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## Competitive Search Equilibrium with Multidimensional Heterogeneity and Two-Sided Ex-ante Investments\*

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### Abstract

We analyze a competitive search environment where heterogeneous workers and firms make costly investments (e.g. in education and physical capital, respectively) before they enter the labor market. A key novelty with respect to existing work is that we allow for multidimensional heterogeneity on both sides of the market. Our environment features transferable utility and symmetric information. As in classical hedonic models, wages depend both on the job's and on the worker's match-relevant characteristics. Yet the presence of search frictions implies that (unlike in those models) markets do not clear. The hedonic wage function and probabilities of finding and filling different jobs are determined endogenously in a competitive search equilibrium. We show that constrained efficient allocations can be determined as optimal solutions to a linear programming problem, whereas the wage function supporting these allocations and associated expected payoffs for workers and firms correspond to the solutions of the 'dual' of that linear program. We use this characterization to show that a competitive search equilibrium exist and is constrained efficient under very general conditions. Jerez (2014) makes a similar point in the context of a model where all the match-relevant characteristics of the traders are exogenous. Here we extend the analysis to allow for two-sided ex-ante investments which are potentially multidimensional. The fact that linear programming techniques have been used for the structural estimation of frictionless matching models suggests that our framework is potentially useful for empirical studies of labor markets and other hedonic markets (like that for housing) where search frictions are prevalent

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# 1 Introduction

Since the seminal work of Diamond (1981, 1982), Mortensen (1982a, 1982b) and Pissarides (1984, 1985), search equilibrium models have become a dominant paradigm to study labor markets. These models abandon the classical assumption of a frictionless trading process that always clears the market. The so-called Diamond-Mortensen-Pissarides search and bargaining model instead introduces a costly trading technology by means of an exogenous matching function which generates simultaneous rationing (and thus trading delays) on both sides of the market. Such a model generates frictional unemployment in equilibrium, and is consistent with the simultaneous coexistence of unemployment and unfilled job vacancies. This kind of two-sided rationing is observed in other markets. In the housing market, for instance, in a given period (e.g. each month) there are owners with property on the market who do not manage to sell, and households searching for a housing unit who do not complete a transaction during the period either.<sup>1</sup>

Competitive search (also referred to as directed search) models have received an increasing attention in recent years. These models deviate from the search and bargaining model in that they abstract away from inefficiencies arising from bilateral monopoly power. The earlier competitive search literature instead focuses solely on search and matching frictions (e.g. see Montgomery (1991), Peters (1991, 1997, 2000), Moen (1997), Shimer (1996,2005), Acemoglu and Shimer (1999a, 1999b), Burdett, Shi, and Wright (2001), Shi (2001), and Mortensen and Wright (2002)). In competitive search models of the labor market, firms typically compete *ex ante* by publicly posting (and committing to) job offers, and workers then direct their search to the more attractive offers. Because firms posting more attractive offers attract more job applicants on average, they are able to fill their vacancies faster.<sup>2</sup> Montgomery (1991), Peters (1991), Moen (1997) and Shimer (1996) were among the first to show that competitive search equilibria are constrained efficient. Whereas these authors consider simple environments with homogeneous buyers and/or sellers, subsequent

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<sup>1</sup>Search models are becoming increasingly popular in quantitative studies of the housing market. See, among others, Díaz and Jerez (2013), Head, Lloyd-Ellis, and Sun (2014) and Ngai and Tenreyro (2014), and the literature cited therein.

<sup>2</sup>By contrast, in search and bargaining models wages are determined *ex post*, once the firm and the worker meet (e.g. by means of the generalized Nash bargaining rule), so they do not affect the traders' rationing probabilities. Competitive search models differ also from the price posting models introduced by Burdett and Judd (1983) and Burdett and Mortensen (1998). The later models feature imperfect competition since, even though firms compete *ex ante* by posting wages, (rather than directing their search to the more attractive offers) workers sample randomly from the wage offer distribution. See Rogerson, Shimer, and Wright (2005) for a comprehensive review of the labor search literature.

work has generalized their results to more general environments. In particular, Shi (2001), Shimer (2005), Eeckhout and Kircher (2010) allow for two-sided one-dimensional heterogeneity.

This paper is particularly related to the work of Acemoglu and Shimer (1999b), Shi (2001) and Masters (2011), who argue that competitive search solves the familiar “hold-up problem”. Acemoglu and Shimer (1999b) analyze a model where firms make investments in physical capital before entering the labor market, so as to increase the value of future production. They show that, in the search and bargaining model, firms’ investments are inefficiently low, partly because some of the surplus these investments generate is appropriated by the firms’ future employees (see also Acemoglu (1996)).<sup>3</sup> With competitive search, however, firms have additional incentives to invest. Because firms making higher investments are more productive, they can also offer higher wages. This in turn allows them to fill their job vacancies faster. In a competitive search equilibrium the share of the surplus that accrues to a firm is determined endogenously, and this share is such that firms receive the social marginal product of their investment. As a result, the firms’ ex-ante investments are efficient. Whereas in Acemoglu and Shimer’s (1999b) model workers and firms are homogeneous and only firms make ex-ante investments, Shi (2001) shows that the efficiency result holds also if workers have heterogeneous skills. Masters (2011) reaches the same conclusion in a model where homogeneous workers and firms make complementary ex-ante investments in physical and human capital, respectively. In all three papers the observability of the agents’ investments and of their match-relevant characteristics –which rules out asymmetric information problems– is crucial for the efficiency result.

To the best of our knowledge, the model in this paper is the first in the search literature which jointly captures the rich two-sided heterogeneity which characterizes the labor market, and the fact that agents on both sides of the market may make ex-ante investments so as to enhance their match-relevant characteristics. It is worth noting that models with these features do exist in the frictionless matching literature (e.g. Cole, Mailath, and Postlewaite (2001), Peters (2009) and Felli and Roberts (2016) allow for one-dimensional two-sided heterogeneity, and Dizdar (2015) allows for multidimensional two-sided heterogeneity).<sup>4</sup> Following the latter literature, we consider an environ-

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<sup>3</sup>Acemoglu (1996) studies a search and bargaining model where workers also make ex-ante investments.

<sup>4</sup>The authors study the efficiency and existence properties of competitive equilibria under full information. See also Peters and Siow (2002), who study an environment with non-transferable utility. Recent work by Mailath, Postlewaite and Samuelson (2013, 2016) extends the analysis of Cole, Mailath, and Postlewaite (2001) to an adverse selection environment where the worker’s investments are private information.

ment with transferable utility and complete information about the match-relevant characteristics of the traders.<sup>5</sup> The class of economies we study is large, as we allow for general production, utility and matching functions, general distributions of worker and firm types, and endogenous market participation.

As is standard in the applied labor literature (e.g. studying how labor market outcomes vary across different groups of workers<sup>6</sup>), in our model workers are heterogeneous in several dimensions, and the same is true for the jobs firms create. As emphasized by Lucas (1977), a distinguishing feature of labor markets is that, just like the payoffs of the firms depend on the match-relevant characteristics of the workers they hire, the workers' payoffs depend on the hedonic attributes of the jobs performed. From this perspective, our framework can be viewed as embedding the classical hedonic model of Rosen (1974) and Lucas (1977) into the competitive search formulation.

We consider an environment where workers differ in several exogenous characteristics (e.g. age, gender, innate talent...), and may invest in a multidimensional vector of hedonic attributes (e.g. years and quality of schooling, skills acquired/enhanced through education,...) before they apply for a job. Jobs are modeled as differentiated goods (e.g. indexed by occupation, tasks and skill requirements, working conditions, location and hours...). Prior to hiring their employees, firms make multidimensional investments (e.g. in technology and computerization, plant and equipment,...). These investments affect the value of employing different kinds of workers. In particular, in certain

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<sup>5</sup>See Acemoglu and Shimer (1999a) for a competitive search model with risk aversion, and Peters and Severinov (1997), Faig and Jerez (2005), Guerrieri (2008), Guerrieri, Shimer, and Wright (2010) and Moen and Rosen (2011) for extensions which feature asymmetric information. See also recent work by Fernández-Blanco and Gomes (2016), who consider a version of Shi (2001)'s model where the worker's productivity is unobservable to the firm.

<sup>6</sup>By way of example, it is worth mentioning a few interesting contributions. For instance, Dustmann, Ludsteck, and Schnberg (2009) challenge previous work claiming that (unlike in the US) the wage distribution in Germany was fairly stable in the 1980s and the 1990s, and instead document a substantial increase in wage inequality. The authors decompose the relative importance of the main drivers of the increase in inequality both for the top and the bottom tails of the wage distribution. These include changes in the workforce composition (in terms of age and education), demand factors such as the polarization of labor demand across occupations requiring different skill levels (which is linked to computerization) and changes in labor market institutions. Another well-known paper by Bover, Arellano, and Bentolila (2002) uses a longitudinal sample of Spanish men in 1987-94 to study the influence benefit duration on the exit rate from unemployment into employment, controlling for observed worker characteristics (e.g. education, age, being or not head of the household,...), for unobserved worker heterogeneity, and for sectoral dummy variables. Carrasco and García-Pérez (2015) perform a similar analysis of the Spanish labor market distinguishing between natives and immigrants (by country of origin of the latter). Finally, a recent paper by Burstein, Morales, and Vogel (2015), seeks to identify the main drivers behind the pronounced changes in relative wages across groups of workers with different characteristics in the US in the past decades. The authors develop and estimate a (frictionless) occupational choice model where workers of different gender and education level are assigned across different jobs and equipment types (one of which is computers). Their results point to computerization and to changes in task productivity as the main drivers of changes in between-education-group inequality and the rise of the skill premium. According to their estimates, these factors also explain roughly half of the rise in the female/male wage.

jobs (e.g. involving routine or easy to automate tasks), investments in computerization make it easier to replace workers with machines (as the two production inputs become highly substitutable). If so, workers in turn will benefit from investing in attributes which are complements (rather than substitutes) to the new capital.<sup>7</sup> The firm’s investments may also affect the hedonic attributes of the jobs firms create. For instance, the firms’ investment in technology/equipment can affect the tasks performed by workers,<sup>8</sup> their working conditions (e.g. the degree of health risk implied by the job), and even the job’s location (e.g. in certain occupations, computerization increasing allows working from home).

In our model, the hedonic wage function and the probabilities of filling and finding different jobs are determined endogenously in a competitive search equilibrium. Under the aforementioned symmetric information assumption, we show that competitive search equilibria are constrained efficient, and provide a general existence theorem. In these sense, our results generalize those of Acemoglu and Shimer (1999b), Shi (2001) and Masters (2011) to a large class of competitive search environments. While our presentation focuses on the labor market, the model applies to other hedonic markets (like that for housing) where search frictions are prevalent.<sup>9</sup>

There is an important methodological difference between this paper and the search literature cited above. Whereas this literature uses strategic (game-theoretic) models, here we adopt the Walrasian (price-taking) approach proposed in Jerez (2014). In a nutshell, that paper shows that the Arrow-Debreu competitive equilibrium notion can be extended to environments with search frictions, essentially by replacing market clearing with a trading technology that is not frictionless (e.g. an exogenous matching technology). The key modeling choice is to incorporate the uncertainty arising from rationing in the definition of a commodity (in the spirit of the Arrow-Debreu theory). Prices of commodities then depend not only on their physical characteristics, but also on the probability that their trade is rationed. In a competitive equilibrium traders take prices as given. They also take as given rationing probabilities, because they are part of the description of a commodity. In equilibrium, the price system adjusts so that the optimal decisions of the agents are consistent with

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<sup>7</sup>Deming (2015) finds evidence that in the past decades social skills have been increasingly rewarded by the labor market, specially when combined with cognitive skills.

