willmot and Lin (2001)).

In this paper we consider the problem of minimization of the ruin probability in a discrete time risk process with proportional reinsurance and investment in a financial market. For the surplus process we consider the usual formulation in Markov decision theory (see Schäli (2004) and Hernández-Lerma and Laserre (1996)). We assume statistical dependency over time for the investment process and following a realistic point of view as is suggested in Cai (2002) and in Cai and Dickson (2004), we modelize the return process as a finite or countable state Markov chain. This model could be in fact a discrete counterpart of the most frequently occurring effect observed in continuous total return processes, e.g. mean-reverting effect. Our aim is to choose the reinsurance-investment control strategies in order to minimize the ruin probability and for sake of simplicity we restrict control policies to be Markovian and stationary. Thus we guarantee optimal control strategies among the admissible control class. First, for this purpose we develop generalized Lundberg inequalities for the ruin probability that depend on the decision or control strategy. Previously we derive recursive and integral equations for the ruin probability. Second, optimality over the admissible control set can be achieved by the monotonic property of the upper bounds that we obtain.

The outline of the paper is as follows. In Section 2 the risk model is formulated. In Section 3 we derive recursive equations for the finite ruin probability and integral equations for the ultimate ruin probability. In Section 4 we obtain probability inequalities for the ultimate probability of ruin. An analysis of the new inequalities and the comparison with the Lundberg’s inequality is also included. In Section 5 we report the outcomes of an illus-
trative application to the ruin probability in a risk process with a heavy tail (PH-type) claims distribution under proportional reinsurance and Markov dependency of the return process. Finally, we studying in Section 6 the optimality of our approximations.

2 Model

A general $X$-valued discrete-time stochastic process $\{X_n\}_{n \geq 0}$ is considered which can be observed and controlled at the beginning of each period. The stochastic development is determined by a sequence of random variables $\{W_n\}_{n \geq 1}$ on some probability space $(\Omega, \mathcal{F}, P)$ with $W_n = (Y_n, Z_n, R_n)$. Let $\{Y_n\}_{n \geq 1}$ and $\{Z_n\}_{n \geq 1}$ be two independent sequences of independent and identically distributed (i.i.d.) nonnegative random variables with probability distribution functions $F$ and $G$ respectively. We assume also independency with respect to the investment return process $\{R_n\}$. In risk theory, $X_n \in X$ (usually $X = \mathbb{R}$) describes the surplus (size of the fund of reserves) of an insurance company after $n$ periods, $Y_n$ denotes the total claims during the $n$-th period, i.e. from time $n-1$ to time $n$, $Z_n$ represents the random of period $n$, and $R_n$ denotes the total retribution of the financial market during the $n$-th period.

A policy is a sequence $a = \{a_n\}_{n \geq 1}$ of decision functions $\varphi_n : X \to A$, with $A = [b, 1] \times [0, 1]^d$ and $0 < b \leq 1$. For the sake of simplicity we can consider a subset $D \subset [0, 1]^d$ for the portfolio values, for example a grid of $[0, 1]^d$. Then $\varphi_n(X_n) = a_n = (b_n, \delta_n)$ will represent the action chosen at the beginning of the period $n+1$. We denote by $\delta_n = (\delta_1^n, \ldots, \delta_d^n)$ the portfolio invest in $d$ risky assets, and $b$ the proportion of the claim to be paid by the insurer. We consider Markovian control policies, which depend only on the current state, and stationary, that is, $\varphi_n = \varphi$. Abusing notation, we will identify functions $\varphi : X \to A$ with stationary strategies. The (measurable) function $h(b, y)$ specifies the part of the claim $y$ paid by the insurer. Then, $h(b, y)$ depends on the retention level $b$ at the beginning of the respective period where $0 \leq h(b, y) \leq y$. In this article, we consider the case of proportional reinsurance, where

$$h(b, y) = b \cdot y \text{ with retention level } 0 < b \leq b \leq 1.$$  \hfill (1)

The premium (income) rate $c$ is fixed. Because the insurer pays a premium rate to the reinsurer which depends on the retention level $b$, we consider the
net income rate $c(b)$ where

$$0 \leq c(b) \leq c = c(b), \quad 0 < b \leq b \leq 1$$

and $c(b)$ is an increasing function and may be calculated according to the expected value principle with and added safety loading $\theta$ from the reinsurer:

$$c(b) = c - (1 + \theta) \cdot E[Y - h(b, Y)]/E[Z].$$

(2)

For an initial state $X_0 = x$ and a control policy $a$, the value of the surplus process at the beginning of the period $n+1$, say $X_n^x,a = X_n$, is given according to

$$X_n = X_{n-1}(1 + \langle \delta_{n-1}, R_n \rangle) + c(b_{n-1})Z_n - h(b_{n-1}, Y_n)$$

(3)

Note that (3) can be rewritten in the usual form in Markov control theory

$$X_n = f(X_{n-1}, a_{n-1}, W_n)$$

(4)

with the random perturbation $W_n = (Y_n, Z_n, R_n)$. In general the process $X_n$ is not homogeneous because its transition probabilities change in time, however we get homogeneity if we consider stationary strategies of the form $\varphi_n = \varphi$.

From this fact we can assume that the transition probabilities of $X_n$ are of the form

$$P(x, a, [x - u, \infty)) = \hat{F}(a, u)$$

(5)

where $\hat{F}(a, u)$ is the probability distribution of the amount of net losses in a single period, provided that $a$ is the action taken by the decision maker at the beginning of the period.

We adopt a realistic point of view, and we let the return process $\{R_n\}_{n \geq 1}$ where $R_n = (R_{n1}, \ldots, R_{nd})$ to be statistically dependent with the price process given by $I_n^a = \langle \delta_{n-1}, R_n \rangle$ for $n \geq 1$ provided the action $a$ is take. Note that $I_n^a$ depends on the decision variable $a$ only through the portfolio $\delta_n = (\delta_{d1}, \ldots, \delta_{dn})$. Moreover $\{I_n^a\}_{n \geq 0}$ is assumed to follow a Markov chain. In addition we assume that for all $n \geq 0$, $I_n^a$ takes a finite or countable number of possible values on $I$. Note that $I_0^a$ is the state of the price process before to make the investment in the financial market and the value $i_0$ in
fact drives the evolution of the process. It is not hard to check that (3) is equivalent to

\[ X_n = x \prod_{i=1}^{n} (1 + \langle \delta_{i-1}^d, R_i^d \rangle) + \sum_{i=1}^{n} (c(b_{i-1})Z_i - h(b_{i-1}, Y_i) \prod_{m=1}^{n} (1 + \langle \delta_{m-1}^d, R_m^d \rangle)) \]

where throughout this article we consider

\[ \prod_{i=n_2}^{n_1} X_i = 1 \quad \text{and} \quad \sum_{i=n_1}^{n_2} X_i = 0 \text{ if } n_1 > n_2 \]

