This is a postprint version of the following published document:

DOI: 10.1016/j.jat.2015.03.005

Proyecto MTM2012-36732-C03-01

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Differential orthogonality: Laguerre and Hermite cases with applications

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Communicated by Francisco Marcellan

Abstract

Let $\mu$ be a finite positive Borel measure supported on $\mathbb{R}$, $L[f] = xf'' + (\alpha + 1 - x)f'$ with $\alpha > -1$, or $L[f] = \frac{1}{2}f'' - xf'$, and $m$ a natural number. We study algebraic, analytic and asymptotic properties of the sequence of monic polynomials $\{Q_n\}_{n \geq m}$ that satisfy the orthogonality relations

$$\int L[Q_n](x)x^kd\mu(x) = 0 \quad \text{for all} \quad 0 \leq k \leq n - 1.$$

We also provide a fluid dynamics model for the zeros of these polynomials.

Keywords: Orthogonal polynomials; Ordinary differential operators; Asymptotic analysis; Weak star convergence; Hydrodynamic

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1. Introduction

Orthogonal polynomials with respect to a differential operator were introduced in [1] as a generalization of the notion of orthogonal polynomials. Analytic and algebraic properties of these classes of polynomials have been considered for some classes of first order differential operators in [2,11], for a Jacobi differential operator in [4], and for differential operators of arbitrary order with polynomials coefficients in [3]. In this paper, we consider orthogonal polynomials with respect to a Laguerre or Hermite operator and a positive Borel measure $\mu$ with unbounded support on $\mathbb{R}$.

We denote by $L_{L}$ the Laguerre and by $L_{H}$ the Hermite differential operators on the linear space $P$ of all polynomials, i.e. for all $f \in P$ and $a \geq -1$

$$L_{L}[f] = xf'' + (1 + a - x)f' = x^{-a} e^{x} \left( x^{a+1} e^{-x} f' \right)' ,$$

$$L_{H}[f] = \frac{1}{2} f'' - xf' = \frac{1}{2} x^{2}\left( e^{-x^2} f' \right)' .$$

Each one of these second order differential operators has a system of monic polynomials which are eigenfunctions of the operator and orthogonal with respect to a measure. Let $\{P_{n}\}_{n=0}^{\infty}$ be the monic Laguerre polynomials with $a > -1$ and $\{H_{n}\}_{n=0}^{\infty}$ the monic Hermite polynomials, then

$$(L_{L}^{n}, L_{L}^{m})_{L} = \int L_{L}^{n}(x) L_{L}^{m}(x) d\mu(x) = 0 \quad \text{if} \quad n \neq m ,$$

$$(H_{n}, H_{m})_{H} = \int H_{n}(x) H_{m}(x) d\mu_{H}(x) = 0 \quad \text{if} \quad n \neq m ,$$

where $d\mu_{L}(x) = x^{a} e^{-x} dx$ and $d\mu_{H}(x) = e^{-x^2} dx$. In addition,

$$L_{L}[L_{L}^{n}] = -n L_{L}^{n} \quad \text{and} \quad L_{H}[H_{n}] = -n H_{n} .$$

(3)

To unify the approach, we will denote by $\mathcal{L}$ the Laguerre or Hermite differential operator ($L_{L}$ or $L_{H}$) in the sequel, by $d\mu$ the Laguerre or Hermite measure ($d\mu_{L}$ or $d\mu_{H}$), by $L_{n}$ the $n$th Laguerre or Hermite monic orthogonal polynomial ($L_{L}^{n}$ or $H_{n}$) and by $\Delta$ the set $\mathbb{R}_{+}$ or $\mathbb{R}$, respectively. We will refer to one or the other depending on the case we are solving.

Let $\mu$ be a finite positive Borel measure, supported on $\Delta$ and $\{P_{n}\}_{n=0}^{\infty}$ the corresponding system of monic orthogonal polynomials, i.e.

$$\langle P_{n}, P_{k} \rangle_{\mu} = \int P_{n}(x) P_{k}(x) d\mu(x) = \begin{cases} 0 & \text{if} \quad n \neq k, \\ \frac{1}{n!} & \text{if} \quad n = k . \\ \end{cases}$$

(4)

We say that $Q_{n}$ is the $n$th monic orthogonal polynomial with respect to the pair $(\mathcal{L}, \mu)$ if $Q_{n}$ has degree $n$ and

$$L[Q_{n}](x) x^{d} d\mu(x) = 0 \quad \text{for all} \quad 0 \leq k \leq n - 1 ,$$

(5)

or, equivalently,

$$L[Q_{n}] = \lambda_{n} P_{n} .$$

(6)

where $\lambda_{n} = -n$. 

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It was shown in [4, Section 2] that it is not always possible to guarantee the existence of a system of polynomials \(\{P_n(x)\}_{n \geq 0}\), orthogonal with respect to the pair \((L, \mu)\), where \(L^{(n, \beta)}\) is the Jacobi differential operator and \(\mu\) an arbitrary positive finite Borel measure. As will be shown later (cf. Propositions 1 and 2), a similar situation occurs for the case of Laguerre and Hermite operators. Let \(m \in \mathbb{N}\) be fixed, a fundamental role in the existence of infinite sequences of polynomials \(\{Q_n(x)\}_{n \geq 0}\), orthogonal with respect to the pair \((L, \mu)\) is played by the class \(P_n(\Delta)\) defined as the family of finite positive Borel measures \(\mu\) supported on \(\Delta\) for which there exists a polynomial \(\rho\) of degree \(m\), such that \(\mu = (\rho)^{-1} w\).

If \(\mu \in P_n(\Delta)\) it is not difficult to see that if \(n > m\), then

\[
P_n(z) = \sum_{k=0}^{n} b_{n,k} L_{n-k}(z), \quad b_{n,k} = \frac{1}{\tau_n} \int P_n(x) L_{n-k}(x) d\mu(x),
\]

\[
\tau_n = \| L_n \|_w^2 = \int L_n^2(x) d\mu(x) = \frac{\Gamma(n + \alpha + 1)}{\alpha \Gamma(n + 2 + \beta) \sqrt{\pi 2^n}}, \quad \mu \in P_n(\mathbb{R}_+), \quad \mu \in P_n(\mathbb{R}).
\]

and from (8) we obtain that the monic polynomial of degree \(n\), for \(n > m\) defined by the formula

\[
\tilde{Q}_n(z) = \sum_{k=1}^{n-1} \lambda_k b_{n,n-k} L_{n-k}(z),
\]

is orthogonal with respect to \((L, \mu)\).