<sup>8</sup>For instance, Deming (2015) shows that advances in information and communication technology tend to increase job rotation and the degree of worker multitasking.

<sup>9</sup>Houses are differentiated goods that are valued for their hedonic characteristics. Whereas some of these characteristics are exogenous, house owners and landlords oftentimes invest in costly renovation activities which alter the houses’ attributes before they put their property on the market.

the trading technology. Jerez (2014) shows that this price-taking equilibrium notion is a reduced form of the strategic notion used in the literature. The point is made in the context of a related model where all the match-relevant characteristics of the traders are exogenous. Here we extend the Walrasian model in Jerez (2014) to allow for two-sided ex-ante investments which are potentially multidimensional.

The strength of the Walrasian formulation is that it allows us to apply the powerful tools of general equilibrium theory to derive our main results. Specifically, both here and in Jerez (2014), we adopt the linear programming approach used by Shapley (1955), Shapley and Shubik (1972), and Grestky, Ostroy and Zame (1992, 1999) to study frictionless matching models with transferable utility. (See also more recent work by Chiappori, McCann, and Nesheim (2010) and Dizdar (2015), among others). Specifically, we show that constrained efficient allocations can be determined as solutions to a linear programming (LP) problem, whereas the hedonic wage function supporting these allocations and the associated expected payoffs for workers and firms correspond to the solutions of the “dual” of that linear program. We use this characterization to show that competitive search equilibria exist and are constrained efficient. Linear programming techniques have not only proved useful to derive the properties of equilibrium allocations in different settings.<sup>10</sup> They have also been used in empirical work. In particular, Galichon and Salanié (2012), Dupuy and Galichon (2014), and Fox (2010) use these techniques for the identification and estimation of frictionless matching models with transferable utility. This suggests that a similar methodology could be used to take competitive search models to the data at a high degree of disaggregation.

The paper is organized as follows. Section 2 describes the environment. In Section 3 we present the general equilibrium model and define a competitive search equilibrium. For the purpose of the exposition, these sections assume that firms are ex-ante symmetric —an assumption which is widely used in the literature (e.g. Shi (2001)). Section 4 describes the LP problem and its dual, and characterizes constrained efficient allocations via the complementary slackness theorem of linear programming. As in Grestky, Ostroy and Zame (1992, 1999), the welfare and existence theorems follow directly from this theorem. Section 5 presents the general version of the model where firms

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<sup>10</sup>Makowski and Ostroy (1996, 2003) analyze a (frictionless) competitive economy with divisible goods. Myerson (1984) highlights the linear programming structure of principal agent models, a structure which has been exploited by Manelli and Vincent (1995) to characterize optimal procurement mechanisms. Jerez (2003) and Song (2012) exploit this linear structure to analyze competitive economies with informational (but no search) frictions. See also the book by Dorfman, Samuelson, and Solow (1987).

are heterogeneous ex-ante, and shows that it is just a twist of the model presented so far. The LP formulation corresponding to the general model and the technical details are presented in the Appendix.

## 2 The Environment

Consider a one-period search model with a measure  $\hat{\xi} \in \mathfrak{R}_{++}$  of ex-ante symmetric firms,<sup>11</sup> and a continuum of heterogeneous workers. Worker types are indexed by  $s \in S$ , where  $S$  is a compact set. The population of workers is described by a Borel measure on  $S$  with full support:  $\xi^S \in M_+(S)$ . As is customary in the literature, we assume that firms are risk neutral and workers have von Neumann-Morgenstern utility functions which are quasilinear in a divisible numeraire. Also, each firm has a single job opening, and workers can have at most one job.

Prior to entering the labor market, both workers and firms can choose to make investments which affect the output that will be produced when a firm employs a worker. Workers of type  $s$  can invest in a list of attributes  $h \in H$  at cost  $c(s, h)$ . Think of  $h$  as a description of the worker's education (e.g. years and quality schooling, skills acquired/enhanced through education,  $\dots$ ), and of  $s$  as a list of exogenous characteristics, some of which may affect the cost of acquiring education. For instance, workers may differ in their innate talent or in their social background. For some types  $s$ , certain values of  $h$  may be simply unattainable because  $c(s, h)$  is too large. Firms can also make multidimensional ex-ante investments  $a \in A$  (e.g. in a specific technology, equipment type,  $\dots$ ) at cost  $C(a)$ . We assume that  $c(s, h)$  and  $C(a)$  are continuous functions. So, for workers with similar exogenous characteristics, the costs of acquiring similar attributes are similar. And, for a given firm, the costs of making similar investments are also similar. Workers and firms may not invest at all; these choices are denoted by  $h_0 \in H$  and  $a_0 \in A$ , respectively, where  $c(\cdot, h_0) = C(a_0) = 0$ . In what follows, we take  $S, H$  and  $A$  to be arbitrary compact metric spaces.

A firm which invests in  $a$  and hires a worker with attributes  $h$  produces  $f(h, a)$  units of output.<sup>12</sup> We assume that  $f$  is continuous, so firms making similar investments have similar production

<sup>11</sup>In this version of the model, all the attributes of a job are endogenous and will depend on the firm's ex-ante investments (see below).

<sup>12</sup>So output depends on the endogenous investments of the worker and the firm, but does not depend on the worker's type (e.g. see also Cole, Mailath, and Postlewaite (2001)). We could instead assume that output depends on both on the worker's type and on the attributes she has acquired (e.g. see Felli and Roberts (2016)). In general,

technologies, and workers with similar attributes have similar productivity at a given job.

The worker's disutility of labor depends on the job's hedonic attributes, which we assume are determined by  $a$ . This disutility, which we denote by  $v(s, h, a)$ , may also depend on the worker's exogenous characteristics  $s$  and/or endogenously acquired attributes  $h$ . For instance, as a recent article in *The Economist* notes, blue-collar male workers are usually reluctant to work in certain jobs, such as health workers or hairdressers, which have been traditionally regarded as feminine.<sup>13</sup> Another illustrative example is that of female workers who have children, who oftentimes have a preference for jobs which are compatible with child rearing (e.g. teaching and part-time jobs, and jobs that allow to work from home).<sup>14</sup> We assume that the function  $v$  is continuous, so the disutility of labor at a given job is similar for workers whose types and attributes are similar. We normalize the disutility of the unemployed to zero. We shall assume that, for each choice  $a \in A$  firms can make, there is a worker type  $s \in S$  and a choice of  $h \in H$  by that worker type that generates a positive bilateral surplus:  $f(h, a) - C(a) - c(s, h) - v(s, h, a) > 0$ .

The ex-ante investments of workers and firms are assumed observable, so firms know the kind of labor they are hire and workers know the kind of jobs they accept. As in the Arrow-Debreu model, the payoffs of the agents are private information, and all the relevant information is transmitted through wages.

Following Lucas (1977), the jobs created in matches where the jobs' attributes  $a$  and/or the worker's attributes  $h$  differ are regarded as differentiated.<sup>15</sup> These differentiated jobs will be created in different (segmented) labor markets.<sup>16</sup> Many differentiated jobs, indexed by  $(h, a) \in H \times A$ , can then potentially be created in our model. Whether or not type- $(h, a)$  jobs are created (i.e., some workers choose  $h$ , some firms choose  $a$ , and some of these workers end up working for some of these firms) will be determined endogenously in equilibrium. The reader may want to think of the special

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some types may be more productive at certain jobs. (For instance, Deming (2015) mentions the female advantage in social skills, and conjectures that it is related with the fall in the wage gender gap.) In our model these differences are embedded in the cost  $c(h, s)$  of acquiring attributes that are required for performing that job effectively. In any case, all our results go through when output is given by  $f(s, h, a)$  provided the components of  $s$  that affect  $f$  are observable.

<sup>13</sup>See article "The Weaker Sex", May 30th 2015.

<sup>14</sup>See Erosa, Fuster, Kambourov, and Rogerson (2016).

<sup>15</sup>Admittedly, for some low-skilled jobs, the worker's investments may not matter in terms of output:  $f(h, a) = f(a)$ . It is easy to devise a simple variant of the model where these jobs are described by  $a$  only.

<sup>16</sup>This is a standard feature of directed search (as opposed to random search) models. See, for instance, Shi (2001) and Menzio and Shi (2010). These authors treat the worker's attributes as exogenous. In particular, in Shi's model, output is given by  $f(a, s)$  and a job is described by a pair  $(a, s)$ , where both  $a$  and  $s$  are one-dimensional objects.

case where the set of jobs that can be created is finite. Indeed, in empirical applications  $S$ ,  $H$  and  $A$  will be finite sets (i.e.,  $s$  will be a finite list of characteristics, each of which can take a finite number of values, and so will  $h$  and  $a$ ).<sup>17</sup>

The trading process in the labor market is characterized by search frictions, meaning that not all workers searching for a job will find one and the same is true for firms who seek to fill a vacancy. Moreover, job filling and job finding probabilities will vary across jobs (e.g. some jobs may be harder to find/fill than others). This is in contrast to a Walrasian labor market where workers and firms can trade instantaneously at the wages that clear the market.

We assume that firms who seek to fill vacancies of type- $(h, a)$  and workers searching for these kind of jobs meet bilaterally and at random. It is standard in the literature to describe this random meeting process by an exogenous matching function with constant returns to scale. To ease notation, suppose that the matching function is the same in all markets. Let  $\beta$  be the measure of firms and  $\sigma$  the measure of workers in a given market. The matching function  $\mathcal{M}(\beta, \sigma)$  determines the total measure of bilateral matches, where  $\mathcal{M} : \mathbb{R}_+^2 \rightarrow \mathbb{R}_+$  is continuous, increasing, and homogeneous of degree one. Since the total number of matches cannot exceed the number of traders in the short side of the market,  $\mathcal{M}(\beta, \sigma) \leq \min\{\beta, \sigma\}$ . In particular,  $\mathcal{M}(0, \sigma) = \mathcal{M}(\beta, 0) = 0$  (e.g. if no workers invest in  $h$  and/or no firms invest in  $a$ , no jobs of this type will be created).

We assume that the Law of Large Numbers holds, so the probability that a worker finds a job in this market is

$$m(\theta) = \frac{\mathcal{M}(\beta, \sigma)}{\sigma} = \mathcal{M}(\theta, 1), \quad (2.1)$$

where  $\theta = \frac{\beta}{\sigma} \in \mathbb{R}_+$  is the ratio of firms to workers searching in the market, or the level of market tightness. Likewise, the probability that a firm finds an employee in this market is

$$\alpha(\theta) = \frac{\mathcal{M}(\beta, \sigma)}{\beta} = \mathcal{M}(1, \theta^{-1}) = m(\theta)\theta^{-1}, \quad (2.2)$$

with the convention that  $\alpha(0) = \lim_{\theta \rightarrow 0} \alpha(\theta)$ . The function  $m(\theta)$  is continuous and increasing, whereas  $\alpha(\theta)$  is continuous and decreasing. It is standard to assume that  $m(0) = 0$ ,  $\lim_{\theta \rightarrow \infty} m(\theta) = 1$ ,  $\alpha(0) = 1$  and  $\lim_{\theta \rightarrow \infty} \alpha(\theta) = 0$ . Intuitively, the higher the market tightness  $\theta$ , the easier it is

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<sup>17</sup>In this case, the continuity of  $c$ ,  $C$ ,  $f$  and  $v$  holds trivially.

for workers to find a job and the harder it is for firms to fill a job in a given market. As  $\theta$  goes to infinity (zero), the job finding probability goes to one (zero) and the job filling probability goes to zero (one). Note that  $\alpha(\theta)$  and  $m(\theta)$  also represent the fractions of firms and workers in the market who are successful in their search for an employment relationship, whereas  $1 - \alpha(\theta)$  and  $1 - m(\theta)$  are the fractions of firms and workers who are rationed.

