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# ON ALPERT MULTIWAVELETS

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ABSTRACT. The multiresolution analysis of Alpert is considered. Explicit for-mulas for the entries in the matrix coefficients of the refinement equation are given in terms of hypergeometric functions. These entries are shown to solve generalized eigenvalue equations as well as partial difference equations. The matrix coefficients in the wavelet equation are also considered and conditions are given to obtain a unique solution.

## 1. INTRODUCTION

The theory of wavelets has had a broad and lasting impact on various areas of mathematics and engineering such as numerical analysis, signal processing, and harmonic analysis [5], [6], [19], [20]. The most well-known wavelet may be the Haar wavelet which is not continuous and one of the great achievements in the area is Daubechies' construction of compactly supported, orthogonal wavelets that are at least continuous [7]. The theory of one variable multiwavelets [1], [9], [11], [17], [18] is an extension of wavelet theory to the case of when there are several scaling functions instead of just one. This extra flexibility allows the construction of piecewise polynomial scaling functions and wavelets that are compactly supported, orthogonal, and at least continuous [10]. The scaling function associated with the Haar wavelet is the constant function supported on  $[0, 1]$  and zero elsewhere and the linear space associated with this function is the space of piecewise constant polynomials with integer knots. The extension of this space to higher degree polynomials gives the space of piecewise polynomials of degree  $n$  with integer knots and an orthogonal basis for this space are the Legendre polynomials restricted to  $[0, 1]$  and their integer translates. Alpert first developed the multiresolution analyses associated with these spaces and applied them to various problems in integral equations [2] and numerical analysis [8] and [21]. For an alternative use of orthogonal polynomials to construct nontraditional "wavelets" see [16]. An important equation in multiresolution analysis is the refinement equation which links the scaling functions on one level to their scaled versions. Here we examine in more detail the coefficients in the refinement equation associated with the Alpert multiresolution analysis with the intent of obtaining formulas for these coefficients as well as recurrence relations. These lead to combinatorial identities and orthogonality relations

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that seem to have been unnoticed. In section 1 we review Alpert's multiresolution analyses and make contact with the Legendre polynomials. In section 2 we derive various representations for the entries of the matrices in the refinement equation and discuss the orthogonality relations satisfied by these coefficients. In section 3 we develop recurrence formulas satisfied by these coefficients and show that they give rise to some generalized eigenvalue problems. In section 4 we investigate the Fourier transform of the scaling functions which turns out to be related to Bessel functions of half integer order. Using some identities satisfied by Bessel functions we arrive at other recurrences satisfied by the entries in the refinement matrices. Finally in section 5 we consider the matrices in the wavelet equation associated with these multiresolution analyses. These matrices must satisfy certain conditions which follow from the orthogonality of the wavelets to the scaling functions and to the other wavelets. We present natural conditions in order for there to be a unique solution to these equations.

## 2. PRELIMINARIES

Let  $\phi^0, \dots, \phi^r$  be compactly supported  $L^2$ -functions, and suppose that  $V_0 = \text{cl}_{L^2} \text{span}\{\phi^i(\cdot - j) : i = 0, 1, \dots, r, j \in \mathbf{Z}\}$ . Then  $V_0$  is called a *finitely generated shift invariant (FSI) space*. Let  $(V_p)_{p \in \mathbf{Z}}$  be given by  $V_p = \{\phi(2^p \cdot) : \phi \in V_0\}$ . Each space  $V_p$  may be thought of as approximating  $L^2$  at a different resolution depending on the value of  $p$ . The sequence  $(V_p)$  is called a *multiresolution analysis* [7, 12, 14] generated by  $\phi^0, \dots, \phi^r$  if (a) the spaces are nested,  $\dots \subset V_{-1} \subset V_0 \subset V_1 \subset \dots$ , and (b) the generators  $\phi^0, \dots, \phi^r$  and their integer translates form a Riesz basis for  $V_0$ . Because of (a) and (b) above, we can write

$$(1) \quad V_{j+1} = V_j \oplus W_j \quad \forall j \in \mathbf{Z}.$$

The space  $W_0$  is called the *wavelet space*, and if  $\psi^0, \dots, \psi^r$  generate a shift-invariant basis for  $W_0$ , then these functions are called *wavelet functions*. If, in addition,  $\phi^0, \dots, \phi^r$  and their integer translates form an orthogonal basis for  $V_0$ , then  $(V_p)$  is called an *orthogonal MRA*. Let  $S_{-1}^n$  be the space of polynomial splines of degree  $n$  continuous except perhaps at the integers, and set  $V_0^n = S_{-1}^n \cap L^2(\mathbf{R})$ . With  $V_p^n$  as above these spaces form a multiresolution analysis. If  $n = 0$  the multiresolution analysis obtained is associated with the Haar wavelet while for  $n > 0$  they were introduced by Alpert [1, 2]. If we let

$$\phi_j(t) = \begin{cases} \hat{p}_j(2t - 1), & 0 \leq t < 1 \\ 0, & \text{elsewhere,} \end{cases}$$

where  $\hat{p}_j(t)$  is the Legendre polynomial [22] of degree  $j$  orthonormal on  $[-1, 1]$  with positive leading coefficient i.e.  $\hat{p}_j(t) = k_j t^j + \text{lower degree terms}$  with  $k_j > 0$  and

$$\int_{-1}^1 \hat{p}_j(t) \hat{p}_k(t) dt = \delta_{k,j},$$

then

$$(2) \quad \Phi_n = [\phi_0 \quad \dots \quad \phi_n]^T,$$

and its integer translates form an orthogonal basis for  $V_0$ . For the convenience in later computations we set

$$(3) \quad P_n(t) = \begin{bmatrix} \hat{p}_0(t) \\ \vdots \\ \hat{p}_n(t) \end{bmatrix} \chi_{[0,1]}.$$