That is,

\[ X_n = x \prod_{i=1}^{n} (1 + I_i^a) + \sum_{i=1}^{n} (c(b_{i-1})Z_i - h(b_{i-1}, Y_i) \prod_{m=1}^{n} (1 + I_m^a)) \]

we define for all \( a \in A \)

\[ Pr\{I_{n+1}^a = j | I_n^a = i, I_{n-1}^a = i_{n-1}, \ldots, I_0^a = i_0, a\} = Pr\{I_{n+1}^a = j | I_n^a = i, a\} = p_{ij}^a \geq 0 \]

where \( \sum_{i=0}^{\infty} p_{ij}^a = 1 \) for \( i = 0, 1, 2, \ldots \). Note that we consider in fact a collection of transition probability matrices \( \{p_{ij}^a\}_{a \in A} \) describing the behavior of \( I_n^a \). The ruin probabilities with infinite and finite horizon, initial surplus \( x \), action \( a \) and given \( I_0^a = i \) are defined as

\[ \psi^a(x, i) = Pr\{\bigcup_{k=1}^{\infty} (X_k^x, a < 0)\} = Pr\{\bigcup_{k=1}^{\infty} (X_k < 0) | I_0^a = i, X_0 = x, a\} \]

and

\[ \psi^a_n(x, i) = Pr\{\bigcup_{k=1}^{n} (X_k^x, a < 0)\} = Pr\{\bigcup_{k=1}^{n} (X_k < 0) | I_0^a = i, X_0 = x, a\} \]

where \( X_k \) is given by (7). Thus,

\[ \psi^a_1(x, i) \leq \psi^a_2(x, i) \leq \cdots \leq \psi^a_n(x, i) \leq \cdots \]

and

\[ \lim_{n \to \infty} \psi^a_n(x, i) = \psi^a(x, i) \]
Note that if $\delta_k = 0$ and $b_k = 1$ for $k = 1, \ldots$ the risk model (7) is reduced to the classical discrete time risk model without investment and reinsurance

$$X_k = x - \sum_{t=1}^{k} (Y_t - cZ_t)$$

If $\delta_k = 0$ and $b_k \in (0, 1]$ for $k = 0, 1, \ldots$ the risk model is

$$X_k = x - \sum_{t=1}^{k} (b_{t-1} Y_t - c(b_{t-1}) Z_t)$$

(12)

and $a_k \equiv b_k$, $k = 0, 1, \ldots$.

Let $\psi^b(x)$ denote the infinite time ruin probability in the risk model given by (12), namely

$$\psi^b(x) = \Pr\left\{ \bigcup_{k=1}^{\infty} \sum_{t=1}^{k} (b_{t-1} Y_t - c(b_{t-1}) Z_t) > x \right\}$$

If we assume stationary strategies $b_t = b_0$, $t \geq 1$ and in addition $E[b_0 Y_1 - c(b_0) Z_1] < 0$, if there is a constant $R_0 > 0$ satisfying

$$E[e^{R_0(b_0 Y_1 - c(b_0) Z_1)}] = 1$$

(13)

we get the classical well-Known Lundberg inequality for the ruin probability

$$\psi^b(x) \leq e^{-R_0 x}.$$  (14)

Furthermore, if all interest rates are non-negative, i.e. $I_n^a \geq 0$ for $n = 0, 1, \ldots$, then it is easy to see that

$$\psi^a(x, i) \leq \psi^b(x), \ x \geq 0$$  (15)

Finally, note that for a model with dividends (constant case), if $d_n$ denotes the short term dividend rate in the $n - th$ period the discrete-time risk model with stochastic investment return and dividends is given by

$$X_n = X_{n-1}(1 + \langle \delta_{n-1}, R_n \rangle) + c(b_{n-1}) Z_n - h(b_{n-1}, Y_n) - d_n X_n$$

and rearranging $X_n$ one obtain

$$X_n = X_{n-1} \left( \frac{(1 + \langle \delta_{n-1}, R_n \rangle)}{(1 + d_n)} \right) + \frac{c(b_{n-1}) Z_n}{(1 + d_n)} - \frac{h(b_{n-1}, Y_n)}{(1 + d_n)}$$

and rearranging $X_n$ one obtain
For $1 + R'_n = \frac{1+R_n}{1+d_n}$, $Z'_n = \frac{Z_n}{1+d_n}$, and $Y'_n = \frac{Y_n}{1+d_n}$, and with conditions that guarantee that $\{R'_n\}$, $\{Z'_n\}$ and $\{Y'_n\}$ are independent processes with similar distributional characteristics as the original processes, then the model becomes

$$X_n = X_{n-1}(1 + \langle \delta_{n-1}, R'_n \rangle) + c(b_{n-1})Z'_n - h(b_{n-1}, Y'_n)$$

which is the same as the model without dividends (3) and it can be analyzed in the same way.

### 3 Recursive and integral equations for ruin probabilities

Throughout this article, we denote the tail of a distribution function $F$ by $\mathcal{T}(x) = 1 - F(x)$. We first give a recursive equation for $\psi^a_n(x,i)$ in (10) and an integral equation for $\psi^a(x,i)$ in (9). These equations hold for any interest rate.

**Lemma 1.** For $n = 1, 2, \ldots$ and any $X_0 = x \geq 0$,

$$\psi^a_{n+1}(x,i) = \sum_{j=0}^{\infty} p_{ij}^a \int_0^{\tau} \int_0^\infty \psi^a_n(x(1+j)-u,j)dF(y)dG(z) + \sum_{j=0}^{\infty} p_{ij}^a \int_0^\infty \mathcal{T}(\tau)dG(z)$$

with $u = b_0y - c(b_0)z$, $b_0$ is the initial retention level, $\tau = \frac{x(1+j) + c(b_0)z}{b_0}$ and $p_{ij}^a$ as in (8),

$$\psi^a_1(x,i) = \sum_{j=0}^{\infty} p_{ij}^a \int_0^\infty \mathcal{T}(\tau)dG(z)$$