Our analysis and all our results extend directly to the case where the matching function differs across jobs. For instance, the use of digital platforms –which allow to match workers with jobs more efficiently– is more widespread in some job markets than in others (e.g. that for computer engineers vs. that for machine operators or domestic workers). In this more general case, the total number of bilateral matches in the market for type- $(h, a)$  jobs is given by  $\hat{\mathcal{M}}(\beta, \sigma; h, a)$ , where  $\hat{\mathcal{M}}$  be continuous (so the random matching process for similar jobs is similar), and where, for a given  $(h, a)$ ,  $\hat{\mathcal{M}}(\cdot, \cdot; h, a)$  has the properties stated above.

### 3 The general equilibrium model

In this section we show that constrained efficient allocations can be characterized as solutions to a LP problem. We then define a competitive (price-taking) search equilibrium. The analysis follows closely that in Jerez (2014). As we have already noted, in that paper the characteristics of the traders on both sides of the market are exogenous. Also, in Jerez (2014) buyers have different valuations depending on the type of seller they trade with (because different seller types offer different goods), but the valuations of the sellers do not depend on the characteristics of the buyers they trade with. By contrast, in the current model, just like the firm’s payoff depends on the worker’s attributes, the worker’s payoff depend on the attributes of the job performed. As we have already noted, this is a distinguishing feature of labor markets.

#### *Allocations*

An allocation is an assignment of workers and firms to different markets. We represent a market by a triple  $(h, a, \theta) \in H \times A \times \mathfrak{R}_+$ , describing the type of job created  $(h, a)$  and the tightness level  $\theta$  prevailing in the market. Consistency requires that only workers investing in  $h$  and firms investing in  $a$  are assigned to market  $(h, a, \theta)$  (see below). Recall that the value of  $\theta$  determines

the probabilities of finding and filling a job, and thus the degree of “trading uncertainty” that workers and firms face in this market. In the spirit of Arrow-Debreu theory, our description of the commodities traded in the different markets,  $(h, a, \theta) \in H \times A \times \mathfrak{R}_+$ , includes this trading uncertainty. To allow for the possibility of non-participation, we introduce an arbitrary real number  $\theta_0 < 0$ , and extend the probability functions in (2.1) and (2.2) to the set  $\Theta \equiv \mathfrak{R}_+ \cup \{\theta_0\}$  by setting  $m(\theta_0) = \alpha(\theta_0) = 0$ . Agents who do not participate in the labor market will be assigned to the “fictitious market”  $x_0 = (h_0, a_0, \theta_0)$ .<sup>18</sup>

As is standard in general equilibrium models with a continuum of heterogeneous agents, we use measures to describe allocations. We first introduce some necessary notation. For a given a metric space  $Y$ , let  $M_c(Y)$  denote the space of signed regular Borel measures on  $Y$  with compact support, endowed with the weak-star topology. If  $Y$  is compact, this is just the space  $M(Y)$  of signed regular Borel measures on  $Y$ . We denote the support of a measure  $\nu \in M_c(Y)$  by  $\text{supp}\nu$ .

Let  $X \equiv (H \times A \times \mathfrak{R}_+) \cup \{x_0\}$  be the set of all markets that can potentially be active (including the fictitious one). An allocation is described by a pair of measures,  $(\mu^{\mathcal{B}}, \mu^{\mathcal{S}}) \in M_{c+}(X) \times M_{c+}(S \times X)$ . The measure  $\mu^{\mathcal{B}}$  describes the assignment of firms across markets, whereas  $\mu^{\mathcal{S}}$  describes the corresponding assignment for workers. That is,  $\mu^{\mathcal{B}}(\Omega)$  is the measure of firms assigned to an arbitrary Borel set of markets  $\Omega \subseteq H \times A \times \mathfrak{R}_+$  (i.e., the aggregate labor demand in these markets). Similarly,  $\mu^{\mathcal{S}}(D \times \Omega)$  is the measure of workers with types  $s$  in the Borel set  $D \subseteq S$  who are assigned to a market in  $\Omega$  (the total supply of labor by workers of type  $s \in D$  in these markets). On the other hand,  $\mu^{\mathcal{S}}(D \times \{x_0\})$  (respectively,  $\mu^{\mathcal{B}}(\{x_0\})$ ) is the measure of these workers (the measure of firms) who do not participate.

The marginals of  $\mu^{\mathcal{S}}$  on  $S$  and  $X$ , which we denote by  $\mu_S^{\mathcal{S}} \in M_+(S)$  and  $\mu_X^{\mathcal{S}} \in M_{c+}(X)$ , provide additional information about the allocation. On the one hand,  $\mu_X^{\mathcal{S}}(\Omega)$  is the total measure of workers assigned under the allocation to market in the set  $\Omega$  (i.e., the aggregate labor supply in these markets). On the other hand,  $\mu_S^{\mathcal{S}}(D)$  is the total measure of workers with types  $s \in D$  who are assigned to an element of  $X$  under the allocation.<sup>19</sup>

With this description, the set of markets where firms participate is given by the support of

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<sup>18</sup>We assume that agents who do not participate in the market make no investments. Clearly, it would be suboptimal for them to do so, because their expected payoff would be negative. Since we focus on constrained efficient allocations, this is without loss of generality.

<sup>19</sup>The marginal of  $\mu^{\mathcal{S}}$  on  $H$  and the marginal of  $\mu^{\mathcal{B}}$  on  $A$  in turn describe the total measure of workers and firms investing in a given set of attributes.

$\mu^{\mathcal{B}}$ , whereas the set of markets where workers participate is given by the support of  $\mu_X^{\mathcal{S}}$ . We say that a market  $(h, a, \theta)$  is active under the allocation if it attracts both workers and firms:  $(h, a, \theta) \in \text{supp}\mu^{\mathcal{B}} \cap \text{supp}\mu_X^{\mathcal{S}}$ .

### *Feasible allocations*

Feasible allocations assign (almost) all the workers and firms in the economy to an element of  $X$ . That is, the total measure of firms assigned under the allocation must be equal to  $\hat{\xi}$ :

$$\mu^{\mathcal{B}}(X) = \hat{\xi}, \quad (3.1)$$

whereas the total measure of assigned workers of a given type  $s$  must be equal to the measure of such types who are present in the population:

$$\mu_S^{\mathcal{S}}(D) = \xi^{\mathcal{S}}(D) \text{ for all Borel } D \subseteq S; \quad (3.2)$$

or, in short,  $\mu_S^{\mathcal{S}} = \xi^{\mathcal{S}}$ .

Feasible allocations must also be consistent with respect to the matching technology, so that in each active market the measure of workers who find jobs is equal to the measure of job vacancies filled by firms:

$$\int_{(h,a,\theta) \in \Omega} \alpha(\theta) d\mu^{\mathcal{B}}(h, a, \theta) = \int_{(h,a,\theta) \in \Omega} m(\theta) d\mu_X^{\mathcal{S}}(h, a, \theta) \quad \text{for all Borel } \Omega \subseteq H \times A \times \mathfrak{R}_+. \quad (3.3)$$

Recall that  $\mu^{\mathcal{B}}$  describes the measure of firms assigned to the different markets, and that the fraction of firms who fill a vacancy in market  $(h, a, \theta)$  is equal to  $\alpha(\theta)$ . The term on the left-hand side of (3.3) thus represents the measure of firms which successfully fill a vacancy in an arbitrary set of markets  $\Omega \subset H \times A \times \mathfrak{R}_+$  (and thus the measure of jobs that are created in those markets). Likewise, since  $\mu_X^{\mathcal{S}}$  describes the measure of workers assigned to the different markets and  $m(\theta)$  is the fraction of workers who find a job in market  $(h, a, \theta)$ , the right-hand side of (3.3) represents the measure of workers who find a job in the set of markets  $\Omega$ .

### *Constrained efficient allocations*

In the presence of search frictions the social planner can assign workers and firms to different markets, but cannot directly assign workers to firms (as in frictionless matching models). Formally, the problem of the planner is to choose a feasible allocation so as to maximize social welfare,

$$\int_X [\alpha(\theta)f(h, a) - C(a)]d\mu^{\mathcal{B}}(h, a, \theta) - \int_{S \times X} [m(\theta)v(s, h, a) + c(s, h)]d\mu^{\mathcal{S}}(s, h, a, \theta). \quad (3.4)$$

The first term in (3.4) measures the economy's aggregate output net of the total cost of the firms' ex-ante investments. Again,  $\mu^{\mathcal{B}}$  describes the measure of firms who are assigned to each market. A fraction  $\alpha(\theta)$  of the firms in market  $(h, a, \theta)$  create jobs and produce  $f(h, a)$  units of output; the rest are rationed and produce nothing. Also, whether or not they end up creating a job, all the firms in this market face the cost  $C(a)$  of their ex-ante investments. The first term in (3.4) aggregates these outputs and costs across all active markets. The second term in (3.4) is interpreted similarly. Recall that  $\mu^{\mathcal{S}}$  describes the measure of workers of a given type  $s$  who are assigned to each market  $(h, a, \theta)$ . All these workers face the cost  $c(s, h)$  of their ex-ante investments. Yet only a fraction  $m(\theta)$  of them find a job, their disutility of labor being  $v(s, h, a)$ . (Recall that the disutility of the unemployed is zero). The second term in (3.4) aggregates the disutilities of the workers and the costs of their ex-ante investments across all markets. Therefore, the expression in (3.4) represents the total welfare gains implied by the allocation.

As in Jerez (2014), the feature of the model we shall emphasize and exploit is the fact that the planner's objective function (3.4) and the feasibility constraints (3.1)–(3.3) are all *linear* in the allocation  $(\mu^{\mathcal{B}}, \mu^{\mathcal{S}})$ . Note that all the (non-constant) terms in these equations are effectively integrals with respect to the measures  $\mu^{\mathcal{B}}$  and/or  $\mu^{\mathcal{S}}$ . (In particular, the marginals  $\mu_S^{\mathcal{S}}$  and  $\mu_X^{\mathcal{S}}$  are defined as integrals with respect to  $\mu^{\mathcal{S}}$ .) Before laying out the details of the linear programming formulation, we define a competitive search equilibrium.

### *Competitive search equilibrium*

Let  $w(h, a, \theta) \in \mathfrak{R}_+$  denote the wage in market  $(h, a, \theta) \in H \times A \times \mathfrak{R}_+$ . As in classical hedonic models, *wages depend both on the attributes of the worker and the attributes of the job*. Moreover, as in competitive search models, *wages also depend on the market tightness level*. The dependence of wages on  $\theta$  is intuitive. When  $\theta$  is lower, firms are more likely to fill a job vacancy, so (other

things equal) they are willing to pay higher wages to complete a transaction. Similarly, since workers are more likely to find a job when  $\theta$  is higher, they are also willing to accept lower wages under these circumstances. It is convenient to extend the hedonic wage function  $w$  to the fictitious market  $x_0$  by setting  $w(x_0) = 0$ . We follow Mas-Colell's (1975) description of the price system for economies with a continuum of differentiated commodities, and assume that wages are described by a continuous function  $w \in C_+(X)$ . The continuity assumption implies that markets for similar jobs where tightness levels are also similar have similar wages.

In a competitive search equilibrium workers and firms take wages as given. In addition, all agents have rational expectations about the tightness level  $\theta$  prevailing in each market, and hence about the probability with which they will complete a transaction if they choose to enter those markets. The expected profit of a firm which first invests in attributes  $a$  and then enters market  $(h, a, \theta)$  is

$$\pi(h, a, \theta; w) = \alpha(\theta) [f(h, a) - w(h, a, \theta)] - C(a). \quad (3.5)$$

Prior to entering the market, the firm pays the cost of investing in  $a$ . Once in the market, the firm fills a job with probability  $\alpha(\theta)$ , in which case it produces  $f(h, a)$  units of output and pays the wage  $w(h, a, \theta)$ . With complementary probability, the firm remains inactive.

