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# Differential orthogonality: Laguerre and Hermite cases with applications

J. Borrego-Morell[a, *], H. Pijeira-Cabrera[b]

[a] *UNESP-Universidade Estadual Paulista, IBILCE, Brazil* [b] *Universidad Carlos III de Madrid, Spain*

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**Abstract**

Let $\mu$ be a finite positive Borel measure supported on $\mathbb{R}$, $\mathcal{L}[f] = xf'' + (\alpha + 1 - x)f'$ with $\alpha > -1$, or $\mathcal{L}[f] = \frac{1}{2}f'' - xf'$, and $m$ a natural number. We study algebraic, analytic and asymptotic properties of the sequence of monic polynomials $\{Q_n\}_{n>m}$ that satisfy the orthogonality relations

$$\int \mathcal{L}[Q_n](x)x^k d\mu(x) = 0 \quad \text{for all } 0 \le k \le n-1.$$

We also provide a fluid dynamics model for the zeros of these polynomials.

*Keywords:* Orthogonal polynomials; Ordinary differential operators; Asymptotic analysis; Weak star convergence; Hydrodynamic

* Corresponding author.
*E-mail addresses:* jbmorell@gmail.com (J. Borrego-Morell), hpijeira@math.uc3m.es (H. Pijeira-Cabrera).

## 1. Introduction

Orthogonal polynomials with respect to a differential operator were introduced in [1] as a generalization of the notion of orthogonal polynomials. Analytic and algebraic properties of these classes of polynomials have been considered for some classes of first order differential operators in [2,11], for a Jacobi differential operator in [4], and for differential operators of arbitrary order with polynomials coefficients in [3]. In this paper, we consider orthogonal polynomials with respect to a Laguerre or Hermite operator and a positive Borel measure $\mu$ with unbounded support on $\mathbb{R}$.

We denote by $\mathcal{L}_L$ the Laguerre and by $\mathcal{L}_H$ the Hermite differential operators on the linear space $\mathbb{P}$ of all polynomials, i.e. for all $f \in \mathbb{P}$ and $\alpha > -1$

$$\mathcal{L}_L[f] = xf'' + (1+\alpha-x)f' = x^{-\alpha}e^{x}\left(x^{\alpha+1}e^{-x}f'\right)', \tag{1}$$

$$\mathcal{L}_H[f] = \frac{1}{2}f'' - xf' = \frac{1}{2}e^{x^2}\left(e^{-x^2}f'\right)'. \tag{2}$$

Each one of these second order differential operators has a system of monic polynomials which are eigenfunctions of the operator and orthogonal with respect to a measure. Let $\{L_n^{\alpha}\}_{n=0}^{\infty}$ be the monic Laguerre polynomials with $\alpha > -1$ and $\{H_n\}_{n=0}^{\infty}$ the monic Hermite polynomials, then

$$\langle L_n^{\alpha}, L_m^{\alpha}\rangle_L = \int L_n^{\alpha}(x)L_m^{\alpha}(x)dw_L^{\alpha}(x) \quad \begin{cases} = 0 & \text{if } n \neq m, \\ \neq 0 & \text{if } n = m, \end{cases}$$
$$\langle H_n, H_m\rangle_H = \int H_n(x)H_m(x)dw_H(x) \quad \begin{cases} = 0 & \text{if } n \neq m, \\ \neq 0 & \text{if } n = m, \end{cases}$$

where $dw_L^{\alpha}(x) = x^{\alpha}e^{-x}dx$ and $dw_H(x) = e^{-x^2}dx$. In addition,

$$\mathcal{L}_L[L_n^{\alpha}] = -nL_n^{\alpha} \quad \text{and} \quad \mathcal{L}_H[H_n] = -nH_n. \tag{3}$$

To unify the approach, we will denote by $\mathcal{L}$ the Laguerre or Hermite differential operator ($\mathcal{L}_L$ or $\mathcal{L}_H$) in the sequel, by $dw$ the Laguerre or Hermite measure ($dw_L^{\alpha}$ or $dw_H$), by $L_n$ the $n$th Laguerre or Hermite monic orthogonal polynomial ($L_n^{\alpha}$ or $H_n$) and by $\Delta$ the set $\mathbb{R}_+$ or $\mathbb{R}$, respectively. We will refer to one or the other depending on the case we are solving.

Let $\mu$ be a finite positive Borel measure, supported on $\Delta$ and $\{P_n\}_{n=0}^{\infty}$ the corresponding system of monic orthogonal polynomials, i.e.

$$\langle P_n, P_k\rangle_{\mu} = \int P_n(x)P_k(x)d\mu(x) \begin{cases} \neq 0 & \text{if } n = k, \\ = 0 & \text{if } n \neq k. \end{cases} \tag{4}$$

We say that $Q_n$ is the $n$th monic orthogonal polynomial with respect to the pair $(\mathcal{L}, \mu)$ if $Q_n$ has degree $n$ and

$$\int \mathcal{L}[Q_n](x)\, x^k d\mu(x) = 0 \quad \text{for all } 0 \le k \le n-1, \tag{5}$$

or, equivalently,

$$\mathcal{L}[Q_n] = \lambda_n P_n, \tag{6}$$

where $\lambda_n = -n$.

It was shown in [4, Section 2] that it is not always possible to guarantee the existence of a system of polynomials $\{Q_n\}_{n\in\mathbb{Z}_+}$ orthogonal with respect to the pair $(\mathcal{L}^{(\alpha,\beta)}, \mu)$, where $\mathcal{L}^{(\alpha,\beta)}$ is the Jacobi differential operator and $\mu$ an arbitrary positive finite Borel measure. As will be shown later (cf. Propositions 1 and 2), a similar situation occurs for the case of Laguerre and Hermite operators. Let $m \in \mathbb{N}$ be fixed, a fundamental role in the existence of infinite sequences of polynomials $\{Q_n\}_{n>m}$ orthogonal with respect to the pair $(\mathcal{L}, \mu)$ is played by the class $\mathcal{P}_m(\Delta)$ defined as the family of finite positive Borel measures $\mu$ supported on $\Delta$ for which there exists a polynomial $\rho$ of degree $m$, such that $\mu = (\rho)^{-1} w$.

If $\mu \in \mathcal{P}_m(\Delta)$ it is not difficult to see that if $n > m$, then

$$P_n(z) = \sum_{k=0}^{m} b_{n,n-k}\, L_{n-k}(z), \qquad b_{n,n-k} = \frac{1}{\tau_{n-k}} \int P_n(x) L_{n-k}(x) dw(x), \tag{7}$$

$$\tau_n = \|L_n\|_w^2 = \int L_n^2(x) dw(x) = \begin{cases} n!\, \Gamma(n+\alpha+1), & \mu \in \mathcal{P}_m(\mathbb{R}_+), \\ n! \sqrt{\pi} 2^{-n}, & \mu \in \mathcal{P}_m(\mathbb{R}), \end{cases} \tag{8}$$

and from (6) we obtain that the monic polynomial of degree $n$, for $n > m$ defined by the formula

$$\widehat{Q}_n(z) = \sum_{k=0}^{m} \frac{\lambda_n}{\lambda_{n-k}} b_{n,n-k}\, L_{n-k}(z), \tag{9}$$

is orthogonal with respect to $(\mathcal{L}, \mu)$.

Notice that from the equivalence between relations (5) and (6), the polynomial $\widehat{Q}_n + c$, $c \in \mathbb{C}$, is orthogonal with respect to $(\mathcal{L}, \mu)$ so that we do not have a unique monic orthogonal polynomial of degree $n$. We had a similar situation when we studied the orthogonality with respect to a Jacobi operator. A natural way to define a unique sequence would be to consider a sequence of complex numbers $\{\zeta_n\}_{n=m+1}^{\infty}$, and define the sequence $\{Q_n\}_{n=m+1}^{\infty}$ satisfying (5), as the polynomial solution of the initial value problem

$$\begin{cases} \mathcal{L}[y] = \lambda_n P_n, & n > m, \\ y(\zeta_n) = 0. \end{cases} \tag{10}$$

We say that $\{Q_n\}_{n=m+1}^{\infty}$ is the sequence of monic orthogonal polynomials with respect to the pair $(\mathcal{L}, \mu)$ such that $Q_n(\zeta_n) = 0$.

Notice that the initial value problem (10) has the unique polynomial solution

$$y(z) = Q_n(z) = \widehat{Q}_n(z) - \widehat{Q}_n(\zeta_n). \tag{11}$$

In this paper, we study some analytic and algebraic properties of the sequence of orthogonal polynomials with respect to a Laguerre or Hermite differential operator. In order to study the asymptotic properties of the sequence of polynomials we shall normalize them with an adequate parameter.

Let $x_n$ be the modulus of the largest zero of the $n$th orthogonal polynomial with respect to $\mu$ (or $w$), from [12, Lemma 11 with $\lambda = 2$] for the Hermite case and [12, Coroll. (p. 191) with $\gamma = 1$] for the Laguerre case, we get

$$\lim_{n\to\infty} c_n^{-1} x_n = 1, \tag{12}$$

where $c_n$ is usually called Mhaskar–Rakhmanov–Saff constant, here with the closed expression

$$c_n = \begin{cases} 4n, & \mu \in \mathcal{P}_m(\mathbb{R}_+) \quad \text{or} \quad w(x) = x^\alpha e^{-x}, \quad x > 0, \\ \sqrt{2n}, & \mu \in \mathcal{P}_m(\mathbb{R}) \quad \text{or} \quad w(x) = e^{-x^2}, \quad x \in \mathbb{R}. \end{cases} \tag{13}$$

Throughout this paper we denote the functions $\varphi(z) = z + \sqrt{z^2 - 1}$ and $\psi(z) = 2z - 1 + 2\sqrt{z(z-1)}$, where the branch of each root is selected from the condition $\varphi(\infty) = \infty$ and $\psi(\infty) = \infty$, respectively. Let $\Delta_c$ be the interval $[0, 1]$ in the Laguerre case and $[-1, 1]$ in the Hermite case. Let $\mathfrak{P}_n(z) = c_n^{-n} P_n(c_n z)$ be the normalized monic orthogonal polynomials with respect to a measure $\mu \in \mathcal{P}_m(\Delta)$.