Notice that from the equivalence between relations (5) and (6), the polynomial \(\tilde{Q}_n + c, c \in \mathbb{C}\), is orthogonal with respect to \((L, \mu)\) so that we do not have a unique monic orthogonal polynomial of degree \(n\). We had a similar situation when we studied the orthogonality with respect to a Jacobi operator. A natural way to define a unique sequence would be to consider a sequence of complex numbers \(\{c_n\}_{n \geq 0}^{\infty}\) and define the sequence \(\{Q_n\}_{n \geq 0}^{\infty}\) satisfying (5), as the polynomial solution of the initial value problem

\[
\begin{align*}
L[y] &= \lambda_n P_n, \quad n > m, \\
y(\xi) &= 0.
\end{align*}
\]

We say that \(\{Q_n\}_{n \geq 0}^{\infty}\) is the sequence of monic orthogonal polynomials with respect to the pair \((L, \mu)\) such that \(Q_n(\xi) = 0\).

Notice that the initial value problem (10) has the unique polynomial solution

\[
y(z) = Q_n(z) = \tilde{Q}_n(z) - \tilde{Q}_n(\xi).
\]

In this paper, we study some analytic and algebraic properties of the sequence of orthogonal polynomials with respect to a Laguerre or Hermite differential operator. In order to study the asymptotic properties of the sequence of polynomials we shall normalize them with an adequate parameter.

Let \(x_n\) be the modulus of the largest zero of the \(n\)th orthogonal polynomial with respect to \(\mu\) (or \(\nu\)), from [12, Lemma 11 with \(\lambda = 2\)] for the Hermite case and [12, Coroll. (p. 191) with \(\gamma = 1\)] for the Laguerre case, we get

\[
\lim_{n \to \infty} c_n x_n = 1,
\]
The set of accumulation points of the zeros of \( \Phi(z) = z + \sqrt{c^2 - 1} \) and \( \Psi(z) = 2z - 1 + 2\sqrt{d/3 - 1} \), where the branch of each root is selected from the condition \( \sqrt{\psi(\infty)} = \infty \) and \( \phi(\infty) = \infty \), respectively. Let \( \Delta_\beta \) be the interval \([0, 1]\) in the Laguerre case and \([-1, 1]\) in the Hermite case. Let \( \Phi_n(z) = e^{\alpha_n} P_n(cz^2) \) be the normalized monic orthogonal polynomials with respect to a measure \( \mu \in P_\Delta(\Delta) \).

To each generic polynomial \( q_n \), let \( \mu_n = n^{-1} \sum_{\omega \in \Delta_n} \delta_\omega \) be the normalized root counting measure, where \( \delta_\omega \) is the Dirac measure with mass 1 at the point \( \omega \). From \([12, Ths. 4 & 4']\) we find that the limit distribution \( \nu_0 \) of the zero counting measure of the normalized Laguerre and Hermite polynomials is

\[
d\nu_n(t) = \begin{cases} \frac{1}{2\pi} \sqrt{\frac{1}{1 - t}}, & t \in [0, 1] \\ \frac{1}{2\pi} \sqrt{1 - t'}, & t \in [-1, 1] \end{cases}
\]

Laguerre case, Hermite case.

From \([14, Chs. III & IV]\) we have that

\[
\lim_{n \to \infty} \| \Phi_n(z)^{\frac{1}{n}} \|_{\mathcal{L}}^2 = \begin{cases} \frac{1}{\sqrt{c}} |\phi(z)| e^{2\sqrt{d}/\psi(z)} \mu \in P_\Delta(\Delta), \\ \frac{1}{\sqrt{c}} |\phi(z)| e^{2\sqrt{d}/\psi(z)} \mu \in P_\Delta(\Delta) \end{cases}
\]

uniformly on compact subsets \( K \subset \mathbb{C} \setminus \Delta_\beta \).

We are interested in asymptotic properties of the normalized monic orthogonal polynomials with respect to a pair \((\mathcal{L}, \mu)\) defined by

\[
\Omega_n(z) = \tilde{\Omega}_n(z) - \tilde{\Omega}_n(cz^2),
\]

where \( \tilde{\Omega}_n(z) = e^{\alpha_n} \tilde{Q}_n(cz^2) \). For these polynomials we prove the following results.

**Theorem 1.** Let \( \mu \in P_\Delta(\Delta) \), where \( \mu \in \mathbb{R} \). Then:

(a) If \( \nu, \sigma \) denote the root counting measure of \( \tilde{\Omega}_n \) and \( \tilde{\Omega}_n' \), respectively then \( \nu_n \to \nu, \sigma \to \sigma \) in the weak star sense.

(b) The set of accumulation points of the zeros of \( \tilde{\Omega}_n \) is \( \Delta_\beta \).

**Theorem 2.** Let \( m \in \mathbb{R}, \mu \in P_\Delta(\Delta) \). Then, for every compact subset \( K \subset \mathbb{C} \setminus \Delta_\beta \) we have uniformly

\[
\lim_{n \to \infty} \frac{\Phi_n(z)}{\tilde{\Omega}_n(z)} = \begin{cases} 1 \mu \in P_\Delta(\Delta), \\ 1 \mu \in P_\Delta(\Delta) \end{cases}
\]

\[
\lim_{n \to \infty} \frac{\Phi_n(z)}{\tilde{\Omega}_n(z)}^{\frac{1}{2}} = \begin{cases} \frac{1}{\sqrt{c}} |\phi(z)| e^{2\sqrt{d}/\psi(z)} \mu \in P_\Delta(\Delta), \\ \frac{1}{\sqrt{c}} |\phi(z)| e^{2\sqrt{d}/\psi(z)} \mu \in P_\Delta(\Delta) \end{cases}
\]
The following result shows that the set of accumulation points of the zeros of the sequence of normalized polynomials, orthogonal with respect to \((\mathcal{L}, \mu)\), is contained in a curve.