The expected utility of a type- $s$  worker who invests in attributes  $h$  and then enters market  $(h, a, \theta)$  is

$$u(s, h, a, \theta; w) = m(\theta) [w(h, a, \theta) - v(s, h, a)] - c(s, h) \quad (3.6)$$

The worker first pays the cost of investing in  $h$ . Once she enters the market, she finds a job with probability  $m(\theta)$ , in which case she receives the wage net of the disutility of working. With complementary probability, the worker is unemployed and his (ex post) utility is zero.

**Definition 1.** *A competitive search equilibrium is a feasible allocation  $(\mu^{B*}, \mu^{S*}) \in M_{c+}(X) \times M_{c+}(S \times X)$  and a wage function  $w^* \in C_+(X)$  such that:*

(i) Firms choose  $(h, a, \theta)$  in  $X$  to maximize their expected profits taking  $w^*$  as given:

$$\Pi(w^*) \equiv \sup_{(h,a,\theta) \in X} \pi(h, a, \theta; w^*) = \pi(h_f^*, a_f^*, \theta_f^*; w^*), \quad (3.7)$$

for almost all  $(h_f^*, a_f^*, \theta_f^*) \in \text{supp}\mu^{\mathcal{B}^*}$ .

(ii) For each type  $s \in S$ , workers choose  $(h, a, \theta)$  in  $X$  to maximize their expected utility taking  $w^*$  as given:

$$v_s(w^*) \equiv \sup_{(h,a,\theta) \in X} u(s, h, a, \theta; w^*) = u(s, h_s^*, a_s^*, \theta_s^*; w^*), \quad (3.8)$$

for almost all  $(s, h_s^*, a_s^*, \theta_s^*) \in \text{supp}\mu^{\mathcal{S}^*}$ .

Condition (i), together with the feasibility condition (3.1), implies that all the firms in the economy choose their ex-ante investments and the markets they enter in order to maximize their expected profits taking wages as given. Note that market  $(h, a, \theta)$  attracts some firms in equilibrium whenever  $(h, a, \theta) \in \text{supp}\mu^{\mathcal{B}^*}$ . Also, since they are ex-ante symmetric, firms make the same profits in all active markets. The firms' equilibrium profits are denoted by  $\Pi(w^*)$ . Condition (ii), combined with the feasibility condition (3.2), is a similar optimization condition for each type of worker. In this condition,  $v_s(w^*)$  denotes the equilibrium indirect utility of a type- $s$  worker. Market  $(h, a, \theta)$  attracts type- $s$  workers in equilibrium whenever  $(s, h, a, \theta) \in \text{supp}\mu^{\mathcal{S}^*}$ . Some workers/firms may not to participate at all if that is optimal for them.

Finally, the rational expectations conditions on the agents' beliefs follows from the aggregate feasibility condition (3.3) imposed on the equilibrium allocation. Note that (3.3) can be written as

$$\int_{(h,a,\theta) \in \Omega} d\mu^{\mathcal{B}}(h, a, \theta) = \int_{(h,a,\theta) \in \Omega} \theta d\mu_X^{\mathcal{S}}(h, a, \theta) \quad \text{for all Borel } \Omega \subseteq H \times A \times \mathfrak{R}_+, \quad (3.9)$$

using (2.2), since  $\alpha(\theta) > 0$  for all  $\theta \in \mathfrak{R}_+$ . Condition (3.9) says that the total measures of workers and firms who enter each active market in equilibrium generate the market tightness levels that the traders take as given when they choose which market to join (as in directed search models). Take, for instance, an allocation which implies an atomless assignment of workers and firms across markets. In this case,  $d\mu^{\mathcal{B}}(h, a, \theta)$  is the density of firms and  $d\mu_X^{\mathcal{S}}(h, a, \theta)$  is the density of workers

in the set of active markets. If the traders' conjectures about the buyer-seller ratio  $\theta$  are correct,  $d\mu^{\mathcal{B}}(h, a, \theta)$  should be equal to  $\theta d\mu_X^{\mathcal{S}}(h, a, \theta)$  in this set. This is what equation (3.9) says.<sup>20</sup> The same interpretation applies if  $\mu^{\mathcal{B}}$  and  $\mu^{\mathcal{S}}$  have a mass point at  $(h, a, \theta)$ , except that in this case we talk about positive masses rather than densities. Note that, if a given market attracts no traders, (3.9) is vacuous since  $d\mu^{\mathcal{B}}(h, a, \theta) = d\mu_X^{\mathcal{S}}(h, a, \theta) = 0$ . In other words, (3.9) is a restriction on active markets only.

A standard feature of general equilibrium models with a continuum of commodities is that the prices of those commodities which are not traded in equilibrium are indeterminate (e.g. see Mas-Colell and Zame (1991) and Gretskey, Ostroy, and Zame (1999)). Note that, if a market is active in our model, so  $(h, a, \theta) \in \text{supp}\mu^{\mathcal{B}} \cap \text{supp}\mu_X^{\mathcal{S}}$ , the firm's optimality condition implies:

$$q^f = \alpha(\theta) [f(h, a) - w^*(h, a, \theta)] - C(a). \quad (3.10)$$

Also, there is some buyer type  $\tilde{s}$  who finds it optimal to invest in  $h$  and join this market:

$$q^{\mathcal{S}}(\tilde{s}) = m(\theta) [w^*(h, a, \theta) - v(\tilde{s}, h, a)] - c(\tilde{s}, h). \quad (3.11)$$

Thus  $w^*(h, a, \theta)$  jointly satisfies (3.10) and (3.11).

On the other hand, for those markets which are inactive, so  $(h, a, \theta) \notin \text{supp}\mu^{\mathcal{B}} \cap \text{supp}\mu_X^{\mathcal{S}}$ ,  $w^*(h, a, \theta)$  satisfies:

$$q^f \geq \alpha(\theta) [f(h, a) - w^*(h, a, \theta)] - C(a), \quad (3.12)$$

$$q^{\mathcal{S}}(s) \geq m(\theta) [w^*(h, a, \theta) - v(s, h, a)] - c(s, h), \quad \forall s \in \mathcal{S}, \quad (3.13)$$

the weak inequality signs being strict in the case of markets which firms (workers) strictly prefer not to join. Hence,  $w^*(h, a, \theta)$  satisfies:<sup>21</sup>

$$f(h, a) - \frac{C(a) + q^f}{\alpha(\theta)} \leq w^*(h, a, \theta) \leq \inf_{s \in \mathcal{S}} \left\{ v(s, h, a) + \frac{q^{\mathcal{S}}(s) + c(s, h)}{m(\theta)} \right\}. \quad (3.14)$$

<sup>20</sup>Formally, (3.9) says that the restriction of  $\mu^{\mathcal{B}}$  to  $H \times A \times \mathfrak{R}_+$  is absolutely continuous with respect to the restriction of  $\mu_X^{\mathcal{S}}$  to the same set, the corresponding Radon-Nikodym derivative being  $f(h, a, \theta) = \theta$ .

<sup>21</sup>We are abusing notation slightly here since, strictly speaking, condition (3.12) applies only when  $\theta > 0$  (because  $\alpha(0) = 0$ ). Yet when  $\theta = 0$ , (3.12) holds trivially, so the lower bound on (3.12) can be ignored.

The term in the left-hand side of (3.14) is the highest wage firms would be willing to pay to participate in this market in equilibrium. The term in the right-hand side of (3.14) is the lowest wage that a worker would accept to participate in this market in equilibrium. A market is inactive whenever the latter term exceeds the former (since opening such a market would imply negative gains from trade). In this case, the wage in this market is indeterminate. It could be as low as  $f(h, a) - \frac{C(a)+q^f}{\alpha(\theta)}$ , as high as  $\inf_{s \in S} \{v(s, h, a) + \frac{q^S(s)+c(s,h)}{m(\theta)}\}$ , or anything in between. All these prices are consistent with the market being inactive.<sup>22</sup> It is common in the general equilibrium literature to use conventions which allow to select a unique supporting price system. We present an example in Section 5.1 below.

## 4 Welfare and existence theorems via Linear programming

The purpose of this section is to bring to light the connection between the planner's LP problem and competitive search equilibria. In doing so, we follow Gretsky, Ostroy and Zame (1992, 1999) and Makowski and Ostroy (1996). We begin by exploiting a well-known duality result in mathematics. Namely, the fact that any linear programming problem has a *dual* problem, which is also linear. Whereas the planner's problem is a maximization problem which will be referred to as the *primal* problem, its dual is a minimization problem. The two problems are related because, as we shall see, (a) their optimal values coincide and (b) the dual variables are the shadow prices of the primal constraints and vice versa.<sup>23</sup>

The primal problem is to find  $(\mu^B, \mu^S) \in M_{c+}(X) \times M_{c+}(S \times X)$  to solve

$$(P) \quad \sup \int_X [\alpha(\theta)f(h, a) - C(a)]d\mu^B(h, a, \theta) - \int_{S \times X} [m(\theta)v(s, h, a) + c(s, h)]d\mu^S(s, h, a, \theta)$$

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<sup>22</sup>A related issue arises in directed search models where out-of-equilibrium beliefs are indeterminate (see Peters (1997)). These models impose refinements to pin down the equilibrium beliefs. For a detailed discussion between the relation between the two equilibrium concepts and the two indeterminacies, see Jerez (2014).

<sup>23</sup>It is well-known in the mathematical optimization literature that, unlike a finite dimensional LP problem, an infinite dimensional LP problem and its dual need not have the same value, and that optimal solutions to these problems may fail to exist. Below we show that the problems in this paper are well-behaved.

s. t.

$$\mu^{\mathcal{B}}(X) = \hat{\xi}, \quad (4.1)$$

$$\mu_S^{\mathcal{S}} = \xi^{\mathcal{S}}, \quad (4.2)$$

$$\int_{(h,a,\theta) \in \Omega} \alpha(\theta) d\mu^{\mathcal{B}}(h, a, \theta) = \int_{(h,a,\theta) \in \Omega} m(\theta) d\mu_X^{\mathcal{S}}(h, a, \theta) \quad \text{for all Borel } \Omega \subset X, \quad (4.3)$$

$$\mu^{\mathcal{B}}, \mu^{\mathcal{S}} \geq 0. \quad (4.4)$$

There is a slight difference between problem (P) and the planner's problem described in Section 3, in that the constraint system (4.3) extends the matching condition (3.3) to the fictitious market  $x_0$ . Replacing (3.3) with (4.3) is convenient for our purposes and does not change the problem. Since  $m(\theta_0) = \alpha(\theta_0) = 0$ , (4.3) is vacuous when  $\theta = \theta_0$ .

Let  $q^f \in \mathfrak{R}$  denote the dual variable associated with constraint (4.1). In the Appendix we show that the dual variables associated with the constraint systems (4.2) and (4.3) are given by two continuous functions:  $q^{\mathcal{S}} \in C(S)$  and  $w \in C(X)$ , respectively.<sup>24</sup> As we shall see, the dual variable  $q^f$  measures the shadow value of having (a small mass of) additional firms enter the economy, whereas  $q^{\mathcal{S}}(s)$  is the corresponding shadow value for workers of type- $s$ . In the terminology of Ostroy (1980, 1984) and Makowski (1980),  $q^f$  represents the “marginal product” (or marginal contribution to social welfare) of a firm, and  $q^{\mathcal{S}}(s)$  is the “marginal product” of a type- $s$  worker.

We abuse notation slightly by denoting the dual variable associated with (4.3) by  $w$ , which is also how we denote the market wage function. This is to emphasize the relationship between both functions. Recall that condition (4.3) says that the total number of jobs created in each market is equal to the total number of workers who find a job in that market. In our model, the dual variable  $w(h, a, \theta)$  measures the shadow value of a job when the worker has attributes  $h$ , the employer has attributes  $a$  and the job is created in market with tightness  $\theta$ . In particular, the presence of search frictions implies that the shadow price of a job (just like market wages) depends not only on the job's description but also on corresponding the market tightness level.