To each generic polynomial $q_n$, let $\mu_n = n^{-1} \sum_{q_n(\omega)=0} \delta_\omega$ be the normalized root counting measure, where $\delta_\omega$ is the Dirac measure with mass 1 at the point $\omega$. From [12, Ths. 4 & 4'] we find that the limit distribution $\nu_w$ of the zero counting measure of the normalized Laguerre and Hermite polynomials is

$$d\nu_w(t) = \begin{cases} 2\pi^{-1}\sqrt{\dfrac{1-t}{t}}\,dt, & t \in [0, 1] \quad \text{Laguerre case}, \\ 2\pi^{-1}\sqrt{1-t^2}\,dt, & t \in [-1, 1] \quad \text{Hermite case}. \end{cases}$$

From [14, Chs. III & IV] we have that

$$\lim_{n\to\infty} |\mathfrak{P}_n(z)|^{\frac{1}{n}} = \begin{cases} \dfrac{1}{e}\ |\psi(z)|\ e^{2\Re[1/\varphi(z)]} & \mu \in \mathcal{P}_m(\mathbb{R}_+), \\ \dfrac{1}{2\sqrt{e}}\ |\varphi(z)|\ e^{\Re[z/\varphi(z)]} & \mu \in \mathcal{P}_m(\mathbb{R}), \end{cases} \tag{14}$$

uniformly on compact subsets $K \subset \mathbb{C} \setminus \Delta_c$.

We are interested in asymptotic properties of the normalized monic orthogonal polynomials with respect to a pair $(\mathcal{L}, \mu)$ defined by

$$\mathfrak{Q}_n(z) = \widehat{\mathfrak{Q}}_n(z) - \widehat{\mathfrak{Q}}_n(\zeta_n), \tag{15}$$

where $\widehat{\mathfrak{Q}}_n(z) = c_n^{-n}\ \widehat{Q}_n(c_n z)$. For these polynomials we prove the following results.

**Theorem 1.** *Let $\mu \in \mathcal{P}_m(\Delta)$, where $m \in \mathbb{N}$. Then:*

(a) *If $\nu_n, \sigma_n$ denote the root counting measure of $\widehat{\mathfrak{Q}}_n$ and $\widehat{\mathfrak{Q}}_n'$ respectively then $\nu_n \xrightarrow{*} \nu_w$ and $\sigma_n \xrightarrow{*} \nu_w$ in the weak star sense.*

(b) *The set of accumulation points of the zeros of $\left\{\widehat{\mathfrak{Q}}_n\right\}_{n=m+1}^{\infty}$ is $\Delta_c$.*

**Theorem 2.** *Let $m \in \mathbb{N}$, $\mu \in \mathcal{P}_m(\Delta)$. Then, for every compact subset $K$ of $\mathbb{C} \setminus \Delta_c$ we have uniformly*

$$\lim_{n\to\infty} \frac{\mathfrak{P}_n(z)}{\widehat{\mathfrak{Q}}_n(z)} = \begin{cases} 1 & \mu \in \mathcal{P}_m(\mathbb{R}_+) \\ 1 & \mu \in \mathcal{P}_m(\mathbb{R}) \end{cases} \tag{16}$$

$$\lim_{n\to\infty} \left|\widehat{\mathfrak{Q}}_n(z)\right|^{\frac{1}{n}} = \begin{cases} \dfrac{1}{e}\ |\psi(z)|\ e^{2\Re[1/\varphi(z)]} & \mu \in \mathcal{P}_m(\mathbb{R}_+), \\ \dfrac{1}{2\sqrt{e}}\ |\varphi(z)|\ e^{\Re[z/\varphi(z)]} & \mu \in \mathcal{P}_m(\mathbb{R}). \end{cases} \tag{17}$$

The following result shows that the set of accumulation points of the zeros of the sequence of normalized polynomials, orthogonal with respect to $(\mathcal{L}, \mu)$ is contained in a curve.

**Theorem 3.** *Let $m \in \mathbb{N}$ and $\mu \in \mathcal{P}_m(\Delta)$. If $\{\zeta_n\}_{n=m+1}^{\infty}$ is a sequence of complex numbers with limit $\zeta \in \mathbb{C} \setminus \Delta_c$. Then:*

(a) *The accumulation points of zeros of the sequence $\{\mathfrak{Q}_n\}_{n=m+1}^{\infty}$ such that $\mathfrak{Q}_n(\zeta_n) = 0$ are located on the set $E = \mathcal{E}(\zeta) \bigcup \Delta_c$, where $\mathcal{E}(\zeta)$ is the curve*

$$\mathcal{E}(\zeta) := \{z \in \mathbb{C} : \Psi(z) = \Psi(\zeta)\}, \tag{18}$$

$\Psi(z) = |\psi(z)|\, e^{2\Re[1/\varphi(z)]}$ *for* $\mu \in \mathcal{P}_m(\mathbb{R}_+)$, *and* $\Psi(z) = |\varphi(z)|\, e^{\Re[z/\varphi(z)]}$ *for* $\mu \in \mathcal{P}_m(\mathbb{R})$.

(b) *If $\mathfrak{d}(\zeta) = \inf_{x \in \Delta_c} |\zeta - x| > 2$ then $E = \mathcal{E}(\zeta)$ and for $n$ sufficiently large are simple.*

The relative asymptotic behavior between the sequences of polynomials $\{\mathfrak{Q}_n\}_{n>m}$ and $\{\mathfrak{P}_n\}_{n>m}$ reads as

**Theorem 4.** *Let $\{\zeta_n\}_{n>m}$ be a sequence of complex numbers with limit $\zeta \in \mathbb{C} \setminus \Delta_c$, $m \in \mathbb{N}$, $\mu \in \mathcal{P}_m(\Delta)$ and $\{\mathfrak{Q}_n\}_{n>m}$ be the sequence of normalized monic orthogonal polynomials with respect to the pair $(\mathcal{L}, \mu)$ such that $\mathfrak{Q}_n(\zeta_n) = 0$, then:*

1. *Uniformly on compact subsets of $\Omega = \{z \in \mathbb{C} : |\Psi(z)| > |\Psi(\zeta)|\}$,*

$$\frac{\mathfrak{Q}_n(z)}{\mathfrak{P}_n(z)} \underset{n\to\infty}{\rightrightarrows} 1. \tag{19}$$

2. *Uniformly on compact subsets of $\Omega = \{z \in \mathbb{C} : |\Psi(z)| < |\Psi(\zeta)|\} \setminus \Delta_c$*

$$\frac{\mathfrak{Q}_n(z)}{\mathfrak{P}_n(\zeta_n)} \underset{n\to\infty}{\rightrightarrows} -1, \tag{20}$$

*where $\Psi$ is as defined in Theorem 3. If $\mathfrak{d}(\zeta) > 2$ then (20) holds for $\Omega = \{z \in \mathbb{C} : |\Psi(z)| < |\Psi(\zeta)|\}$.*

The paper continues as follows. Section 2 is dedicated to the study of existence, uniqueness and some results concerning the properties of the zeros of orthogonal polynomials with respect to the Laguerre or Hermite operators. In Sections 3 and 4 we study the asymptotic behavior of the polynomials $\widehat{\mathfrak{Q}}_n$ and $\mathfrak{Q}_n$ respectively. Finally, in Section 5 we show a fluid dynamics model for the zeros of these polynomials.

## 2. The polynomial $Q_n$

First of all, we are interested in discussing systems of polynomials such that for some $m \in \mathbb{N}$, for all $n > m$, they are solutions of (6). In order to classify those measures $\mu$ for which the existence of such sequences of orthogonal polynomials with respect to $(\mathcal{L}, \mu)$ can be guaranteed, we prove a preliminary lemma.

**Lemma 5.** *Let $\mu$ be a finite positive Borel measure with support contained on $\mathbb{R}$ and let $n \in \mathbb{N}$ be fixed. Then, the differential equation (6) has a monic polynomial solution $Q_n$ of degree $n$, which is unique up to an additive constant, if and only if*

$$\int P_n(x) dw(x) = 0, \quad \text{where } P_n \text{ is as (4).} \tag{21}$$

**Proof.** Suppose that there exists a polynomial $Q_n$ of degree $n$, such that $\mathcal{L}[Q_n] = -n\,P_n$. Then, integrating (1) or (2) with respect to the Laguerre measure on $\mathbb{R}_+$ or Hermite measure on $\mathbb{R}$ respectively we have (21).

Conversely, suppose that $P_n$ satisfies (21). Let $Q_n$ be the polynomial of degree $n$ defined by $Q_n(z) = L_n(z) + \sum_{k=0}^{n-1} a_{n,k} L_k(z)$, where $a_{n,0}$ is an arbitrary constant and $a_{n,k} = \frac{\lambda_n}{\lambda_k \tau_k} \int P_n(x) L_k(x) dw(x)$, $k = 1, \ldots, n-1$. From the linearity of $\mathcal{L}[\cdot]$ and (3) we get that $\mathcal{L}[Q_n] = -n\,P_n$. $\square$

From the preceding lemma, as in [4, Coroll. 2.2], we obtain

**Proposition 1.** *Let $w$ be the Laguerre or Hermite measure and $\mu$ a finite positive Borel measure on $\Delta$, such that $d\mu(x) = r(x)dw(x)$ with $r \in L^2(w)$. Then, $m$ is the smallest natural number such that for each $n > m$ there exists a monic polynomial $Q_n$ of degree $n$, unique up to an additive constant and orthogonal with respect to $(\mathcal{L}, \mu)$ if and only if $r^{-1}$ is a polynomial of degree $m$.*

**Proof.** Suppose that $m$ is the smallest natural number such that for each $n > m$ there exists a monic polynomial $Q_n$ of degree $n$, unique up to an additive constant and orthogonal with respect to $(\mathcal{L}, \mu)$. According to Lemma 5

$$\int L_n(x) \frac{d\mu(x)}{r(x)} = \int L_n(x) dw(x) \begin{cases} = 0 & \text{if } n > m, \\ \neq 0 & \text{if } n = m. \end{cases}$$

But this is equivalent to saying that $\frac{1}{r(x)} = \sum_{k=0}^{m} c_k L_k(x)$ with $c_m \neq 0$. The converse is straightforward. $\square$

It is possible to give another characterization, in terms of the quasi orthogonality concept, for the existence of a system of polynomials such that for all $n > m$, for some $m \in \mathbb{N}$, they are solutions of (6).