Equation (1) implies the existence of the *refinement* equation,

$$(4) \quad \Phi_n\left(\frac{t}{2}\right) = C_{-1}^n \Phi_n(t) + C_1^n \Phi_n(t-1),$$

where the  $C_i^n$ ,  $i = -1, 1$ , are  $(n+1) \times (n+1)$  matrices. The orthonormality of the entries in  $\Phi_n\left(\frac{t}{2}\right)$  implies that

$$(5) \quad 2I_{n+1} = C_{-1}^n C_{-1}^{nT} + C_1^n C_1^{nT},$$

where  $I_n$  is the  $n \times n$  identity matrix and  $A^T$  is the transpose of  $A$ . In terms of the entries of  $P_n$  we see

$$(6) \quad \hat{p}_i(t) = \sum_{j=0}^i (C_{-1}^n)_{i,j} \hat{p}_j(2t+1)|_{[-1,0]} + \sum_{j=0}^i (C_1^n)_{i,j} \hat{p}_j(2t-1)|_{[0,1]},$$

for  $-1 \leq t \leq 1$ . In order to exploit the symmetry of the Legendre polynomials we shift  $t \rightarrow t+1$  so that

$$(7) \quad \begin{aligned} \Phi_n\left(\frac{t+1}{2}\right) &= P_n(t) = C_{-1}^n \Phi_n(t+1) + C_1^n \Phi_n(t) \\ &= C_{-1}^n P_n(2t+1) + C_1^n P_n(2t-1). \end{aligned}$$

The monic polynomials have the following representation in terms of a  ${}_2F_1$  hypergeometric function [22, p. 80],

$$(8) \quad p_n(t) = \frac{2^n n!}{(n+1)_n} {}_2F_1\left(\begin{matrix} -n, n+1 \\ 1 \end{matrix}; \frac{1-t}{2}\right),$$

where formally,

$${}_pF_q\left(\begin{matrix} a_1, \dots, a_p \\ b_1, \dots, b_q \end{matrix}; t\right) = \sum_{i=0}^{\infty} \frac{(a_1)_i \dots (a_p)_i}{(b_1)_i \dots (b_q)_i (1)_i} t^i$$

with  $(a)_0 = 1$  and  $(a)_i = (a)(a+1)\dots(a+i-1)$  for  $i > 0$ . Since one of the numerator parameters in the definition of  $p_n$  is a negative integer the series in equation (8) has only finitely many terms. The relation between  $\hat{p}_n$  and  $p_n$  is given by

$$(9) \quad \hat{p}_n(t) = \frac{\sqrt{2n+1}(2n-1)!!}{\sqrt{2}n!} p_n(t).$$

A representation that makes the symmetry of the Legendre polynomials manifest is [22, p. 83]

$$(10) \quad p_{2n}(x) = (-1)^n \frac{(1/2)_n}{(n+1/2)_n} {}_2F_1\left(\begin{matrix} -n, n+1/2 \\ 1/2 \end{matrix}; x^2\right),$$

and

$$(11) \quad p_{2n+1}(x) = (-1)^n \frac{(3/2)_n x}{(n+3/2)_n} {}_2F_1\left(\begin{matrix} -n, n+3/2 \\ 3/2 \end{matrix}; x^2\right).$$

Finally we recall the well-known recurrence formula satisfied by the monic Legendre polynomial,

$$(12) \quad p_{n+1}(t) = tp_n(t) - \frac{n^2}{(2n+1)(2n-1)}p_{n-1}(t).$$

**2.1. Coefficient representations.** Since the Legendre polynomials are symmetric or antisymmetric we need only compute  $C_1$  which equation (7) shows is given by

$$(13) \quad C_1^n = \int_0^1 P_n(t)P_n(2t-1)^T dt,$$

so that

$$(14) \quad (C_1^n)_{i,j} = \int_0^1 \hat{p}_i(t)\hat{p}_j(2t-1)dt.$$

In the above equation we index the entries in  $C_1^n$  beginning with  $i = 0, j = 0$ . Because of the orthogonality of the Legendre polynomials to powers of  $t$  less than their degree the above integral is equal to zero for  $i < j$ . Summarizing we find

**Lemma 1.** *Let  $C_1^n$  and  $C_{-1}^n$  be the matrix coefficients in the above refinement equation. Then  $C_1^n$  is a lower triangular matrix with positive diagonal entries. Furthermore*

$$(15) \quad (C_{-1}^n)_{i,j} = (-1)^{i+j}(C_1^n)_{i,j}, \quad i, j \geq 0,$$

which gives the orthogonality relations

$$(16) \quad 0 = ((-1)^{i+k} + 1) \sum_{j=0}^i (C_1^n)_{i,j}(C_1^n)_{k,j} \quad k > i,$$

and

$$(17) \quad 1 = \sum_{j=0}^i (C_1^n)_{i,j}(C_1^n)_{i,j}.$$

We examine the above integral using monic polynomials  $p_i$  which in terms of hypergeometric functions is

$$(18) \quad I_{i,j}^1 = \int_0^1 p_i(t)p_j(2t-1)dt$$

$$(19) \quad = \frac{2^{i+j}(1)_i(1)_j}{(i+1)_i(j+1)_j} I_{i,j}^2,$$

where

$$I_{i,j}^2 = (-1)^j \int_0^1 {}_2F_1 \left( \begin{matrix} -i, i+1 \\ 1 \end{matrix}; \frac{1-t}{2} \right) {}_2F_1 \left( \begin{matrix} -j, j+1 \\ 1 \end{matrix}; t \right) dt.$$

The symmetry of the Legendre polynomials has been used to obtain the last expression. From the definition of the hypergeometric functions we find after integration,

$$I_{i,j}^2 = (-1)^j \sum_{k=0}^i \sum_{n=0}^j \frac{(-i)_k(i+1)_k}{(1)_k(1)_k 2^k} \frac{(-j)_n(j+1)_n}{(1)_n(1)_n} \frac{k!n!}{(n+k+1)!}.$$