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<sup>24</sup>  $q^{\mathcal{S}}$  and  $w$  lie in the topological duals of the spaces  $M(S)$  and  $M_c(X)$  (as these are the spaces where the measures  $\xi^{\mathcal{S}}$  and  $\mu^{\mathcal{B}}$  lie). Since  $C(S)$  is endowed with the uniform norm topology and  $C(X)$  is endowed with the topology of uniform convergence on compact sets,  $M(S)$  and  $M_c(X)$  are the respective topological duals of these spaces. When  $M(S)$  and  $M_c(X)$  are endowed with the weak-star topology, the converse statement also holds.

The Lagrangian associated with problem (P) is

$$\begin{aligned}\mathcal{L} &= \int_X [\alpha(\theta)f(h, a) - C(a)]d\mu^{\mathcal{B}}(h, a, \theta) - \int_{S \times X} [m(\theta)v(s, h, a) + c(s, h)]d\mu^{\mathcal{S}}(s, h, a, \theta) \\ &+ q^f [\hat{\xi} - \int_X d\mu^{\mathcal{B}}(h, a, \theta)] + \int_S q^{\mathcal{S}}(s) [d\xi^{\mathcal{S}}(s) - d\mu_S^{\mathcal{S}}(s)] \\ &+ \int_X w(h, a, \theta)m(\theta)d\mu_X^{\mathcal{S}}(h, a, \theta) - \int_X w(h, a, \theta)\alpha(\theta)d\mu^{\mathcal{B}}(h, a, \theta),\end{aligned}$$

where we set  $w(x_0) = 0$  without loss of generality (since  $m(\theta_0) = \alpha(\theta_0) = 0$ ). The Lagrangian can be rearranged as follows:

$$\begin{aligned}\mathcal{L} &= q^f \hat{\xi} + \int_S q^{\mathcal{S}}(s)d\xi^{\mathcal{S}}(s) - \int_X \left( q^f - \alpha(\theta)[f(h, a) - w(h, a, \theta)] + C(a) \right) d\mu^{\mathcal{B}}(h, a, \theta) \\ &- \int_{S \times X} (q^{\mathcal{S}}(s) - m(\theta)[w(h, a, \theta) - v(s, h, a)] + c(s, h)) d\mu^{\mathcal{S}}(s, h, a, \theta).\end{aligned}$$

Finally, we may use the payoff functions of firms and workers,  $\pi(h, a, \theta; w)$  and  $u(s, h, a, \theta; w)$ , defined in (3.5) and (3.6) to write:

$$\begin{aligned}\mathcal{L} &= q^f \hat{\xi} + \int_S q^{\mathcal{S}}(s)d\xi^{\mathcal{S}}(s) - \int_X [q^f - \pi(h, a, \theta; w)]d\mu^{\mathcal{B}}(h, a, \theta) \\ &- \int_{S \times X} [q^{\mathcal{S}}(s) - u(s, h, a, \theta; w)] d\mu^{\mathcal{S}}(s, h, a, \theta).\end{aligned}\tag{4.5}$$

The dual problem (D) is to find  $(q^f, q^{\mathcal{S}}, w) \in \mathfrak{R} \times C(S) \times C(X)$  to solve

$$(D) \quad \inf \quad q^f \hat{\xi} + \int_S q^{\mathcal{S}}(s)d\xi^{\mathcal{S}}(s)$$

s.t.

$$q^f \geq \pi(h, a, \theta; w) \quad \text{for all } (h, a, \theta) \in X,\tag{4.6}$$

$$q^{\mathcal{S}}(s) \geq u(s, h, a, \theta; w) \quad \text{for all } (s, h, a, \theta) \in S \times X.\tag{4.7}$$

It is easy to see that problem (D) is linear, once it is noted that—because utility is transferable— $w$  enters linearly the payoff functions  $\pi(h, a, \theta)$  and  $u(s, h, a, \theta)$ . The linear constraint systems (4.6) and (4.7) have the following interpretation. The term in the right-hand side of (4.6) gives the firms' expected profits in each market  $(h, a, \theta) \in X$  as a function of the shadow price of the jobs created

in that market,  $w(h, a, \theta)$ . Constraint (4.6) says that  $q^f$  ought to be an upper bound for the firms' expected profits in all markets. Similarly, (4.7) says that  $q^S(s)$  is an upper bound for the expected utility of a type- $s$  worker in the different markets (given the shadow price of the jobs created in those markets).

The above implies that the dual constraint systems (4.6) and (4.7) can also be written as

$$q^f \geq \Pi(w) \tag{4.8}$$

$$q^S(s) \geq v_s(w) \quad \text{for all } s \in S, \tag{4.9}$$

where  $\Pi(w)$  is the firms' profit function, and  $v_s(w)$  is the indirect utility function of type- $s$  workers in Definition 2. Because the objective of problem (D) is to minimize  $q^f \hat{\xi} + \int_S q^S(s) d\xi^S(s)$ , (4.8) and (4.9) bind at an optimum. Thus, the optimal value of  $w$  minimizes the sum of firms' profits and the indirect utilities of all the workers who live in the economy:

$$w^\circ = \arg \left\{ \min_{w \in C(X)} \left[ q^f \Pi(w) + \int_S v_s(w) d\xi^S(s) \right] \right\}. \tag{4.10}$$

The optimal values of  $q^f$  and  $q^S$  in turn give firms' profits and the workers' indirect utilities at prices  $w^\circ$ :  $q^{f^\circ} = \Pi(w^\circ)$  and  $q^{S^\circ}(s) = v_s(w^\circ)$ .<sup>25</sup>

Denote the optimal values for problems (P) and (D) by  $\nu(P)$  and  $\nu(D)$ , respectively. Theorem 1 states that these problems have optimal solutions and the same optimal value. This in turn implies that the dual variables are indeed the shadow prices of the primal constraints, and vice versa.

**Theorem 1.** *Problems (P) and (D) have optimal solutions, and  $\nu(P) = \nu(D)$ .*

We may then appeal to the complementary slackness theorem of linear programming to characterize optimal solutions for problems (P) and (D) (see Anderson and Nash 1987, Theorem 3.2).

**Theorem 2.** *(Complementary Slackness Theorem) Feasible solutions  $(\mu^B, \mu^S)$  and  $(q^f, q^S, w)$  for*

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<sup>25</sup>Makowski and Ostroy (1996) show how the fact that the constraints of the dual problem can be incorporated into the objective function is characteristic of the LP version of general equilibrium. They make this point in the context of a frictionless exchange economies with transferable utility and a finite number of divisible goods. In the Appendix, we show that feasible dual solutions satisfy  $q^f \geq 0$  and  $q^S \geq 0$ , so individual rationality holds (see Lemma A.2). This is indeed a general property of the LP formulation of Makowski and Ostroy (1996) and Gretskey, Ostroy and Zame (1992, 1999), where individual rationality constraints appear in the dual (rather than in the primal problem). We also show that  $w \geq 0$  without loss of generality.

problems (P) and (D) are optimal if and only if they satisfy the complementary slackness conditions:

$$q^f = \pi(h_f, a_f, \theta_f; w) \quad \text{for almost all } (h_f, a_f, \theta_f) \in \text{supp}\mu^{\mathcal{B}}, \quad (4.11)$$

$$q^S(s) = u(s, h_s, a_s, \theta_s; w) \quad \text{for almost all } (s, h_s, a_s, \theta_s) \in \text{supp}\mu^{\mathcal{S}}. \quad (4.12)$$

As in Gretsky, Ostroy and Zame (1992) and Makowski and Ostroy (1996), the complementary slackness theorem allows to establish the equivalence between constrained efficient allocations and competitive equilibria. Specifically, Theorem 2 implies that equilibrium allocations coincide with the optimal solutions to problem (P), whereas the firms' profits, the workers' indirect utilities and the wages in equilibrium coincide with the optimal solutions to problem (D).

**Theorem 3.** (*Decentralization of constrained efficient allocations*)

- (I) Let  $(\mu^{\mathcal{B}*}, \mu^{\mathcal{S}*}, w^*)$  be a competitive equilibrium. Define  $q^{f*} = \Pi(w^*)$  and  $q^{S*}(s) = v_s(w^*)$  for each  $s \in S$ . Then  $(\mu^{\mathcal{B}*}, \mu^{\mathcal{S}*})$  solves problem (P), and  $(q^{f*}, q^{S*}, w^*)$  solves problem (D).
- (II) Suppose  $(\mu^{\mathcal{B}o}, \mu^{\mathcal{S}o})$  and  $(q^{fo}, q^{So}, w^o)$  are optimal solutions for problems (P) and (D). Then  $(\mu^{\mathcal{B}o}, \mu^{\mathcal{S}o}, w^o)$  is a competitive equilibrium. Moreover,  $q^{fo}$  gives the firms' equilibrium profits and  $q^{So}(s)$  gives the equilibrium indirect utilities of type- $s$  workers.

According to Theorem 2, an allocation  $(\mu^{\mathcal{B}}, \mu^{\mathcal{S}})$  is constrained efficient if it is feasible and there exists shadow prices  $(q^f, q^S, w)$  which satisfy the dual feasibility constraints (4.6) and (4.7) and the complementary slackness conditions (4.11) and (4.12). Yet (4.6) and (4.11) are equivalent to the firms' profit maximization condition in the definition of a competitive equilibrium (condition (i)). Indeed, (4.6) and (4.11) say that the measure  $\mu^{\mathcal{B}}$  assigns firms to market  $(h_f, a_f, \theta_f)$  if and only if entering such a market is a profit maximizing choice for firms at wages  $w$ . The optimal value of  $q^f$  then gives the firms' profits in equilibrium. Similarly, (4.7) and (4.12) are equivalent of condition (ii) in Definition 1, since they say that  $\mu^{\mathcal{S}}$  assigns type- $s$  workers to market  $(h_s, a_s, \theta_s)$  if and only if participating in this market is an optimal choice for these workers given  $w$ . Hence, the optimal value of  $q^S(s)$  gives the indirect utility of type- $s$  workers in equilibrium. Finally, the optimal value of  $w$  describes the transfers that firms need to pay workers in each market in order to decentralize

the allocation  $(\mu^{\mathcal{B}}, \mu^{\mathcal{S}})$ .<sup>26</sup>

The existence of a competitive equilibrium follows directly from Theorem 1 and Part II in Theorem 3.

**Theorem 4.** *A competitive equilibrium exists.*

## 5 Extensions and discussion

There are several variants of the model presented so far that involve straightforward variations of the linear programming formulation in Section 4, and where essentially the same argument allows to establish the equivalent constrained efficiency and existence results. Here we briefly discuss two of them: the variant of the model with free entry, and the more general model where firms are heterogeneous *ex-ante*.

### 5.1 Free entry

With free entry of firms, the feasibility constraint (4.1) disappears, as the mass of firms that enter the economy is now endogenous. Also, an entry cost  $\kappa$  is typically introduced, and so the firms' profits are now given by:

$$\pi(h, a, \theta; w) = \alpha(\theta) [f(h, a) - w(h, a, \theta)] - C(a) - \kappa. \quad (5.1)$$

Apart from this, the only other change in the definition of a competitive search equilibrium is that the zero profit condition must hold:  $\Pi(w^*)$  is set to zero in condition (i). In terms of the linear programming formulation, eliminating constraint (4.1) from the primal is essentially equivalent to setting  $q^f = 0$  in the dual problem. In proving the constrained efficiency and existence results, virtually the same arguments go through.

