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Amenable traces and Følner C^* -algebras

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Abstract

In the present article we review an approximation procedure for amenable traces on unital and separable C^* -algebras acting on a Hilbert space in terms of Følner sequences of non-zero finite rank projections. We apply this method to improve spectral approximation results due to Arveson and Bédos. We also present an abstract characterization in terms of unital completely positive maps of unital separable C^* -algebras admitting a non-degenerate representation which has a Følner sequence or, equivalently, an amenable trace. This is analogous to Voiculescu's abstract characterization of quasidiagonal C^* -algebras. We define Følner C^* -algebras as those unital separable C^* -algebras that satisfy these equivalent conditions. Finally we also mention some permanence properties related to these algebras.

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1. Introduction

There are two well-known important characterizations of discrete amenable groups, one given in terms of the existence of an invariant mean, and the other in terms of the existence of Følner nets of finite subsets of the group. In his seminal article [17, Section V], Alain Connes gave an analogue of these in the context of von Neumann algebras introducing amenable traces and Følner nets for operators, respectively (see also [18,29,30] as well as Sections 2 and 3 for precise definitions and additional results). Følner nets for operators are given in terms of non-zero finite rank orthogonal projections $\{P_\lambda\}_\lambda$ in the corresponding Hilbert space and satisfying natural approximation conditions (see Definition 2.1 for details). Connes used these concepts as a crucial tool in the classification of injective type II_1 factors. Recently, this circle of ideas has been used to define a new invariant for a general separable type II_1 factor that measures how badly the factor fails to satisfy Connes' Følner type condition (cf. [4]).

In addition to these theoretical developments, Følner sequences for operators have been also used in spectral approximation problems: given a sequence of linear operators $\{T_n\}_{n \in \mathbb{N}}$ acting on a complex Hilbert space \mathcal{H} that approximates an operator T in a suitable sense, it is natural to ask how the spectral objects of T relate to those of T_n when $n \rightarrow \infty$. We recall next the following classical approximation result for scalar spectral measures of Toeplitz operators due to Szegő: denote by \mathbb{T} the unit circle with normalized Haar measure $d\theta$ and consider the real-valued functions g in $L^\infty(\mathbb{T})$ which can be thought as (self-adjoint) multiplication operators on the complex Hilbert space $\mathcal{H} := L^2(\mathbb{T})$, i.e., $M_g \varphi = g\varphi$, $\varphi \in \mathcal{H}$. Denote by P_n the finite-rank orthogonal projection onto the linear span of $\{z^l \mid z \in \mathbb{T}, l = 0, \dots, n\}$ and let $M_g^{(n)} := P_n M_g P_n$ be the corresponding finite section matrix. Write the corresponding eigenvalues (repeated according to multiplicity) as $\{\lambda_{0,n}, \dots, \lambda_{n,n}\}$. Then, for any continuous $f: \mathbb{R} \rightarrow \mathbb{R}$ one has

$$\lim_{n \rightarrow \infty} \frac{1}{n+1} (f(\lambda_{0,n}) + \dots + f(\lambda_{n,n})) = \int_{\mathbb{T}} f(g(\theta)) d\theta \quad (1.1)$$

(see [33, Section 8], [20, Chapter 5] and [37] for a careful analysis of this result; a recent standard book analyzing many aspects of Toeplitz operators and containing a large number of references is [10]). Eq. (1.1) may be also reformulated in terms of weak-* convergence of the corresponding spectral measures and it allows the numerical approximation of the spectrum of M_g in terms of the eigenvalues of its finite sections (see [2] as well as Chapter 7 in [31] and references cited therein). These classical approximation results motivated Arveson to consider spectral approximations in a more general context than Toeplitz operators and uses techniques from operator algebras. Among other results,

Arveson gave conditions that guarantee that the essential spectrum of a self-adjoint operator T may be recovered from the sequence of eigenvalues of certain finite dimensional compressions T_n (cf. [3,2]). These results were then extended by Bédos who systematically applied the concept of Følner sequence to spectral approximation problems [5–7] (see also [25] and references therein). In general, operator algebraic techniques have also contributed to address these approximation problems (some examples are [11,21,23]). The notion of Følner sequences has turned to be also interesting in the context of single operator theory. In [26] Yakubovich and the second author show that several classes of non-normal operators have a Følner sequence and analyze the relation to the class of finite operators introduced by Williams in [38]. See also [1] for a review of the notion of Følner sequence of projections in operator algebras and operator theory that includes a new proof that any essentially normal operator has an increasing Følner sequence strongly converging to $\mathbb{1}$. In addition, Følner sequences in operator algebras are also important in the study of growth conditions, i.e., conditions involving the asymptotic growth, as $n \rightarrow \infty$, of the spaces linearly spanned by the sets X^n , where X is a finite generating set of a C^* -algebra (see, e.g., [34,16]). We also refer to [12,13,27] for a thorough description of the relations of amenable traces and Følner sequences to other important areas like, e.g., Connes’ embedding problem.

An important step in the proof of the Arveson–Bédos spectral approximation result mentioned above is the compatibility between the choice of the Følner sequence in the Hilbert space and the amenable trace. In fact, if the unital and separable concrete C^* -algebra $\mathcal{A} \subset \mathcal{L}(\mathcal{H})$ has an amenable trace τ and $\{P_n\}_n$ is a Følner sequence of non-zero finite rank projections for \mathcal{A} it is needed that the projections approximate the amenable trace in the following natural sense

$$\tau(A) = \lim_{n \rightarrow \infty} \frac{\text{Tr}(AP_n)}{\text{Tr}(P_n)}, \quad A \in \mathcal{A}, \quad (1.2)$$

where $\text{Tr}(\cdot)$ denotes the canonical trace on $\mathcal{L}(\mathcal{H})$. Now given $\mathcal{A} \subset \mathcal{L}(\mathcal{H})$ with an amenable trace τ , it is possible to construct a Følner sequence in different ways. As observed by Bédos in [6] one way to obtain a Følner sequence $\{P_n\}$ for $\mathcal{A} \subset \mathcal{L}(\mathcal{H})$ is essentially contained in [17,18]. In these articles Connes adapts the group theoretic methods by Day and Namioka to the context of operators. Using this technique one loses track of the initial amenable trace τ , in the sense that the sequence $\{P_n\}$ does not necessarily satisfy (1.2). To avoid this problem one may assume in addition that \mathcal{A} has a unique tracial state. This is sufficient to guarantee a good spectral approximation behavior of relevant examples like almost Mathieu operators, which are contained in the irrational rotation algebra (cf. [9]).

In contrast with the previous method, the construction of a Følner sequence given in [27, Theorem 6.1] (see also [15, Theorem 6.2.7]) allows the approximation of the original trace as in Eq. (1.2). We will review this method in Section 3 and apply it to prove a spectral approximation result in the spirit of Arveson and Bédos, but removing the hypothesis of a unique trace (see Theorem 3.2 for details as well as [3, p. 354], [6, Theorem 1.3] or [7, Theorem 6(iii)]).