**Proposition 2.** *Let $\mu$ be a finite positive Borel measure on $\mathbb{R}$ and $\{P_n\}_{n=0}^{\infty}$ the sequence of monic orthogonal polynomials with respect to $\mu$. Then, $m$ is the smallest natural number such that for each $n > m$ there exists, except for an additive constant, a unique monic polynomial $Q_n$, orthogonal with respect to the pair $(\mathcal{L}, \mu)$, if and only if for all $n > m$*

$$\int P_n(x)\, x^k dw(x) = 0, \quad \textit{for } k = 0, 1, \ldots, n-m,$$

*i.e. the polynomial $P_n$ is quasi-orthogonal of order $n - m + 1$ with respect to the measure $w$.*

**Proof.** Assume that $m$ is the smallest natural number such that for each $n > m$ there exists a monic polynomial $Q_n$ of degree $n$, unique up to an additive constant and orthogonal with respect to $(\mathcal{L}, \mu)$. From Lemma 5 we have that (21) holds for $n > m$. From the three term recurrence relation for $\{P_n\}_{n=0}^{\infty}$

$$\begin{aligned} x P_n(x) &= P_{n+1}(x) + \beta_n P_n(x) + \alpha_n^2 P_{n-1}(x), \quad n \geq 1, \\ &\quad P_0(x) = 1,\ P_{-1}(x) = 0,\ \alpha_n, \beta_n \in \mathbb{R} \text{ and } \alpha_n \neq 0, \end{aligned} \tag{22}$$

$$\text{thus} \int P_n(x) x^k dw(x) = 0 \quad \text{for all } 0 \leq k < n - m, \tag{23}$$

which implies that the polynomial $P_n$ is quasi-orthogonal of order $n - m + 1$ with respect to the measure $w$ (Laguerre or Hermite).

Conversely, assume that $m$ is the smallest natural number such that for $n > m$, the polynomial $P_n$ is quasi-orthogonal of order $n-m+1$ with respect to the measure $dw$. Then we have that

$$P_n(x) = L_n(x) + \sum_{k=1}^{m} d_{n-k} L_{n-k}(x),$$

which implies that for all integers $n > m$ the polynomials $P_n$ satisfy the condition (21). From Lemma 5 we have that there exists a monic polynomial $Q_n$ of degree $n$, unique up to an additive constant and orthogonal with respect to $(\mathcal{L}, \mu)$, for all $n > m$. □

From the above proposition, we deduce in particular that the differential equation (6) has, except for an additive constant, a unique monic polynomial solution $Q_n$ of degree *$n$ for all the natural numbers* only if $P_n = L_n$ and $d\mu = dw$. Hence $Q_n = L_n$, the polynomial eigenfunctions of $\mathcal{L}$, whose properties are well known.

Let us continue by noting that the polynomials $Q_n$ and $\widehat{Q}_n$ (see (9) and (11)) are primitives of the same polynomial $Q_n'$ (or $\widehat{Q}_n'$) and

$$\int \widehat{Q}_n(x)\, x^k \, dw(x) = 0, \quad k = 0, 1, \ldots, n-m-1. \tag{24}$$

Applying classical arguments [17], it is not difficult to prove the following result, which will be used in the sequel.

**Proposition 3.** *The polynomial $\widehat{Q}_n$ defined by (9) for all $n > m$, has at least $(n-m)$ zeros and $(n-m-1)$ critical points of odd multiplicity on $\Delta$.*

For $m = 2$ we denote by $\widetilde{\mathcal{P}}_2[\mathbb{R}]$ the class of measures of the form $d\mu = \frac{e^{-x^2}}{x^2+x_1^2} dx$, $x_1 \neq 0$ in the Hermite case. The following proposition shows some results concerning the zeros of $\widehat{Q}_n$ and $\widehat{Q}_n'$ for measures $\mu \in \mathcal{P}_1[\mathbb{R}_+]$ or $\mu \in \widetilde{\mathcal{P}}_2[\mathbb{R}]$.

**Proposition 4.** *Assume that $\mu \in \mathcal{P}_1[\mathbb{R}_+]$ or $\mu \in \widetilde{\mathcal{P}}_2[\mathbb{R}]$, then the zeros of $\widehat{Q}_n$ and $\widehat{Q}_n'$ are real and simple. The critical points of $Q_n$ interlace the zeros of $P_n$.*

**Proof.** 1. *Laguerre case.* If $m = 1$ and $\mu \in \mathcal{P}_1[\mathbb{R}_+]$ from Proposition 3 the polynomial $\widehat{Q}_n$ has at least $(n-1)$ real zeros of odd multiplicity on $\mathbb{R}_+$. But, $\widehat{Q}_n$ is a polynomial with real coefficients and degree $n$, consequently the zeros of $\widehat{Q}_n$ are real and simple. As $Q_n' = \widehat{Q}_n'$, from Rolle's theorem all the critical points of $Q_n$ are real, simple, and $(n-2)$ of them are contained on $\mathbb{R}_+^* = ]0, \infty[$.

Denote $G(x) = x^{\alpha+1} e^{-x} Q_n'(x)$, with $\alpha \in ]-1, \infty[$. Notice that $G$ is a real-valued, continuous and differentiable function on $\mathbb{R}_+^*$. Suppose that there exists $x \in \mathbb{R}_+^*$ such that $G(x) = 0$. As $G(0) = 0$ from Rolle's Theorem there exists $x' \in \mathbb{R}_+^*$ such that $G'(x') = 0$. But, $G'(x) = x^{\alpha} e^{-x} \mathcal{L}_L[Q_n] = \lambda_n x^{\alpha} e^{-x} P_n(x)$ and all the critical points of $G$ are contained on $\mathbb{R}_+^*$. Hence all the critical points of $Q_n$ belong to $\mathbb{R}_+^*$.

2. *Hermite case.* Consider now $\mu \in \widetilde{\mathcal{P}}_2[\mathbb{R}]$, that is, $m = 2$ and $d\mu(x) = \frac{e^{-x^2}}{x^2+x_1^2} dx$, $x_1 \neq 0$. Using the relations (9) and [18, 5.6.1] we have that for $k > 1$

$$\widehat{Q}_{2k}(z) = L_k^{-1/2}(z^2) + \frac{k}{k-1} \frac{\langle P_{2k}, H_{2k-2}, \rangle_H}{\langle H_{2k-2}, H_{2k-2}, \rangle_H} L_{k-1}^{-1/2}(z^2), \tag{25}$$

$$\widehat{Q}_{2k+1}(z) = zL_k^{1/2}(z^2) + \frac{2k+1}{2k-1}\frac{\langle P_{2k+1}, H_{2k-1},\rangle_H}{\langle H_{2k-1}, H_{2k-1},\rangle_H} zL_{k-1}^{1/2}(z^2).$$

As $L_n^{-1/2}(z^2), zL_n^{1/2}(z^2)$ are the $2n$ and $2n+1$ monic orthogonal polynomials of degree $2n$ and $2n+1$ respectively with respect to the measure $d\mu(x) = e^{-x^2}dx$, from (25) and [18, Th. 3.3.4] we have that the zeros of $\widehat{Q}_n, n > 2$ are real.

The statement that critical points of $Q_n$ interlace the zeros of $P_n$ follows by applying Rolle's theorem to the functions $G(x) = x^{\alpha+1} e^{-x} Q_n'(x)$ and $G(x) = e^{-x^2} Q_n'(x)$, for both the Laguerre and Hermite cases. □

We conjecture that Proposition 4 is still valid for any measure in the class $\mathcal{P}_m(\Delta)$, $m > 1$, for the Laguerre case or $m > 2$, $m$ even, for the Hermite case.

Finally, we find asymptotic bounds for the coefficients $b_{n,n-k}$ that define the polynomial $\widehat{Q}_n$.

**Proposition 5.** *Let $m \in \mathbb{N}$ and $\mu \in \mathcal{P}_m(\Delta)$. Then for $n$ large enough, there exist constants $C_\rho^L$ and $C_\rho^H$ such that*

$$|b_{n,n-k}| = \frac{|\langle P_n, L_{n-k}\rangle_w|}{\|L_{n-k}\|_w^2} < \begin{cases} C_\rho^L\, n^k & \textit{Laguerre case,} \\ C_\rho^H\, \sqrt{n^k} & \textit{Hermite case,} \end{cases}$$

*for $k = 1, \dots, m$.*

**Proof.** Let $\rho(x) = \sum_{j=1}^m \rho_j x^j$ and $\rho_+ = \max_{0\le j\le m} |\rho_j|$. From the Cauchy–Schwarz inequality we have

$$\begin{aligned} |b_{n,n-k}| &\le \frac{\|P_n\|_\mu}{\|L_{n-k}\|_w^2}\sqrt{\langle \rho L_{n-k}, L_{n-k}\rangle_w} \le \frac{|\rho L_{n-m}|_\mu}{|\rho_m|\,\|L_{n-k}\|_w^2}\sqrt{\langle \rho L_{n-k}, L_{n-k}\rangle_w} \\ &\le \frac{\rho_+}{|\rho_m|\,\|L_{n-k}\|_w^2}\sqrt{\sum_{j=0}^m \left|\langle x^j, L_{n-m}^2\rangle_w\right|}\sqrt{\sum_{j=0}^m \left|\langle x^j, L_{n-k}^2\rangle_w\right|}. \end{aligned} \tag{26}$$

We analyze separately the Laguerre and Hermite cases. Without loss of generality we can assume that $n > 2m$.