then

$$\psi^a(x,i) = \sum_{j=0}^{\infty} p_{ij}^a \int_0^{\tau} \int_0^\infty \psi^a_n(x(1+j)-u,j)dF(y)dG(z) + \sum_{j=0}^{\infty} p_{ij}^a \int_0^\infty \mathcal{T}(\tau)dG(z)$$
Proof. Let \( U_k = b_{k-1}Y_k - c(b_{k-1})Z_k \), given \( Y_1 = y \), \( Z_1 = z \), the control strategy \( a \), and \( I^n_1 = j \), from (7), we have \( U_1 = b_0y - c(b_0)z = u \), then \( X_1 = x(1 + I^n_1) - U_1 = h_1 - u \), where \( h_1 = x(1 + j) \). Thus, if \( u > h_1 \) then
\[
Pr\{X_1 < 0| Z_1 = z, Y_1 = y, I^n_1 = j, I^n_0 = i, a\} = 1,
\]
which implies that for \( u > h_1 \)
\[
Pr\left\{ \bigcup_{k=1}^{n+1} (X_k < 0)| Z_1 = z, Y_1 = y, I^n_1 = j, I^n_0 = i, a\right\} = 1
\]
while if \( 0 \leq u \leq h_1 \), then
\[
Pr\{X_1 < 0| Z_1 = z, Y_1 = y, I^n_1 = j, I^n_0 = i, a\} = 0 \tag{18}
\]
Let \( \{\widetilde{Y}_n\}_{n \geq 1} \), \( \{\widetilde{Z}_n\}_{n \geq 1} \), and \( \{\widetilde{I}^n_i\}_{n \geq 0} \) be independent copies of \( \{Y_n\}_{n \geq 1} \), \( \{Z_n\}_{n \geq 1} \), and \( \{I^n_i\}_{n \geq 0} \), respectively. Suppose that \( \tilde{U}_k = b_{k-1}\widetilde{Y}_k - c(b_{k-1})\widetilde{Z}_k \). Thus, (18) and (7) imply that for \( 0 \leq u \leq h_1 \),
\[
Pr\left\{ \bigcup_{k=1}^{n+1} (X_k < 0)| Z_1 = z, Y_1 = y, I^n_1 = j, I^n_0 = i, a\right\} 
\]
\[
= Pr\left\{ \bigcup_{k=2}^{n+1} (X_k < 0)| Z_1 = z, Y_1 = y, I^n_1 = j, I^n_0 = i, a\right\} 
\]
\[
= Pr\left\{ \bigcup_{k=2}^{n+1} (h_1 - u) \sum_{l=1}^{k} (1 + I^n_l) - \sum_{l=1}^{k} U_l \sum_{m=l+1}^{k} (1 + I^n_m) < 0)| I^n_1 = j, a\right\} \tag{19}
\]
\[
= Pr\left\{ \bigcup_{k=2}^{n} (h_1 - u) \sum_{l=1}^{k} (1 + \tilde{I}^n_l) - \sum_{l=1}^{k} \tilde{U}_l \sum_{m=l+1}^{k} (1 + \tilde{I}^n_m) < 0)| \tilde{I}^n_0 = j, a\right\} 
\]
\[
= \psi^n_n(h_1 - u, j) = \psi^n_n(x(1 + j) - u, j)
\]
where the second equality follows from the Markov property of \( \{I^n_i\}_{n \geq 0} \), and independence of \( \{Y_n\}_{n \geq 1} \), \( \{Z_n\}_{n \geq 1} \) and \( \{I^n_i\}_{n \geq 0} \).

Let consider condition \( A = \{Z_1 = z, Y_1 = y, I^n_1 = j, I^n_0 = i, a\} \). Suppose
that \( F(y) = \Pr(Y \leq y) \) and \( G(z) = \Pr(Z \leq z) \). From equation (7) we obtain

\[
\psi_{n+1}^a(x, i) = \Pr\left\{ \bigcup_{k=1}^{n+1} (X_k < 0) | I_0^a = i, a \right\}
\]

\[
= \sum_{j=0}^{\infty} p_{ij}^a \int_0^\infty \int_0^\infty \Pr\left\{ \bigcup_{k=1}^{n+1} (X_k < 0) | \text{condition A} \right\} F(y) dG(z).
\]

Then

\[
\psi_{n+1}^a(x, i) = \sum_{j=0}^{\infty} p_{ij}^a \left\{ \int_0^\tau \int_0^\tau \psi_{n}^a(x(1 + j) - u, j) dF(y) dG(z) + \int_0^\infty dF(y) dG(z) \right\}
\]

\[
= \sum_{j=0}^{\infty} p_{ij}^a \left\{ \int_0^\tau \int_0^\tau \psi_{n}^a(x(1 + j) - u, j) dF(y) dG(z) + \int_0^\infty F(\tau) dG(z) \right\}
\]

where \( \tau = \frac{x(1+j)+c(b_0)z}{b_0} \). Next by letting \( n \to \infty \) in (20), and by the Lebesgue dominated convergence theorem, we obtain \( \lim_{n \to \infty} \psi_{n+1}^a(x, i) = \psi^a(x, i) \), i.e.

\[
\psi^a(x, i) = \sum_{j=0}^{\infty} p_{ij}^a \left\{ \int_0^\tau \int_0^\tau \psi^a(x(1 + j) - u, j) dF(y) dG(z) + \int_0^\infty F(\tau) dG(z) \right\}
\]

In particular,

\[
\psi_1^a(x, i) = \sum_{j=0}^{\infty} p_{ij}^a \int_0^\infty F((x(1 + j) + c(b_0)z)/b_0) dG(z).
\]

Note that if we consider the classical risk model without investment and reinsurance, i.e. \( \delta_n = 0 \) and \( b = 1 \) for \( n \geq 0 \), we obtain similar results as in Cai & Dickson [2].
4 Inequalities for ruin probabilities

In this section we assume that the return process is nonnegative, i.e. $I_n^a \geq 0$ for $n = 0, 1, \ldots$ and all $a \in A$. We will use the results obtained in the last section to find upper bounds for the ruin probability taking into account the information contributed by the Markov chain of the price process.

4.1 Inequalities for ruin probabilities by the inductive approach

Theorem 1. Suppose that $R_0$ is a constant satisfying (13) then

$$
\psi^a(x, i) \leq \beta \sum_{j=0}^{\infty} p_{ij}^a E[e^{-R_0x(1+I_1)}|I_0^a = i, a], \quad x \geq 0, \ i \in I
$$

where

$$
\beta^{-1} = \inf_{\theta \geq 0} \frac{\int_0^\theta e^{R_0y} dF(y)}{e^{R_0\theta} F(\theta)} \text{ with } b \in (0, 1].
$$

Proof. We apply induction method. For $n = 1$ we can write

$$
\bar{F}(\theta) = \left(\frac{\int_\theta^\infty e^{R_0b\theta} dF(y)}{e^{R_0b\theta} F(\theta)}\right)^{-1} e^{-R_0b\theta} \int_\theta^\infty e^{R_0b\theta} dF(y)
$$

$$
\leq \beta e^{-R_0b\theta} \int_\theta^\infty e^{R_0b\theta} dF(y) \leq \beta e^{-R_0b\theta} \bar{E}[e^{R_0bY_1}|a],
$$

for any $\theta \geq 0$. Which implies that for any $x \geq 0$, $i \geq 0$ and $b_0 \in [0, 1]$

$$
\psi^a_1(x, i) = Pr\{X_1 < 0|I_0^a = i, a\} = Pr\{Y_1 > \frac{x(1+j)+c(b_0)z}{b_0} |I_0^a = i, a\}
$$

$$
= \sum_{j=0}^{\infty} p_{ij}^a \int_0^\infty \bar{F}(\frac{x(1+j)+c(b_0)z}{b_0}) dG(z),
$$