In the last section of this article we will also give an abstract characterization of unital separable C^* -algebras admitting a non-degenerate representation π on a Hilbert space such

that there is a Følner sequence for $\pi(\mathcal{A})$ or, equivalently, such that $\pi(\mathcal{A})$ has an amenable trace (see [Theorem 4.3](#)). More precisely, we obtain that these conditions are equivalent to the existence of a sequence of unital completely positive (u.c.p.) maps $\varphi_n: \mathcal{A} \rightarrow M_{k(n)}(\mathbb{C})$ which is asymptotically multiplicative with respect to the normalized Hilbert–Schmidt norm $\|\cdot\|_{2,\text{tr}}$ on $M_{k(n)}(\mathbb{C})$. Motivated by this relationship, we call the C^* -algebras admitting such finite dimensional approximations *Følner C^* -algebras* ([Definition 4.1](#)). It turns out that this is the same class as the *weakly hypertracial C^* -algebras* studied by Bédos in [6]. Our result is inspired by Voiculescu’s abstract characterization of quasidiagonal C^* -algebras (cf. [35]), which asserts that a unital separable C^* -algebra \mathcal{A} is quasidiagonal if and only if there is a sequence of u.c.p. maps $\varphi_n: \mathcal{A} \rightarrow M_{k(n)}(\mathbb{C})$ which is asymptotically multiplicative and asymptotically isometric with respect to the operator norm on $M_{k(n)}(\mathbb{C})$. We end the paper by recalling some known permanence properties of weakly hypertracial C^* -algebras, proved by Bédos in [6].

We refer the reader to [15], especially Chapter 6, for background material. The paper is basically self-contained modulo some of the results contained in that book.

Notation. We will denote by $\mathcal{L}(\mathcal{H})$ the C^* -algebra of bounded linear operators on the complex separable Hilbert space \mathcal{H} , and by $\mathcal{K}(\mathcal{H})$ the ideal of compact operators on \mathcal{H} . The unitary group of a unital C^* -algebra \mathcal{A} is denoted by $\mathcal{U}(\mathcal{A})$. We will assume that any representation of a unital C^* -algebra preserves the unit (i.e. it is non-degenerate). To simplify expressions we will sometime use the standard notation for the commutator of two operators: $[A, B] := AB - BA$.

2. Følner type conditions for operators

The notion of Følner sequences for operators has its origins in group theory. Recall that a discrete countable group Γ is amenable if it has an invariant mean, i.e. there is a positive linear functional ψ on $\ell^\infty(\Gamma)$ with norm one such that

$$\psi(\gamma f) = \psi(f), \quad \gamma \in \Gamma, f \in \ell^\infty(\Gamma),$$

where $(\gamma f)(\gamma_0) := f(\gamma^{-1}\gamma_0)$. A Følner sequence for Γ is a sequence of non-empty finite subsets $\Gamma_i \subset \Gamma$ that satisfy

$$\lim_i \frac{|(\gamma \Gamma_i) \Delta \Gamma_i|}{|\Gamma_i|} = 0 \quad \text{for all } \gamma \in \Gamma, \tag{2.1}$$

where Δ denotes the symmetric difference and $|X|$ is the cardinality of X for any set X . Then, Γ has a Følner sequence if and only if Γ is amenable (cf. Chapter 4 in [28]). If Γ has a Følner sequence one can always find another Følner sequence which, in addition to Eq. (2.1), is also increasing and complete, i.e. $\Gamma_i \subset \Gamma_j$ if $i \leq j$ and $\Gamma = \cup_i \Gamma_i$.

The counterpart of the previous definition in the context of operator algebras is given as follows.

Definition 2.1. Let $\mathcal{A} \subset \mathcal{L}(\mathcal{H})$ be a C^* -algebra of bounded operators on a complex separable Hilbert space \mathcal{H} .

- (i) A sequence of non-zero finite rank orthogonal projections $\{P_n\}_{n \in \mathbb{N}} \subset \mathcal{L}(\mathcal{H})$ is called a *Følner sequence for \mathcal{A}* if

$$\lim_n \frac{\|AP_n - P_nA\|_2}{\|P_n\|_2} = 0, \quad A \in \mathcal{A}, \quad (2.2)$$

where $\|\cdot\|_2$ denotes the Hilbert–Schmidt norm.

The Følner sequence $\{P_n\}_n$ is said to be a *proper Følner sequence* if it is an increasing sequence of projections converging to $\mathbb{1}$ in the strong operator topology.

- (ii) \mathcal{A} satisfies the *Følner condition* if for any finite set $\mathcal{F} \subset \mathcal{A}$ and any $\varepsilon > 0$ there exists a finite rank orthogonal projection P such that

$$\frac{\|AP - PA\|_2}{\|P\|_2} < \varepsilon, \quad A \in \mathcal{F}. \quad (2.3)$$

We will state next some immediate consequences of the definition that will be used later on. Note that the definition of a Følner sequence can be extended in an obvious way to any set \mathcal{T} of bounded operators in $\mathcal{L}(\mathcal{H})$ requiring condition (2.2) for all $A \in \mathcal{T}$ (see also Sections 1 and 2 in [26]).

Proposition 2.2. *Let $\mathcal{T} \subset \mathcal{L}(\mathcal{H})$ be a set of operators and $\{P_n\}_{n \in \mathbb{N}}$ a sequence of non-zero finite rank orthogonal projections.*

- (i) $\{P_n\}_{n \in \mathbb{N}}$ is a Følner sequence for \mathcal{T} if and only if it is a Følner sequence for $C^*(\mathcal{T}, \mathbb{1})$ (the C^* -algebra generated by \mathcal{T} and $\mathbb{1}$).
- (ii) Let \mathcal{T} be a self-adjoint set (i.e. $\mathcal{T}^* = \mathcal{T}$). Then $\{P_n\}_{n \in \mathbb{N}}$ is a Følner sequence for \mathcal{A} if and only if one of the four following equivalent conditions holds for all $A \in \mathcal{A}$:

$$\lim_n \frac{\|AP_n - P_nA\|_p}{\|P_n\|_p} = 0, \quad p \in \{1, 2\} \quad (2.4)$$

or

$$\lim_n \frac{\|(I - P_n)AP_n\|_p}{\|P_n\|_p} = 0, \quad p \in \{1, 2\}, \quad (2.5)$$

where $\|\cdot\|_1$ and $\|\cdot\|_2$ are the trace-class and Hilbert–Schmidt norms, respectively.

Proof. Part (i) is straightforward and part (ii) is Lemma 1 in [7]. \square

The following proposition is shown by a standard argument.

Proposition 2.3. *Let $\mathcal{A} \subset \mathcal{L}(\mathcal{H})$ be a separable C^* -algebra. Then, \mathcal{A} has a Følner sequence if and only if \mathcal{A} satisfies the Følner condition.*

2.1. Quasidiagonality

The existence of a Følner sequence for a set of operators \mathcal{T} is a weaker notion than quasidiagonality. Recall that a (separable) set of operators $\mathcal{T} \subset \mathcal{L}(\mathcal{H})$ is said to be quasidiagonal if there exists an increasing sequence of finite-rank projections $\{P_n\}_{n \in \mathbb{N}}$ converging strongly to $\mathbb{1}$ and such that

$$\lim_n \|TP_n - P_nT\| = 0, \quad T \in \mathcal{T}. \quad (2.6)$$

The existence of proper Følner sequences can be understood as a quasidiagonality condition, but relative to the growth of the dimension of the underlying spaces. It can be easily shown that if $\{P_n\}_n$ quasidiagonalizes a family of operators \mathcal{T} , then this sequence of non-zero finite rank orthogonal projections is also a Følner sequence for \mathcal{T} . In [35], Voiculescu characterized abstractly quasidiagonality for unital separable C^* -algebras in terms of u.c.p. maps (see also [36]). This has become by now the standard definition of quasidiagonality for operator algebras (see, for example, [15, Definition 7.1.1]).