Since  $(n+k+1)! = (k+2)_n(1)_{k+1}$  the sum on  $n$  equals  ${}_2F_1\left(\begin{matrix} -j, j+1 \\ k+2 \end{matrix}; 1\right) = \frac{(k-j+1)_j}{(k+2)_j}$  by the Chu-Vandermonde formula [4, p. 3], so

$$I_{i,j}^2 = (-1)^j \sum_{k=j}^i \frac{(-i)_k(i+1)_k(k-j+1)_j}{(1)_k(1)_{k+1}2^k(k+2)_j},$$

where the fact that  $(k-j+1)_j = 0$  for  $k < j$  has been used to obtain the equality. Shifting  $k$  by  $k-j$ , then using the identities  $(a+j)_k = (a)_k(a+j)_k$  with  $a = -i, i+1$ ,  $(k+1)_j = \frac{(1)_j(j+1)_k}{(1)_k}$ , and  $(k+j+2)_j = \frac{(1)_{2j+1}(2j+2)_k}{(1)_{j+1}(j+2)_k}$ , yields

$$\begin{aligned} I_{i,j}^2 &= (-1)^j \frac{(-i)_j(i+1)_j}{(1)_{2j+1}2^j} \sum_{k=0}^{-i+j} \frac{(-i+j)_k(i+j+1)_k}{(1)_k(2j+2)_k2^k} \\ &= (-1)^j \frac{(-i)_j(i+1)_j}{(1)_{2j+1}2^j} {}_2F_1\left(\begin{matrix} -i+j, i+j+1 \\ 2j+2 \end{matrix}; \frac{1}{2}\right). \end{aligned}$$

Substituting this into equation (18) yields

$$(20) \quad I_{i,j}^1 = \frac{2^i(1)_i(i+1)_j(1)_j}{(1)_{i-j}(i+1)_i(j+1)_j(1)_{2j+1}} {}_2F_1\left(\begin{matrix} -i+j, i+j+1 \\ 2j+2 \end{matrix}; \frac{1}{2}\right),$$

where we have used the identity  $(-1)^j(-i)_j = (1)_i/(1)_{i-j}$ . This shows that

$$\begin{aligned} (C_1^n)_{i,j} &= \frac{(2i-1)!!(2j-1)!!\sqrt{(2i+1)(2j+1)}}{(1)_j(1)_i} I_{i,j}^1 \\ (21) \quad &= l_{i,j} {}_2F_1\left(\begin{matrix} -i+j, i+j+1 \\ 2j+2 \end{matrix}; \frac{1}{2}\right), \end{aligned}$$

where

$$(22) \quad l_{i,j} = \sqrt{\frac{2i+1}{2j+1} \frac{(i+j)!}{2^j(2j)!(i-j)!}}.$$

When the parity of  $i$  and  $k$  are the same, the sum in equation (16) must be equal to zero and it is easy to check that the sum in (16) is not in general equal to zero when  $i$  and  $k$  are of different parities. If we set  $n = i - j$  in the hypergeometric function above, the function becomes

$$(23) \quad 2^n \frac{(2j+2)_n}{(n+2j+1)_n} {}_2F_1\left(\begin{matrix} -n, n+2j+1 \\ 2j+2 \end{matrix}; \frac{1}{2}\right) = p_n^{(2j+1, -1)}(0),$$

where  $p_n^{(\alpha, \beta)}(x)$  is the monic Jacobi polynomial. Since  $\beta = -1$ ,  $p_n^{(2j+1, -1)}(x)$  is not in the standard class of Jacobi orthogonal polynomials, furthermore in the discrete orthogonality above both the degree and the order are changing. The representation given in equation (21) suggests an easy recurrence formula in  $i$  but not so simple in  $j$ . A useful representation for the above hypergeometric function that simplifies the dependence on  $j$  maybe obtained by using the transformation  ${}_2F_1\left(\begin{matrix} -n, b \\ c \end{matrix}; x\right) = \frac{(b)_n}{(c)_n}(-x)^n {}_2F_1\left(\begin{matrix} -n, -c-n+1 \\ -b-n+1 \end{matrix}; \frac{1}{x}\right)$  which yields,

$$\begin{aligned} \hat{l}_{i,j} {}_2F_1\left(\begin{matrix} -(i-j), i+j+1 \\ 2j+2 \end{matrix}; \frac{1}{2}\right) &= {}_2F_1\left(\begin{matrix} -(i-j), -i-j-1 \\ -2i \end{matrix}; 2\right) \\ (24) \quad &= {}_2F_1\left(\begin{matrix} -n, -2i+n-1 \\ -2i \end{matrix}; 2\right), \end{aligned}$$

where  $\hat{l}_{i,j} = (-2)^{i-j} \frac{(i+j+1)!(i+j)!}{(2j+2)!(2i)!}$  and  $n = i - j$ . The last equality shows that the hypergeometric function is related to Krawtchouk polynomials [3, p. 347].

The orthogonality relation (16) is nontrivial only among the even and odd rows of  $C_1^n$ . To take this into account we use the expressions (10) and (11). Furthermore in order to make apparent the polynomial character in  $j$  of the resulting hypergeometric function we use the transformation leading to equation (24). In this case

$$\begin{aligned} I_{2i,j}^1 &= (-1)^{i+j} 2^j \frac{(1)_i (1)_j}{(-i)_i (j+1)_j} \int_0^1 t^{2i} {}_2F_1 \left( \begin{matrix} -i, -i+1/2 \\ -2i+1/2 \end{matrix}; 1/t^2 \right) {}_2F_1 \left( \begin{matrix} -j, j+1 \\ 1 \end{matrix}; t \right) \\ (25) \quad &= (-1)^{i+j} 2^j \frac{(1)_i (1)_j}{(-i)_i (j+1)_j} \sum_{k=0}^i \frac{(-i)_k (-i+1/2)_k}{(1)_k (-2i+1/2)_k} S_{j,n}^e, \end{aligned}$$

where

$$\begin{aligned} S_{j,j}^e &= \sum_{n=0}^j \frac{(-j)_n (j+1)_n}{(1)_n (1)_n} \int_0^1 t^{2(i-k)+n} dt \\ &= \frac{1}{2(i-k)+1} \sum_{n=0}^j \frac{(-j)_n (j+1)_n (2(i-k)+1)_n}{(1)_n (1)_n (2(i-k)+2)_n}. \end{aligned}$$

The last sum is  ${}_3F_2 \left( \begin{matrix} -j, j+1, 2(i-k)+1 \\ 1, 2(i-k)+2 \end{matrix}; 1 \right) = \frac{(-j)_j (2(i-k)-j+1)_j}{(1)_j (2(i-k)+2)_j}$  where the Pfaff-Saalschutz formula [4, p.9] has been used since the hypergeometric function is balanced (i.e the sum of the numerator parameter is one less than the sum of the denominator parameters). Substitution of the above result in equation (25) yields

$$(26) \quad I_{2i,j}^1 = (-1)^{i+j} 2^j \frac{(1)_i (-j)_j}{(-i)_i (j+1)_j} \hat{S}_{2i,j},$$

where

$$(27) \quad \hat{S}_{2i,j} = \sum_{k=0}^i \frac{(-i)_k (-i+1/2)_k (2(i-k)-j+1)_j}{(1)_k (-2i+1/2)_k (2(i-k)+1)_{j+1}}.$$