10
which, in turn, implies by (22) that

\[
\psi^a_1(x, i) \leq \sum_{j=0}^{\infty} p^a_{ij} \left( \beta E[e^{R_0 b Y_1|a} \cdot \int_0^{\infty} e^{-R_0 b_0 \left( \frac{x(1+j) + c(b_0)z}{b_0} \right)} dG(z) \right)
\]

\[
= \beta E[e^{R_0 b Y_1|a}] \cdot \sum_{j=0}^{\infty} p^a_{ij} \int_0^{\infty} e^{-R_0 [x(1+j) + c(b_0)z]} dG(z)
\]

\[
= \beta E[e^{R_0 b Y_1|a}] \cdot \sum_{j=0}^{\infty} p^a_{ij} E[e^{-R_0 [x(1+j) + c(b_0)z]}|I_0^a = i, a]
\]

\[
= \beta E[e^{R_0 b Y_1|a}] E[e^{-R_0 c(b)Z_1}|a] E[e^{-R_0 x(1+I_1^a)}|I_0^a = i, a]
\]

\[
= \beta E[e^{-R_0 x(1+I_1^a)}|I_0^a = i, a].
\]

Under an inductive hypotheses, we assume for any \( x \geq 0 \) and any \( i \in I \),

\[
\psi^a_1(x, i) \leq \beta E[e^{-R_0 x(1+I_1^a)}|I_0^a = i, a] \tag{23}
\]

thus, for \( 0 \leq y \leq \frac{x(1+j) + c(b_0)z}{b_0} \), with \( x \) and \( i \) replaced by \( x(1+j) + c(b_0)z - b_0y \) and \( j \) respectively in (23), and \( I_1^a \geq 0 \), we have

\[
\psi^a_1(x(1 + j) + c(b_0)z - b_0y, j) \leq \beta E[e^{-R_0 [x(1+j) + c(b_0)z - b_0y]}|(1+I_1^a)|I_0^a = j, a]
\]

\[
\leq \beta e^{-R_0 [x(1+j) + c(b_0)z - b_0y]} \tag{24}
\]
therefore, replacing (24) in (16), we get

\[
\psi_{n+1}^a(x, i) \leq \sum_{j=0}^{\infty} p_{ij}^a \left( \beta \int_0^\infty e^{-R_0[z(1+j)+c(b_0)z]} \int_\tau^\infty e^{R_0b_0y} dF(y) dG(z) \right) 
+ \sum_{j=0}^{\infty} p_{ij}^a \left( \beta \int_0^\infty e^{-R_0[z(1+j)+c(b_0)z]} \int_0^\tau e^{R_0b_0y} dF(y) dG(z) \right)
\]

\[
= \sum_{j=0}^{\infty} p_{ij}^a \left( \beta \int_0^\infty e^{-R_0[z(1+j)+c(b_0)z]} \int_0^\infty e^{R_0b_0y} dF(y) dG(z) \right)
\]

\[
= \beta E[e^{R_0bY_1}|a] \sum_{j=0}^{\infty} p_{ij}^a \int_0^\infty e^{-R_0[z(1+j)+c(b_0)z]} dG(z)
\]

\[
= \beta E[e^{R_0bY_1}|a] \cdot E[e^{-R_0c(b)Z_1}|a] \cdot E[e^{-R_0x(1+I_0^a)}|I_0^a = i, a]
\]

\[
= \beta E[e^{-R_0x(1+I_0^a)}|I_0^a = i, a].
\]

Hence, for any \( n = 1, 2, \ldots \), (23) holds. Therefore, (21) follows by letting \( n \to \infty \) in (23). \( \square \)

Next we consider the case when the claim distribution belongs to the particular class of NWUC distributions (see (35) in appendix A for details).

**Corollary 1.** Under the hypothesis of Theorem 1, and assuming that \( E[e^{R_0bY_1}|a] < \infty \) for all \( b \in (0, 1) \) and in addition if \( F \) is a NWUC distributions, then

\[
\psi^a(x, i) \leq (E[e^{R_0bY_1}|a])^{-1} E[e^{-R_0x(1+I_0^a)}|I_0^a = i, a], \quad x \geq 0.
\]

**Proof.** Following Willmot & Lin [19] in pp. 96–97, and defining \( r = R_0b_0 > 0 \), we have

\[
\beta^{-1} = \inf_{t \geq 0} \frac{\int_t^\infty e^{ry} dF(y)}{e^{rt} F(t)} = \int_0^\infty e^{ry} dF(y)
\]

that is, \( \beta^{-1} = E[e^{R_0bY_1}] = E[e^{R_0bY_1}|a] \). Finally replacing this equality in (21), then (25) holds. \( \square \)
4.2 Inequalities for ruin probabilities by the martingale approach

Another tool for deriving inequalities for ruin probabilities is the martingale approach. The ruin probabilities associated with the discounted risk process \( \{V_n, n = 1, 2, \ldots \} \), i.e.

\[
\psi_n^a(x, i) = \Pr\left\{ \bigcup_{k=1}^{n} (X_k < 0) \mid I_0^a = i, a \right\} = \Pr\left\{ \bigcup_{k=1}^{n} (V_k < 0) \mid I_0^a = i, a \right\}
\]

where \( V_n = X_n \prod_{l=1}^{n} (1 + I_l^a)^{-1}, n = 1, 2, \ldots \).

In the classical risk model, \( \{e^{-R_0 X_n}, n = 1, 2, \ldots \} \) is a martingale. However, for model (7), there is no constant \( r > 0 \) such that \( \{e^{-rX_n}, n = 1, 2, \ldots \} \) is a martingale. Still, there exists a constant \( r > 0 \) such that \( \{e^{-rV_n}, n = 1, 2, \ldots \} \) is a supermartingale, which allows us to derive probability inequalities by the optional stopping theorem. Such a constant is defined in the following proposition.