Definition 2.4. A unital separable C^* -algebra \mathcal{A} is called *quasidiagonal* if there exists a sequence of u.c.p. maps $\varphi_n: \mathcal{A} \rightarrow M_{k(n)}(\mathbb{C})$ which is both asymptotically multiplicative (i.e. $\|\varphi_n(AB) - \varphi_n(A)\varphi_n(B)\| \rightarrow 0$ for all $A, B \in \mathcal{A}$) and asymptotically isometric (i.e. $\|A\| = \lim_{n \rightarrow \infty} \|\varphi_n(A)\|$ for all $A \in \mathcal{A}$).

The unilateral shift is a prototype that shows the difference between the notions of Følner sequences and quasidiagonality. On the one hand, it is a well-known fact that the unilateral shift S is not a quasidiagonal operator. (This was shown by Halmos in [22]; in fact, in this reference it is shown that S is not even quasi-triangular.) In the setting of abstract C^* -algebras it can also be shown that a C^* -algebra containing a proper (i.e. non-unitary) isometry is not quasidiagonal (see, e.g. [14,15]). It can be shown, though, that certain weighted shifts are quasidiagonal (cf. [32]).

On the other hand, it is easy to give a Følner sequence for S . In fact, define S on $\mathcal{H} := \ell^2(\mathbb{N}_0)$ by $Se_i := e_{i+1}$, where $\{e_i \mid i = 0, 1, 2, \dots\}$ is the canonical basis of \mathcal{H} and consider for any n the orthogonal projections P_n onto $\text{span}\{e_i \mid i = 0, 1, 2, \dots, n\}$. Then

$$\|[P_n, S]\|_2^2 = \sum_{i=1}^{\infty} \|[P_n, S]e_i\|^2 = \|e_{n+1}\|^2 = 1$$

and

$$\frac{\|[P_n, S]\|_2}{\|P_n\|_2} = \frac{1}{\sqrt{n+1}} \xrightarrow{n \rightarrow \infty} 0.$$

3. Approximations of amenable traces

The existence of Følner sequences for a concrete C^* -algebra \mathcal{A} has several operator algebraic consequences. The most prominent one is the existence of an amenable trace on \mathcal{A} . In this section we will review a particularly useful approximation of an amenable trace and consider its application to a spectral approximation problem.

We refer to Chapter 3 of [13] for a careful analysis of subsets of the amenable traces that can be related to suitable finite-dimensional approximation properties of the corresponding C^* -algebra. Fundamental results related to amenable traces were obtained by Kirchberg (using the name *liftable tracial state*) in [24].

Let $\mathcal{A} \subset \mathcal{L}(\mathcal{H})$ be a unital C^* -algebra. A state τ on \mathcal{A} is called an *amenable trace* if there exists a state ψ on $\mathcal{L}(\mathcal{H})$ such that $\psi|_{\mathcal{A}} = \tau$ and

$$\psi(XA) = \psi(AX), \quad X \in \mathcal{L}(\mathcal{H}), A \in \mathcal{A}.$$

The state ψ is also referred to in the literature as a hypertrace on $\mathcal{L}(\mathcal{H})$. Amenable traces are the operator algebraic analogues of the invariant means for groups mentioned at the beginning of the preceding section (cf. [17,18,7,13]). Part (ii) of the following result is known to experts (see e.g. Exercise 6.2.6 in [15]); part (i) is well known and stated in several places in the literature. Since the result is very important for this paper, and for convenience of the reader, we give a complete proof of it.

Proposition 3.1. *Let $\mathcal{A} \subset \mathcal{L}(\mathcal{H})$ be a unital separable C^* -algebra.*

- (i) *If \mathcal{A} has a Følner sequence $\{P_n\}_n$, then \mathcal{A} has an amenable trace.*
- (ii) *Assume that $\mathcal{A} \cap \mathcal{K}(\mathcal{H}) = \{0\}$, and let τ be an amenable trace on \mathcal{A} . Then \mathcal{A} has a Følner sequence $\{P_n\}_n$ satisfying*

$$\tau(A) = \lim_{n \rightarrow \infty} \frac{\text{Tr}(AP_n)}{\text{Tr}(P_n)}, \quad A \in \mathcal{A}, \quad (3.1)$$

where Tr denotes the canonical trace on $\mathcal{L}(\mathcal{H})$.

Proof. The proof of part (i) is a standard argument. Let $\{P_n\}_n$ be a Følner sequence for \mathcal{A} . Consider the following canonical sequence of states of $\mathcal{L}(\mathcal{H})$

$$\psi_n(X) := \frac{\text{Tr}(XP_n)}{\text{Tr}(P_n)}, \quad X \in \mathcal{L}(\mathcal{H}).$$

Using Eq. (2.4) with $p = 1$ it is an $\frac{\varepsilon}{2}$ -argument to show that any weak-* cluster point of the sequence ψ_n (which exists by weak-* compactness) defines a hypertrace on \mathcal{A} .

Part (ii) requires several steps. It is enough to show that for any finite self-adjoint set $\mathcal{F} \subset \mathcal{A}$ and any $1 > \varepsilon > 0$ there exists a finite rank orthogonal projection $Q \in \mathcal{L}(\mathcal{H})$ such that

$$\frac{\|BQ - QB\|_2}{\|Q\|_2} < \varepsilon \quad \text{and} \quad \left| \tau(B) - \frac{\text{Tr}(BQ)}{\text{Tr}(Q)} \right| < \varepsilon, \quad B \in \mathcal{F} \quad (3.2)$$

(cf. Definition 2.1 and Proposition 2.3).