• *Laguerre case* ($L_n = L_n^\alpha$, $\Delta = \mathbb{R}_+$ and $dw(x) = x^\alpha e^{-x}dx$). From [13, (III.4.9) and (I.2.9)] we have the connection formula

$$L_{n-k}^\alpha(z) = \sum_{\nu=k}^{k+j}\binom{j}{\nu-k}\frac{(n-k)!}{(n-\nu)!}L_{n-\nu}^{\alpha+j}(z),$$

then from (8) and the orthogonality

$$\begin{aligned} \langle x^j, \left(L_{n-k}^\alpha\right)^2\rangle_L &= \sum_{\nu=k}^{k+j}\binom{j}{\nu-k}\frac{(n-k)!}{(n-\nu)!}\int \left(L_{n-\nu}^{\alpha+j}(x)\right)^2 x^{\alpha+j}e^{-x}dx, \\ &= \sum_{\nu=k}^{k+j}\binom{j}{\nu-k}(n-k)!\,\Gamma(n-\nu+j+\alpha+1), \\ &\le 2^j(n-k)!\,\Gamma(n-k+j+\alpha+1), \end{aligned}$$

$$\begin{aligned} \text{and} \quad \sum_{j=0}^m \langle x^j, L_{n-k}^2\rangle_w &\le (n-k)!\sum_{j=0}^m 2^j\,\Gamma(n-k+j+\alpha+1), \\ &\le (2^{m+1}-1)(n-k)!\,\Gamma(n-k+m+\alpha+1). \end{aligned}$$

Hence, from (26), (8) and $n$ large enough

$$|b_{n,n-k}| \leq \frac{\rho_+(2^{m+1}-1)}{|\rho_m|}\sqrt{\frac{(n-m)!\Gamma(n+\alpha+1)\Gamma(n+m-k+\alpha+1)}{(n-k)!\Gamma^2(n-k+\alpha+1)}},$$
$$\leq \frac{\rho_+(2^{m+1}-1)}{|\rho_m|}\sqrt{\frac{(n+\alpha)^{k+m}}{(n-m)^{m-k}}} \leq \frac{\rho_+ 2^m(2^{m+1}-1)}{|\rho_m|}\, n^k.$$

• *Hermite case* ($L_n = H_n$, $\Delta = \mathbb{R}$ and $dw(x) = e^{-x^2}dx$). By the symmetry property of the Hermite polynomials, if $\nu$ is an odd number

$$\int x^\nu H_{n-k}^2(x)dw(x) = 0.$$

Hence, from (26)

$$|b_{n,n-k}| \leq \frac{\rho_+}{|\rho_m|\,\|H_{n-k}\|_w^2}\sqrt{\sum_{j=0}^{\lfloor \frac{m}{2}\rfloor}\|x^j H_{n-m}\|_w^2}\sqrt{\sum_{j=0}^{\lfloor \frac{m}{2}\rfloor}\|x^j H_{n-k}\|_w^2},$$

where for all $x \in \mathbb{R}$, the symbol $\lfloor x \rfloor$ denotes the largest integer less than or equal to $x$. As it is well known (cf. [18, (5.5.6) and (5.5.8)]), the Hermite polynomials satisfy the recurrence relation $zH_n(z) = H_{n+1}(z) + \frac{n}{2}H_{n-1}(z)$, from which we get by induction on $j$

$$z^j H_n(z) = \sum_{\nu=0}^{j}\sigma_{j,\nu}(n)H_{n+j-2\nu}(z), \tag{27}$$

where $\sigma_{j,\nu}(n)$ is a polynomial in $n$ of degree equal to $\nu$ and leading coefficient $2^{-\nu}\binom{j}{\nu}$ (i.e. $\sigma_{j,\nu}(n) = 2^{-\nu}\binom{j}{\nu}n^\nu + \cdots$). Hence, from (8), for $n$ large enough

$$\|x^j H_{n-k}\|_w^2 = \sum_{\nu=0}^{j}\sigma_{j,\nu}^2(n-k)\|H_{n-k+j-2\nu}\|_w^2,$$
$$\leq \frac{\sqrt{\pi}\,(n-k-j)!}{2^{n-k+j}}\left(\sum_{\nu=0}^{j}2^{2\nu}\sigma_{j,\nu}^2(n-k)(n-k+j)^{2j-2\nu}\right),$$
$$\leq \frac{2\sqrt{\pi}\,(n-k-j)!(n-k)^{2j}}{2^{n-k}}\binom{2j}{j}$$

with $j = 0, 1, \ldots, \lfloor \frac{m}{2}\rfloor$, therefore

$$|b_{n,n-k}| \leq \frac{\rho_+\,2^{n-k}}{\sqrt{\pi}|\rho_m|\,(n-k)!}\sqrt{\sum_{j=0}^{\lfloor \frac{m}{2}\rfloor}\|x^j H_{n-m}\|_w^2}\sqrt{\sum_{j=0}^{\lfloor \frac{m}{2}\rfloor}\|x^j H_{n-k}\|_w^2}$$
$$\leq \frac{2m!\rho_+}{|\rho_m|}\sqrt{\sum_{j=0}^{\lfloor \frac{m}{2}\rfloor}(n-m)^{2j}\frac{(n-m-j)!}{(n-k)!}}\sqrt{\sum_{j=0}^{\lfloor \frac{m}{2}\rfloor}(n-k)^{2j}\frac{(n-k-j)!}{(n-k)!}}$$

$$\leq \frac{2m!\rho_+}{|\rho_m|}\sqrt{\sum_{j=0}^{\lfloor\frac{m}{2}\rfloor}\frac{(n-m)^{2j}}{(n-m-j)^{m+j-k}}}\sqrt{\sum_{j=0}^{\lfloor\frac{m}{2}\rfloor}\frac{(n-k)^{2j}}{(n-m-j)^{j}}}$$

$$\leq \frac{2m!\rho_+}{|\rho_m|}\sqrt{8m(n-k)^{-\lfloor\frac{m}{2}\rfloor}}\sqrt{2m(n-k)^{\lfloor\frac{m}{2}\rfloor}}\, n^k = \frac{8m(m)!\rho_+}{|\rho_m|}\, n^k. \quad \square$$

## 3. The polynomial $\widehat{\mathfrak{Q}}_n$

In this section we prove asymptotic properties of the normalized monic orthogonal polynomials with respect to a Laguerre or Hermite differential operator. We recall that as in Section 1, $\Delta_c$ denotes the interval [0, 1] in the Laguerre case and [−1, 1] in the Hermite case, and the sequence of real numbers $\{c_n\}_{n=1}^{\infty}$ is given by (13). Set $\mathfrak{L}_{n,\nu}(z) = c_n^{-\nu}\, L_\nu(c_n z)$; $\mathfrak{L}_n(z) \equiv \mathfrak{L}_{n,n}(z)$ and $\mathfrak{P}_{n,\nu}(z) = c_n^{-\nu}\, P_\nu(c_n z)$; $\mathfrak{P}_n(z) \equiv \mathfrak{P}_{n,n}(z)$.

We prove now some preliminary lemmas.

**Lemma 6.** *Let $m \in \mathbb{N}$, $\mu \in \mathcal{P}_m(\Delta)$ and $\zeta$ such that $\widehat{\mathfrak{Q}}_n(\zeta) = 0$. Then for all n sufficiently large $d_c(\zeta) < 2\varpi_c$, where*

$$\varpi_c = \begin{cases} 1 + 2^{-1}\, C_\rho^L & \textit{Laguerre case},\\ 1 + \sqrt{2}\, C_\rho^H & \textit{Hermite case}, \end{cases}$$

*$d_c(z) = \min_{x\in\Delta_c} |z - x|$, and $C_\rho^L$ and $C_\rho^H$ are the same constants of Proposition 5.*

**Proof.** For each fixed $n > m$, we have that

$$x_n^{-n}\widehat{Q}_n(x_n z) = \sum_{k=0}^{m} \frac{\lambda_n\, b_{n,n-k}}{x_n^k \lambda_{n-k}}\, x_n^{-n+k} L_{n-k}(x_n z),$$

where $x_n$ is the zero of the largest modulus of $L_n$. It follows that the smallest interval containing the zeros of $\{x_n^{-k} L_k(x_n z)\}_{k=0}^{n}$ is $\Delta_c$. Hence, if $\zeta$ is such that $\widehat{Q}_n(x_n\zeta) = 0$, from [15, Coroll. 1], Proposition 5, (13) and (12) we have,

$$d_c(\zeta) \leq 1 + \max_{1\leq k\leq m}\left|\frac{\lambda_n\, b_{n,n-k}}{x_n^k \lambda_{n-k}}\right| < 1 + 2\max_{1\leq k\leq m}\left|\frac{b_{n,n-k}}{x_n^k}\right| \leq \varpi_c, \tag{28}$$

where

$$\varpi_c = \begin{cases} 1 + 2^{-1}\, C_\rho^L & \text{Laguerre case},\\ 1 + \sqrt{2}\, C_\rho^H & \text{Hermite case}. \end{cases}$$

Notice that $\widehat{\mathfrak{Q}}_n\left(\frac{x_n}{c_n} z\right) = c_n^{-n}\widehat{Q}_n(x_n z)$; therefore, if $\zeta$ is such that $\widehat{Q}_n(x_n\zeta) = 0$ then $\zeta^* = \frac{x_n}{c_n}\zeta$ is such that $\widehat{\mathfrak{Q}}_n(\zeta^*) = 0$. From (12) and (13) we have that for $n$ large, $\left|\frac{x_n}{c_n}\right| < 2$. Using now (28) we obtain the lemma. $\square$

If $\{\Pi_n\}_{n=0}^{\infty}$ is a sequence of orthogonal polynomials with respect to either the measures $\mu$ or $w$ we denote by $\{\mathfrak{t}_n\}_{n=0}^{\infty}$ the sequence of monic normalized polynomials, that is,

$$\mathfrak{t}_n(z) = c_n^{-n}\Pi_n(c_n z) \quad \text{and} \quad \mathfrak{t}_{n,\nu}(z) = c_n^{-\nu}\Pi_\nu(c_n z). \tag{29}$$

From the interlacing property of the zeros of consecutive orthogonal polynomials, if $K$ is a compact subset of $\mathbb{C} \setminus \Delta_c$ it follows that there exist a constant $M_*$ such that for $n$ large enough

$$\left| \frac{\mathfrak{t}_{n,n-k}(z)}{\mathfrak{t}_n(z)} \right| < M_k \leq M_*, \quad k = 1, \ldots, m, \tag{30}$$

uniformly on $z \in K$, where $M_k = 2 \sup_{\substack{z \in K \\ x \in \Delta_c}} |z - x|^{-k}$, $M_* = \max\{M_1, \ldots, M_m\}$.