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<sup>26</sup>To be precise, one also needs to show is that the function  $v_s(w^*)$  is continuous in  $s$ , so it lies in the same space as the dual variable  $q^{\mathcal{S}}$ . This follows trivially from Berge's Maximum Theorem since condition (ii) in Definition 1 implies

$$v_s(w^*) \equiv \sup_{(h, a, \theta) \in \text{supp} \mu_X^{\mathcal{S}^*}} u(s, h, a, \theta; w^*) = u(s, h_s^*, a_s^*, \theta_s^*; w^*), \quad (4.13)$$

$u$  is continuous, and, because  $\mu^{\mathcal{S}^*}$  has compact support, so does  $\mu_X^{\mathcal{S}^*}$ .

The free entry assumption highly simplifies the model when firms are ex-ante symmetric. For instance, one may tackle the price indeterminacy by selecting the *highest* prices that support the equilibrium allocation. In our model this is equivalent to selecting a supporting wage function which satisfies

$$w(h, a, \theta) = f(h, a) - \frac{C(a) + \kappa}{\alpha(\theta)}. \quad (5.2)$$

At these prices, firms make zero profits in all markets, whether active or not.<sup>27</sup> The reason why this selection criterion is particularly useful is that it directly pins down the equilibrium wage function.

## 5.2 Two-sided ex-ante heterogeneity

Consider a generalization of the model in Sections 2-4 where firms are of different types  $b \in B$  (and  $B$  is an arbitrary compact metric space). The population of firms is described by a measure  $\xi^B \in M_+(B)$  with full support. Let  $\hat{C}(b, a)$  denote the cost of investing in  $a \in A$  for a type- $b$  firm, where  $\hat{C}$  is continuous and  $\hat{C}(\cdot, a_0) = 0$ . The output generated by a type- $b$  firms which invest in  $a$  and hires a worker who invest in  $h$  is  $\hat{f}(b, h, a)$ . The worker's disutility of labor may depend on some of the exogenous characteristics of the firms they work for (e.g. such as sector), and is denoted by  $\hat{v}(s, b, h, a)$ . Suppose that, for each firm type  $b \in B$ , there is an investment  $a$  by the firm, a worker type  $s \in S$  and a choice of  $h \in H$  by the latter that generates a positive bilateral surplus:  $\hat{f}(b, h, a) - \hat{C}(b, a) - c(s, h) - \hat{v}(s, b, h, a) > 0$ .

In this version of the model, jobs are described by a triple  $(b, h, a) \in B \times H \times A$ . As before, the description of a market further includes the associated degree of trading uncertainty, as described by the corresponding tightness level  $\theta$ . A market is thus represented by an element  $(b, h, a, \theta) \in B \times H \times A \times \mathfrak{R}_+$ . The set of markets that can potentially be active is  $\hat{X} = \{B \times H \times A \times \mathfrak{R}_+\} \cup \hat{x}_0$ , where  $\hat{x}_0$  represents non-participation.

Let  $\hat{w}(b, h, a, \theta)$  denote the wage in market  $(b, h, a, \theta)$ , where  $\hat{w}(\hat{x}_0) = 0$ . Consistency requires that only type- $b$  firms who invest in  $a \in A$  participate in this market, their expected payoff being

$$\hat{\pi}(b, h, a, \theta; \hat{w}) = \alpha(\theta) \left[ \hat{f}(b, h, a) - \hat{w}(b, h, a, \theta) \right] - \hat{C}(b, a). \quad (5.3)$$

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<sup>27</sup>The refinement on out-of-equilibrium beliefs which is typically imposed in directed search models has precisely this same implication. Such a refinement is the equivalent of this price selection criterion.

Similarly, only workers who invest on  $h$  are allowed to participate in this market, their expected payoff being a function of the worker's type  $s$ :

$$\hat{u}(s, b, h, a, \theta; \hat{w}) = m(\theta) [\hat{w}(b, h, a, \theta) - \hat{v}(s, b, h, a)] - c(s, h). \quad (5.4)$$

### 5.2.1 General equilibrium model

As before, an allocation assigns firms and workers of different types across markets. Formally, an allocation is described by a pair  $(\hat{\mu}^B, \hat{\mu}^S) \in M_{c+}(\hat{X}) \times M_{c+}(S \times \hat{X})$ . In particular,  $\hat{\mu}^B(\hat{\Omega})$  is the measure of firms assigned to an arbitrary Borel set of markets  $\hat{\Omega} \subseteq \hat{X}$  (i.e., the aggregate labor demand in these markets).

Feasibility requires that the total measure of firms of a given type  $b$  who are assigned under an allocation be equal to the measure of such types who are present in the population. So (4.1) is replaced by  $\hat{\mu}_B^B = \xi^B$ . The second feasibility constraint (3.2) is essentially unaltered (except for the fact that  $X$  has been replaced by  $\hat{X}$ ):  $\hat{\mu}_S^S = \xi^S$ . The third feasibility constraint, which says that the total number of jobs created in an arbitrary set of markets is equal to the total number of workers who find a job in those markets, is a simple variant of (3.3):

$$\int_{(b,h,a,\theta) \in \hat{\Omega}} \alpha(\theta) d\hat{\mu}^B(b, h, a, \theta) = \int_{(b,h,a,\theta) \in \hat{\Omega}} m(\theta) d\hat{\mu}_{\hat{X}}^S(b, h, a, \theta) \quad \text{for all Borel } \hat{\Omega} \subseteq B \times H \times A \times \mathfrak{R}_+. \quad (5.5)$$

In the definition of a competitive equilibrium, the only substantial change regards condition (i), which now says that all firm types  $b \in B$  make profit maximizing choices.

**Definition 2.** *A competitive search equilibrium is a feasible allocation  $(\hat{\mu}^B, \hat{\mu}^S) \in M_{c+}(\hat{X}) \times M_{c+}(S \times \hat{X})$  and a wage function  $\hat{w}^* \in C_+(\hat{X})$  such that:*

(i) *For almost all  $(b, h_f^*, a_f^*, \theta_f^*) \in \text{supp} \hat{\mu}^{B*}$ ,*

$$\hat{\Pi}_b(\hat{w}^*) \equiv \sup_{(h,a,\theta) \in H \times A \times \Theta} \hat{\pi}(b, h, a, \theta; \hat{w}^*) = \hat{\pi}(b, h_f^*, a_f^*, \theta_f^*; \hat{w}^*), \quad (5.6)$$

where  $\hat{\Pi}_b$  denotes the profit function of a type- $b$  firm.

(ii) For almost all  $(s, b_s^*, h_s^*, a_s^*, \theta_s^*) \in \text{supp}\mu^{S^*}$ .

$$v_s(w^*) \equiv \sup_{(b,h,a,\theta) \in X} u(s, h, a, \theta; w^*) = u(s, b_s^*, h_s^*, a_s^*, \theta_s^*; w^*). \quad (5.7)$$

Again, the feasibility condition (5.5) implies a rational expectations condition on equilibrium beliefs:

$$\int_{(b,h,a,\theta) \in \hat{\Omega}} d\hat{\mu}^B(b, h, a, \theta) = \int_{(b,h,a,\theta) \in \hat{\Omega}} \theta d\hat{\mu}_X^S(b, h, a, \theta) \quad \text{for all Borel } \hat{\Omega} \subseteq B \times H \times A \times \mathfrak{R}_+. \quad (5.8)$$

(The interpretation of this condition is essentially as before).

### 5.2.2 LP formulation and main results

The primal LP problem in this general model is to find  $(\hat{\mu}^B, \hat{\mu}^S) \in M_{c_+}(\hat{X}) \times M_{c_+}(S \times \hat{X})$  to solve

$$(P_G) \quad \sup \int_{\hat{X}} [\alpha(\theta)\hat{f}(b, h, a) - \hat{C}(b, a)] d\hat{\mu}^B - \int_{S \times \hat{X}} [m(\theta)\hat{v}(s, b, h, a) + c(s, h)] d\hat{\mu}^S \quad \text{s.t.}$$

$$\hat{\mu}_B^B = \xi^B, \quad (5.9)$$

$$\hat{\mu}_S^S = \xi^S, \quad (5.10)$$

$$\int_{\hat{\Omega}} \alpha(\theta) d\hat{\mu}^B = \int_{\hat{\Omega}} m(\theta) d\hat{\mu}_X^S \quad \text{for all Borel } \hat{\Omega} \subseteq \hat{X}, \quad (5.11)$$

$$\hat{\mu}^B, \hat{\mu}^S \geq 0, \quad (5.12)$$

where (as before) the objective function describes the economy's total welfare.

The shadow price of constraint (5.9) (rather than a real number) is a continuous function  $\hat{q}^B \in C(B)$ . This function describes the marginal contribution to social welfare of the different firm types. The spaces where the other dual variables lie are essentially unaltered (except that, since  $X$  has been replaced by  $\hat{X}$ , the shadow price of constraint (5.11) now lies in  $C(\hat{X})$ ).

The dual problem  $(D_G)$  is to find  $(\hat{q}^B, \hat{q}^S, \hat{w}) \in C(B) \times C(S) \times C(\hat{X})$  to solve

$$(D_G) \quad \inf \int_B \hat{q}^B(b) d\xi^B(b) + \int_S \hat{q}^S(s) d\xi^S(s)$$

s.t.

$$\hat{q}^{\mathcal{B}}(b) \geq \hat{\pi}(b, h, a, \theta; \hat{w}) \quad \text{for all } (b, h, a, \theta) \in \hat{X}, \quad (5.13)$$

$$\hat{q}^{\mathcal{S}}(s) \geq \hat{u}(s, b, h, a, \theta; \hat{w}) \quad \text{for all } (s, b, h, a, \theta) \in S \times \hat{X}. \quad (5.14)$$

One can again show that problems  $(P_G)$  and  $(D_G)$  have optimal solutions and the same optimal value. The proof of this result is a simple variant of that of Theorem 2 in Jerez (2014), which (for completeness) is presented in the Appendix.<sup>28</sup> In the light of this result, it is direct to show that the complementary slackness theorem again implies an equivalence between competitive search equilibria and the optimal solutions to the LP problems, and the existence of a competitive search equilibrium.

## Appendix A

### A.1 Notation

We begin with some preliminary notation.

Take a metric space  $Z$  which is locally compact and separable. Let  $C(Z)$  denote the space of continuous real-valued functions on  $Z$ , endowed with the topology of uniform convergence on compact sets. The topological dual of  $C(Z)$  is the space  $M_c(Z)$  of signed regular Borel measures on  $Z$  with compact support (see Hewitt (1959)). We let  $M_c(Z)$  be endowed with the weak-star topology, so  $C(Z)$  is also the dual of  $M_c(Z)$ . The dual pair of spaces  $(C(Z), M_c(Z))$  is endowed with the standard bilinear form:

$$\langle f, \gamma \rangle = \int_{z \in Z} f(z) d\gamma(z), \quad f \in C(Z), \quad \gamma \in M_c(Z),$$

where the bracket notation highlights the infinite dimensional nature of the spaces in the pairing.

In the special case where  $Z$  is compact, the topological dual of  $C(Z)$  is the space  $M(Z)$  of signed regular Borel measures on  $Z$ .<sup>29</sup> (If  $Z$  is finite, both  $C(Z)$  and  $M(Z)$  are isomorphic to the Euclidean space). We write  $C_+(Z)$ ,  $M_{c+}(Z)$  and  $M_+(Z)$  for the positive cones of the three spaces.

For any integer  $n$ , the product spaces  $\prod_{j=1, \dots, n} C(Z_j)$  and  $\prod_{j=1, \dots, n} M_c(Z_j)$  are endowed with

<sup>28</sup>The argument is slightly different from that in Gretsky, Ostroy, and Zame (1992) and Makowski and Ostroy (1996) mainly because, unlike in their papers, the measures describing an allocation are defined over a non-compact set.