**Proposition 1.** Assume that \( E[bY_1 - c(b)Z_1] < 0 \). In addition we suppose that for \( i \in I \), there exists \( \rho_i > 0 \) satisfying

\[
E \left[ e^{\rho_i [bY_1 - c(b)Z_1] (1 + I_l^a)} \mid I_0^a = i, a \right] = 1.
\]

Then,

\[
R_1 = \min_{i \in I} \rho_i \geq R_0
\]

and for all \( i \in I \)

\[
E \left[ e^{R_1 [bY_1 - c(b)Z_1] (1 + I_l^a)} \mid I_0^a = i, a \right] \leq 1.
\]

**Proof.** Note that the function

\[
l_i(r) = E \left[ e^{r [bY - c(b)Z] (1 + I_l^a)} \mid I_0^a = i, a \right] - 1
\]

\( r > 0 \), for each \( i \in I \) is a convex function with \( l_i(0) = 0 \)

\[
l_i'(0) = E \left[ (bY - c(b)Z) \right] E \left[ (1 + I_l^a)^{-1} \mid I_0^a = i, a \right] < 0.
\]
Therefore, \( \rho_i \) is the unique positive root \( \rho \) of the equation \( l_i(r) = 0 \) on \((0, \infty)\). Further,

\[
E \left[ e^{R_0|bY-c(b)Z|(1+I_1^a)^{-1}} \right] = \sum_{j=0}^{\infty} p_{ij}^a E \left[ e^{e^{R_0|b_0Y-c(b_0)Z|(1+j)^{-1}}} \right]
\]

by Jensen’s inequality

\[
= \sum_{j=0}^{\infty} p_{ij}^a \cdot E \left[ e^{R_0|bY_1-c(b_0)Z_1|(1+j)^{-1}} \right] \leq \sum_{j=0}^{\infty} p_{ij}^a E \left[ e^{e^{R_0|bY-c(b)Z|(1+j)^{-1}}} \right] \]

\[
= \sum_{j=0}^{\infty} p_{ij}^a = 1 \implies E \left[ e^{e^{R_0|bY-c(b)Z|(1+j)^{-1}}} \right] \leq 1
\]

which implies that \( l_i(R_0) \leq 0 \). Here \( R_0 \leq \rho_i \) and

\[
R_1 = \min_{i \in I} \rho_i \geq R_0.
\]

Thus, \((28)\) holds. In addition \( R_i \leq \rho_i \), for all \( i \in I \), which implies that \( l_i(R_1) \leq 0 \), i.e. \((29)\) holds. \( \square \)

**Theorem 2.** Under the hypothesis of Proposition 1, for all \( i \in I \),

\[
\psi^a(x, i) \leq e^{-R_1 x}, \quad x \geq 0.
\] (30)

**Proof.** For the surplus process \( \{X_k\} \) given by \((7)\), we have

\[
V_k = X_k \prod_{l=1}^{k} (1 + I_l^a)^{-1} = x - \sum_{l=1}^{k} \left( (b_{l-1}Y_l - c(b_{l-1})Z_l) \prod_{t=1}^{l} (1 + I_t^a)^{-1} \right) \]
(31)

and \( S_n = e^{-R_n V_n} \). Then \( S_{n+1} = S_n e^{-R_1(b_{n+1}Y_{n+1} - c(b_n)Z_{n+1}) \prod_{t=1}^{n+1} (1+I_t^a)^{-1}} \). Thus, for
any $n \geq 1$,
\[
E[S_{n+1} \mid Y_1, \ldots Y_n, Z_1, \ldots, Z_n, I_1^a, \ldots I_n^a]
\]
\[
= S_n E \left[ e^{-R_1(b_n Y_{n+1} - c(b_n)Z_{n+1})} \prod_{t=1}^{n+1} (1+I_t^a)^{-1} \mid Y_1, \ldots Y_n, Z_1, \ldots, Z_n, I_1^a, \ldots I_n^a \right]
\]
\[
= S_n E \left[ e^{-R_1(b_n Y_{n+1} - c(b_n)Z_{n+1})} \prod_{t=1}^{n+1} (1+I_t^a)^{-1} \mid I_1^a, \ldots I_n^a \right]
\]
\[
\leq S_n E \left( e^{-R_1(b_n Y_{n+1} - c(b_n)Z_{n+1})} (1+I_{n+1}^a)^{-1} \mid I_1^a, \ldots I_n^a \right) \prod_{t=1}^{n+1} (1+I_t^a)^{-1}
\]
\[
S_n E \left( e^{-R_1(b_n Y_{n+1} - c(b_n)Z_{n+1})} (1+I_{n+1}^a)^{-1} \mid I_n^a \right) \prod_{t=1}^{n+1} (1+I_t^a)^{-1} \leq S_n
\]
which implies that \( \{S_n, n = 1, 2, \ldots\} \) is a supermartingale.

Let \( T_i = \min\{n : V_n < 0 \mid I_0^a = i\} \) where \( V_n \) is given by (31). Then \( T_i \) is a stopping time and \( n \land T_i = \min\{n, T_i\} \) is a finite stopping time. Thus, by the optimal stopping theorem for martingales, we get
\[
E(S_n \land T_i) \leq E(S_0) = e^{-R_1 x}.
\]

Hence,
\[
e^{-R_1 x} \geq E(S_n \land T_i) \geq E((S_n \land T_i) I_{(T_i \leq n)}) \geq E((S_{T_i}) I_{(T_i \leq n)})
\]
\[
= E(e^{-R_1 V_{T_i}} I_{(T_i \leq n)}) \geq E(I_{(T_i \leq n)}) = \psi_n^a(x, i),
\]
where (32) follows from \( V_{T_i} < 0 \). Thus, (30) follows by letting \( n \to \infty \) in (32).

5 Numerical results

We present a numerical example to illustrate the bounds given by Theorems 1 and 2 and for the purpose Matlab and Maple implementations are developed.
We consider claim distribution of the PH type because they and their moments can be written in a closed form, various quantities of interest can be evaluated with relative ease, and finally, the set of PH distributions is dense in the set of all distributions with support in \([0, \infty)\).

Suppose that the claim size \(Y\) has a phase-type density \(PH(\alpha, T)\) with \(\alpha = (1/2, 1/2), e = (1, 1), T = \begin{pmatrix} -1 & 0 \\ 0 & -2 \end{pmatrix}, I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\), and \(t = -Te = \begin{pmatrix} 1 \\ 2 \end{pmatrix}\) then

\[
MY(s) = E[e^{sY}] = \alpha (-sI - T)^{-1} t
\]

thus, \(E[Y] = \frac{d}{ds}MY(s) \bigg|_{s=0} = \alpha(T)^{-2}t = 0.75\), and \(Y\) is NWUC.

We denote the length of periods by \(Z_k\). Let \(Z \sim Exp(1), E[Z] = 1\), thus

\[
M_Z(s) = E[e^{sZ}] = (1 - s)^{-1}.
\]

We consider an interest model with three possible interest rates: \(I = \{6\%, 8\%, 10\%\}\). In addition we suppose two portfolio investment \(\delta \in \{1, 2\}\). Thus the corresponding transition probability matrices of the price process \(I_n^a\) for \(a_1 = (b, 1), a_2 = (b, 2)\) are

\[
P_1 = \begin{pmatrix} 0 & 0.9 & 0.1 \\ 0.8 & 0.2 & 0 \\ 0.9 & 0.1 & 0 \end{pmatrix} \quad \text{and} \quad P_2 = \begin{pmatrix} 0.3 & 0.7 & 0 \\ 0 & 0.2 & 0.8 \\ 0 & 0.1 & 0.9 \end{pmatrix}.
\]

We fixed the premium income rate (2) \(c = 0.975\) and the safety loading of the reinsurer \(\theta = 0.1\).