The following lemma is needed to study the modulus of the sequence $\left\{ \frac{\mathfrak{P}_n}{\mathfrak{L}_n} \right\}_{n=0}^{\infty}$.

**Lemma 7.** *Suppose that $m \in \mathbb{N}$ is fixed, and $K \subset \mathbb{C} \setminus \Delta_c$ a compact subset. Then, for $n$ sufficiently large*

$$\left| \left( \frac{c_{n+m}}{c_n} \right)^n \frac{\mathfrak{t}_n(z)}{\mathfrak{t}_n(\frac{c_{n+m}}{c_n} z)} \right| < 3^{\frac{2m}{d}}, \quad n > n_0, \ \forall z \in K, \tag{31}$$

*where $d = \inf_{\substack{z \in K \\ x \in \Delta_c}} |z - x|$ and $\mathfrak{t}_n$ as in (29).*

**Proof.** Let us define the monic polynomial $\mathfrak{t}_n^*(z) = \left( \frac{c_n}{c_{n+m}} \right)^n \mathfrak{t}_n \left( \frac{c_{n+m}}{c_n} z \right)$. We have that (31) is equivalent to proving that

$$\left| \frac{\mathfrak{t}_n(z)}{\mathfrak{t}_n^*(z)} \right| \leq 3^{\frac{2m}{d}}, \quad n > n_0, \ \forall z \in K.$$

If $\{z_{k,n}^*\}_{k=1}^n$, $\{z_{k,n}\}_{k=1}^n$ denotes the zeros of the polynomials $\mathfrak{t}_n^*$, $\mathfrak{t}_n$ respectively, we have the relation $z_{k,n}^* = \frac{c_n}{c_{n+m}} z_{k,n}$, $k = 1, \ldots, n$. If we denote $k_n = \frac{c_n}{c_{n+m}}$, we have, for all $n$ sufficiently large

$$\begin{aligned} \left| \frac{\mathfrak{t}_n(z)}{\mathfrak{t}_n^*(z)} \right| &\leq \left| \prod_{k=1}^n \left( 1 + \frac{(k_n - 1) z_{k,n}}{z - k_n z_{k,n}} \right) \right| \leq \prod_{k=1}^n \left( 1 + |k_n - 1| \left| \frac{z_{k,n}}{z - k_n z_{k,n}} \right| \right) \\ &\leq \prod_{k=1}^n \left( 1 + \frac{2|k_n - 1|}{d} \right) \leq \left( 1 + \frac{2|k_n - 1|}{d} \right)^n < 3^{\frac{2n|k_n - 1|}{d}} \leq 3^{\frac{2m}{d}}, \end{aligned} \tag{32}$$

where $d = \inf_{\substack{z \in K \\ x \in \Delta_c}} |z - x|$. $\square$

We prove now that the modulus of the sequence $\left\{ \frac{\mathfrak{P}_n}{\mathfrak{L}_n} \right\}_{n=0}^{\infty}$ is uniformly bounded from above and below in the interior of $\mathbb{C} \setminus \Delta_c$.

**Lemma 8.** *Let $\mu \in \mathcal{P}_m(\Delta)$, where $m \in \mathbb{N}$ and $K \subset \mathbb{C} \setminus \Delta_c$ a compact subset. Then, for all $n$ sufficiently large there exists a constant $C^*$ such that*

$$\left| \frac{\mathfrak{P}_n(z)}{\mathfrak{L}_n(z)} \right| \leq C^*, \quad n > n_0, \ \forall z \in K.$$

**Proof.** From Relation (7) we deduce that $\frac{\mathfrak{P}_n(z)}{\mathfrak{L}_n(z)} = 1 + \sum_{k=1}^m \frac{b_{n,n-k}}{c_n^k} \frac{\mathfrak{L}_{n,n-k}(z)}{\mathfrak{L}_n(z)}$. Hence, from Proposition 5, and Lemma 7 we deduce that for $n$ large enough

$$\left| \frac{\mathfrak{P}_n(z)}{\mathfrak{L}_n(z)} \right| \leq 1 + \sum_{k=1}^m C_\rho \left| \frac{\mathfrak{L}_{n,n-k}(z)}{\mathfrak{L}_n(z)} \right|, \tag{33}$$

Using (33) and (30) we deduce the lemma. $\square$

**Lemma 9.** *Let $\mu \in \mathcal{P}_m(\Delta)$, where $m \in \mathbb{N}$ and $K \subset \mathbb{C} \setminus \Delta_c$ is a compact subset. Then, for all $n$ sufficiently large there exists a constant $C$ such that*

$$C \leq \left| \frac{\mathfrak{P}_n(z)}{\mathfrak{L}_n(z)} \right|, \quad n > n_0, \ \forall z \in K.$$

**Proof.** We have that $\rho(z)L_n(z) = \sum_{k=0}^{m} \mathfrak{b}_{n,n-k} P_{n+m-k}(z)$, where $\mathfrak{b}_{n,n-k} = \frac{\int L_n(x)P_{n+m-k}(x)\rho(x)d\mu(x)}{\|P_{n+m-k}(x)\|_\mu^2}$, or equivalently,

$$\frac{\rho(c_{n+m}z)}{c_{n+m}^m} \left( \frac{c_n}{c_{n+m}} \right)^n \frac{\mathfrak{L}_n(\frac{c_{n+m}}{c_n}z)}{\mathfrak{L}_n(z)} \frac{\mathfrak{L}_n(z)}{\mathfrak{P}_{n+m}(z)} = \sum_{k=0}^{m} \frac{\mathfrak{b}_{n,n-k}}{c_{n+m}^k} \frac{\mathfrak{P}_{n+m,n+m-k}(z)}{\mathfrak{P}_{n+m}(z)}. \tag{34}$$

From the Cauchy Schwartz inequality we have that

$$|\mathfrak{b}_{n,n-k}| \leq \frac{\left(\int L_n^2(x)dw(x)\right)^{1/2} \left(\int P_{n+m-k}^2(x)dw(x)\right)^{1/2}}{\|P_{n+m-k}\|_\mu^2} = \frac{\|L_n\|_w \|P_{n+m-k}\|_w}{\|P_{n+m-k}\|_\mu^2}.$$

Using an infinite–finite range inequality for the case in which $w$ is a Laguerre weight, cf. [14], we have that there exists a constant $k_L$ such that for all $n$ large enough

$$\begin{aligned} \frac{k_L}{n^m} \int_0^\infty L_n^2(x)dw(x) &\leq \frac{k_{0,L}}{(4n)^m} \int_0^\infty P_n^2(x)dw(x) \leq \frac{1}{\rho_+(4n)^m} \int_0^{4n} P_n^2(x)dw(x) \\ &\leq \int_0^\infty P_n^2(x)d\mu(x), \end{aligned}$$

where $\rho_+ = \max_{0 \leq j \leq m} |\rho_j|$. Analogously, for the case of an Hermite weight, for all $n$ large enough, we have that there exists a constant $k_H$ such that

$$\begin{aligned} \frac{k_H}{n^{m/2}} \int_{-\infty}^\infty L_n^2(x)dw(x) &\leq \frac{k_{0,H}}{(2n)^{m/2}} \int_{-\infty}^\infty P_n^2(x)dw(x) \leq \frac{1}{\rho_+(2n)^{m/2}} \int_{-\sqrt{2n}}^{\sqrt{2n}} P_n^2(x)dw(x) \\ &\leq \int_{-\infty}^\infty P_n^2(x)d\mu(x). \end{aligned}$$

Hence, for all $n$ large enough

$$\begin{aligned} \|P_n\|_\mu^2 &\geq k_L n^{-m} \|L_n\|_w^2, \quad \text{Laguerre case}, \\ \|P_n\|_\mu^2 &\geq k_H n^{-m/2} \|L_n\|_w^2, \quad \text{Hermite case}. \end{aligned} \tag{35}$$

From (7) and Proposition 5 we deduce that for $n$ large enough, there exists a constant $k_1$ such that

$$\|P_n\|_w \leq k_1 \|L_n\|_w. \tag{36}$$

Inequalities (35) and (36) give us that there exists a constant $M^*$ such that for all $n$ large enough

$$\frac{|\mathfrak{b}_{n,n-k}|}{c_{n+m}^k} \leq M^*, \quad 1 \leq k \leq m. \tag{37}$$

From (30) it follows that there exists a constant $M_*$ such that for all $z \in K$

$$\left| \frac{\mathfrak{P}_{n+m,n+m-k}(z)}{\mathfrak{P}_{n+m}(z)} \right| < M_*, \quad k = 1, \ldots, m. \tag{38}$$

Using Lemma 7, (34), (37) and (38) we obtain

$$\left|\frac{\rho(c_{n+m}z)}{c_{n+m}^m}\right|\left|\frac{\mathfrak{L}_n(z)}{\mathfrak{P}_{n+m}(z)}\right| \leq 3^{\frac{2m}{d}}\left(1+m\,M^*\,M_*\right), \tag{39}$$

with $d$ as in Lemma 7. Hence, from (30), (38), (39) and Lemma 7 we obtain that for all $n$ sufficiently large there exists $M > 0$ such that

$$\left|\frac{\rho(c_{n+m}z)}{c_{n+m}^m}\right|\left|\frac{\mathfrak{L}_n(z)}{\mathfrak{P}_n(z)}\right| \leq M, \quad \forall z \in K. \tag{40}$$