<sup>29</sup>As noted by Hewitt (1959), the topology of uniform convergence on compact sets coincides with the uniform norm topology in this case.

the corresponding product topologies, and are also paired in duality with bilinear form:

$$\sum_{j=1}^n \langle f_j, \gamma_j \rangle, \quad (f_1, f_2, \dots, f_n) \in \prod_{j=1, \dots, n} C(Z_j), \quad (\gamma_1, \gamma_2, \dots, \gamma_n) \in \prod_{j=1, \dots, n} M_c(Z_j).$$

## A.2 The primal and dual linear programming problems

This section includes the formal description of the primal and dual LP problems in Section 5.2. To simplify notation, we drop all the “hats” (that were used in the production and cost functions and in all the variables to differentiate that Section from the model in Section 3).

Before stating the primal problem, a remark about constraint (5.5) is in order. We have written this constraint as:

$$\int_{(b, h, a, \theta) \in \Omega} \alpha(\theta) d\mu^{\mathcal{B}}(b, h, a, \theta) = \int_{(b, h, a, \theta) \in \Omega} m(\theta) d\mu_X^{\mathcal{S}}(b, h, a, \theta) \quad \text{for all Borel } \Omega \subset X$$

where  $X \equiv (B \times H \times A \times \mathfrak{R}_+) \cup \{x_0\}$ . The left-hand side of this constraint describes a measure which is absolutely continuous with respect to  $\mu^{\mathcal{B}}$  with Radon-Nikodym derivative  $\tilde{\alpha}(b, a, h, \theta) = \alpha(\theta)$ . Similarly, the right-hand side of (4.3) describes a measure which is absolutely continuous with respect to  $\mu_X^{\mathcal{S}}$  with Radon-Nikodym derivative  $\tilde{m}(b, a, h, \theta) = m(\theta)$ . Constraint (4.3) says that these two measures, which lie in  $M_c(X)$  and which we denote by  $\eta(\tilde{\alpha}, \mu^{\mathcal{B}})$  and  $\eta(\tilde{m}, \mu_X^{\mathcal{S}})$  respectively, are equal:

$$\eta(\tilde{\alpha}, \mu^{\mathcal{B}}) = \eta(\tilde{m}, \mu_X^{\mathcal{S}}).$$

Using the standard (compact) notation, the *primal* problem is to find  $x = (\mu^{\mathcal{B}}, \mu^{\mathcal{S}}) \in M_c(X) \times M_c(S \times X)$  to solve

$$\begin{aligned} (P_G) \quad & \sup && \langle x, c \rangle \\ & \text{s.t.} && Ax = b, \\ & && x \geq 0. \end{aligned}$$

Here  $c = (c^{\mathcal{B}}, c^{\mathcal{S}}) \in C(X) \times C(S \times X)$  where

$$\begin{aligned} c^{\mathcal{B}}(b, h, a, \theta) &= \alpha(\theta)f(b, h, a) - C(b, a) \\ c^{\mathcal{S}}(s, b, h, a, \theta) &= -m(\theta)v(s, b, h, a) - c(s, h). \end{aligned}$$

Also,  $b = (\xi^{\mathcal{B}}, \xi^{\mathcal{S}}, 0) \in M(B) \times M(S) \times M_c(X)$ . Finally,  $A : M_c(X) \times M_c(S \times X) \rightarrow M(B) \times M(S) \times M_c(X)$  is a continuous linear map defined by

$$A(\mu^{\mathcal{B}}, \mu^{\mathcal{S}}) = (\mu_B^{\mathcal{B}}, \mu_S^{\mathcal{S}}, \eta(\tilde{\alpha}, \mu^{\mathcal{B}}) - \eta(\tilde{m}, \mu_X^{\mathcal{S}})).$$

Formally, problem (P) is an equality constrained LP problem (see Anderson and Nash 1987).

The *dual* problem is to find  $y = (q^B, q^S, w) \in C(B) \times C(S) \times C(X)$  to solve

$$(D_G) \quad \begin{aligned} & \inf && \langle b, y \rangle \\ & \text{s.t.} && A^*y \geq c, \end{aligned}$$

where  $A^* : C(B) \times C(S) \times C(X) \rightarrow C(X) \times C(S \times X)$  is the adjoint of  $A$ . That is,  $A^*$  is defined by the relation

$$\langle x, (A^*y) \rangle = \langle Ax, y \rangle, \text{ for all } x \in M_c(X) \times M_c(S \times X) \text{ and all } y \in C(B) \times C(S) \times C(X).$$

This is precisely the dual problem stated in Section 5.2.

### A.3 Existence of optimal solutions and absence of a duality gap

In this section we prove that problems  $(P_G)$  and  $(D)_G$  have optimal solutions and the same optimal value:  $\nu(P_G) = \nu(D_G)$  (i.e., the general version of Theorem 1). In doing so, we rely on the assumptions that  $f$ ,  $v$ ,  $C$  and  $c$  and the matching function  $\mathcal{M}$  are continuous, and  $B$ ,  $S$ ,  $H$  and  $A$  are compact sets. This, combined with the fact that trading probabilities are bounded, is all we need to prove these results.

We begin by showing that both problems are consistent (i.e. their feasible sets are not empty) and bounded (i.e.  $\nu(P_G)$  and  $\nu(D_G)$  are finite).

**Lemma A. 1.** *Problems  $(P_G)$  and  $(D_G)$  are consistent and bounded.*

*Proof.* An allocation where workers make no investments (so they all choose  $h = h_0$ ), neither do firms (who choose  $a = a_0$ ) and all agents are assigned to a fictitious market where  $\theta = \theta_0$  is a feasible solution for problem  $(P_G)$ . Hence, problem  $(P_G)$  is consistent. Also, since total welfare is zero under autarky,  $\nu(P_G) \geq 0$ .

In problem  $(D_G)$ , set  $w = w_1 \in C(X)$  where  $w_1(b, h, a, \theta) = 0$  for all  $(b, h, a, \theta) \in X$ . In the constraint systems (4.6) and (4.7),  $\alpha(\theta)$  and  $m(\theta)$  are bounded above by one and below by zero (since they are probabilities). One then can find a feasible dual solution where  $w = w_1$  by choosing  $q_1^B \in C(B)$  and  $q_1^S \in C(S)$  constant so that

$$\begin{aligned} q_1^B(b) &= \bar{q}_1^B \equiv \sup_{(b,h,a) \in B \times H \times A} f(b, h, a) - C(b, a) + \epsilon > 0, \\ q_1^S(s) &= \epsilon, \quad s \in S, \end{aligned}$$

where  $\epsilon$  is a positive real number. Since  $f$  and  $C$  are continuous and  $B \times H \times A$  is compact, the above supremum  $\bar{q}_1^B$  is attained. So problem  $(D_G)$  is consistent. (The supremum  $\bar{q}_1^B$  is clearly positive: we have assumed that for each firm type  $b$  there is a value of  $a$ , a value of  $h$  and a worker type  $s$  such that  $f(b, h, a) - C(b, a) - c(s, h) - v(s, b, h, a) > 0$ , and so  $f(b, h, a) - C(b, a) > 0$ ). Moreover,

$$\nu(D_G) \leq \int_B q_1^B(b) d\xi^B(b) + \int_S q_1^S(s) d\xi^S(s) = \bar{q}_1^B \xi^B(B) + \epsilon \xi^S(S) < \infty.$$

Finally, by the weak duality theorem (Anderson and Nash 1987, Theorem 2.1),  $\nu(P_G) \leq \nu(D_G)$ ,

so the primal and dual problems are bounded:

$$0 \leq \nu(P_G) \leq \nu(D_G) \leq \bar{q}_1^{\mathcal{B}} \xi^{\mathcal{B}}(B) + \epsilon \xi^{\mathcal{S}}(S) < \infty. \quad \square$$

Next, we show that problem  $(P_G)$  is solvable.

**Theorem A. 1.** *Problem  $(P_G)$  has optimal solutions.*

*Proof.* The feasible set of problem  $(P_G)$  is bounded, and the constraint map and objective function are weak-star continuous, so the result follows from Theorem 3.20 in Anderson and Nash (1987).  $\square$

We also show that problems  $(P_G)$  and  $(D_G)$  have the same optimal value.

**Theorem A. 2.** *There is no duality gap:  $\nu(P_G) = \nu(D_G)$ .*

*Proof.* The positive cone of  $C(X) \times C(S \times X)$  has a non-empty interior, denoted by  $Y_0$ . Also,  $(q_1^{\mathcal{B}}, q_1^{\mathcal{S}}, w_1) \in C_+(B) \times C_+(S) \times C_+(X)$  in the proof of Lemma A.1 is a Slater point in the feasible set of problem  $(D_G)$ . Since  $\nu(D_G)$  is finite, Theorem 3.13 in Anderson and Nash (1987) implies that  $\nu(P_G) = \nu(D_G)$ .  $\square$

We still need to show that problem  $(D_G)$  is solvable. We begin by stating two preliminary results. Lemma A.2 shows that the set of feasible dual solutions can be taken to be bounded without loss of generality. The proof uses the fact that  $f, v, c$  and  $C$  are continuous,  $\alpha$  and  $m$  are bounded, and  $B, S, H$  and  $A$  are compact sets. Lemma A.3 shows that the tightness level  $\theta$  can be restricted without loss of generality to lie on a compact subset of  $\mathfrak{R}_+$  (e.g. to be bounded above).

**Lemma A. 2.** *The set of feasible dual solutions can be taken to be bounded without loss of generality. In particular, feasible dual solutions satisfy  $q^{\mathcal{B}} \geq 0$  and  $q^{\mathcal{S}} \geq 0$ . Also, we may assume that  $q^{\mathcal{B}}$  and  $q^{\mathcal{S}}$  are bounded above and that*

$$\min_{(s,b,h,a) \in S \times B \times H \times A} v(s, b, h, a) \leq w(b, h, a, \theta) \leq \max_{(b,h,a) \in B \times H \times A} f(b, h, a) \text{ for all } (b, h, a, \theta) \in X.$$

*Proof.* Substituting (5.3) into the dual constraint system (5.13) and setting  $\theta = \theta_0, h = h_0$  and  $a = a_0$  implies that  $q^{\mathcal{B}} \geq 0$ . Likewise, substituting (5.4) into the dual constraint system (5.14) and setting  $\theta = \theta_0, h = h_0$  and  $a = a_0$  implies that  $q^{\mathcal{S}} \geq 0$ .