Let $\mathcal{F} \subset \mathcal{A}$ and $1 > \varepsilon > 0$ be given as before. First, from Stinespring's theorem and the proof of Theorem 6.2.7 in [15] there exists a u.c.p. map

$$\varphi: \mathcal{A} \rightarrow M_k(\mathbb{C})$$

(where $M_k(\mathbb{C}) \cong \mathcal{L}(\mathcal{H}_k)$ and $\dim \mathcal{H}_k = k$), an isometry

$$V: \mathcal{H}_k \rightarrow \mathfrak{h}$$

and a representation

$$\pi: \mathcal{A} \rightarrow \mathcal{L}(\mathfrak{h})$$

satisfying

$$\varphi(A) = V^* \pi(A) V, \quad A \in \mathcal{A}. \quad (3.3)$$

$$|\text{tr}(\varphi(B^*B) - \varphi(B^*)\varphi(B))| < \varepsilon, \quad B \in \mathcal{F}. \quad (3.4)$$

$$|\tau(B) - \text{tr}(\varphi(B))| < \varepsilon, \quad B \in \mathcal{F}, \quad (3.5)$$

where $\text{tr}(\cdot)$ is the unique tracial state on the matrix algebra. We introduce next Stinespring's projection

$$P := VV^*,$$

which is a finite rank projection in $\mathcal{L}(\mathfrak{h})$. Using the relation $P\pi(A)P = V\varphi(A)V^*$, $A \in \mathcal{A}$, it is straightforward to show that

$$\frac{\|(\mathbb{1} - P)\pi(B)P\|_2}{\|P\|_2} < \sqrt{\varepsilon}, \quad B \in \mathcal{F}. \quad (3.6)$$

The second step in the proof makes use of Voiculescu's theorem as stated, e.g., in [15, Section 1.7]. Consider the inclusion $\iota: \mathcal{A} \rightarrow \mathcal{L}(\mathcal{H})$. Since $\mathcal{A} \cap \mathcal{K}(\mathcal{H}) = \{0\}$ we have that ι and $\iota \oplus \pi$ are approximately unitarily equivalent relative to compacts. In particular, there is a unitary $W: \mathcal{H} \rightarrow \mathcal{H} \oplus \mathfrak{h}$ such that

$$\|B - W^*(B \oplus \pi(B))W\| < \varepsilon, \quad B \in \mathcal{F}. \quad (3.7)$$

Define the orthogonal projection Q on \mathcal{H} by

$$Q := W^*(0 \oplus P)W$$

and note that $\|Q\|_2 = \|P\|_2$, where the Hilbert–Schmidt norms are considered on the Hilbert spaces \mathcal{H} and \mathfrak{h} , respectively. Putting $Q^\perp := \mathbb{1} - Q$ and using Eqs. (3.6) and (3.7) we have the following estimates for any $B \in \mathcal{F}$:

$$\begin{aligned} \|Q^\perp B Q\|_2 &\leq \|Q^\perp(B - W^*(B \oplus \pi(B))W)Q\|_2 + \|Q^\perp(W^*(B \oplus \pi(B))W)Q\|_2 \\ &\leq \|B - W^*(B \oplus \pi(B))W\| \|Q\|_2 + \|(W^*(\mathbb{1} \oplus P^\perp)(B \oplus \pi(B))(0 \oplus P)W)\|_2 \\ &\leq \varepsilon \|Q\|_2 + \|P^\perp \pi(B) P\|_2 \\ &\leq 2\sqrt{\varepsilon} \|Q\|_2. \end{aligned}$$

Since $\mathcal{F}^* = \mathcal{F}$, we obtain the first condition of Eq. (3.2) (with $4\sqrt{\varepsilon}$ instead of ε).

We still have to show the second condition in Eq. (3.2). Note that for any $B \in \mathcal{F}$ we have

$$\begin{aligned} \text{tr}(\varphi(B)) &= \text{tr}(V^*\pi(B)V) = \frac{\text{Tr}(P\pi(B))}{\text{Tr}(P)} \\ &= \frac{\text{Tr}((0 \oplus P)(B \oplus \pi(B)))}{\text{Tr}(P)} \\ &= \frac{\text{Tr}(Q(W^*(B \oplus \pi(B))W))}{\text{Tr}(Q)}. \end{aligned}$$

Finally, we can use the previous relation as well as Eqs. (3.5) and (3.7) to show the estimates

$$\begin{aligned} \left| \tau(B) - \frac{\text{Tr}(QB)}{\text{Tr}(Q)} \right| &\leq |\tau(B) - \text{tr}(\varphi(B))| + \left| \text{tr}(\varphi(B)) - \frac{\text{Tr}(QB)}{\text{Tr}(Q)} \right| \\ &\leq \varepsilon + \left| \frac{\text{Tr}(Q(W^*(B \oplus \pi(B))W - B))}{\text{Tr}(Q)} \right| \\ &\leq \varepsilon + \|W^*(B \oplus \pi(B))W - B\| < 2\varepsilon, \end{aligned}$$

and the proof is concluded. \square

3.1. Approximation of spectral measures

We will now present an application of [Proposition 3.1\(ii\)](#) to spectral approximation. The argument in the proof of [Theorem 3.2](#) below will be used later in the proof of our main characterization result ([Theorem 4.3](#)).

We need to recall from [7] the definition of Szegő pairs for a concrete C^* -algebra $\mathcal{A} \subset \mathcal{L}(\mathcal{H})$. This notion incorporates the good spectral approximation behavior of scalar spectral measures of self-adjoint elements in \mathcal{A} and is motivated by Szegő's classical approximation results mentioned in the introduction.

Let \mathcal{A} be a unital C^* -algebra acting on \mathcal{H} and let τ be a tracial state on \mathcal{A} . For any self-adjoint element $T \in \mathcal{A}$ we denote by μ_T the spectral measure associated with the trace τ of \mathcal{A} . Consider a sequence $\{P_n\}_n$ of non-zero finite rank projections on \mathcal{H} and write the corresponding (self-adjoint) compressions as $T_n := P_n T P_n$. Denote by μ_T^n the probability measure on \mathbb{R} supported on the spectrum of T_n , i.e.,

$$\mu_T^n(\Delta) := \frac{N_T^n(\Delta)}{d_n}, \quad \Delta \subset \mathbb{R} \text{ Borel},$$

where $N_T^n(\Delta)$ is the number of eigenvalues of T_n (multiplicities counted) contained in Δ and d_n is the dimension of the subspace $P_n \mathcal{H}$.

We say that $\{\{P_n\}_n, \tau\}$ is a *Szegő pair* for \mathcal{A} if $\mu_T^n \rightarrow \mu_T$ weakly for all self-adjoint elements $T \in \mathcal{A}$, i.e.,

$$\lim_{n \rightarrow \infty} \frac{1}{d_n} (f(\lambda_{1,n}) + \dots + f(\lambda_{d_n,n})) = \int f(\lambda) d\mu_T(\lambda), \quad f \in C_0(\mathbb{R}),$$

where $\{\lambda_{1,n}, \dots, \lambda_{d_n,n}\}$ are the eigenvalues (repeated according to multiplicity) of T_n .

By [7, Theorem 6(i), (ii)], if $\{\{P_n\}_n, \tau\}$ is a Szegő pair for \mathcal{A} , then $\{P_n\}_n$ must be a Følner sequence for \mathcal{A} , τ must be an amenable trace, and [Eq. \(3.1\)](#) must hold for every $A \in \mathcal{A}$. [Proposition 3.1\(ii\)](#) allows to complete *any* amenable trace τ on \mathcal{A} with a Følner sequence so that the pair $\{\{P_n\}_n, \tau\}$ is a *Szegő pair* for \mathcal{A} , as follows.