Let us denote by $\{z_k\}_{k=1}^m$ the roots of the polynomial $\rho$, and $d^* = \inf_{z\in K}|z|$. Let us choose $\varepsilon$ so that for $n$ large enough $\left|\frac{z_k}{c_{n+m}}\right| < \varepsilon < d^*$, $k = 1, \ldots, m$. Hence,

$$\left(d^* - \varepsilon\right)^m \leq \prod_{k=1}^m \left(|z| - \left|\frac{z_k}{c_{n+m}}\right|\right) \leq \prod_{k=1}^m \left|\left(z - \frac{z_k}{c_{n+m}}\right)\right| = \left|\frac{\rho(c_{n+m}z)}{c_{n+m}^m}\right|. \tag{41}$$

Therefore, from (40) and (41), for all $n$ large enough we have that

$$\left|\frac{\mathfrak{L}_n(z)}{\mathfrak{P}_n(z)}\right| \leq \frac{M}{(d^*-\varepsilon)^m}, \quad \forall z \in K,$$

and this proves the lemma. □

**Lemma 10.** *Let $\mu \in \mathcal{P}_m(\Delta)$, where $m \in \mathbb{N}$ and $K \subset \mathbb{C} \setminus \Delta_c$ is a compact subset. Then,*

$$\left|\frac{\widehat{\mathfrak{Q}}_n(z)}{\mathfrak{L}_n(z)} - \frac{\mathfrak{P}_n(z)}{\mathfrak{L}_n(z)}\right| \rightrightarrows 0, \quad \forall z \in K.$$

**Proof.** For each fixed $n > m$, we have that

$$\frac{\widehat{\mathfrak{Q}}_n(z) - \mathfrak{P}_n(z)}{\mathfrak{L}_n(z)} = \sum_{k=0}^m \left(\frac{\lambda_n}{\lambda_{n-k}} - 1\right) \frac{b_{n,n-k}}{c_n^k}\,\frac{\mathfrak{L}_{n,n-k}(z)}{\mathfrak{L}_n(z)}. \tag{42}$$

As $\lambda_n = -n$ in the Laguerre case and $\lambda_n = -2n$ in the Hermite case, then for each $k$ fixed, $k = 1, \ldots, m$,

$$\lim_{n\to\infty} \frac{\lambda_n}{\lambda_{n-k}} = 1. \tag{43}$$

From (30), (42), (43) and Proposition 5 we deduce the lemma. □

**Proof** (Theorem 1)**.** (a) From [18, (5.1.14), (5.5.10)] we have that $\mathfrak{L}'_{n,n-k} = (n-k)\widetilde{\mathfrak{L}}_{n,n-1-k}$, where

$$\widetilde{\mathfrak{L}}_{n,n-1-k} = \begin{cases} c_n^{-(n-1-k)} L_{n-1-k}^{\alpha+1}(c_n z), & \text{Laguerre case,} \\ c_n^{-(n-1-k)} H_{n-1-k}(c_n z), & \text{Hermite case.} \end{cases}$$

Let us define

$$d\widetilde{w}(x) = \begin{cases} dw_L^{\alpha+1}(x), & \text{Laguerre case,} \\ dw_H(x), & \text{Hermite case.} \end{cases}$$

$$d w_n(x) = \begin{cases} c_n^{-1} d w_L^{\alpha}(c_n x), & \text{Laguerre case,} \\ c_n^{-1} d w_H(c_n x), & \text{Hermite case.} \end{cases}$$

$$d \widetilde{w}_n(x) = \begin{cases} c_n^{-1} d w_L^{\alpha+1}(c_n x), & \text{Laguerre case,} \\ c_n^{-1} d w_H(c_n x), & \text{Hermite case.} \end{cases}$$

Notice that $\{\mathfrak{L}_{n,n-k}\}_{k=0}^{n}$ and $\{\widetilde{\mathfrak{L}}_{n,n-k}\}_{k=0}^{n}$ are monic orthogonal polynomials with respect to $w_n, \widetilde{w}_n$ respectively, hence, from [8, (11)], we have that the sequences $\{\mathfrak{L}_{n,n-k}\}_{n=0}^{\infty}$ and $\{\widetilde{\mathfrak{L}}_{n,n-k}\}_{n=0}^{\infty}$ for every $k = 0, \ldots, m$ satisfy that

$$\lim_{n\to\infty} \|w_n \mathfrak{L}_{n,n-k}\|_{L^2(\Delta)}^{1/n} = e^{-F_w}, \qquad \lim_{n\to\infty} \|\widetilde{w}_n \widetilde{\mathfrak{L}}_{n,n-k}\|_{L^2(\Delta)}^{1/n} = e^{-F_w}, \tag{44}$$

where $F_w$ is the modified Robin constant for the weights $w, \widetilde{w}$ (or the extremal constant according to the terminology of [8]) and $\|.\|_{L^2(\Delta)}$ denotes the $L^2$-norm with the Lebesgue measure with support on $\Delta$.

From [9, Ths. 1 & 2] we have that

$$\begin{aligned} \|w_n \mathfrak{L}_{n,n-k}\|_{L^\infty(\Delta)} &\le K_1 n^{\beta} \|w_n \mathfrak{L}_{n,n-k}\|_{L^2(\Delta)}, \\ \|\widetilde{w}_n \widetilde{\mathfrak{L}}_{n,n-k}\|_{L^\infty(\Delta)} &\le K_2 n^{\beta} \|\widetilde{w}_n \widetilde{\mathfrak{L}}_{n,n-k}\|_{L^2(\Delta)}, \end{aligned} \tag{45}$$

where $K_1, K_2$ are constants that do not depend on $n$, $\beta = 1/2$ for the Laguerre case, and $\beta = 1/4$ for the Hermite case. Using (44), (45), and [8, (11)] we obtain that

$$\lim_{n\to\infty} \|w_n \mathfrak{L}_{n,n-k}\|_{L^\infty(\Delta)}^{1/n} = e^{-F_w}, \qquad \lim_{n\to\infty} \|\widetilde{w}_n \widetilde{\mathfrak{L}}_{n,n-k}\|_{L^\infty(\Delta)}^{1/n} = e^{-F_w}. \tag{46}$$

Then we have

$$\begin{aligned} \|w_n \widehat{\mathfrak{Q}}_n\|_{L^\infty(\Delta)} &\le \sum_{k=0}^{m} \left| \frac{\lambda_n b_{n,n-k}}{c_n^k \lambda_{n-k}} \right| \|w_n \mathfrak{L}_{n,n-k}\|_{L^\infty(\Delta)} \\ &\le \left| \frac{(m+1)\lambda_n b_{n,n-k^*(n)}}{c_n^{k^*(n)} \lambda_{n-k^*(n)}} \right| \|w_n \mathfrak{L}_{n,n-k^*(n)}\|_{L^\infty(\Delta)}, \end{aligned}$$

and

$$\begin{aligned} \|\widetilde{w}_n \widehat{\mathfrak{Q}}'_n\|_{L^\infty(\Delta)} &\le \sum_{k=0}^{m} \left| \frac{(n-k)\lambda_n b_{n,n-k}}{c_n^k \lambda_{n-k}} \right| \|\widetilde{w}_n \widetilde{\mathfrak{L}}_{n,n-1-k}\|_{L^\infty(\Delta)} \\ &\le \left| \frac{(m+1)(n-k^{**}(n))\lambda_n b_{n,n-k^{**}(n)}}{c_n^{k^{**}(n)} \lambda_{n-k^{**}(n)}} \right| \|\widetilde{w}_n \widetilde{\mathfrak{L}}_{n,n-1-k^{**}(n)}\|_{L^\infty(\Delta)}, \end{aligned}$$

where $\|.\|_{L^\infty(\Delta)}$ denotes the sup norm and $k^*(n), k^{**}(n)$ denote positive integer numbers such that the following equalities hold

$$\left| \frac{\lambda_n b_{n,n-k^*(n)}}{c_n^{k^*(n)} \lambda_{n-k^*(n)}} \right| \|w_n \mathfrak{L}_{n,n-k}\|_{L^\infty(\Delta)} = \max_{k=0,\ldots,m} \left| \frac{\lambda_n b_{n,n-k}}{c_n^k \lambda_{n-k}} \right| \|w_n \mathfrak{L}_{n,n-k}\|_{L^\infty(\Delta)},$$

$$\begin{aligned} &\left| \frac{(n-k^{**}(n))\lambda_n b_{n,n-k^{**}(n)}}{c_n^{k^{**}(n)} \lambda_{n-k^{**}(n)}} \right| \|\widetilde{w}_n \widetilde{\mathfrak{L}}_{n,n-1-k^{**}(n)}\|_{L^\infty(\Delta)} \\ &\quad = \max_{k=0,\ldots,m} \left| \frac{(n-k)\lambda_n b_{n,n-k}}{c_n^k \lambda_{n-k}} \right| \|\widetilde{w}_n \widetilde{\mathfrak{L}}_{n,n-k}\|_{L^\infty(\Delta)}. \end{aligned}$$

From these last inequalities and (46) we deduce that

$$\lim_{n\to\infty}\left(\|w_n\widehat{\mathfrak{Q}}_n\|_{L^\infty(\Delta)}\right)^{1/n}=e^{-F_w},\qquad \lim_{n\to\infty}\left(\|\widetilde{w}_n\widehat{\mathfrak{Q}}_n'\|_{L^\infty(\Delta)}\right)^{1/n}=e^{-F_w}.$$

Therefore, if $\nu_n, \delta_n$ denote the root counting measure of $\widehat{\mathfrak{Q}}_n$ and $\widehat{\mathfrak{Q}}_n'$ respectively, from [5, Th. 1.1] we deduce that $\nu_n \xrightarrow{*} \nu_w$, $\delta_n \xrightarrow{*} \nu_w$ in the weak star sense.