Now it is most convenient to consider  $j$  even or odd. For  $j \rightarrow 2j$  the above sum is equal to zero for  $i - j < k$ . Thus,

$$\hat{S}_{2i,2j} = \sum_{k=0}^{i-j} \frac{(-i)_k (-i+1/2)_k (2(i-k)-2j+1)_{2j}}{(1)_k (-2i+1/2)_k (2(i-k)+1)_{2j+1}}.$$

For  $m = 0, 1$  we have the equations,

$$\begin{aligned} (2(i-k-j)+1+m)_{2j-m} &= 2^{2j-m} \frac{(i-k-j+\frac{m+1}{2})(i-k-j+1+\frac{m}{2}) \cdots (i-\frac{1}{2})(i)}{(-i+\frac{1}{2})_k (-i)_k} \\ &= (-1)^m 2^{2j-m} \frac{((-i+j+\frac{-m+1}{2})_k (-i+j-\frac{m}{2})_k (-i)_j (-i+\frac{1}{2})_{j-m}}{(-i+1/2)_k (-i)_k}, \end{aligned}$$

and  $(2(i-k)+1)_{2j+1-m} = 2^{2j+1-m} \frac{(-i+1/2)_k (i+1/2)_{j+1-m} (-i)_k (i+1)_j}{(-i-j+m-1/2)_k (-i-j)_k}$ . Thus with  $m=0$ ,

$$\begin{aligned} \hat{S}_{2i,2j} &= \frac{1}{2} \frac{(-i)_j (-i+1/2)_j}{(i+1/2)_{j+1} (i+1)_j} \sum_{k=0}^{i-j} \frac{(-i+j)_k (-i+j+1/2)_k (-i-j-1/2)_k (-i-j)_k}{(1)_k (-2i+1/2)_k (-i)_k (-i+1/2)_k} \\ &= \frac{1}{2} \frac{(-i)_j (-i+1/2)_j}{(i+1/2)_{j+1} (i+1)_j} {}_4F_3 \left( \begin{matrix} -i+j, -i+j+1/2, -i-j-1/2, -i-j \\ -2i+1/2, -i, -i+1/2 \end{matrix}; 1 \right). \end{aligned}$$

Substitution of this into equation (27) yields

$$(28) \quad I_{2i,2j}^1 = 2^{2j-1} \frac{(-i)_j (-i+1/2)_j (2j)!}{(i+1/2)_{j+1} (i+1)_j (2j+1)_{2j}} {}_4F_3 \left( \begin{matrix} -(i-j), -i+j+1/2, -i-j-1/2, -i-j \\ -2i+1/2, -i, -i+1/2 \end{matrix}; 1 \right).$$

With  $j \rightarrow 2j-1$  in equation (26) and  $m=1$  in the above identities we obtain

$$(29) \quad I_{2i,2j-1}^1 = 2^{2j-2} \frac{(-i)_j (-i+1/2)_{j-1} (2j-1)!}{(i+1/2)_j (2j)_{2j-1} (i+1)_j} {}_4F_3 \left( \begin{matrix} -(i-j), -i+j-1/2, -i-j+1/2, -i-j \\ -2i+1/2, -i, -i+1/2 \end{matrix}; 1 \right).$$

Similar manipulations for  $i$  odd lead to

$$(30) \quad I_{2i+1,2j}^1 = 2^{2j-1} \frac{(-i)_j (-i-1/2)_j (2j)!}{(i+3/2)_j (i+1)_{j+1} (2j+1)_{2j}} {}_4F_3 \left( \begin{matrix} -(i-j), -i+j-1/2, -i-j-1, -i-j-1/2 \\ -2i-1/2, -i, -i-1/2 \end{matrix}; 1 \right),$$

and

$$(31) \quad I_{2i+1,2j+1}^1 = 2^{2j} \frac{(-i)_j (-i-1/2)_{j+1} (2j+1)!}{(i+3/2)_{j+1} (i+1)_{j+1} (2j+2)_{2j+1}} {}_4F_3 \left( \begin{matrix} -(i-j), -i+j+1/2, -i-j-1, -i-j-3/2 \\ -2i-1/2, -i, -i-1/2 \end{matrix}; 1 \right).$$

Collecting the above computations gives:

**Theorem 2.** *The entries in the matrix  $C_1^n$  have the following representations:*

$$(32) \quad \begin{aligned} (C_1^n)_{i,j} &= \frac{\sqrt{(2i+1)(2j+1)}(i+j)!}{2^j(2j+1)!(i-j)!} {}_2F_1 \left( \begin{matrix} -i+j, i+j+1 \\ 2j+2 \end{matrix}; \frac{1}{2} \right) \\ &= (-1)^{i-j} \frac{\sqrt{(2i+1)(2j+1)}(2i)!}{2^i(i+j+1)!(i-j)!} {}_2F_1 \left( \begin{matrix} -i+j, -i-j-1 \\ -2i \end{matrix}; 2 \right). \end{aligned}$$

Alternatively,

$$(33) \quad (C_1^n)_{2i,j} = W_{2i,j} {}_4F_3 \left( \begin{matrix} -i + \lceil \frac{j}{2} \rceil, -i + \lfloor \frac{j}{2} \rfloor + 1/2, -i - \lceil \frac{j}{2} \rceil, -i - \lfloor \frac{j}{2} \rfloor - 1/2 \\ -2i + 1/2, -i, -i + 1/2 \end{matrix}; 1 \right)$$



and

$$(34) \quad (C_1^n)_{2i+1,j} = W_{2i+1,j} {}_4F_3 \left( \begin{matrix} -i + \lfloor \frac{j}{2} \rfloor, & -i + \lceil \frac{j}{2} \rceil - 1/2, & -i - \lfloor \frac{j}{2} \rfloor - 1, & -i - \lceil \frac{j}{2} \rceil - 1/2; & 1 \end{matrix} \right),$$