Theorem 3.2. *Let \mathcal{A} be a unital, separable C^* -algebra acting on a separable Hilbert space \mathcal{H} , and assume that $\mathcal{A} \cap \mathcal{K}(\mathcal{H}) = \{0\}$. If τ is an amenable trace on \mathcal{A} , then there exists a proper Følner sequence $\{P_n\}_n$ such that $\{\{P_n\}_n, \tau\}$ is a Szegő pair for \mathcal{A} .*

Proof. By using the same arguments as in the proof of [Proposition 3.1\(ii\)](#), we get that the following local condition is satisfied: for every finite self-adjoint set \mathcal{F} of \mathcal{A} , and for every $\varepsilon > 0$, there exists a finite rank orthogonal projection $Q \in \mathcal{L}(\mathcal{H})$ such that

$$\frac{\|[Q, A]\|_2}{\|Q\|_2} < \varepsilon \quad \text{and} \quad \left| \tau(A) - \frac{\text{Tr}(QA)}{\text{Tr}(Q)} \right| < \varepsilon \quad \text{for all } A \in \mathcal{F}.$$

From this local condition, we are going to construct an increasing sequence $\{P_n\}_n$ such that $P_n \nearrow \mathbb{1}$ in the strong operator topology and such that

$$\lim_n \frac{\|[P_n, A]\|_2}{\|P_n\|_2} = 0, \quad \tau(A) = \lim_n \frac{\text{Tr}(P_n A)}{\text{Tr}(P_n)} \quad \text{for all } A \in \mathcal{A}.$$

Take a countable dense subset $\{A_1, A_2, \dots\}$ of \mathcal{A} , with $A_i \neq 0$ for all i . Take $\varepsilon_n = 2^{-n}$ for all $n \geq 1$ and let Q_n be a finite rank orthogonal projection such that

$$\frac{\| [Q_n, A_i] \|_2}{\| Q_n \|_2} < \varepsilon_n \quad \text{and} \quad \left| \tau(A_i) - \frac{\text{Tr}(Q_n A_i)}{\text{Tr}(Q_n)} \right| < \varepsilon_n \quad \text{for } i = 1, \dots, n.$$

We will show next that we may also assume that

$$\dim(Q_n(\mathcal{H})) \xrightarrow[n \rightarrow \infty]{} \infty.$$

In fact, recall from the proof of [Proposition 3.1\(ii\)](#) that the dimension of Q_n coincides with the dimension of Stinespring's projection associated to the corresponding u.c.p. map $\varphi_n: \mathcal{A} \rightarrow M_{k(n)}(\mathbb{C})$. Since we can replace φ_n with a finite direct sum of n copies of φ_n , without changing the fundamental estimates [\(3.4\)](#) and [\(3.5\)](#), we obtain our claim.

Now consider a sequence $\{R_n\}_n$ of finite-rank orthogonal projections such that $R_n \nearrow \mathbb{1}$. Take $P_1 = Q_1$, $R_1 = Q_1$ and assume that P_1, \dots, P_n have been constructed so that the following conditions hold.

- (1) $R_i \leq P_i$ for $i = 1, \dots, n$.
- (2) $P_1 \leq P_2 \leq \dots \leq P_n$.
- (3) $\| [P_i, A_j] \|_2 < \varepsilon_i \| P_i \|_2$ for $1 \leq j \leq i \leq n$.
- (4) $\left| \tau(A_i) - \frac{\text{Tr}(P_n A_i)}{\text{Tr}(P_n)} \right| < \varepsilon_n$ for $1 \leq i \leq n$.

Since $\dim(Q_l(\mathcal{H})) \xrightarrow[l \rightarrow \infty]{} \infty$, we may take $m > n + 1$ such that¹

$$\| Q_m \|_2 \geq \frac{4 \| R_{n+1} \vee P_n \|_2}{\varepsilon_{n+1}} \max\{1, \| A_1 \|, \dots, \| A_{n+1} \| \}. \quad (3.8)$$

Set $P_{n+1} := R_{n+1} \vee P_n \vee Q_m$ and we have to show that the corresponding equations (1)–(4) above are also true for step $n + 1$. Clearly $R_{n+1} \leq P_{n+1}$ and $P_n \leq P_{n+1}$. We can write $P_{n+1} = Q_m \oplus P'_{n+1}$ with $\| P'_{n+1} \|_2 \leq \| R_{n+1} \vee P_n \|_2$. For $i = 1, \dots, n + 1$, we have

$$\begin{aligned} \frac{\| [P_{n+1}, A_i] \|_2}{\| P_{n+1} \|_2} &\leq \frac{\| [Q_m, A_i] \|_2}{\| P_{n+1} \|_2} + \frac{\| [P'_{n+1}, A_i] \|_2}{\| P_{n+1} \|_2} \\ &\leq \frac{\| [Q_m, A_i] \|_2}{\| Q_m \|_2} + \frac{2 \cdot \| A_i \| \cdot \| P'_{n+1} \|_2}{\| Q_m \|_2} \\ &< \frac{\varepsilon_{n+1}}{2} + \frac{\varepsilon_{n+1}}{2} = \varepsilon_{n+1}, \end{aligned}$$

where for the last estimate we have used [\(3.8\)](#).

Finally, we still have to show condition (4) that implies P_{n+1} is also a good approximation of the amenable trace. Write $\alpha := \frac{\text{Tr}(Q_m)}{\text{Tr}(P_{n+1})} < 1$. Then using again [\(3.8\)](#) note that

$$|1 - \alpha| = \frac{\| P'_{n+1} \|_2^2}{\| P_{n+1} \|_2^2} \leq \frac{\varepsilon_{n+1}^2}{16} \left(\max\{1, \| A_1 \|^2, \dots, \| A_{n+1} \|^2\} \right)^{-1}.$$

¹ If P, Q are orthogonal projections on \mathcal{H} we denote by $P \vee Q$ the orthogonal projection onto the closure of $\text{span}\{P\mathcal{H} \cup Q\mathcal{H}\}$.

Hence using again the decomposition $P_{n+1} = Q_m \oplus P'_{n+1}$ we have for $i = 1, \dots, n+1$:

$$\begin{aligned}
\left| \tau(A_i) - \frac{\text{Tr}(A_i P_{n+1})}{\text{Tr}(P_{n+1})} \right| &\leq \left| \tau(A_i) - \frac{\text{Tr}(A_i Q_m)}{\text{Tr}(P_{n+1})} \right| + \frac{|\text{Tr}(A_i P'_{n+1})|}{\text{Tr}(P_{n+1})} \\
&\leq \left| \tau(A_i) - \frac{\text{Tr}(A_i Q_m)}{\text{Tr}(Q_m)} \right| \\
&\quad + \left| \frac{\text{Tr}(A_i Q_m)}{\text{Tr}(Q_m)} (1 - \alpha) \right| + \frac{|\text{Tr}(A_i P'_{n+1})|}{\text{Tr}(P_{n+1})} \\
&\leq \varepsilon_m + 2\|A_i\| \cdot \frac{\varepsilon_{n+1}^2}{16} \cdot \left(\max_i \{1, \|A_i\|^2\} \right)^{-1} \\
&\leq \varepsilon_m + \frac{\varepsilon_{n+1}^2}{8} \\
&\leq \frac{\varepsilon_{n+1}}{2} + \frac{\varepsilon_{n+1}^2}{8} < \varepsilon_{n+1}.
\end{aligned}$$

It follows that $P_n \nearrow \mathbb{1}$ and that $\lim_n \frac{\|P_n A\|_2}{\|P_n\|_2} = 0$ for all $A \in \mathcal{A}$.