For a given  $(b, h, a, \theta) \in B \times H \times A \times \mathfrak{R}_{++}$ , if an optimal primal solution satisfies  $(b, h, a, \theta) \in \text{supp} \mu^{\mathcal{B}}$  then  $(\tilde{s}, b, h, a, \theta) \in \text{supp} \mu^{\mathcal{S}}$  for some  $\tilde{s} \in S$ . This is because the restrictions of  $\mu^{\mathcal{B}}$  and  $\mu_X^{\mathcal{S}}$  to  $B \times H \times A \times \mathfrak{R}_{++}$  are mutually absolutely continuous measures, and so they have the same support.<sup>30</sup> (There is always such a  $(b, h, a, \theta)$ , when one focuses on the interesting case where autarky is not an

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<sup>30</sup>The corresponding Radon-Nikodym derivatives are  $f$  and  $1/f$  where  $f(b, h, a, \theta) = \theta$ . This follows from equations (2.2) and (3.3) since  $\alpha(\theta), m(\theta) > 0$  for all  $\theta > 0$ .

optimal allocation.) By the complementary slackness theorem, in this case, optimal dual solutions satisfy

$$q^{\mathcal{B}}(b) = \alpha(\theta) [f(b, h, a) - w(b, h, a, \theta)] - C(b, a), \quad (\text{A.1})$$

$$q^{\mathcal{S}}(\tilde{s}) = m(\theta) [w(b, h, a, \theta) - v(\tilde{s}, b, h, a)] - c(\tilde{s}, h). \quad (\text{A.2})$$

Now, since  $q^{\mathcal{B}}(b), q^{\mathcal{S}}(\tilde{s}) \geq 0$  and  $\alpha(\theta), m(\theta) > 0$  (and  $C$  and  $c$  are also positive functions), it follows that  $v(\tilde{s}, b, h, a) \leq w(b, h, a, \theta) \leq f(b, h, a)$ . Hence,

$$0 \leq \inf_{s \in S} v(s, b, h, a) \leq w(b, h, a, \theta) \leq f(b, h, a). \quad (\text{A.3})$$

The continuity of  $v$  and the compactness of  $S$  imply that the infimum in (A.3) is attained. Since  $w$  is continuous, (A.3) also holds for  $(b, h, a, 0) \in X$ . Equation (A.3), together with (5.3) and (5.4), in turn implies that the terms on right-hand side of (5.13) and (5.14) are bounded above (since matching probabilities are bounded,  $f, v, c$  and  $C$  are continuous functions, and  $B, S, A$  and  $H$  are compact). So there is no loss of generality in assuming that  $q^{\mathcal{B}}$  and  $q^{\mathcal{S}}$  are bounded above.

On the other hand, if  $(b, h, a, \theta) \notin \text{supp}\mu^{\mathcal{B}}$  then  $(b, h, a, \theta) \notin \text{supp}\mu_X^{\mathcal{S}}$ . In this case, market  $(b, h, a, \theta)$  is inactive, and its shadow price  $w(b, h, a, \theta)$  can be chosen arbitrarily among all the values that satisfy

$$q^{\mathcal{B}}(b) \geq \alpha(\theta) [f(b, h, a) - w(b, h, a, \theta)] - C(b, a), \quad (\text{A.4})$$

$$q^{\mathcal{S}}(s) \geq m(\theta) [w(b, h, a, \theta) - v(s, b, h, a)] - c(s, h) \quad \forall s \in S. \quad (\text{A.5})$$

In particular, we may restrict without loss of generality to values of  $w(b, h, a, \theta)$  satisfying (A.3). Indeed, if (A.4) and (A.5) hold for  $w(b, h, a, \theta) > f(b, h, a)$  then they must hold for  $w(b, h, a, \theta) = f(b, h, a)$  since  $q^{\mathcal{B}}(b) \geq 0$ . Likewise, if these equations hold for  $w(b, h, a, \theta) < v(\underline{s}, b, h, a) = \min_{s \in S} v(s, b, h, a)$  then they must also hold for  $w(b, h, a, \theta) = v(\underline{s}, b, h, a)$  since  $q^{\mathcal{S}} \geq 0$ . Finally, since (A.3) holds for all  $(b, h, a, \theta) \in X$  and (again  $B, S, A$ , and  $H$  are compact and  $f$  and  $v$  are continuous functions),

$$\min_{(s, b, h, a) \in S \times B \times H \times A} v(s, b, h, a) \leq w(b, h, a, \theta) \leq \max_{(b, h, a) \in B \times H \times A} f(b, h, a), \quad \forall (b, h, a, \theta) \in X. \quad (\text{A.6})$$

□

**Lemma A. 3.** *There exists a sufficiently large  $\bar{\theta} \in \mathfrak{R}_+$  such that, if all the constraints which are associated with elements  $\theta > \bar{\theta}$  are eliminated from problem  $(D_G)$ , the set of optimal dual solutions does not change.*

*Proof.* Suppose the statement in Lemma A.3 were not true. Let  $(\mu^{\mathcal{B}}, \mu^{\mathcal{S}})$  be an optimal primal solution. Take an increasing sequence  $\{\theta_j\} \subset \mathfrak{R}_+$  with  $\theta_j \rightarrow \infty$ . For each  $j$  there then exists  $(b_j, h_j, a_j, \hat{\theta}_j) \in \text{supp}\mu^{\mathcal{B}}$  with  $\hat{\theta}_j > \theta_j$ . Equivalently,  $(s_j, b_j, h_j, a_j, \hat{\theta}_j) \in \text{supp}\mu^{\mathcal{S}}$  for some  $s_j \in S$ , since the restrictions of  $\mu^{\mathcal{B}}$  and  $\mu_X^{\mathcal{S}}$  to  $B \times H \times A \times \mathfrak{R}_{++}$  are mutually exclusive measures which have the same support. (If not, the complementary slackness theorem would imply that the dual constraints associated to all  $\theta > \theta_j$  can be ignored without loss of generality, since they do not bind). But then the support of  $\mu^{\mathcal{B}}$  contains the sequence  $\{(b_j, h_j, a_j, \hat{\theta}_j)\}$  where  $\lim \hat{\theta}_j \rightarrow \infty$ , leading to a contradiction since this support is compact by definition. □

The solvability of problem  $(D_G)$  cannot be settled using an argument similar to that in Theorem A.1 because the space of continuous functions on a compact set is not the dual of any normed space. We follow the approach used in Anderson and Nash (1987) for the continuous transportation problem (see their Theorem 5.2) and rephrase problem  $(D_G)$  in an enlarged space which does have this property. Then we appeal to the continuity of the  $f$ ,  $v$ ,  $C$  and  $c$  and the matching function, and the compactness of  $B$ ,  $H$  and  $A$  to show that an optimal solution in the enlarged space lies in the original space.

**Theorem A. 3.** *Problem  $(D_G)$  has optimal solutions.*

*Proof.* Let us rephrase problem  $(D_G)$  with  $(q^B, q^S, w)$  in  $L^\infty(\xi^B) \times L^\infty(\xi^S) \times L^\infty(\mu^B)$ , where  $\mu^B$  is optimal for problem  $(P)$ . (This space is the dual of  $L^1(\xi^B) \times L^1(\xi^S) \times L^1(\mu^B)$ ). The new dual problem is solvable by Theorem 3.20 in Anderson and Nash (1987) since Lemma A.2 implies that its feasible set can be taken to be bounded without loss of generality.

We now show that there exists an optimal solution of this new problem where the functions  $q^B$ ,  $q^S$  and  $w$  are continuous. Suppose  $(q^B, q^S, w)$  is optimal for the new dual problem. Feasibility requires that

$$\alpha(\theta)w(b, h, a, \theta) \geq \alpha(\theta)f(b, h, a) - C(b, a) - q^B(b), \quad \forall (b, h, a, \theta) \in B \times H \times A \times \mathfrak{R}_+, \quad (\text{A.7})$$

and that

$$q^S(s) + m(\theta)v(s, b, h, a) + c(s, h) \geq m(\theta)w(b, h, a, \theta), \quad \forall (s, b, h, a, \theta) \in S \times B \times H \times A \times \mathfrak{R}_+. \quad (\text{A.8})$$

Recall that  $w(\cdot, \theta_0) = 0$ . Define  $w_2 \in L^\infty(\mu^B)$  so

$$m(\theta)w_2(b, h, a, \theta) = \inf_{s \in S} \{q^S(s) + m(\theta)v(s, b, h, a) + c(s, h)\} \quad (\text{A.9})$$

for  $(b, h, a, \theta) \in B \times H \times A \times \mathfrak{R}_+$ , and  $w_2(\cdot, \theta_0) = 0$ . Then,  $(q^B, q^S, w_2)$  is another optimal solution.

We now show that the restriction of  $w_2$  to  $B \times H \times A \times (0, \bar{\theta}]$  is continuous. Take a sequence  $\{s_i\}$  in  $S$  such that  $(q^S(s_i) + m(\theta)v(s_i, b, h, a) + c(s_i, h))$  converges to  $\alpha(\theta)w_2(b, h, a, \theta)$ . Since  $B \times H \times A \times [0, \bar{\theta}]$  is compact,  $m(\theta)v(s, b, h, a) + c(s, h)$  is uniformly continuous on  $B \times S \times H \times A \times [0, \bar{\theta}]$ . For any  $\epsilon > 0$  there then exists  $\delta$  such that

$$|m(\theta)v(s_i, b, h, a) + c(s_i, h) - m(\theta')v(s_i, b', h', a') - c(s_i, h')| < \epsilon, \quad i = 1, 2, \dots \quad (\text{A.10})$$

whenever  $(b', h', a', \theta')$  lies in a  $\delta$ -neighborhood of  $(b, h, a, \theta)$ , and both elements lie in  $B \times H \times A \times [0, \bar{\theta}]$ . Equations (A.8) and (A.10) then imply that

$$m(\theta')w_2(b', h', a', \theta') \leq q^S(s_i) + m(\theta')v(s_i, b', h', a') + c(s_i, h') \quad (\text{A.11})$$

$$< q^S(s_i) + m(\theta)v(s_i, b, h, a) + c(s_i, h) + \epsilon, \quad i = 1, 2, 3, \dots \quad (\text{A.12})$$

Taking the limit yields

$$m(\theta')w_2(b', h', a', \theta') - \epsilon \leq \alpha(\theta)w_2(b, h, a, \theta). \quad (\text{A.13})$$

A symmetric argument implies that

$$m(\theta)w_2(b, h, a, \theta) - \epsilon \leq \alpha(\theta')w_2(b', h', a', \theta'). \quad (\text{A.14})$$

for any such  $(b', h', a', \theta')$  and  $(b, h, a, \theta)$ . Hence, the restriction of  $\alpha(\theta)w_2(b, h, a, \theta)$  to  $B \times H \times A \times [0, \bar{\theta}]$  is continuous. Since  $m$  is continuous and strictly positive when  $\theta \in (0, \bar{\theta}]$ , the restriction of  $w_2(b, h, a, \theta)$  to  $B \times H \times A \times (0, \bar{\theta}]$  (the quotient of two continuous functions) is continuous.

To see that  $w_2$  is continuous, take an increasing sequence of compact sets  $\{\Theta_j\}$  converging to  $\Theta \cup \{\theta_0\}$ ; e.g.  $\Theta_j = [\epsilon_j, \theta_j] \cup \{\theta_0\}$  with  $\epsilon_j \downarrow 0$  and  $\theta_j \uparrow \infty$ . Consider the sequence of functions  $\{f_j\}$  where  $f_j = \chi_{\Theta_j} w_2$ , where  $\chi_{\Theta_j}$  denotes the characteristic function on  $\Theta_j$  (so  $w_2$  and  $f_j$  coincide on  $\Theta_j$ ). Since  $f_j$  is continuous on  $\Theta_j$  and  $w_2 = \lim_{j \rightarrow \infty} f_j$ , it follows that  $w_2$  is continuous.

Finally, defining

$$q_2^{\mathcal{B}}(b) = \max_{(h, a, \theta) \in H \times A \times ([0, \bar{\theta}] \cup \{\theta_0\})} \alpha(\theta) [f(b, h, a) - w_2(b, h, a, \theta)] - C(b, a), \quad (\text{A.15})$$

$$q_2^{\mathcal{S}}(h) = \max_{(b, h, a, \theta) \in B \times H \times A \times ([0, \bar{\theta}] \cup \{\theta_0\})} m(\theta) [w_2(b, h, a, \theta) - v(s, b, h, a)] - c(s, h), \quad (\text{A.16})$$

yields yet another optimal solution  $(q_2^{\mathcal{B}}, q_2^{\mathcal{S}}, w_2)$  since, by Lemma A.3, the constraints associated with elements  $\theta > \bar{\theta}$  can be ignored without loss of generality in (4.6)-(4.7). By Berge's Maximum Theorem,  $q_2^{\mathcal{B}}$  and  $q_2^{\mathcal{S}}$  are continuous.  $\square$

The former results imply the equivalent results to Theorems 2–4 in Section 4. The argument is essentially the same, and is thus omitted.

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