(b) From Lemma 9, if $\varepsilon$ is sufficiently small and $K\subset\mathbb{C}\setminus\Delta_c$ is a compact subset, for all $n$ sufficiently large we have that, for some positive constant $C$,

$$C-\varepsilon\le\left|\frac{\mathfrak{P}_n(z)}{\mathfrak{L}_n(z)}\right|-\varepsilon\le\left|\frac{\widehat{\mathfrak{Q}}_n(z)}{\mathfrak{L}_n(z)}\right|.$$

From this fact and from Lemma 6 we deduce that the set of accumulation points is contained on $\Delta_c$ and from (a) of the present theorem we deduce that the set of accumulation points of the zeros of $\widehat{\mathfrak{Q}}_n$ is $\Delta_c$. □

**Proof** (Theorem 2)**.** From (b) of Theorem 1 we deduce that for the Laguerre case

$$\lim_{n\to\infty}\frac{\widehat{Q}_n'(c_nz)}{\widehat{Q}_n(c_nz)}=\lim_{n\to\infty}\frac{\widehat{Q}_n''(c_nz)}{\widehat{Q}_n'(c_nz)}=\frac{1}{2\pi}\int_0^1\frac{1}{z-t}\sqrt{\frac{1-t}{t}}dt=\frac{1}{2}\left(1-\sqrt{1-1/z}\right),$$

and for the Hermite case

$$\lim_{n\to\infty}\frac{\widehat{Q}_n'(c_nz)}{c_n\widehat{Q}_n(c_nz)}=\lim_{n\to\infty}\frac{\widehat{Q}_n''(c_nz)}{c_n\widehat{Q}_n'(c_nz)}=\frac{1}{\pi}\int_{-1}^1\frac{\sqrt{1-t^2}}{z-t}dt=z\left(1-\sqrt{1-1/z^2}\right),$$

on compact subsets $K\subset\mathbb{C}\setminus\Delta_c$. From (6) and the preceding relations we have for the Laguerre case

$$\frac{P_n(c_nz)}{\widehat{Q}_n(c_nz)}=\frac{zc_n}{\lambda_n}\frac{\widehat{Q}_n''(c_nz)}{\widehat{Q}_n'(c_nz)}\frac{\widehat{Q}_n'(c_nz)}{\widehat{Q}_n(c_nz)}+\left(\frac{1+\alpha-c_nz}{\lambda_n}\right)\frac{\widehat{Q}_n'(c_nz)}{\widehat{Q}_n(c_nz)},\tag{47}$$

and for the Hermite case

$$\frac{P_n(c_nz)}{\widehat{Q}_n(c_nz)}=\frac{1}{2}\frac{1}{\lambda_n}\frac{\widehat{Q}_n''(c_nz)}{\widehat{Q}_n'(c_nz)}\frac{\widehat{Q}_n'(c_nz)}{\widehat{Q}_n(c_nz)}-\left(\frac{c_nz}{\lambda_n}\right)\frac{\widehat{Q}_n'(c_nz)}{\widehat{Q}_n(c_nz)}.\tag{48}$$

Taking limits in (47) and (48) we obtain (16). Relation (17) follows from (14) and (16). □

## 4. The polynomial $\mathfrak{Q}_n$

Some basic properties of the zeros of $\mathfrak{Q}_n$ follow directly from (1) and (2). For example, the multiplicity of the zeros of $\mathfrak{Q}_n$ is at most 3, a zero of multiplicity 3 is also a zero of $\mathfrak{P}_n$ and a zero of multiplicity 2 is a critical point of $\widehat{\mathfrak{Q}}_n$. In the next lemma we prove conditions for the boundedness of the zeros of $\mathfrak{Q}_n$ and determine their asymptotic behavior.

**Lemma 11.** *Let $\mu\in\mathcal{P}_m(\Delta)$, where $m\in\mathbb{N}$ and define for $z\in\mathbb{C}$, $\mathfrak{D}(z)=\sup_{x\in\Delta_c}|z-x|$. If $\{\zeta_n\}_{n=m+1}^{\infty}$ is a sequence of complex numbers with limit $\zeta\in\mathbb{C}$, then for every $d>1$ there is a positive number $N_d$, such that $\{z\in\mathbb{C}:\mathfrak{Q}_n(z)=0\}\subset\{z\in\mathbb{C}:|z|\le\mathfrak{D}(\zeta)+d\}$ whenever $n>N_d$.*

**Proof.** As $\mathfrak{Q}_n(z) = 0$ then $\widehat{\mathfrak{Q}}_n(z) = \widehat{\mathfrak{Q}}_n(\zeta_n)$. From the Gauss–Lucas theorem (cf. [16, Section 2.1.3]), it is known that the critical points of $\widehat{\mathfrak{Q}}_n$ are in the convex hull of its zeros and from (b) of Theorem 1 the zeros of the polynomials $\{\widehat{\mathfrak{Q}}_n\}_{n=m+1}^{\infty}$ accumulate on $\Delta_c$. Hence, from the *bisector theorem* (see [16, Section 5.5.7]) $|z| \leq \mathfrak{D}(\zeta_n) + 1$ and the lemma is established. $\square$

We are now ready to prove Theorem 3.

**Proof** (Theorem 3)**.** From Lemma 11 we have that the zeros of $\mathfrak{Q}_n$ are located in a compact set. From (15) the zeros of $\mathfrak{Q}_n$ satisfy the equation

$$\left|\widehat{\mathfrak{Q}}_n(z)\right|^{\frac{1}{n}} = \left|\widehat{\mathfrak{Q}}_n(\zeta_n)\right|^{\frac{1}{n}}. \tag{49}$$

If $z \in \mathbb{C} \setminus \Delta_c$, taking limit when $n \to \infty$, from Lemma 11, and using (17) of Theorem 2 on both sides of (49), we have that the zeros of the sequence of polynomials $\{\mathfrak{Q}_n\}_{n=m+1}^{\infty}$ cannot accumulate outside the set

$$\{z \in \mathbb{C} : \Psi(z) = \Psi(\zeta)\} \bigcup \Delta_c.$$

To verify the second statement of the theorem, note that if $z$ is a zero of $\mathfrak{Q}_n$, from (15) we get

$$\prod_{k=1}^{n} \left| \frac{z - \widehat{x}_{n,k}}{\zeta_n - \widehat{x}_{n,k}} \right| = 1, \quad \text{where } \widehat{x}_{n,k} \text{ are the zeros of } \widehat{\mathfrak{Q}}_n. \tag{50}$$

Let $\mathcal{V}_\varepsilon(\Delta_c)$ be the $\varepsilon$-neighborhood of $\Delta_c$ defined as $\mathcal{V}_\varepsilon(\Delta_c) = \{z \in \mathbb{C} : \mathfrak{d}(z) < \epsilon\}$, as $\lim_{n\to\infty} \zeta_n = \zeta$, then for all $\varepsilon > 0$ there is a $N_\varepsilon > 0$ such that $|\mathfrak{d}(\zeta_n) - \mathfrak{d}(\zeta)| < \varepsilon$ whenever $n > N_\varepsilon$.

If $\mathfrak{d}(\zeta) > 2$, let us choose $\varepsilon = \varepsilon_\zeta = \frac{1}{2}(\mathfrak{d}(\zeta) - 2)$ and suppose that there is a $z_0 \in \mathcal{V}_{\varepsilon_\zeta}(\Delta_c)$ such that $\mathfrak{Q}_n(z_0) = 0$ for some $n > N_{\varepsilon_\zeta}$. Hence

$$\prod_{k=1}^{n} \left| \frac{z_0 - \widehat{x}_{n,k}}{\zeta_n - \widehat{x}_{n,k}} \right| < \left( \frac{2 + \varepsilon_\zeta}{\mathfrak{d}(\zeta_n)} \right)^n < 1, \tag{51}$$

which is a contradiction with (50), hence $\{z \in \mathbb{C} : \mathfrak{Q}_n(z) = 0\} \bigcap \mathcal{V}_{\varepsilon_n}(\Delta_c) = \varnothing$ for all $n > N_{\varepsilon_\zeta}$, i.e. the zeros of $\mathfrak{Q}_n$ cannot accumulate on $\mathcal{V}_{\varepsilon_\zeta}(\Delta_c)$.

From (15) it is straightforward that a multiple zero of $\mathfrak{Q}_n$ is also a critical point of $\widehat{\mathfrak{Q}}_n$. But, from (b) of Theorem 1 and the Gauss–Lucas theorem the set of accumulation points of $\widehat{\mathfrak{Q}}_n$ is $\Delta_c$, where we have that for $n$ sufficiently large the zeros of $\mathfrak{Q}_n$ are simple. $\square$

**Proof** (Theorem 4)**.** 1. Let us prove first that

$$\frac{\mathfrak{Q}_n(z)}{\widehat{\mathfrak{Q}}_n(z)} = 1 - \frac{\widehat{\mathfrak{Q}}_n(\zeta_n)}{\widehat{\mathfrak{Q}}_n(z)} \underset{n\to\infty}{\rightrightarrows} 1, \tag{52}$$

uniformly on compact subsets $K$ of the set $\{z \in \mathbb{C} : |\Psi(z)| > |\Psi(\zeta)|\}$. In order to prove (52) it is sufficient to show that

$$\frac{\widehat{\mathfrak{Q}}_n(\zeta_n)}{\widehat{\mathfrak{Q}}_n(z)} \underset{n\to\infty}{\rightrightarrows} 0, \tag{53}$$

uniformly on $K$.

From [7] and Lemmas 8, 9, we have that for all $n$ large enough it is possible to find constants $c^*$, $c$ such that

$$c^* \leq \left| \frac{\mathfrak{P}_n(z)}{\Psi^n(z)} \right| \leq c, \tag{54}$$

uniformly on compact subsets of $\mathbb{C} \setminus \Delta_c$. Then we have

$$\left| \frac{\widehat{\mathfrak{Q}}_n(\zeta_n)}{\widehat{\mathfrak{Q}}_n(z)} \right| = \left| \frac{\widehat{\mathfrak{Q}}_n(\zeta_n)}{\mathfrak{P}_n(\zeta_n)} \right| \left| \frac{\mathfrak{P}_n(z)}{\widehat{\mathfrak{Q}}_n(z)} \right| \left| \frac{\mathfrak{P}_n(\zeta_n)}{\Psi^n(\zeta_n)} \right| \left| \frac{\Psi^n(z)}{\mathfrak{P}_n(z)} \right| \left| \left( \frac{\Psi(\zeta_n)}{\Psi(z)} \right) \right|^n .$$

From (16) of Theorem 2 and (54) the first four factors on the right hand side of the previous formula are bounded; meanwhile, the last factor tends to 0 when $n \to \infty$, and we get (53). Finally, the assertion 1 is straightforward from (16) of Theorem 2.