**Theorem 3.** Let \(m \in \mathbb{N}\) and \(\mu \in \mathcal{P}_m(\Delta)\). If \(\{\zeta_n\}_{n=0}^{\infty}\) is a sequence of complex numbers with limit \(\zeta \notin \Delta\), then:

(a) The accumulation points of zeros of the sequence \(\{\Omega_n\}_{n=0}^{\infty}\) such that \(\Omega_n(\zeta_n) = 0\) are located on the set \(E = \mathcal{E}(\zeta) \cup \Delta\), where \(\mathcal{E}(\zeta)\) is the curve

\[
\mathcal{E}(\zeta) := \{z \in \mathbb{C} : \Phi(z) = \Psi(z)\},
\]

\[
\Phi(z) = \left|\phi(z)\right|^{2m}e^{\phi(z)}\quad \text{for} \quad \mu \in \mathcal{P}_m(\mathbb{R}), \quad \text{and} \quad \Psi(z) = \left|\psi(z)\right|^{2\psi(z)}\quad \text{for} \quad \mu \in \mathcal{P}_m(\mathbb{R}).
\]

(b) If \(\Phi(z) = \inf_{z \in \Delta} |z - \zeta| > 2\) then \(E = \mathcal{E}(\zeta)\) and for \(n\) sufficiently large are simple.

The relative asymptotic behavior between the sequences of polynomials \(\Omega_n\) and \(\Psi_n\) reads as

**Theorem 4.** Let \(\{\zeta_n\}_{n=0}^{\infty}\) be a sequence of complex numbers with limit \(\zeta \notin \Delta\), \(m \in \mathbb{N}\), \(\mu \in \mathcal{P}_m(\Delta)\) and \(\{\Omega_n\}_{n=0}^{\infty}\) and \(\{\Psi_n\}_{n=0}^{\infty}\) be the sequence of normalized monic orthogonal polynomials with respect to the pair \((\mathcal{L}, \mu)\) such that \(\Omega_0(\zeta_0) = 0\), then:

1. Uniformly on compact subsets of \(\Omega = \{z \in \mathbb{C} : |\Phi(z)| > |\Psi(z)|\}

\[
\frac{\Omega_n(z)}{\Psi_n(z)} \xrightarrow{n \to \infty} 1.
\]

2. Uniformly on compact subsets of \(\Omega = \{z \in \mathbb{C} : |\Phi(z)| < |\Psi(z)|\} \setminus \Delta.

\[
\frac{\Omega_n(z)}{\Psi_n(z)} \xrightarrow{n \to \infty} -1,
\]

where \(\Phi\) is as defined in Theorem 3. If \(\Phi(\zeta) > 2\) then (20) holds for \(\Omega = \{z \in \mathbb{C} : |\Phi(z)| < |\Psi(z)|\}.

The paper continues as follows. Section 2 is dedicated to the study of existence, uniqueness and some results concerning the properties of the zeros of orthogonal polynomials with respect to the Laguerre or Hermite operators. In Sections 3 and 4 we study the asymptotic behavior of the polynomials \(\Omega_n\) and \(\Psi_n\) respectively. Finally, in Section 5 we show a fluid dynamics model for the zeros of these polynomials.

2. The polynomial \(Q_n\)

First of all, we are interested in discussing systems of polynomials such that for some \(m \in \mathbb{N}\), for all \(n > m\), they are solutions of (6). In order to classify those measures \(\mu\) for which the existence of such sequences of orthogonal polynomials with respect to \((\mathcal{L}, \mu)\) can be guaranteed, we prove a preliminary lemma.

**Lemma 5.** Let \(\mu\) be a finite positive Borel measure with support contained on \(\mathbb{R}\) and let \(n \in \mathbb{N}\) be fixed. Then, the differential equation (6) has a monic polynomial solution \(Q_n\) of degree \(n\), which is unique up to an additive constant, if and only if

\[
\int P_n(x)dw(x) = 0, \quad \text{where} \quad P_n \text{ is as (4)}.
\]
such that for each \( n > m \) there exists a monic polynomial \( Q_n \) of degree \( n \), unique up to an additive constant and orthogonal with respect to \( (\mathcal{L}, \mu) \) if and only if \( r^{-1} \) is a polynomial of degree \( m \).

Proof. Suppose that \( m \) is the smallest natural number such that for each \( n > m \) there exists a monic polynomial \( Q_n \) of degree \( n \), unique up to an additive constant and orthogonal with respect to \( (\mathcal{L}, \mu) \). According to Lemma 5

\[
\int L_n(x) \frac{d\mu(x)}{r(x)} = \int L_n(x) dw(x) = 0 \quad \text{if } n > m,
\]

\[
\neq 0 \quad \text{if } n = m.
\]

But this is equivalent to saying that

\[
\frac{1}{r(x)} = \sum_{k=0}^m c_k L_k(x) \quad \text{with } c_m \neq 0.
\]

The converse is straightforward. \( \square \)

It is possible to give another characterization, in terms of the quasi orthogonality concept, for the existence of a system of polynomials such that for all \( n > m \), for some \( m \in \mathbb{N} \), they are solutions of (6).

Proposition 2. Let \( \mu \) be a finite positive Borel measure on \( \mathbb{R} \) and \( \{P_n\}_0^\infty \) the sequence of monic orthogonal polynomials with respect to \( \mu \). Then, \( m \) is the smallest natural number such that for each \( n > m \) there exists, except for an additive constant, a unique monic polynomial \( Q_n \), orthogonal with respect to the pair \( (\mathcal{L}, \mu) \), if and only if for all \( n > m \)

\[
\int P_n(x) x^k dw(x) = 0, \quad \text{for } k = 0, 1, \ldots, n - m,
\]

i.e. the polynomial \( P_n \) is quasi-orthogonal of order \( n - m + 1 \) with respect to the measure \( w \).

Proof. Assume that \( m \) is the smallest natural number such that for each \( n > m \) there exists a monic polynomial \( Q_n \) of degree \( n \), unique up to an additive constant and orthogonal with respect to \( (\mathcal{L}, \mu) \). From Lemma 5 we have that (21) holds for \( n > m \). From the three term recurrence relation for \( \{P_n\}_0^\infty \)

\[
x P_n(x) = P_{n-1}(x) + \beta_n P_{n-2}(x), \quad n \geq 1,
\]

\[
P_0(x) = 1, \quad P_1(x) = 0, \quad \beta_n, \beta \in \mathbb{R} \text{ and } \beta_n \neq 0,
\]

(22)

thus

\[
\int P_n(x) x^k dw(x) = 0 \quad \text{for all } 0 \leq k < n - m.
\]

(23)

which implies that the polynomial \( P_n \) is quasi-orthogonal of order \( n - m + 1 \) with respect to the measure \( w \) (Laguerre or Hermite).
Conversely, assume that \( m \) is the smallest natural number such that for \( n > m \), the polynomial \( P_n \) is quasi-orthogonal of order \( n - m + 1 \) with respect to the measure \( d\mu \). Then we have that

\[
P_n(x) = L_n(x) + \sum_{k=1}^{m} d_{n-k} L_{n-k}(x),
\]

which implies that for all integers \( n > m \) the polynomials \( P_n \) satisfy the condition (21). From Lemma 5 we have that there exists a monic polynomial \( Q_n \) of degree \( n \), unique up to an additive constant and orthogonal with respect to \( (\mathcal{L}, \mu) \), for all \( n > m \).