Now the proof of Theorem 6(iii) in [7] gives that $\{\{P_n\}, \tau\}$ is a Szegő pair for \mathcal{A} . \square

Remark 3.3. The preceding theorem is a contribution to the study of Szegő-type theorems in the context of C^* -algebras. Note, nevertheless, that the existence a Følner sequence approximating nicely the amenable trace is established in abstract terms. This gives in general no clue of what the matrix approximations of concrete operators are. It would be interesting to construct in concrete cases explicit Følner sequences of this type to address spectral approximation problems in this more general context (see, e.g., Chapter 7 in [21]).

4. Følner C^* -algebras

In this section, we introduce the abstract definition of a Følner C^* -algebra and we obtain our main result characterizing Følner C^* -algebras in terms of Følner sequences and also of amenable traces. Moreover, we state some consequences for tensor products and nuclear C^* -algebras.

We denote by $\text{tr}(\cdot)$ the unique tracial state on a matrix algebra $M_n(\mathbb{C})$.

Definition 4.1. Let \mathcal{A} be a unital, separable C^* -algebra.

- (i) We say that \mathcal{A} is a *Følner C^* -algebra* if there exists a sequence of u.c.p. maps $\varphi_n: \mathcal{A} \rightarrow M_{k(n)}(\mathbb{C})$ such that

$$\lim_n \|\varphi_n(AB) - \varphi_n(A)\varphi_n(B)\|_{2,\text{tr}} = 0, \quad A, B \in \mathcal{A}, \quad (4.1)$$

where $\|F\|_{2,\text{tr}} := \sqrt{\text{tr}(F^*F)}$, $F \in M_n(\mathbb{C})$.

- (ii) We say that \mathcal{A} is a *proper Følner C^* -algebra* if there exists a sequence of u.c.p. maps $\varphi_n: \mathcal{A} \rightarrow M_{k(n)}(\mathbb{C})$ satisfying Eq. (4.1) and which, in addition, are asymptotically isometric, i.e.,

$$\|A\| = \lim_n \|\varphi_n(A)\|, \quad A \in \mathcal{A}. \quad (4.2)$$

It is clear that if \mathcal{A} is a separable, unital and quasidiagonal C^* -algebra (cf. Definition 2.4), then \mathcal{A} is a proper Følner algebra. Moreover, let \mathcal{B} be a unital C^* -subalgebra of \mathcal{A} . Clearly, if \mathcal{A} is a (proper) Følner algebra, then \mathcal{B} is again a (proper) Følner algebra. This is not true if \mathcal{B} is a non-unital C^* -subalgebra (i.e. $\mathbb{1}_{\mathcal{A}} \notin \mathcal{B}$).

Although, in principle, the two concepts – Følner and properly Følner – seem to be different, we can show that they indeed define the same class of unital, separable C^* -algebras.

Proposition 4.2. *Let \mathcal{A} be a unital separable C^* -algebra. Then \mathcal{A} is a Følner C^* -algebra if and only if \mathcal{A} is a proper Følner C^* -algebra.*

Proof. Assume that \mathcal{A} is a Følner C^* -algebra, and let $\varphi_n: \mathcal{A} \rightarrow M_{k(n)}(\mathbb{C})$ be a sequence of u.c.p maps such that (4.1) holds. Considering the direct sum of a sufficiently large number of copies of φ_n , for each n , we may assume that

$$\lim_{n \rightarrow \infty} \frac{n}{k(n)} = 0. \quad (4.3)$$

Let $\pi: \mathcal{A} \rightarrow \mathcal{L}(\mathcal{H})$ be a faithful representation of \mathcal{A} on a separable Hilbert space \mathcal{H} . Let $\{P_n\}_n$ be an increasing sequence of orthogonal projections on \mathcal{H} , converging to $\mathbb{1}$ in the strong operator topology and such that $\dim(P_n(\mathcal{H})) = n$ for all n . Then for all $A \in \mathcal{A}$ we have $\|A\| = \lim_n \|P_n \pi(A) P_n\|$. Let $\psi_n: \mathcal{A} \rightarrow M_{k(n)+n}(\mathbb{C})$ be given by:

$$\psi_n(A) = \varphi_n(A) \oplus P_n \pi(A) P_n,$$

for $A \in \mathcal{A}$. Then ψ_n is a u.c.p. map. For $A, B \in \mathcal{A}$, set $X_n = P_n \pi(A) (1 - P_n) \pi(B) P_n$. Then we have

$$\begin{aligned} \|\psi_n(AB) - \psi_n(A)\psi_n(B)\|_{2,\text{tr}}^2 &\leq \|\varphi_n(AB) - \varphi_n(A)\varphi_n(B)\|_{2,\text{tr}}^2 + \frac{\text{Tr}(X_n^* X_n)}{k(n) + n} \\ &\leq \|\varphi_n(AB) - \varphi_n(A)\varphi_n(B)\|_{2,\text{tr}}^2 + \frac{n \cdot \|A\|^2 \cdot \|B\|^2}{k(n) + n}. \end{aligned}$$

Using (4.3) we get

$$\lim_n \|\psi_n(AB) - \psi_n(A)\psi_n(B)\|_{2,\text{tr}} = 0.$$

On the other hand, for $A \in \mathcal{A}$, we have

$$\|A\| - \|\psi_n(A)\| \leq \|A\| - \|P_n \pi(A) P_n\| \rightarrow 0$$

so that (4.2) holds for the sequence (ψ_n) . This concludes the proof. \square

For the next result recall that a representation π of an abstract C^* -algebra \mathcal{A} on a Hilbert space \mathcal{H} is called *essential* if $\pi(\mathcal{A})$ contains no non-zero compact operators.

Theorem 4.3. *Let \mathcal{A} be a unital separable C^* -algebra. Then the following conditions are equivalent.*

- (i) *There exists a faithful representation $\pi: \mathcal{A} \rightarrow \mathcal{L}(\mathcal{H})$ such that $\pi(\mathcal{A})$ has a Følner sequence.*

- (ii) *There exists a faithful essential representation $\pi: \mathcal{A} \rightarrow \mathcal{L}(\mathcal{H})$ such that $\pi(\mathcal{A})$ has a Følner sequence.*
- (iii) *Every faithful essential representation $\pi: \mathcal{A} \rightarrow \mathcal{L}(\mathcal{H})$ satisfies that $\pi(\mathcal{A})$ has a proper Følner sequence.*
- (iv) *There exists a non-zero representation $\pi: \mathcal{A} \rightarrow \mathcal{L}(\mathcal{H})$ such that $\pi(\mathcal{A})$ has an amenable trace.*
- (v) *Every faithful representation $\pi: \mathcal{A} \rightarrow \mathcal{L}(\mathcal{H})$ satisfies that $\pi(\mathcal{A})$ has an amenable trace.*
- (vi) *\mathcal{A} is a Følner C^* -algebra.*

Proof. The implications (iii) \Rightarrow (ii) \Rightarrow (i) and (v) \Rightarrow (iv) are obvious. To show that (i) implies (ii) suppose that $\pi: \mathcal{A} \rightarrow \mathcal{L}(\mathcal{H})$ is faithful and denote by $\{P_n\}_n$ a Følner sequence for $\pi(\mathcal{A})$. Define the representation

$$\widehat{\pi}: \mathcal{A} \rightarrow \mathcal{L}\left(\bigoplus_{\infty} \mathcal{H}\right), \quad \widehat{\pi}(A) := \bigoplus_{\infty} \pi(A), \quad A \in \mathcal{A},$$

which, by construction, is essential. Moreover, choose the sequence of finite-rank projections $\widehat{P}_n := P_n \oplus 0 \oplus 0 \cdots$. Since

$$\lim_n \frac{\|\widehat{\pi}(A)\widehat{P}_n - \widehat{P}_n\widehat{\pi}(A)\|_2}{\|\widehat{P}_n\|_2} = \lim_n \frac{\|\pi(A)P_n - P_n\pi(A)\|_2}{\|P_n\|_2} = 0, \quad A \in \mathcal{A},$$

we conclude that $\{\widehat{P}_n\}_n$ is a Følner sequence for $\widehat{\pi}(\mathcal{A})$.