with

$$W_{2i,j} = K_{2i,j} \frac{2^{j-1} j! (-i)_{\lceil \frac{j}{2} \rceil} (-i - \frac{1}{2})_{\lfloor \frac{j}{2} \rfloor}}{(i + \frac{1}{2})_{\lfloor \frac{j}{2} \rfloor + 1} (i+1)_{\lceil \frac{j}{2} \rceil} (j+1)_j},$$

$$W_{2i+1,j} = K_{2i+1,j} \frac{2^{j-1} j! (-i)_{\lceil \frac{j}{2} \rceil} (-i + \frac{1}{2})_{\lfloor \frac{j}{2} \rfloor}}{(i + \frac{1}{2})_{\lfloor \frac{j}{2} \rfloor + 1} (i+1)_{\lceil \frac{j}{2} \rceil} (j+1)_j},$$

and

$$K_{i,j} = \frac{(2i-1)!!(2j-1)!!\sqrt{(2i+1)(2j+1)}}{(1)_i(1)_j}.$$

In all cases the above hypergeometric functions are balanced. Also the above functions satisfy the orthogonality relations given by equations (16) and (17).

The values of  $(C_1^n)_{i,j}$  for  $j = i, i-1$ , and  $i-2$  with  $n > 2$  are simple and given by

$$(35) \quad (C_1^n)_{i,i} = \frac{1}{2^i}, \quad (C_1^n)_{i,i-1} = \frac{\sqrt{(2i+1)(2i-1)}}{2^i},$$

and

$$(36) \quad (C_1^n)_{i,i-2} = \frac{(i-2)\sqrt{(2i+1)(2i-1)}}{2^i}.$$

For  $n > 1$  we find using Kummer's theorem [4, p.9],

$$(C_1^n)_{i,0} = \sqrt{2i+1} \frac{\Gamma(3/2)}{\Gamma((2-i)/2)\Gamma((i+3)/2)}$$

$$= \begin{cases} 0, & i \text{ even, } i > 0, \\ (-1)^{\frac{i-1}{2}} \frac{\sqrt{2i+1}}{2} \left(\frac{1}{2}\right)_{\frac{i-1}{2}} / ((i+1)/2)!, & i \text{ odd, } i > 0, \end{cases}$$

where  $\Gamma$  is the Gamma function. That  $(C_1^n)_{2i,0} = 0$  also follows from the symmetry and orthogonality of the Legendre polynomials. For the simplest case when  $n = 0$  i.e. piecewise constant scaling functions we find that

$$C_1^0 = 1.$$

For other  $n$  we find,

$$C_1^1 = \begin{pmatrix} 1 & 0 \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}, \quad C_1^2 = \begin{pmatrix} 1 & 0 & 0 \\ \frac{\sqrt{3}}{2} & \frac{1}{2} & 0 \\ 0 & \frac{\sqrt{15}}{4} & \frac{1}{4} \end{pmatrix}, \quad \text{and } C_1^3 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ \frac{\sqrt{3}}{2} & \frac{1}{2} & 0 & 0 \\ 0 & \frac{\sqrt{15}}{4} & \frac{1}{4} & 0 \\ -\frac{\sqrt{7}}{8} & \frac{\sqrt{21}}{8} & \frac{\sqrt{35}}{8} & \frac{1}{8} \end{pmatrix}.$$

### 3. RECURRENCE FORMULAS AND GENERALIZED EIGENVALUE PROBLEM

The contiguous relations for hypergeometric functions give recurrence formulas among the entries in the matrix  $C_1^n$  which we now study. A useful and well-known relation ([3, equation (2.5.15)]) that  ${}_2F_1$  hypergeometric functions satisfy is the following:

$$(37) \quad e_1 {}_2F_1 \left( \begin{matrix} a-1, b+1 \\ c \end{matrix}; x \right) = e_2 {}_2F_1 \left( \begin{matrix} a, b \\ c \end{matrix}; x \right) + e_3 {}_2F_1 \left( \begin{matrix} a+1, b-1 \\ c \end{matrix}; x \right),$$

where

$$e_1 = 2b(c-a)(b-a-1), \quad e_3 = 2a(b-c)(b-a+1),$$

and

$$e_2 = (b-a)[(1-2x)(b-a-1)(b-a+1) + (b+a-1)(2c-b-a-1)].$$

With  $x = 1/2$ ,  $a = -i+j$ ,  $b = i+j+1$ ,  $c = 2j+2$  and the definition of  $l_{i,j}$  we find

$$\frac{(i+j+2)(i+1-j)i}{\sqrt{(2i+3)(2i+1)j(j+1)}} (C_1^n)_{i+1,j} = (C_1^n)_{i,j} - \frac{(i+j)(i-j-1)(i+1)}{\sqrt{(2i-1)(2i+1)j(j+1)}} (C_1^n)_{i-1,j},$$

where the top line of equation (32) has been used. Since  $(i+j+2)(i-j+1) = (i+1)(i+2) - j(j+1)$  we see that the above equation can be recast as a generalized eigenvalue equation,

$$(38) \quad A_i (C_1^n)_{i,j} = j(j+1) B_i (C_1^n)_{i,j}, \quad 0 \leq j \leq i < n,$$

where

$$(39) \quad A_i = \frac{i(i+1)(i+2)}{\sqrt{(2i+3)(2i+1)}} E_+ + \frac{(i-1)(i)(i+1)}{\sqrt{(2i+1)(2i-1)}} E_-,$$

and

$$(40) \quad B_i = \frac{i}{\sqrt{(2i+3)(2i+1)}} E_+ + 1 + \frac{i+1}{\sqrt{(2i+1)(2i-1)}} E_-.$$

Here  $E_{\pm}$  are the forward and backward shifts in  $i$  respectively. In the above equation the fact that  $(C_1^n)_{i,j} = 0$  for  $i < j$  has been used. To obtain a recurrence for fixed  $i$  substitute  $x = 2$ ,  $a = -i+j$ ,  $b = -i-j-1$  and  $c = -2i$  in equation (37) which when coupled with the second line in equation (32) yields

$$\begin{aligned} & \frac{(i+j)(j+1)(i-j+1)}{\sqrt{(2j+1)(2j-1)}} (C_1^n)_{i,j-1} - 3(j+1)j(C_1^n)_{i,j} \\ & + \frac{(i+j+2)j(i-j-1)}{\sqrt{(2j+1)(2j+3)}} (C_1^n)_{i,j+1} = -i(i+1)(C_1^n)_{i,j}. \end{aligned}$$