From the above proposition, we deduce in particular that the differential equation (6) has, except for an additive constant, a unique monic polynomial solution \( Q_n \) of degree \( n \) for all the natural numbers only if \( P_n = L_n \) and \( d\mu = dw \). Hence \( Q_n = L_n \), the polynomial eigenfunctions of \( \mathcal{L} \), whose properties are well known.

Let us continue by noting that the polynomials \( Q_n \) and \( \tilde{Q}_n \) (see (9) and (11)) are primitives of the same polynomial \( Q_n^\prime \) (or \( \tilde{Q}_n^\prime \)) and

\[
\int \tilde{Q}_n(x)x^k \, dw(x) = 0, \quad k = 0, 1, \ldots, n - m - 1.
\]  

Applying classical arguments [17], it is not difficult to prove the following result, which will be used in the sequel.

**Proposition 3.** The polynomial \( \tilde{Q}_n \), defined by (9) for all \( n > m \), has at least \((n - m)\) zeros and \((n - m - 1)\) critical points of odd multiplicity on \( \Delta \).

For \( m = 2 \) we denote by \( \tilde{T}_2[\mathbb{R}] \) the class of measures of the form \( d\mu = \frac{e^{-x}}{1 + e^{-x}} \, dx, \, x_1 \neq 0 \) in the Hermite case. The following proposition shows some results concerning the zeros of \( \tilde{Q}_n \) and \( \tilde{Q}_m \), for measures \( \mu \in \tilde{T}_1[\mathbb{R}] \) or \( \mu \in \tilde{T}_2[\mathbb{R}] \).

**Proposition 4.** Assume that \( \mu \in \tilde{T}_1[\mathbb{R}] \) or \( \mu \in \tilde{T}_2[\mathbb{R}] \), then the zeros of \( \tilde{Q}_n \) and \( \tilde{Q}_m \) are real and simple. The critical points of \( \tilde{Q}_n \) interlace the zeros of \( P_n \).

**Proof.**

1. *Laguerre case.* If \( m = 1 \) and \( \mu \in \tilde{T}_1[\mathbb{R}] \) from Proposition 3 the polynomial \( \tilde{Q}_n \) has at least \((n - 1)\) real zeros of odd multiplicity on \( \mathbb{R}^+ \). But, \( \tilde{Q}_n \) is a polynomial with real coefficients and degree \( n \), consequently the zeros of \( \tilde{Q}_n \) are real and simple. As \( \tilde{Q}_n = \tilde{Q}_n^\prime \), from Rolle’s theorem all the critical points of \( \tilde{Q}_n \) are real, simple, and \((n - 2)\) of them are contained on \( \mathbb{R}^+ \).

Denote \( G(x) = x^{n-1}e^{-x} \tilde{Q}_n^\prime(x) \), with \( x \in [1, \infty[ \). Notice that \( G(x) \) is a real-valued, continuous and differentiable function on \( \mathbb{R}^+ \). Suppose that there exists \( x \in \mathbb{R}^+ \) such that \( G(x) = 0 \). As \( G(0) = 0 \) from Rolle’s Theorem there exists \( x' \in \mathbb{R}^+ \) such that \( G(x') = 0 \). But, \( G(x) = x^n e^{-x} \mathcal{L}[\mu] \) and all the critical points of \( G \) are contained on \( \mathbb{R}^+ \). Hence all the critical points of \( \tilde{Q}_n \) belong to \( \mathbb{R}^+ \).

2. *Hermite case.* Consider now \( \mu \in \tilde{T}_2[\mathbb{R}] \), that is, \( m = 2 \) and \( d\mu(x) = \frac{e^{-x}}{2 + e^{-x}} \, dx, \, x_1 \neq 0 \). Using the relations (9) and [18, 5.6.1] we have that for \( k > 1 \)

\[
\tilde{Q}_{2k}(x) = L_k x^{1/2} + \frac{k}{k - 1} \frac{\langle P_k, H_{2k-2} \rangle}{\langle H_{2k-2}, H_{2k-2} \rangle} L_{k-1} x^{1/2}.
\]  

(25)
Proof. Let \( P_{n-1}(z) = x^{n-1} e^{-x^2} \) and \( C_n = \frac{1}{2^n n!} \sum_{k=0}^{n} \binom{n}{k} x^k \). Then, from (8) and the orthogonality of the Laguerre and Hermite cases.

Proof of Proposition 5. Let \( m \in \mathbb{N} \) and \( \mu \in \mathcal{P}_m(\Delta) \). Then for \( n \) large enough, there exist constants \( C^L_m \) and \( C^H_m \) such that

\[
|b_{n,\mu}| = \frac{|P_n L_{n-\mu}|}{|L_{n-\mu}|} ≤ \begin{cases} C^L_m \cdot n^3 & \text{Laguerre case,} \\ C^H_m \cdot n^2 & \text{Hermite case,} \end{cases}
\]

for \( k = 1, \ldots, m \).

Proof. Let \( \rho(z) = \sum_{j=1}^{\infty} \rho_j z^j \) and \( \rho_{\infty} = \max_{j \in \mathbb{N}} |\rho_j| \). From the Cauchy–Schwarz inequality we have

\[
|b_{n,\mu}| ≤ \frac{|P_n|}{|L_{n-\mu}|} \sqrt{\langle \rho L_{n-\mu}, L_{n-\mu} \rangle} ≤ \frac{|\rho| L_{n-\mu} \mu}{|L_{n-\mu}|} \sqrt{\langle \rho L_{n-\mu}, L_{n-\mu} \rangle} \leq \frac{\rho_{\infty}}{|\mu|} \sum_{k=0}^{\infty} |(x^{1}, L^{n}_{n-k})| w_k \sum_{k=0}^{m} |(x^{1}, L^{n}_{n-k})| w_k.
\]

We analyze separately the Laguerre and Hermite cases. Without loss of generality we can assume that \( n > 2m \).