The implication (ii) \Rightarrow (iv) follows from [Proposition 3.1\(i\)](#).

We now show that (iv) \Rightarrow (v), following the proof of [[15](#), Proposition 6.2.2]: let $\pi_0: \mathcal{A} \rightarrow \mathcal{L}(\mathcal{H}_0)$ be a faithful representation and identify \mathcal{A} with $\pi_0(\mathcal{A})$. Let $\pi: \mathcal{A} \rightarrow \mathcal{L}(\mathcal{H})$ be a non-zero representation such that $\pi(\mathcal{A})$ has an amenable trace τ which extends to a hypertrace ψ on $\mathcal{L}(\mathcal{H})$. From Arveson's extension theorem (see, e.g., Theorem 1.6.1 in [[15](#)]), there exists a u.c.p. map $\Phi: \mathcal{L}(\mathcal{H}_0) \rightarrow \mathcal{L}(\mathcal{H})$ extending π . Defining $\psi_0 := \psi \circ \Phi$ it remains to show that ψ_0 is a hypertrace on $\mathcal{L}(\mathcal{H}_0)$ (or $\tau_0 := \psi_0|_{\mathcal{A}}$ is an amenable trace on \mathcal{A}). By construction it is immediate that ψ_0 is a state on $\mathcal{L}(\mathcal{H}_0)$ extending the trace τ_0 . It remains to show that ψ_0 is centralized by \mathcal{A} : for any $X \in \mathcal{L}(\mathcal{H}_0)$ and $A \in \mathcal{A}$ we have

$$\begin{aligned} \psi_0(AX) &= \psi(\Phi(AX)) = \psi(\Phi(A)\Phi(X)) = \psi(\Phi(X)\Phi(A)) \\ &= \psi_0(XA), \end{aligned}$$

where for the second and fourth equalities we have used that \mathcal{A} is a multiplicative domain for Φ .

(v) \Rightarrow (vi) follows from [[15](#), Theorem 6.2.7].

(vi) \Rightarrow (iii): Let $\iota: \mathcal{A} \rightarrow \mathcal{L}(\mathcal{H})$ be a faithful essential representation, and identify \mathcal{A} with its image $\iota(\mathcal{A})$ under ι . Let $\varphi_n: \mathcal{A} \rightarrow M_{k(n)}(\mathbb{C})$ be a sequence of u.c.p. maps such that $\lim_n k(n) = \infty$ and such that [\(4.1\)](#) holds. (Use the trick at the beginning of the proof of [Proposition 4.2](#) to show that we can always get such a sequence $k(n)$.)

By using the same arguments as in the proof of [Theorem 3.2](#) (disregarding the part concerning the approximation of the amenable trace τ), we get that there is a proper Følner sequence $\{P_n\}$ for \mathcal{A} , as desired. \square

- Remark 4.4.** (i) The class of C^* -algebras introduced in this section has been considered before by Bédos. In [6] the author defines a C^* -algebra \mathcal{A} to be weakly hypertracial if \mathcal{A} has a non-degenerate representation π such that $\pi(\mathcal{A})$ has a hypertrace. In this sense, the preceding theorem gives a new characterization of weakly hypertracial C^* -algebras in terms of u.c.p. maps.
- (ii) The equivalences between (i), (iv) and (v) in [Theorem 4.3](#) are essentially known (see [6]).
- (iii) The relation between amenable traces and the asymptotic multiplicativity condition needed in [Definition 4.1\(i\)](#) has been considered before (see, e.g., [Theorem 2.3](#) in [12]).
- (iv) The Toeplitz algebra (i.e., the C^* -algebra generated by the unilateral shift) is an example that shows that not every Følner algebra is a quasidiagonal C^* -algebra.

In the final part of this section we recall some operator algebraic properties of the class of Følner C^* -algebras. Most of them have already been proved in Section 2 of [6] in a more general context. For convenience of the reader and to make this exposition partly self-contained we give short proofs in some cases.

Corollary 4.5. *Let \mathcal{A} be a unital separable C^* -algebra. If a non-zero quotient of \mathcal{A} is a Følner C^* -algebra, then \mathcal{A} is a Følner C^* -algebra. In particular, any C^* -algebra admitting a finite-dimensional representation is a Følner C^* -algebra.*

Proof. This follows from condition (iv) in [Theorem 4.3](#). Indeed, let $\rho: \mathcal{A} \rightarrow \mathcal{B}$ be a surjective $*$ -homomorphism onto a Følner C^* -algebra \mathcal{B} , and let $\pi: \mathcal{B} \rightarrow \mathcal{L}(\mathcal{H})$ be a non-zero representation such that $\pi(\mathcal{B})$ has an amenable trace ([Theorem 4.3\(iv\)](#)). Then $\pi \circ \rho$ is a non-zero representation of \mathcal{A} such that $\pi \circ \rho(\mathcal{A})$ has an amenable trace. By [Theorem 4.3\(iv\)](#) we conclude that \mathcal{A} is a Følner C^* -algebra. \square

Let $\mathcal{A} \subset \mathcal{L}(\mathcal{H})$ be a Følner C^* -algebra. Note that it can happen that \mathcal{A} has no *proper* Følner sequence (recall [Definition 2.1\(i\)](#)) as the following simple example shows: let $\mathcal{B} \subset \mathcal{L}(\mathcal{H}_0)$ be a unital separable C^* -algebra which is not a Følner C^* -algebra acting on an infinite dimensional Hilbert space and define $\mathcal{A} := \mathbb{C} \oplus \mathcal{B}$ on $\mathcal{H} := \mathbb{C} \oplus \mathcal{H}_0$. By the previous corollary \mathcal{A} is a Følner C^* -algebra, and it is readily checked that \mathcal{A} has no proper Følner sequence in $\mathcal{L}(\mathcal{H})$ (although, by [Theorem 4.3\(iii\)](#), it will have a proper Følner sequence in a different representation).

For the next result we recall some standard notation. We will denote by $\mathcal{A} \odot \mathcal{B}$ the algebraic tensor product of two C^* -algebras \mathcal{A} and \mathcal{B} , and by $\mathcal{A} \otimes \mathcal{B}$ its minimal tensor product. The fact that any C^* -tensor product of two Følner C^* -algebras is a Følner C^* -algebra was proved by Bédos in [6, Proposition 2.13]. However the nice interplay between amenable traces and Følner sequences shown in our proof is a genuine application of our approach.