**Lundberg bound:** In this example we can guarantee that (14) holds for each \(b \in (0, 1]\). First, it is easy to see that for \(b\) fixed \(E[BY_1 - c(b)Z_1] < 0\). Second, for each \(b \in (0, 1]\) there exist a constant \(R_0\) such that (13) is achieved. Note that solving

\[
E[e^{R_0Y}] : E[e^{-R_0c(b)Z_1}] = 1
\]
is equivalent to solve

\[(1 + c(b)R_0) = \alpha (-bR_0 I - T)^{-1} t\]

The Lundberg bound for the ruin probability is

\[\psi^b(x) \leq e^{-R_0 x}, \text{ for } x \geq 0.\]

Figure 1 shows the non-linear relation between \(R_0\) and \(b\) which is the solution of the preceding equation. Table 1 present numerical values of the bounds obtained for several admissible decision policies.

**Induction bound:** In this example the claim distribution is a NWUC (see [19], page 24) and such that \(E[e^{R_0 b Y_1 | a}] = M_Y(R_0 b) < \infty\) for each \(b \in (0, 1]\). Then Corollary 1 applies and for each \(i \in I\)

\[\psi^a(x, i) \leq \left(E[e^{R_0 b Y_1 | a}]^{-1} E[e^{-R_0 x(1 + f_k) | I_0^a = i, a}]\right)^{-1} \sum_{k \in \{6,8,10\}} p_{ik} e^{-R_0 x(1+k)} \text{ for } x \geq 0.\]

Numerical values of this bound obtained for several admissible decision policies are presented in Table 1. As it is expected we get induction bounds.
smaller than the Lundberg bounds for the same decision policies.

**Martingale bound:** In this example we can guarantee that hypotheses of Proposition 1 follow. Then Theorem 2 holds and we get bound (30). Condition (27) of Proposition 1

\[ E \left[ e^{\rho_i (by_1 - c(b)z_1)(1 + r^i)^{-1}} \mid I_0^a = i, a \right] = 1, \]

is equivalent to the following condition for each \( i \in I \)

\[ \sum_{k \in \{6, 8, 10\}} p_{ik} e^{\rho_i (1+k)^{-1}} My \left( \frac{b \rho_i}{1 + k} \right) Mz \left( \frac{-c(b) \rho_i}{1 + k} \right) = 1 \]

or

\[ \sum_{k \in \{6, 8, 10\}} p_{ik} e^{\rho_i (1+k)^{-1}} \alpha \left( -b \rho_i (1 + k)^{-1} I - T \right)^{-1} t \cdot \left( 1 + \frac{c(b) \rho_i}{1 + k} \right)^{-1} = 1. \]

In our example we solve \( R_1 = \min_{i \in I} \rho_i \geq R_0 \) and then we obtain \( \psi^a(x, i_1) \leq e^{-R_1x}, x \geq 0 \). Numerical results of this bound are reported in Table 1. And it is obvious that this martingale bound improves the results for the induction bound.

Finally, we find of special interest the case of small reinsurer for which the retention level could be restricted by economic considerations. Thus we run numerical experiments in order to compare for a fixed retention level \( b \) the ruin probability bounds that could be achieved, or conversely depending on the type of the claim distribution for a fixed ruin probability level we can evaluate the admissible investment/reinsurance policies.

Figure 2 shows the bounds for different approaches with \( x = 5 \) and \( i = 8\% \) while \( b \in (0, 1] \).
Figure 2: Bounds of ruin probability by different approaches vs $b$.

<table>
<thead>
<tr>
<th>$P_1$</th>
<th>$b$</th>
<th>Lundberg bound</th>
<th>Induction bound</th>
<th>Martingale bound</th>
<th>$R_0$</th>
<th>$R_1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$P_1$</td>
<td>0.01</td>
<td>$0.223e-427$</td>
<td>$0.934e-453$</td>
<td>$\rightarrow 0$</td>
<td>$196.94$</td>
<td>$\rightarrow \infty$</td>
</tr>
<tr>
<td>$P_1$</td>
<td>0.25</td>
<td>$0.516e-15$</td>
<td>$0.125e-15$</td>
<td>$0.199e-18$</td>
<td>$7.039$</td>
<td>$8.611$</td>
</tr>
<tr>
<td>$P_1$</td>
<td>0.5</td>
<td>$0.323e-7$</td>
<td>$0.239e-7$</td>
<td>$0.448e-9$</td>
<td>$3.449$</td>
<td>$4.304$</td>
</tr>
<tr>
<td>$P_1$</td>
<td>0.75</td>
<td>$0.111e-4$</td>
<td>$0.118e-4$</td>
<td>$0.586e-6$</td>
<td>$2.280$</td>
<td>$2.869$</td>
</tr>
<tr>
<td>$P_1$</td>
<td>0.85</td>
<td>$0.434e-4$</td>
<td>$0.504e-4$</td>
<td>$0.317e-5$</td>
<td>$2.008$</td>
<td>$2.532$</td>
</tr>
<tr>
<td>$P_1$</td>
<td>0.95</td>
<td>$0.1261e-3$</td>
<td>$0.157e-3$</td>
<td>$0.120e-4$</td>
<td>$1.794$</td>
<td>$2.265$</td>
</tr>
<tr>
<td>$P_1$</td>
<td>1</td>
<td>$0.2e-3$</td>
<td>$0.255e-3$</td>
<td>$0.212e-4$</td>
<td>$1.703$</td>
<td>$2.152$</td>
</tr>
<tr>
<td>$P_2$</td>
<td>0.01</td>
<td>$0.22e-427$</td>
<td>$0.653e-462$</td>
<td>$\rightarrow 0$</td>
<td>$196.94$</td>
<td>$\rightarrow \infty$</td>
</tr>
<tr>
<td>$P_2$</td>
<td>0.25</td>
<td>$0.5166e-15$</td>
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<td>$7.039$</td>
<td>$8.675$</td>
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<td>$2.282$</td>
</tr>
<tr>
<td>$P_2$</td>
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<td>$0.194e-3$</td>
<td>$0.195e-4$</td>
<td>$1.703$</td>
<td>$2.168$</td>
</tr>
</tbody>
</table>

Table 1: Numerical bounds of ruin probability.
6 Optimality

In the preceding Section probability inequalities for the ultimate probability of ruin are derived. Our goal is to minimize the ruin probability $\psi_*(x) = \inf_a \psi^a(x)$, that is hitting $\Delta = (-\infty, 0]$. Following Gajek (2005) the idea is to use our Lundberg-type upper bounds as initial approximations and then iteratively apply an integral operator. Groniowska and Niemiro (2005) use a similar approximation provided that the risk model follows a random walk. Our approximations are closely related to the well-known in stochastic control theory Policy Iteration Algorithm, (see Hernandez-Lerma and Laserre (1996), Section 4.4), which provides decreasing approximations to the minimum, over all possible decision strategies, of the ultimate probability of ruin.