• Laguerre case \((L_n = L^L_n, \Delta = \mathbb{R}_+, d\omega(x) = x^n e^{-x} dx)\). From [13, (III.4.9) and (I.2.9)] we have the connection formula

\[
L^L_{n-k}(x) = \sum_{j=k}^{\infty} \frac{j}{(n-k)!} \binom{n-k}{j} x^j L^L_j(x),
\]

then from (8) and the orthogonality

\[
\langle L^L_{n-k}, \tilde{L}^L_k \rangle = \sum_{j=k}^{\infty} \frac{j}{(n-k)!} \binom{n-k}{j} \int_{\mathbb{R}} (L^L_j(x))^2 x^j e^{-x} dx,
\]

\[
= \sum_{j=k}^{\infty} \frac{1}{(n-k)!} \Gamma(n-k+j+\alpha+1),
\]

\[
≤ 2^{n-k} \Gamma(n-k+j+\alpha+1),
\]

and

\[
\sum_{k=0}^{m} \langle L^L_{n-k}, L^L_k \rangle w_k ≤ (n-k) \sum_{k=0}^{m} 2^{n-k} \Gamma(n-k+m+\alpha+1),
\]

\[
≤ (2^{m+1} - 1) (n-k) \Gamma(n-k+m+\alpha+1).
\]
Hence, from (26), (8) and $n$ large enough

$$|b_{n,x,k}| \leq \frac{\rho_{n}(2^{n+1} - 1)}{|\rho_{n}|} \sqrt{(n - m)\Gamma(n + \alpha + 1) \Gamma(n + m - k + \alpha + 1)},$$

$$\leq \frac{\rho_{n}(2^{n+1} - 1)}{|\rho_{n}|} \frac{(n + \alpha + 1)^{2}(n - k)^{2}(n - k + \alpha + 1)}{(n - m)(n - k)^{2}},$$

Hence, from (26), (8) and $n$ large enough

$$|b_{n,x,k}| \leq \frac{\rho_{n}(2^{n+1} - 1)}{|\rho_{n}|} \frac{(n + \alpha + 1)^{2}(n - k)^{2}(n - k + \alpha + 1)}{(n - m)(n - k)^{2}},$$

where for all $x \in \mathbb{R}$, the symbol $|x|$ denotes the largest integer less than or equal to $x$. As it is well known (cf. [18, (5.5.6) and (5.5.8)]), the Hermite polynomials satisfy the recurrence relation

$$\sigma^{(j)} (x) = \frac{\rho_{n}(2^{n+1} - 1)}{|\rho_{n}|} \frac{(n + \alpha + 1)^{2}(n - k)^{2}(n - k + \alpha + 1)}{(n - m)(n - k)^{2}},$$

Hence, from (26), (8) and $n$ large enough

$$|b_{n,x,k}| \leq \frac{\rho_{n}(2^{n+1} - 1)}{|\rho_{n}|} \frac{(n + \alpha + 1)^{2}(n - k)^{2}(n - k + \alpha + 1)}{(n - m)(n - k)^{2}},$$

where $\sigma_{j,s} (n)$ is a polynomial in $n$ of degree equal to $\nu$ and leading coefficient $2^{-\nu}(\nu!)$. Hence, from (8), for $n$ large enough

$$|z^{j} H_{n,x} (z)| \leq \frac{\rho_{n}(2^{n+1} - 1)}{|\rho_{n}|} \frac{(n + \alpha + 1)^{2}(n - k)^{2}(n - k + \alpha + 1)}{(n - m)(n - k)^{2}},$$

with $j = 0, 1, \ldots, \left[\frac{m}{2}\right]$, therefore

$$|b_{n,x,k}| \leq 2^{2\nu} \left[\frac{\rho_{n}(2^{n+1} - 1)}{|\rho_{n}|} \frac{(n + \alpha + 1)^{2}(n - k)^{2}(n - k + \alpha + 1)}{(n - m)(n - k)^{2}}\right].$$
Lemma 6. Let \( m \in \mathbb{N} \), \( \mu \in P_m(\Delta) \) and \( \xi \) such that \( \widehat{\Delta}_\mu(\xi) = 0 \). Then for all \( n \) sufficiently large \( d_n(\xi) < 2m' \), where

\[
\begin{align*}
\sigma_n &= \begin{cases} 
1 + \frac{1}{2} C_n^L & \text{Laguerre case,} \\
1 + \sqrt{\frac{C_n^H}{C_n^L}} & \text{Hermite case,}
\end{cases} \\
d_n(\xi) &= \min_{1 \leq k \leq n} \left( | \xi - x_k(\xi) | \right),
\end{align*}
\]

Proof. For each fixed \( n > m \), we have that

\[
x_n^{-1} Q_n(x_n \xi) = \sum_{k=1}^{n} b_{n,k} x_{k,n-1} \xi^{n-k} L_{n-k}(x_n \xi),
\]

where \( x_k \) is the zero of the largest modulus of \( L_k \). It follows that the smallest interval containing the zeros of \( \{x_n^{-1} L_k(x_n \xi)\}_{k=1}^{\infty} \) is \( \Delta \). Hence, if \( \xi \) is such that \( \widehat{\Delta}_\mu(\xi) = 0 \), from [15, Coroll. 1], Proposition 5, (13) and (12) we have,

\[
d_n(\xi) \leq 1 + \max_{1 \leq k \leq n} \left| \frac{b_{n,k,n-1}}{x_{n,k,n-1}} \right| \leq \sigma_n,
\]

where

\[
\begin{align*}
\sigma_n &= \begin{cases} 
1 + \frac{1}{2} C_n^L & \text{Laguerre case,} \\
1 + \sqrt{\frac{C_n^H}{C_n^L}} & \text{Hermite case.}
\end{cases}
\end{align*}
\]

Notice that \( \widehat{\Delta}_\mu \left( \frac{\xi}{x_n \xi} \right) = c_n^{-1} \widehat{\Delta}_\mu(x_n \xi) \); therefore, if \( \xi \) is such that \( \widehat{\Delta}_\mu(x_n \xi) = 0 \) then \( \xi^* = \frac{x_n \xi}{\xi} \) is such that \( \widehat{\Delta}_\mu(\xi^*) = 0 \). From (12) and (13) we have that for \( n \) large, \( \left| \frac{\xi}{x_n \xi} \right| < 2 \). Using now (28) we obtain the lemma. \( \square \)

If \( \{P_n\}_{n=0}^{\infty} \) is a sequence of orthogonal polynomials with respect to either the measures \( \mu \) or \( \nu \) we denote by \( \{\tau_n\}_{n=0}^{\infty} \) the sequence of monic normalized polynomials, that is,

\[
\begin{align*}
\tau_n(\xi) &= c_n^{-1} P_n(x_n \xi), \\
\tau_n(\xi) &= c_n^{-1} P_n(x_n \xi).
\end{align*}
\]

(29)
From the interlacing property of the zeros of consecutive orthogonal polynomials, if $K$ is a compact subset of $\mathbb{C} \setminus \Delta_e$, it follows that there exist a constant $M_e$ such that for $n$ large enough
\[
\left| \frac{t_{k,n-4}(c)}{t_{n}(c)} \right| < M_e, \quad k = 1, \ldots, m, \tag{30}
\]
uniformly on $z \in K$, where $M_e = 2 \sup_{z \in K} |z - x|^{-4}, M_e = \max\{M_1, \ldots, M_m\}.