This also can be recast as the generalized eigenvalue equation,

$$(41) \quad \hat{A}_j (C_1^n)_{i,j} = i(i+1) \hat{B}_j (C_1^n)_{i,j}, \quad 0 < j \leq i < n,$$

where

$$(42) \quad \hat{A}_j = \frac{j(j+1)(j+2)}{\sqrt{(2j+3)(2j+1)}} \hat{E}_+ + 3j(j+1)1 + \frac{(j-1)(j)(j+1)}{\sqrt{(2j+1)(2j-1)}} \hat{E}_-,$$

and

$$(43) \quad \hat{B}_j = \frac{j}{\sqrt{(2j+3)(2j+1)}} \hat{E}_+ + 1 + \frac{j+1}{\sqrt{(2j+1)(2j-1)}} \hat{E}_-.$$

Here  $\hat{E}_\pm$  are the forward, backward shifts in  $j$  respectively. As above we use the condition that  $(C_1^n)_{i,j} = 0$  for  $i < j$ . An interesting formula may be found by eliminating  $p_j(2t-1)$  in (18) using equation (12) which gives

$$I_{i,j}^1 = -I_{i,j-1}^1 - \frac{(j-1)^2}{(2j-1)(2j-3)} I_{i,j-2}^1 + 2 \int_0^1 t p_i(t) p_{j-1}(2t-1) dt.$$

Now eliminating  $tp_i(t)$  yields

$$I_{i,j}^1 = -I_{i,j-1}^1 - \frac{(j-1)^2}{(2j-1)(2j-3)} I_{i,j-2}^1 + 2I_{i+1,j-1}^1 + \frac{2i^2}{(2i+1)(2i-1)} I_{i-1,j-1}^1.$$

The first line of equation (21) yields after increasing  $j$  by one,

$$(44) \quad \tilde{A}_j(C_1^n)_{i,j} = \tilde{B}_i(C_1^n)_{i,j}, \quad 0 \leq j \leq i < n,$$

where

$$(45) \quad \tilde{A}_j = \frac{j}{\sqrt{(2j+1)(2j-1)}} \hat{E}_- + 1 + \frac{j+1}{\sqrt{(2j+3)(2j+1)}} \hat{E}_+,$$

and

$$(46) \quad \tilde{B}_i = \frac{2i}{\sqrt{(2i+1)(2i-1)}} E_- + \frac{2(i+1)}{\sqrt{(2i+3)(2i+1)}} E_+.$$

With the above we formulate:

**Theorem 3.** *Let  $C_1^n$  and  $C_{-1}^n$  be as in Theorem (2). Then they satisfy the generalized eigenvalue problems given in equations (38) and (41) and the difference equation (44).*

#### 4. THE FOURIER TRANSFORM

An important object in wavelet theory is the Fourier transform of the scaling functions. To exploit the symmetry of the Legendre polynomials we will use equation (7) and define

$$\tilde{P}_n(a) = \int_{-\infty}^{\infty} e^{iat} \Phi_n\left(\frac{t+1}{2}\right) dt = \int_{-1}^1 e^{iat} P_n(t) dt,$$

so that

$$(47) \quad \tilde{P}_n(a) = T_n(a) \tilde{P}_n(a/2),$$

where

$$(48) \quad T_n(a) = (C_{-1}^n e^{ia/2} + C_1^n e^{-ia/2})/2.$$

Since (see [15])

$$(49) \quad \int_{-1}^1 e^{iat} \hat{p}_n(t) dt = \sqrt{2n+1} \sqrt{2\pi} i^n J_{n+1/2}(a) / \sqrt{a},$$

where  $J_\nu$  is the Bessel function of order  $\nu$ , we obtain the addition formula,

$$(50) \quad \begin{aligned} & \sqrt{2j+1} i^j \frac{J_{j+1/2}(a)}{\sqrt{a}} \\ &= \frac{1}{2} \sum_{k=0}^j (C_1^n)_{j,k} ((-1)^{j+k} e^{ia/2} + e^{-ia/2}) i^k \sqrt{2k+1} \frac{J_{k+1/2}(a/2)}{\sqrt{\frac{a}{2}}}, \end{aligned}$$

where the symmetry properties of entries of  $C_{-1}^n$  have been used. Thus for  $j \rightarrow 2j$  in the above formula we find

$$\begin{aligned} \sqrt{4j+1} (-1)^j \frac{J_{2j+1/2}(a)}{\sqrt{a}} &= \cos(a/2) \sum_{k=0}^j (-1)^k (C_1^n)_{2j,2k} \sqrt{4k+1} \frac{J_{2k+1/2}(a/2)}{\sqrt{\frac{a}{2}}} \\ &\quad + \sin(a/2) \sum_{k=0}^{j-1} (-1)^k (C_1^n)_{2j,2k+1} \sqrt{4k+3} \frac{J_{2k+3/2}(a/2)}{\sqrt{\frac{a}{2}}}, \end{aligned}$$

while for  $j \rightarrow 2j+1$ ,

$$\begin{aligned} \sqrt{4j+3} (-1)^j \frac{J_{2j+3/2}(a)}{\sqrt{a}} &= -\sin(a/2) \sum_{k=0}^j (-1)^k (C_1^n)_{2j+1,2k} \sqrt{4k+1} \frac{J_{2k+1/2}(a/2)}{\sqrt{\frac{a}{2}}} \\ &\quad + \cos(a/2) \sum_{k=0}^{j-1} (-1)^k (C_1^n)_{2j+1,2k+1} \sqrt{4k+3} \frac{J_{2k+3/2}(a/2)}{\sqrt{\frac{a}{2}}}. \end{aligned}$$