Assume that the transition probabilities of $X_n$ are given by (5).

We consider the Bellman operator defined as follows. For $v \in V$, the set of all measurable functions $v : X \setminus \Delta \to [0, 1],

$$Bv(x) = \inf_{a \in A} [P(x, a, \Delta) + \int_{X \setminus \Delta} P(x, a, dx') v(x')]$$

If $\varphi : X \to A$, we consider the following operator

$$T^\varphi v(x) = P(\varphi(x), x, \Delta) + \int_{X \setminus \Delta} P(\varphi(x), x, dx') v(x')$$

We state now some results essentially standard but necessary for the sequel. They are known as the measurable selection conditions for the Bellman's Principle of Optimality.

Proposition 2. Let $A$ be a compact set and let assume that the stochastic kernel $P$ is strongly continuous with respect to $a$

$$u(x, a) = \int_X P(x, a, dx') w(x')$$

i.e. the measurable function $u : X \times A \to [0, 1]$ is continuous in $a$ for every fixed $x \in X$ and for every measurable function $w : X \to [0, 1]$.

Then the Bellman operator has the following properties:
• i) Attainability: for every \( v \in V \) there exist a measurable \( \varphi : X \to A \) such that \( Bv = T^\varphi v \). Therefore, \( Bv \in V \).

• ii) Monotonicity: if \( v_1 \leq v_2 \) then \( Bv_1 \leq Bv_2 \).

• iii) Monotone continuity: if \( v_k \nearrow v \) then \( Bv_k \nearrow Bv \).

Proof:

Note that in the definition of operator \( B \), the function under the infimum is continuous with respect to \( a \). Because the kernel is assumed to be strongly continuous. Attainability of the infimum over the compact set \( A \) holds, and consequently the infimum is also measurable. Result ii) is trivial. The Lebesgue monotone convergence theorem and interchange results of limits and minima on continuous functions over compact sets imply iii).

For further details see Hernandez-Lerma and Laserre (1996).

\( \square \)

We turn now to the Lundberg-type inequality probabilities for the ruin probability developed in Section 4. The existence of this bounds means that for at least one strategy the probability distribution \( \hat{F}(a, .) \) has the adjustment coefficient as we assume in the following.

Proposition 3. Let consider the upper bound \( v(x) = \exp\{-R_1 x\} \) given by (30) in Theorem 2. Assuming that there exist \( a \in A \) such that \( \int_X e^{R_1 x}, \hat{F}(a, dy) = 1 \), where the distribution function \( \hat{F} \) is given by (5) then the function \( v(x) \) satisfies \( T^a v \leq v \).
Proof:

Because the function \( v(x) \in V \) and satisfies Proposition 2, we can state

\[
T^a v(x) = P(x, a, \Delta) + \int_{X \setminus \Delta} P(x, a, dx') v(x')
\]

\[
= \int_{[x, \infty)} \hat{F}(a, dx') + \int_{(-\infty, x]} \hat{F}(a, dx') v(x - x')
\]

\[
= \hat{F}(a, dx') + \int_{(-\infty, x]} \exp(-R_1(x - x')) \cdot \hat{F}(a, dx')
\]

\[
\leq \int_{[x, \infty)} \exp(-R_1(x - x')) \cdot \hat{F}(a, dx') + \int_{(-\infty, x]} \exp(-R_1(x - x')) \cdot \hat{F}(a, dx')
\]

\[
= \exp(-R_1(x)) \int_R \exp(R_1 x') \cdot \hat{F}(a, dx')
\]

\[
= \exp(-R_1(x)) = v(x)
\]

**Theorem 3.** Let assume that Proposition 3 holds. Let the distribution function \( \hat{F} \) verify that there exist \( h > 0 \) and \( \zeta > 0 \) such that \( (1 - \hat{F}(a, h)) \geq \xi \), for every \( a \in A \). Then

- i) If we define recursively \( \tilde{v}_k = B \tilde{v}_{k-1} \), starting with \( \tilde{v}_0 = v \), we obtain the monotone-decreasing convergence \( \tilde{v}_k \downarrow \psi_* \).

- ii) Moreover we have that \( \tilde{v}_k \rightarrow \psi_* \) where the convergence may be non-monotonic.

**Remark:** Note that the condition for \( \hat{F} \) seems to be satisfied for all reasonable risk models. It is worth noting that if the conditions is fulfilled, then for every strategy \( a \) and every initial state \( x \) the surplus process verify \( P^a_x(T < \infty \text{ or } X_n \rightarrow \infty) = 1 \) and consequently almost every trajectory either goes infinity or falls below zero at least once.

**Proof:**
• i) By induction it can be proved that \( \bar{v}_{k-1} \geq \bar{v}_k \), starting with \( \bar{v}_0 \geq T^a \bar{v}_0 \geq B \bar{v}_0 = \bar{v}_1 \). Because \( B \psi_a = \psi \) by induction we obtain \( \bar{v}_k \geq \psi \).

• ii) Note that \( \psi \leq \psi^a \leq \bar{v}_0 \) implies \( \psi \leq \bar{v}_k \) because \( B \) is a monotonic operator and has a fixed point which is \( \psi \). Because \( P^a_x(T < \infty \text{ or } X_n \rightarrow \infty) = 1 \) holds, then provided that \( v(x) = \exp\{-R_t x\} \) verifies \( \lim_{x \rightarrow \infty} v(x) = 0 \) and the dominated convergence theorem we obtain

\[
\int_{X \setminus \Delta} P(x, a, dx') v(x') = \int_{(-\infty, x]} \hat{F}(a, dx') v(x - x')
\]

\[
= E^a_x(I(T > k)\bar{v}_0(X_x) \rightarrow 0
\]

This is true for every strategy \( a \), in particular for \( a_* \). Then we have

\[
\bar{v}_k(x) \leq P^a_\ast(T < \infty) + E^a_\ast[I(T > k)\bar{v}_0(X_x)]
\]

The first term tends to \( \psi_{a_*} \) and the second term goes to zero. It follows that \( \lim \bar{v}_k \leq \psi_{a_*} = \psi \). Combining with i) then the convergence \( \bar{v}_k \rightarrow \psi \) is obtained.