Proposition 4.6. *Let \mathcal{A} and \mathcal{B} be two Følner C^* -algebras, and let $\pi_A: \mathcal{A} \rightarrow \mathcal{L}(\mathcal{H}_A)$ and $\pi_B: \mathcal{B} \rightarrow \mathcal{L}(\mathcal{H}_B)$ be faithful essential representations of \mathcal{A} and \mathcal{B} . Then $\mathcal{A} \otimes \mathcal{B}$ is a Følner C^* -algebra. Moreover if τ_A and τ_B are amenable traces on \mathcal{A} and \mathcal{B} , then there exists a hypertrace on $\mathcal{L}(\mathcal{H}_A \otimes \mathcal{H}_B)$ extending the state $\tau_A \otimes \tau_B$ on $\mathcal{A} \otimes \mathcal{B}$.*

Proof. By Proposition 3.1(ii), there are Følner sequences $\{P_n\}_n$ and $\{Q_n\}_n$ for \mathcal{A} and \mathcal{B} , acting on \mathcal{H}_A and \mathcal{H}_B respectively, such that

$$\tau_{\mathcal{A}}(A) = \lim_{n \rightarrow \infty} \frac{\text{Tr}(AP_n)}{\text{Tr}(P_n)}, \quad A \in \mathcal{A}; \quad \tau_{\mathcal{B}}(B) = \lim_{n \rightarrow \infty} \frac{\text{Tr}(BQ_n)}{\text{Tr}(Q_n)}, \quad B \in \mathcal{B}.$$

We show that $\{P_n \otimes Q_n\}_n$ is a Følner sequence for $\mathcal{A} \otimes \mathcal{B} \subset \mathcal{L}(\mathcal{H}_A \otimes \mathcal{H}_B)$. Let $A \in \mathcal{A}$ and $B \in \mathcal{B}$. Then we have

$$\begin{aligned} & \frac{\|(\mathbb{1} \otimes \mathbb{1} - P_n \otimes Q_n)(A \otimes B)(P_n \otimes Q_n)\|_2^2}{\|P_n \otimes Q_n\|_2^2} \\ & \leq \frac{\|((\mathbb{1} - P_n) \otimes \mathbb{1})(A \otimes B)(P_n \otimes Q_n)\|_2^2}{\|P_n \otimes Q_n\|_2^2} \\ & \quad + \frac{\|(P_n \otimes (\mathbb{1} - Q_n))(A \otimes B)(P_n \otimes Q_n)\|_2^2}{\|P_n \otimes Q_n\|_2^2} \\ & = \frac{\|(\mathbb{1} - P_n)AP_n\|_2^2}{\|P_n\|_2^2} \cdot \frac{\|BQ_n\|_2^2}{\|Q_n\|_2^2} + \frac{\|P_nAP_n\|_2^2}{\|P_n\|_2^2} \cdot \frac{\|(\mathbb{1} - Q_n)BQ_n\|_2^2}{\|Q_n\|_2^2} \\ & \leq \|B\|^2 \cdot \frac{\|(\mathbb{1} - P_n)AP_n\|_2^2}{\|P_n\|_2^2} + \|A\|^2 \cdot \frac{\|(\mathbb{1} - Q_n)BQ_n\|_2^2}{\|Q_n\|_2^2} \xrightarrow{n \rightarrow \infty} 0. \end{aligned}$$

Since the set $\{A \otimes B \mid A \in \mathcal{A}, B \in \mathcal{B}\}$ is a self-adjoint, generating set for $\mathcal{A} \otimes \mathcal{B}$, it follows from Proposition 2.2 that $\{P_n \otimes Q_n\}$ is a Følner sequence for $\mathcal{A} \otimes \mathcal{B}$. This shows that $\mathcal{A} \otimes \mathcal{B}$ is a Følner C^* -algebra.

Let ψ be a weak- $*$ cluster point of the sequence of states $\{\psi_n\}_n$ defined by

$$\psi_n(X) = \frac{\text{Tr}(X(P_n \otimes Q_n))}{\text{Tr}(P_n \otimes Q_n)}, \quad X \in \mathcal{L}(\mathcal{H}_A \otimes \mathcal{H}_B).$$

Then $\psi(A \otimes B) = \tau_{\mathcal{A}}(A)\tau_{\mathcal{B}}(B)$ for all $A \in \mathcal{A}$ and $B \in \mathcal{B}$. Hence, ψ extends the state $\tau_{\mathcal{A}} \otimes \tau_{\mathcal{B}}$ on $\mathcal{A} \otimes \mathcal{B}$. Moreover, ψ is a hypertrace for $\mathcal{A} \otimes \mathcal{B}$ on $\mathcal{L}(\mathcal{H}_A \otimes \mathcal{H}_B)$ (see the proof of Proposition 3.1(i)). This concludes the proof. \square

Corollary 4.7 ([6, Proposition 2.13]). *Let \mathcal{A} and \mathcal{B} be two Følner C^* -algebras, and let α be any C^* -norm on the algebraic tensor product $\mathcal{A} \odot \mathcal{B}$. Then $\mathcal{A} \otimes_{\alpha} \mathcal{B}$ is a Følner C^* -algebra.*

Proof. Let α be any C^* -norm on the algebraic tensor product $\mathcal{A} \odot \mathcal{B}$. Then there is a surjective $*$ -homomorphism $\mathcal{A} \otimes_{\alpha} \mathcal{B} \rightarrow \mathcal{A} \otimes \mathcal{B}$. So the result follows from Proposition 4.6 and Corollary 4.5. \square

The relation with nuclearity is as follows. Recall that there are non-nuclear Følner C^* -algebras, such as $C^*(\mathbb{F}_2)$, the full C^* -algebra of the free group on two generators, which is even quasidiagonal.

Corollary 4.8. *Let \mathcal{A} be a unital nuclear C^* -algebra. Then \mathcal{A} is a Følner C^* -algebra if and only if \mathcal{A} admits a tracial state. In particular, every stably finite unital nuclear C^* -algebra is Følner.*

Proof. The first part follows from [Theorem 4.3](#) and [[15](#), Proposition 6.3.4]. If \mathcal{A} is a stably finite unital nuclear C^* -algebra then \mathcal{A} admits a tracial state by [[8](#), Corollary V.2.1.16]. \square

Note that the Cuntz algebras \mathcal{O}_n are nuclear but not Følner.

Finally, we characterize Følner reduced crossed products. The proof of the following result follows from Proposition 2.12 in [[6](#)]. Let us remark that it is possible to give a variation of Bédos' proof using Day's fixed point theorem (cf., [[19](#)]).

Proposition 4.9. *Let Γ be a countable discrete group and let α be an action of Γ on a separable C^* -algebra \mathcal{A} . Then the following conditions are equivalent.*

- (i) $\mathcal{A} \rtimes_{\alpha,r} \Gamma$ is a Følner C^* -algebra.
- (ii) Γ is amenable and \mathcal{A} has a Γ -invariant amenable trace.
- (iii) \mathcal{A} is a Følner C^* -algebra and Γ is an amenable group.

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