2. For the assertion 2 of the theorem it is sufficient to prove that

$$\frac{\mathfrak{Q}_n(z)}{\widehat{\mathfrak{Q}}_n(\zeta_n)} = \frac{\widehat{\mathfrak{Q}}_n(z)}{\widehat{\mathfrak{Q}}_n(\zeta_n)} - 1 \underset{n\to\infty}{\rightrightarrows} -1, \tag{55}$$

uniformly on compact subsets $K$ of the set $\{z \in \mathbb{C} : |\Psi(z)| < |\Psi(\zeta)|\} \setminus \Delta_c$. Note that

$$\frac{\widehat{\mathfrak{Q}}_n(z)}{\widehat{\mathfrak{Q}}_n(\zeta_n)} = \frac{\widehat{\mathfrak{Q}}_n(z)}{\mathfrak{P}_n(z)} \frac{\mathfrak{P}_n(\zeta_n)}{\widehat{\mathfrak{Q}}_n(\zeta_n)} \frac{\mathfrak{P}_n(z)}{\Psi^n(z)} \frac{\Psi^n(\zeta_n)}{\mathfrak{P}_n(\zeta_n)} \left( \frac{\Psi(z)}{\Psi(\zeta_n)} \right)^n .$$

Now, the first part of the assertion 2 is straightforward from (16) of Theorem 2 and (54).

If $\mathfrak{d}(\zeta) > 2$, let $\mathcal{V}_\varepsilon(\Delta_c) = \{z \in \mathbb{C} : \mathfrak{d}(z) < \epsilon\}$ be a $\varepsilon$-neighborhood of $\Delta_c$, where $\varepsilon = \varepsilon_\zeta = \frac{\mathfrak{d}(\zeta)}{2} - 1$. By the same reasoning used to deduce (51) we get that

$$\left| \frac{\widehat{\mathfrak{Q}}_n(z)}{\widehat{\mathfrak{Q}}_n(\zeta_n)} \right| < \kappa^n, \quad \text{for all } z \in \mathcal{V}_\varepsilon(\Delta_c),\ \kappa < 1. \tag{56}$$

Hence from the first part of the assertion 2 and (56) we get the second part of the assertion 2. □

## 5. A fluid dynamics model

In this section we show a hydrodynamical model for the zeros of the orthogonal polynomials with respect to the pair $(\mathcal{L}, \mu)$. In [4], we gave a hydrodynamic interpretation for the critical points of orthogonal polynomials with respect to a Jacobi differential operator.

Let us consider a flow of an incompressible fluid in the complex plane, due to a system of $n$ *different* points ($n > 1$) fixed at $w_i$, $1 \leq i \leq n$. At each point $w_i$ of the system there is defined a complex potential $\mathcal{V}_i$, which for the Laguerre case equals to the sum of a *source (sink)* with a fixed strength $\Re[c_i]$ plus a *vortex* with a fixed strength $\Im[c_i]$ plus a *uniform stream* $U_i$ at infinity. Here $c_i$ and $d_i$ are fixed complex numbers which depend on the position of the remaining points $\{w_i\}_{i=1}^n$, see [10, Ch. VIII] for the terminology. We shall call *n system* to the set of the $n$ points fixed at $w_i$ with its respective potential of velocities.

Define the functions

$$f_i(w_1, \ldots, w_n) = \frac{R_n''(w_i)}{R_n'(w_i)}, \quad i = 1, \ldots, n \text{ where } R_n(z) = \prod_{i=1}^{n}(z - w_i).$$

The complex potentials $\mathcal{V}_L$ (Laguerre case) or $\mathcal{V}_H$ (Hermite case) at any point $z$ (see [6, Ch. 10]), by the principle of superposition of solutions, are given by

$$\mathcal{V}_L(z) = \sum_{i=1}^{n} \mathcal{V}_{L,i} = \sum_{i=1}^{n} (-z + (1+\alpha - w_i)\log(z - w_i) \\ + (z + w_i \log(z - w_i)) f_i(w_1, \dots, w_n)), \tag{57}$$

and

$$\mathcal{V}_H(z) = \sum_{i=1}^{n} \mathcal{V}_{H,i} = \sum_{i=1}^{n} \left( -z + \frac{1}{2}(f_i(w_1, \dots, w_n) - 2w_i)\log(z - w_i) \right). \tag{58}$$

From a complex potential $\mathcal{V}$, a complex velocity $\mathbf{V}$ can be derived by differentiation ($\mathbf{V}(z) = \frac{d\mathcal{V}}{dz}$). A standard problem associated with the complex velocity is to find the zeros, that correspond to the set of *stagnation points*, i.e. points where the fluid has zero velocity.

We are interested in the problem: Build an $n$ system (location of the points $w_1, \dots, w_n$) such that the stagnation points are at preassigned points with *nice* properties. As it is well known, the zeros of the orthogonal polynomials with respect to a finite positive Borel measures on $\mathbb{R}$ have a rich set of *nice* properties (cf. in [18, Ch. VI]), and will be taken as preassigned stagnation points. Here we consider $\mu \in \mathcal{P}_1[\mathbb{R}_+]$ or $\mu \in \widetilde{\mathcal{P}}_2[\mathbb{R}]$. In the next paragraph we establish the statement of the problem for both Laguerre and Hermite cases.

**Problem.** Let $\{x_1, \dots, x_n\}$ be the set of zeros of the $n$th orthogonal polynomial $P_n$ ($n > 1$ for the Laguerre case and $n > 2$ for the Hermite) with respect to a measure $\mu \in \mathcal{P}_1[\mathbb{R}_+]$ or $\mu \in \widetilde{\mathcal{P}}_2[\mathbb{R}]$. Suppose a flow is given, with complex potential $\mathcal{V}_L$ (Laguerre case) or $\mathcal{V}_H$ (Hermite). Build an $n$ system (location of the points $w_1, \dots, w_n$) such that the stagnation points are attained at the points $z = x_i$, with $i = 1, 2, \dots, n$.

Consider first the Laguerre case. If $x_k$ ($k = 1, \dots, n$.) are stagnation points then

$$\frac{\partial \mathcal{V}_L}{\partial z}(x_k) = (1+\alpha - x_k)\sum_{i=1}^{n} \frac{1}{x_k - w_i} + x_k \sum_{i=1}^{n} \frac{R_n''(w_i)}{R_n'(w_i)(x_k - w_i)} = 0. \tag{59}$$

We are looking for a solution $R_n(z) = \prod_{i=1}^{n}(z - w_i)$, with $w_i \neq w_j \neq x_k, \forall i, j, k, i \neq j$, such that (59) holds. This assumption implies that the sum in the second term of the left hand side of expression (59) is the partial-fraction decomposition of $\frac{R_n''}{R_n}$ evaluated at $x = x_k$. Therefore, (59) is equivalent to

$$x_k R_n''(x_k) + (1+\alpha - x_k)R_n'(x_k) = 0, \quad k = 1,\ 2,\ \dots, n.$$

Note that $xR_n''(x) + (1+\alpha - x)R_n'(x)$ is a polynomial of degree $n$, with leading coefficient $\lambda_n$ that vanishes at the zeros of $P_n$, i.e.

$$xR_n''(x) + (1+\alpha - x)R_n'(x) = \lambda_n P_n(x). \tag{60}$$

Observe that expression (60) is equivalent to (6). From Proposition 4, the zeros of $\widehat{Q}_n, \widehat{Q}_n'$ are real, simple and $Q_n'(x_k) \neq 0$. Therefore, $R_n = \widehat{Q}_n$ is a solution. Hence, an answer to our problem yields the $n$ points as the $n$ zeros of the polynomial $\widehat{Q}_n$.

For the Hermite case we have a similar situation. Thus, if $x_k$ is a stagnation point, $\frac{\partial \mathcal{V}_H}{\partial z}(x_k) = 0$, which gives

$$x_k \sum_{i=1}^{n} \frac{1}{x_k - w_i} - \frac{1}{2} \sum_{i=1}^{n} \frac{R_n''(w_i)}{R_n'(w_i)(x_k - w_i)} = 0, \quad k = 1, 2, \ldots, n. \tag{61}$$

Again, we can deduce that the expression (61) equals to $\frac{1}{2} R_n''(x_k) - x_k R_n'(x_k) = 0$, for $k = 1, \ldots, n$.

Note that $\frac{1}{2} R_n''(x) - x R_n'(x)$ is a polynomial of degree $n$, with leading coefficient $\lambda_n$ that vanishes at the zeros of $P_n$, i.e.

$$\frac{1}{2} R_n''(x) - x R_n'(x) = \lambda_n P_n(x). \tag{62}$$

Therefore, the expression (62) is equivalent to (6). From Proposition 4, the zeros of $\widehat{Q}_n$, $\widehat{Q}_n'$ are real and simple and $Q_n'(x_k) \neq 0$, which implies that $R_n = \widehat{Q}_n$ is a solution to our problem. As a conclusion,

**Answer.** *The flow of an incompressible two-dimensional fluid, due to n points with complex potential $\mathcal{V}_L$ or $\mathcal{V}_H$, located at the zeros of the nth orthogonal polynomial $\widehat{Q}_n$ with respect to $(\mathcal{L}, \mu)$, with $\mu \in \mathcal{P}_1[\mathbb{R}_+]$ or $\mu \in \widetilde{\mathcal{P}}_2[\mathbb{R}]$ has its n stagnation points at the n zeros of the nth orthogonal polynomial $\widehat{Q}_n$.*

It would be interesting to consider the uniqueness of the solution obtained. In other words, what could be said about the solutions of the form $Q_n(z) = \widehat{Q}_n(z) - \widehat{Q}_n(\zeta_n)$ and to extend this model to more general classes of measures $\mu$. It would be also of interest to decide if these stagnation or equilibrium points are stable.

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