Recurrence formulas may also be obtained using the fact that Bessel functions satisfy a differential difference equation. Multiply equation (47) by  $\sqrt{a}$  for  $a > 0$  and set

$$(51) \quad \hat{P}_n(a) = \sqrt{2\pi} [J_{1/2}(a) \quad \cdots \quad i^n \sqrt{2n+1} J_{n+1/2}(a)]^T = G_n \mathbf{J}_n(a),$$

where

$$(52) \quad G_n = \sqrt{2\pi} \text{diagonal}(1, \dots, i^n \sqrt{2n+1}),$$

and

$$(53) \quad \mathbf{J}_n(a) = [J_{1/2}(a) \quad \cdots \quad J_{n+1/2}(a)]^T.$$

With the above substitutions equation (47) becomes

$$(54) \quad \hat{P}_n(a) = \sqrt{2} T_n(a) \hat{P}_n(a/2).$$

Differentiation of  $\hat{P}_n$  and the use of the differential difference relation  $2J_{n+1/2}(a)' = J_{n-1/2}(a) - J_{n+3/2}(a)$  yields

$$\begin{aligned} 2\hat{P}_n(a)' &= 2G_n \mathbf{J}_n(a)' = G_n L_n \mathbf{J}_n(a) + G_n [J_{-1/2}(a), 0, \dots, 0, -J_{n+3/2}(a)]^T \\ &= G_n L_n G_n^{-1} \sqrt{2} T_n(a) \hat{P}_n(a/2) + G_n [J_{-1/2}(a), 0, \dots, 0, -J_{n+3/2}(a)]^T, \end{aligned}$$

where  $L_n$  is an  $(n+1) \times (n+1)$  tridiagonal matrix which is  $-1$  on the upper diagonal,  $0$  on the diagonal, and  $1$  on the lower diagonal, and equations (54) and (51) have been used to obtain the last equality. Differentiation of the right hand

side of equation (54) using similar manipulations as above yields

$$(55) \quad T_n(a)' \hat{P}_n(a/2) = \left( H_n T_n(a) - \frac{1}{2} T_n(a) H_n \right) \hat{P}_n(a/2) \\ + \frac{1}{\sqrt{2}} G_n [J_{-1/2}(a), 0, \dots, 0, -J_{n+3/2}(a)]^T \\ - \frac{1}{2} T_n(a) G_n [J_{-1/2}(a/2), 0, \dots, 0, -J_{n+3/2}(a/2)]^T,$$

where  $H_n = G_n L_n G_n^{-1}$ . Examination of the above equation for  $a$  small and positive shows that for fixed  $j$  the sequence  $((-1)^{j+k} e^{-ia/2} + e^{ia/2}) J_k(a/2)$ ,  $k = 0, \dots, n$  is linearly independent. Thus the above equation yields the difference equation,

$$(56) \quad K_i (C_1^n)_{i,j} = J_j (C_1^n)_{i,j}, \quad 0 < i < j < n,$$

where

$$K_i = \sqrt{\frac{2i+1}{2i-1}} E_- + \sqrt{\frac{2i+1}{2i+3}} E_+,$$

and

$$J_j = \frac{1}{2} \sqrt{\frac{2j-1}{2j+1}} \hat{E}_- + \frac{1}{2} \sqrt{\frac{2j+3}{2j+1}} \hat{E}_+ + 1.$$

## 5. WAVELETS

We now develop equations to compute a set of orthogonal wavelets associated with the above scaling functions. We are interested in finding wavelet functions that form a basis for  $L^2(\mathbf{R})$  and are obtained by integer translates and dilations by 2 of a fixed set of functions. From equation (1) with the change of variable that leads to (7), then for approximation order  $n$  it is enough to find  $(n+1) \times (n+1)$  matrices  $D_{-1}$  and  $D_1$ , and functions

$$\Psi_n = (\psi_0^n \quad \dots \quad \psi_n^n)^T$$

given by

$$(57) \quad \Psi_n\left(\frac{t+1}{2}\right) = D_{-1}^n \Phi_n(t+1) + D_1^n \Phi_n(t) \\ = D_{-1}^n P_n(2t+1) + D_1^n P_n(2t-1),$$

where the last equality holds for  $-1 \leq t \leq 1$ . The imposed orthogonality implies

$$(58) \quad C_{-1}^n D_{-1}^{nT} + C_1^n D_1^{nT} = 0,$$

and

$$(59) \quad D_{-1}^n D_{-1}^{nT} + D_1^n D_1^{nT} = 2I_{n+1}.$$

From (57) we find

$$D_1^n = \int_0^1 \Psi_n(t) P_n(2t-1) dt,$$

and

$$D_{-1}^n = \int_{-1}^0 \Psi_n(t) P_n(2t+1) dt.$$

For general  $n$  there is an infinite number of solutions to the above equations even if we ask that the wavelet functions in  $\Psi_n$  be symmetric or antisymmetric. If we solve equations (58) and (59) with  $n = 0$ , we find that  $(D_{-1}^0)_{0,0} = -(D_1^0)_{0,0} = (C_1^0)_{0,0}$

so that the first wavelet function is the Haar wavelet which is antisymmetric. Thus to obtain symmetry set  $(D_{-1}^n)_{i,j} = (-1)^{i+j+1}(D_1^n)_{i,j}$ ,  $0 \leq i, j \leq n$ . For  $n = 1$  we find

$$D_1^1 = \begin{pmatrix} (D_1^1)_{0,0} & (D_1^1)_{0,1} \\ (D_1^1)_{1,0} & (D_1^1)_{1,1} \end{pmatrix},$$

and

$$D_{-1}^1 = \begin{pmatrix} (-D_1^1)_{0,0} & (D_1^1)_{0,1} \\ (D_1^1)_{1,0} & (-D_1^1)_{1,1} \end{pmatrix}.$$

If we insist that  $D_1^1$  has positive diagonal entries, there is a unique solution to equations (58) and (59) given by

$$D_1^1 = \begin{pmatrix} (C_1^1)_{1,1} & -(C_1^1)_{1,0} \\ 0 & 1 \end{pmatrix}.$$

This suggests that a unique solution can be found for which  $D_1^n$  is upper triangular with positive diagonal entries.