The following lemma is needed to study the modulus of the sequence $\left\{ \frac{P_n}{L_n} \right\}_{n \in \mathbb{N}}$.

**Lemma 7.** Suppose that $m \in \mathbb{N}$ is fixed, and $K \subset \mathbb{C} \setminus \Delta_e$ a compact subset. Then, for $n$ sufficiently large
\[
\left( \frac{z e^{-n} - \lambda}{n} \right)^n \left| \frac{t_n(z)}{t_{n}(c)} \right| < 3^n, \quad n > n_0, \quad \forall z \in K, \tag{31}
\]
where $d = \inf_{z \in K \setminus \Delta_e} |z - x|$ and $t_n$ as in (29).

**Proof.** Let us define the monic polynomial $t'_n(z) = \left( \frac{z e^{-n} - \lambda}{n} \right)^n t_n(z)$. We have that (31) is equivalent to proving that
\[
\left| \frac{t_n(z)}{t'_n(z)} \right| \leq 3^n, \quad n > n_0, \quad \forall z \in K.
\]
If $\{\lambda_k, \alpha_k\}_{k=1}^n$, $\{\alpha_k, \beta_k\}_{k=1}^n$ denotes the zeros of the polynomials $t'_n, t_n$ respectively, we have the relation $\lambda_k = \frac{\alpha_k}{\beta_k}$, $k = 1, \ldots, n$. If we denote $\lambda_k = \frac{\alpha_k}{\beta_k}$, we have, for all $n$ sufficiently large
\[
\left| \frac{t_n(z)}{t'_n(z)} \right| \leq \prod_{k=1}^n \left( 1 + \frac{(\lambda_k - 1)\alpha_k}{\beta_k} \right) \leq \prod_{k=1}^n \left( 1 + \frac{|\lambda_k - 1|\alpha_k}{\beta_k} \right)
\]
\[
\leq \prod_{k=1}^n \left( 1 + \frac{2|\lambda_k - 1|\alpha_k}{\beta_k} \right) \leq \left( 1 + \frac{2|\lambda_k - 1|\alpha_k}{\beta_k} \right)^n < 3^{\frac{2n|\lambda_k - 1|\alpha_k}{\beta_k}} \leq 3^n, \tag{32}
\]
where $d = \inf_{z \in K \setminus \Delta_e} |z - x|$. □

We prove now that the modulus of the sequence $\left\{ \frac{P_n}{L_n} \right\}_{n \in \mathbb{N}}$ is uniformly bounded from above and below in the interior of $\mathbb{C} \setminus \Delta_e$.

**Lemma 8.** Let $\mu \in \mathcal{P}_m(\Delta), \text{ where } m \in \mathbb{N} \text{ and } K \subset \mathbb{C} \setminus \Delta_e, \text{ a compact subset. Then, for all } n \text{ sufficiently large there exists a constant } C^* \text{ such that}
\[
\left| \frac{P_n(z)}{L_n(z)} \right| \leq C^*, \quad n > n_0, \quad \forall z \in K.
\]

**Proof.** From Relation (7) we deduce that $P_n(z) = 1 + \sum_{k=1}^n \frac{\lambda_k z^{k-1}}{n}$. Hence, from Proposition 5, and Lemma 7 we deduce that for $n$ large enough
\[
\left| \frac{P_n(z)}{L_n(z)} \right| \leq 1 + \sum_{k=1}^n \frac{\lambda_k z^{k-1}}{n} < 1 + \sum_{k=1}^n \frac{M_k}{n} \frac{\lambda_k}{x} \leq \sum_{k=1}^n \frac{M_k}{n} \frac{\lambda_k}{x}, \tag{33}
\]
Using (33) and (30) we deduce the lemma. □
Lemma 9. Let \( \mu \in \mathcal{P}_m(\Delta) \), where \( m \in \mathbb{N} \) and \( K \subset \mathbb{C} \setminus \Delta_\rho \). Then, for all \( n \) sufficiently large there exists a constant \( C \) such that

\[
C \leq \frac{|\mathcal{P}_n(\Delta)|}{|\mathcal{E}_n(\Delta)|}, \quad n > n_0, \forall \zeta \in K.
\]

Proof. We have that \( \rho(z) L_n(z) = \sum_{k=0}^{\infty} b_{n,k} P_{n+k}(z) \), where

\[
b_{n,k} = \frac{\rho(z) L_n(z)}{\rho(z) L_n(z)} = \frac{\mathcal{P}_n(z) \mathcal{E}_n(z)}{\mathcal{E}_n(z) + \mathcal{P}_n(z)} = \sum_{k=0}^{\infty} b_{n,k} \frac{\mathcal{P}_{n+k}(z)}{\mathcal{E}_n(z)} \tag{33}
\]

From the Cauchy Schwartz inequality we have that

\[
|b_{n,k}| \leq \left( \int_0^\infty L_n^2(x) dw(x) \right)^{1/2} \left( \int_0^\infty \frac{p_{n+k}^2(x)}{\rho(x) \rho(z)} dw(x) \right)^{1/2} = \left[ \mathcal{P}_n \mathcal{E}_n \right]^{1/2} \left[ \mathcal{P}_{n+k} \mathcal{E}_{n+k} \right]^{1/2} \tag{34}
\]