Remark: By the attainability property of the Bellman operator there is a stationary strategy \( a \) such that \( B \bar{v}_{k-1} = T^a \bar{v}_{k-1} \). It is not difficult to prove that considering this stationary strategy \( T^a \bar{v}_{k-1} \leq \bar{v}_{k-1} \) implies \( \psi^a \leq \bar{v}_k \). If we compute the stationary strategy simultaneously with \( \bar{v}_k \), we obtain a stationary strategy for which at each stage the ruin probability is at most \( \bar{v}_k \).
References


A Stochastic orders and phase-type distributions

DFR \quad \Rightarrow \quad NWU
\downarrow
\downarrow
IMRL \quad \Rightarrow \quad 2-
NWU \quad \Rightarrow \quad NWUC

\diamond \text{The df } F \text{ is DFR (Decreasing Failure Rate): if } \frac{F(x+y)}{F(y)} \text{ is non-decreasing in } y \text{ for fixed } x \geq 0, \text{ i.e. } F(y) \text{ is log-convex. Then } \mu(y) = \frac{f(y)}{F(y)} \text{ is nonincreasing in } y.

Let \( F(y) \) phase-type distribution with parameters \((\alpha, T)\), we want to see under what condition the phase-type distributions are DFR. That is equivalent to satisfy that \( \frac{\partial}{\partial y} \frac{F(x+y)}{F(y)} \) is non-negative in \( y \) for fixed \( x \geq 0 \). In this case \( \frac{F(x+y)}{F(y)} = \frac{\alpha e^{T(x+y)}e^{T_y}e}{\alpha e^{T_y}e}, \) then

\[
\frac{\partial}{\partial y} \frac{F(x+y)}{F(y)} = \frac{\alpha T e^{T(x+y)}\hat{\alpha} e^{T_y}e - \alpha e^{T(x+y)}\hat{\alpha} T e^{T_y}e}{(\alpha e^{T_y}e)^2}
\]

is non-negative, if only if the numerator is non-negative, i.e,

\[
\alpha T e^{T(x+y)}\hat{\alpha} e^{T_y}e - \alpha e^{T(x+y)}\hat{\alpha} T e^{T_y}e = \alpha (T e^{T(x+y)}\hat{\alpha} - e^{T(x+y)}\hat{\alpha} T) e^{T_y}e = \alpha (T A - A T) e^{T_y}e \geq 0
\]  \( (33) \)

where \( A = e^{T(x+y)}\hat{\alpha} \) and \( \hat{\alpha} = e \cdot \alpha = \overline{1} \cdot (\alpha_1, \ldots, \alpha_n) = (\alpha_1 \cdot \overline{1}, \ldots, \alpha_n \cdot \overline{1}) \). The previous condition (33) is only possible if \( T A - A T \) is non-negative definite. The equality is when \( T \) commute with \( A = e^{T(x+y)}e\alpha \).

So that, The phase-type distribution are DFR if only if \( T A - A T \) is non-negative definite.

\diamond \text{The df } F \text{ is IMRL (Increasing Mean Residual Life): if}

\[
r(y) = E[T_y] = \frac{\int_{y}^{\infty} (t-y) dF(t)}{F(y)} dt
\]

\[
= \frac{\int_{y}^{\infty} F(t+y) dt}{F(y)} = \frac{E(Y) F_1(y)}{F(y)}
\]

is increasing. Where \( F_1(y) = \frac{\int_{y}^{\infty} F(x) dx}{E[Y]} \) with \( y \geq 0 \) and \( E[Y] = \int_{0}^{\infty} F(y) dy \).

Let \( F(x) = PH(\alpha, T) \), always \( T^{-1} \) exist. En this case \( E[Y] = -\alpha T^{-1}e \), then
\[ F_1(y) = \frac{\int_0^y e^{Tx}e^{d_x}}{E[Y]} = \frac{\alpha T^{-1}(e^{Ty} - I)e}{E[Y]} \] and \( F_1(y) = \frac{\alpha T^{-1}e^{Tx}e}{\alpha T^{-1}e} \). So \( r(y) = \frac{-\alpha T^{-1}e^{Ty}e}{\alpha T^{-1}e} \). So \( F_1(y) = \frac{-\alpha T^{-1}e^{Ty}e}{\alpha T^{-1}e} \).

\( F(y) \) is IMRL if only if \( \frac{\partial}{\partial y} r(y) \) is non-negative, i.e., \( F(y) \) is IMRL iff \(-e^{Ty}\hat{\alpha} + T^{-1}e^{Ty}\hat{\alpha}T\) is non-negative definite.

\( \diamond \) The df \( F \) is NWU (New Worse than Used): if

\[ \overline{F}(x + y) \geq \overline{F}(y) \cdot \overline{F}(x), \ \forall x \geq 0, \ \forall y \geq 0. \] (34)

Let \( F(x) = PH(\alpha, T) \), we say that \( F \) is NWU if only if \( \overline{F}(x+y) - \overline{F}(y) \cdot \overline{F}(x) = \alpha e^{Tx}(I - e\alpha)e^{Ty} \geq 0 \), i.e., the phase-type distribution is NWU iff \( I - \hat{\alpha} \) is non-negative definite.

\( \diamond \) The df \( F \) is 2-NWU (Second New Worse than Used): if

\[ \overline{F}_1(x + y) \geq \overline{F}_1(y) \cdot \overline{F}_1(x), \ \forall x \geq 0, \ \forall y \geq 0. \]

Let \( F(x) = PH(\alpha, T) \), in this case \( \overline{F}_1(x) = 1 - F_1(x) = \frac{\alpha T^{-1} e^{Tx}e}{\alpha T^{-1} e} \). We say that \( F \) is 2-NWU iff

\[ \overline{F}_1(x + y) - \overline{F}_1(y) \cdot \overline{F}_1(x) = \frac{\alpha T^{-1}(e\alpha T^{-1} e^{Ty} - e^{Ty} e\alpha T^{-1})e^{Tx}e}{(\alpha T^{-1} e)^2} \geq 0. \]

The phase-type distribution is 2-NWU iff \( T^{-1}(Be^{Ty} - e^{Ty}B) \) is non-negative definite, where \( B = e\alpha T^{-1} \).

\( \diamond \) The df \( F_1 \) is NWUC (New Worse than Used in Convex ordering): if

\[ \overline{F}_1(x + y) \geq \overline{F}_1(y) \cdot \overline{F}(x), \ \forall x \geq 0, \ \forall y \geq 0. \] (35)

Let \( F(x) = PH(\alpha, T) \), we say that \( F \) is NWUC iff

\[ \overline{F}_1(x + y) - \overline{F}_1(y) \cdot \overline{F}(x) = \frac{\alpha T^{-1} e^{Ty}(I - e\alpha)e^{Tx}e}{\alpha T^{-1} e} \geq 0. \]

The phase-type distribution is NWUC iff \( T^{-1} \) and \( T^{-1} e^{Ty}(I - e\alpha) \) are non-negative or non-positive definite both simultaneously.