**Theorem 4.** *Let  $C_1$  be a lower triangular matrix with positive diagonal entries satisfying  $C_{-1}C_{-1}^T + C_1C_1^T = 2I$  where  $C_{-1}$  be obtained from  $C_1$  by the symmetry relation  $(C_{-1})_{i,j} = (-1)^{i+j}(C_1)_{i,j}$ . Then for  $n \geq 1$  there is a unique upper triangular  $(n+1) \times (n+1)$  matrix  $D_1$  with positive diagonal entries that satisfies equations (58) and (59) where  $D_{-1}$  has the symmetry relations  $(D_{-1})_{i,j} = (-1)^{i+j+1}(D_1)_{i,j}$  and  $(D_1)_{n,n} = 1$*

*Proof.* We note that the result is true for  $n = 1$  so we suppose it is true by induction for  $n - 1$ . Consider the  $n \times n$  matrices  $\hat{C}_1$  obtained from  $C_1$  by deleting the first row and column. Then from the induction hypothesis there is a unique upper triangular  $\hat{D}_1$  associated with  $\hat{C}_1$  which satisfies equations (58) and (59) and  $(\hat{D}_1)_{n-1,n-1} = 1$ . Let  $\mathbf{c}_0$  be the first column of  $C_1$ ,  $\hat{\mathbf{c}}_i, i = 1, \dots, n$  be the rows of  $\hat{C}_1$ , and write

$$C_1 = \begin{pmatrix} \mathbf{c}_0 & 0 \\ & \hat{C}_1 \end{pmatrix}.$$

Likewise let  $\mathbf{d}_0$  be the first row of  $D_1$  and  $\hat{\mathbf{d}}_i, i = 1, \dots, n$  be the rows of  $\hat{D}_1$ . Using the symmetry equations we see that (58) and (59) yield the equations,

$$(60) \quad ((\mathbf{c}_0)_i, \hat{\mathbf{c}}_i) \mathbf{d}_0^T = 0, \quad i = 1, 3, \dots,$$

$$(61) \quad (0, \hat{\mathbf{d}}_i) \mathbf{d}_0^T = 0, \quad i = 2, 4, \dots,$$

and

$$(62) \quad \mathbf{d}_0 \mathbf{d}_0^T = 1.$$

Since  $\hat{C}_1$  is lower triangular with positive diagonal elements, the vectors  $(\mathbf{c}_0)_i, \hat{\mathbf{c}}_i$  in equation (60) are independent. The equations  $\hat{C}_{-1}(\hat{D}_{-1})^T + \hat{C}_1(\hat{D}_1)^T = 0$  and  $\hat{D}_{-1}(\hat{D}_{-1})^T + \hat{D}_1(\hat{D}_1)^T = 0$  show that the vectors  $(0, \hat{\mathbf{d}}_i)$  in (61) are orthogonal to each other and to the vectors in (60). Thus the rank of the matrix whose rows are the equations (60) and (61) is  $n$ . With  $i = 1$  in (60) we find  $(C_1)_{0,1}(D_1)_{0,0} + (C_1)_{1,1}(D_1)_{0,1} = 0$ , which implies that  $(D_1)_{0,0}$  is the free variable which is made unique by the choice of the positive solution to equation (62).  $\square$

We now show

**Lemma 5.** *Given  $C_i^n$ ,  $i = \{-1, 1\}$ , suppose  $D_1^n$  and  $D_1^1$  satisfy the hypothesis of Theorem (4). Then  $(D_1^n)_{n-2j,n} = 0$  and  $(D_1^n)_{n-2j,n-k} = (D_1^{n-1})_{n-2j,n-k}$ .*

*Proof.* We begin with the observation that from the symmetry relations we find  $(D_{-1}^n)_{n-2j,n} = -(D_1^n)_{n-2j,n}$ . Thus the last row of equation (59) shows that  $(D_{-1}^n)_{n-2j,n} = 0$ . The proof of Theorem (4) also shows that in order to compute  $(D_1^n)_{n-2k,i}$ ,  $i = n - 2k, \dots, n - 1$ , we can choose  $\hat{D}_1$  and  $\hat{D}_{-1}$  so that they start with the row  $n - 2j + 1$  (starting from zero) of  $D_1^n$  and  $D_{-1}^n$ . Examination of the equations (60) and (61) yields

$$(C_1^m)_{n-i,n-i-1}(D_1^n)_{n-i-1,n-i-1} + (C_1^n)_{n-i,n-i}(D_1^n)_{n-i-1,n-i} = 0,$$

and

$$(D_1^n)_{n-i-1,n-i-1}^2 + (D_1^n)_{n-i-1,n-i}^2 = 1,$$

for  $i = 0, 1$ . The unique solutions of these equations from Theorem (4) and the entries of  $(C_1^n)$  in equation (35) above are given respectively by equations (63) and (64) below and show explicitly that the result for  $(D_1^n)_{n-2,n-k}$ ,  $k = 1, 2$ . Using that  $(C_1^n)_{n-j,n-k} = (C_1^{n-1})_{n-j+1,n-k+1}$  for  $j > 0, k > 0$  in equations (60) and the induction hypothesis in equations (61) implies that the entries  $(D_1^n)_{n-2j,n-k}$  solve the same equations as  $(D_1^{n-1})_{n-2j,n-k}$  for  $k = 1, \dots, 2j$ . The uniqueness of the solutions given by Theorem (4) above proves the Lemma.  $\square$

Using Theorem (4) allows us to compute some of the matrix elements in  $D_1^n$ . To this end we find for row  $n + 1$ ,  $(D_1^n)_{n,n} = 1$ , for row  $n$ ,

$$(63) \quad (D_1^n)_{n-1,n-1} = \frac{1}{2n}, \quad (D_1^n)_{n-1,n} = -\frac{\sqrt{(2n+1)(2n-1)}}{2n},$$

for row  $n - 1$

$$(64) \quad (D_1^n)_{n-2,n-2} = \frac{1}{2n-2}, \quad (D_1^n)_{n-2,n-1} = -\frac{\sqrt{(2n-1)(2n-3)}}{2n-2}, \quad (D_1^n)_{n-2,n} = 0,$$

for row  $n - 2$ ,

$$(D_1^n)_{n-3,n-3} = \frac{3}{4(n-1)(n-2)}, \quad (D_1^n)_{n-3,n-2} = -\frac{3\sqrt{(2n-3)(2n-5)}}{4(n-1)(n-2)},$$

$$(D_1^n)_{n-3,n-1} = \frac{(2n+1)\sqrt{(2n-1)(2n-5)}}{4(n)(n-1)}, \quad (D_1^n)_{n-3,n} = \frac{\sqrt{(2n+1)(2n-5)}}{4(n)(n-1)}.$$

NOTE ADDED IN PROOF

In [13], it is shown that the nonzero entries in  $D_1^n$  are balance  ${}_4F_3$  hypergeometric functions evaluated at 1 and the wavelet functions associated with these matrices are piecewise hypergeometric polynomials.

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