Using an infinite–finite range inequality for the case in which \( w \) is a Laguerre weight, cf. [14], we have that there exists a constant \( k_L \) such that for all \( n \) large enough

\[
k_L \int_0^\infty L_n^2(x) dw(x) \leq k_{L,1} \left[ \int_0^\infty \rho(x) \rho(z) dw(x) \right] \int_0^\infty p_n^2(x) dw(x) \leq \frac{1}{\rho_1(2\mu)^{2n}} \int_0^\infty p_n^2(x) dw(x) \leq \int_0^\infty p_n^2(x) d\mu(x),
\]

where \( \rho_+ = \max_{0 \leq |z| \leq 1} \rho_0 \). Analogously, for the case of an Hermite weight, for all \( n \) large enough, we have that there exists a constant \( k_H \) such that

\[
k_H \int_{-\infty}^{\infty} L_n^2(x) dw(x) \leq k_{H,1} \left[ \int_{-\infty}^{\infty} \rho(x) \rho(z) dw(x) \right] \int_{-\infty}^{\infty} p_n^2(x) dw(x) \leq \frac{1}{\rho_1(2\mu)^{2n+2}} \int_{-\infty}^{\infty} p_n^2(x) dw(x) \leq \int_{-\infty}^{\infty} p_n^2(x) d\mu(x).
\]

Hence, for all \( n \) large enough

\[
||P_n|| \leq k_n \rho_+ \left[ \mathcal{P}_n \right]^{1/2}, \quad \text{Laguerre case}, \tag{35}
\]

\[
||P_n|| \geq k_n \rho_+ \left[ \mathcal{P}_n \right]^{1/2}, \quad \text{Hermite case}.
\]

From (7) and Proposition 5 we deduce that for \( n \) large enough, there exists a constant \( k_1 \) such that

\[
||P_n|| \leq k_1 \left[ \mathcal{P}_n \right]^{1/2} \tag{36}
\]

Inequalities (35) and (36) give us that there exists a constant \( M^* \) such that for all \( n \) large enough

\[
\frac{|b_{n,k}|}{\mathcal{E}_n(z)} \leq M^*, \quad 1 \leq k \leq m. \tag{37}
\]

From (30) it follows that there exists a constant \( M_2 \) such that for all \( z \in K \)

\[
\frac{\mathcal{P}_{n+k}(z)}{\mathcal{E}_n(z)} < M_2, \quad k = 1, \ldots, m. \tag{38}
\]
Using Lemma 7, (34), (37) and (38) we obtain
\[
\frac{\rho(\epsilon_{n+1})}{c_{n+1}} \leq 3 M_\epsilon (1 + n^\epsilon M_\epsilon),
\]
with \(d\) as in Lemma 7. Hence, from (30), (38), (39) and Lemma 7 we obtain that for all \(n\)
sufficiently large there exists \(M > 0\) such that
\[
\frac{\rho(\epsilon_{n+1})}{c_{n+1}} \leq M, \quad \forall \epsilon \in K.
\]
Let us denote by \(\{z_k\}_{k=1}^\infty\) the roots of the polynomial \(\rho\), and \(d^* = \inf_{\epsilon \in K} |\epsilon|\). Let us choose \(\epsilon\) so that for \(n\) large enough \(\Delta_k = \epsilon < d^*, k = 1, \ldots, m\). Hence,
\[
(d^* - \epsilon)^m \leq \prod_{k=1}^m \left| z_k - \frac{2 \Delta_k}{c_{n+1}} \right| \leq \prod_{k=1}^m \left| z_k - \frac{2 \Delta_k}{c_{n+1}} \right| = \frac{\rho(\epsilon_{n+1})}{c_{n+1}}.
\]
Therefore, from (40) and (41), for all \(n\) large enough we have that
\[
\frac{\rho(\epsilon_{n+1})}{c_{n+1}} \leq M, \quad \forall \epsilon \in K,
\]
and this proves the lemma. \(\Box\)

Lemma 10. Let \(\mu \in \mathcal{F}_m(\Delta), \) where \(m \in \mathbb{N}\) and \(K \subset \mathbb{C} \setminus \Delta_k\) is a compact subset. Then,
\[
\left| \frac{\Delta_\mu(z)}{\Delta_\lambda(z)} \right| \to 0, \quad \forall \epsilon \in K.
\]

Proof. For each fixed \(n > m, \) we have that
\[
\frac{\Delta_\mu(z) - \Delta_\lambda(z)}{\Delta_\lambda}(z) = \sum_{k=1}^m \left( \frac{\lambda_k}{\lambda_{n,k}} - 1 \right) \frac{\Delta_\mu(z) - \Delta_{\lambda_k}(z)}{\Delta_\lambda}(z).
\]
As \(\lambda_k = -m\) in the Laguerre case and \(\lambda_k = -2n\) in the Hermite case, then for each \(k\) fixed, \(k = 1, \ldots, m, \)
\[
\lim_{n \to \infty} \frac{\lambda_k}{\lambda_{n,k}} = 1.
\]
From (30), (42), (43) and Proposition 5 we deduce the lemma. \(\Box\)

Proof (Theorem 1), (a) From [18, (5.1.14), (5.5.10)] we have that \(\Delta_n^{(n-k)}(z)\), where
\[
\Delta_n^{(n-k)}(z) = \left[ z^{(n-k)} - \frac{\alpha_{n-k}}{\alpha_{n-1-k}}(z) \right] \frac{\alpha_{n-k}}{\alpha_{n-1-k}}(z), \quad \text{Laguerre case,}
\]
\[
\frac{\alpha_{n-k}}{\alpha_{n-1-k}}(z) \frac{\alpha_{n-k}}{\alpha_{n-1-k}}(z), \quad \text{Hermite case.}
\]
Let us define
\[
d\bar{w}(z) = \frac{d\bar{w}^{(n-k)}(z)}{d\bar{w}^{(n-k)}(z)}, \quad \text{Laguerre case,}
\]
\[
d\bar{w}(z) = \frac{d\bar{w}^{(n-k)}(z)}{d\bar{w}^{(n-k)}(z)}, \quad \text{Hermite case.}
\]
where from [9, Ths. 1 & 2] we have that
\[ \|d\| \leq \|w\| \max \|\lambda\| = n, \]
respectively, hence, from [8, (11)], we have that the sequences \(n_0, \ldots, n_m\) for every \(k \in \mathbb{N}\) are constants that do not depend on \(\lambda\).

Notice that \(\{\xi_{n,x-k}\}_{n=0}^\infty\) and \(\{\xi_{n,y-k}\}_{n=0}^\infty\) are monic orthogonal polynomials with respect to \(w, \bar{w}\) respectively, hence, from [8, (11)], we have that the sequences \(\{\xi_{n,x-k}\}_{n=0}^\infty\) and \(\{\xi_{n,y-k}\}_{n=0}^\infty\) for every \(k = 0, \ldots, m\) satisfy that
\[ \lim_{n \to \infty} \|w_k \xi_{n,x-k}\|_{L^1(\Delta)}^{1/n} = e^{-F_w}, \]
where \(F_w\) is the modified Robin constant for the weights \(w, \bar{w}\) (or the extremal constant according to the terminology of [11]) where \(\|\cdot\|_{L^1(\Delta)}\) denotes the \(L^1\)-norm with the Lebesgue measure with support on \(\Delta\).

From [9, Ths. 1 & 2] we have that
\[ \|w_k \xi_{n,x-k}\|_{L^1(\Delta)} \leq K_1 \|w_k \xi_{n,x-k}\|_{L^1(\Delta)}, \]
\[ \|\tilde{w}_k \xi_{n,y-k}\|_{L^1(\Delta)} \leq K_2 \|w_k \xi_{n,y-k}\|_{L^1(\Delta)}, \]
where \(K_1, K_2\) are constants that do not depend on \(n, \beta = 1/2\) for the Laguerre case, and \(\beta = 1/4\) for the Hermite case. Using (44), (45), and [5, (11)] we obtain that
\[ \lim_{n \to \infty} \|w_k \xi_{n,x-k}\|_{L^1(\Delta)}^{1/n} = e^{-F_w}, \]
\[ \lim_{n \to \infty} \|\tilde{w}_k \xi_{n,y-k}\|_{L^1(\Delta)}^{1/n} = e^{-F_\bar{w}}. \]

Then we have
\[ \|w_k \xi_{n,x-k}\|_{L^\infty(\Delta)} \leq \sum_{k=0}^{m} \|w_k \xi_{n,x-k}\|_{L^\infty(\Delta)} \]
\[ \|\tilde{w}_k \xi_{n,y-k}\|_{L^\infty(\Delta)} \leq \sum_{k=0}^{m} \|\tilde{w}_k \xi_{n,y-k}\|_{L^\infty(\Delta)}. \]

where \(\|\cdot\|_{L^\infty(\Delta)}\) denotes the sup norm and \(k^{*}(a), k^{**}(a)\) denote positive integer numbers such that the following equalities hold
\[ \frac{\lambda_{n-k} \xi_{n,x-k}\|_{L^\infty(\Delta)}}{c_{n-k}^{k^{*}(a)} \xi_{n,x-k}\|_{L^\infty(\Delta)}} = \max_{k=0, \ldots, m} \frac{\lambda_{n-k} \xi_{n,x-k}\|_{L^\infty(\Delta)}}{c_{n-k}^{k^{*}(a)} \xi_{n,x-k}\|_{L^\infty(\Delta)}} \]
\[ \frac{\lambda_{n-k} \xi_{n,y-k}\|_{L^\infty(\Delta)}}{c_{n-k}^{k^{**}(a)} \xi_{n,y-k}\|_{L^\infty(\Delta)}} = \max_{k=0, \ldots, m} \frac{\lambda_{n-k} \xi_{n,y-k}\|_{L^\infty(\Delta)}}{c_{n-k}^{k^{**}(a)} \xi_{n,y-k}\|_{L^\infty(\Delta)}}. \]
From these last inequalities and (46) we deduce that
\[
\lim_{n \to \infty} \left( \left\| w_n \frac{\Delta_n}{\Delta_n(\Delta_n)} \right\| \right)^{1/\alpha} = e^{-p_d}, \quad \lim_{n \to \infty} \left( \left\| w_n \frac{\Delta_n}{\Delta_n(\Delta_n)} \right\| \right)^{1/\alpha} = e^{-p_d}.
\]
Therefore, if \( \nu_d, \lambda_d \) denote the root counting measure of \( \Delta_n \) and \( \Delta_n^\infty \) respectively, from \([5, \text{Th. 1}].) we deduce that \( \nu_d \xrightarrow[n \to \infty]{} \nu_d, \lambda_d \xrightarrow[n \to \infty]{} \lambda_d \) in the weak star sense.

(b) From Lemma 9, if \( \varepsilon \) is sufficiently small and \( K \subset C \setminus \Delta \) is a compact subset, for all \( n \) sufficiently large we have that, for some positive constant \( C \),
\[
C - \varepsilon \leq \frac{\mu_n(z)}{\Delta_n(z)} - \varepsilon \leq \frac{\mu_n(z)}{\Delta_n(z)}.
\]
From this fact and from Lemma 6 we deduce that the set of accumulation points is contained on \( \Delta \) and from (a) of the present theorem we deduce that the set of accumulation points of the zeros of \( \Omega_n \) is \( \Delta \). □

**Proof (Theorem 2).** From (b) of Theorem 1 we deduce that for the Laguerre case
\[
\lim_{n \to \infty} \frac{P_n(z_n)}{Q_n(z_n)} = \lim_{n \to \infty} \frac{P_n'(z_n)}{Q_n'(z_n)} = \frac{1}{2} \int_0^1 \frac{1}{t^2} \left( \frac{1}{2 - t} \right) dt = \frac{1}{2} \left( 1 - \sqrt{1 - 1/2} \right),
\]
and for the Hermite case
\[
\lim_{n \to \infty} \frac{P_n(z_n)}{Q_n(z_n)} = \lim_{n \to \infty} \frac{P_n'(z_n)}{Q_n'(z_n)} = \frac{1}{2} \int_0^1 \frac{1}{t^2} \left( \frac{1}{2 - t} \right) dt = \frac{1}{2} \left( 1 - \sqrt{1 - 1/2} \right).
\]
on compact subsets \( K \subset C \setminus \Delta \). From (6) and the preceding relations we have for the Laguerre case
\[
\frac{P_n(z_n)}{Q_n(z_n)} = \frac{c_n}{x_n} \frac{Q_n'(z_n)}{Q_n(z_n)} \frac{Q_n(z_n)}{P_n(z_n)} \frac{1 + x_n - x_n}{} \frac{Q_n'(z_n)}{Q_n(z_n)}
\]
(47)
and for the Hermite case
\[
\frac{P_n(z_n)}{Q_n(z_n)} = \frac{1}{2} \frac{Q_n'(z_n)}{Q_n(z_n)} \frac{Q_n(z_n)}{P_n(z_n)} \frac{c_n}{x_n} \frac{Q_n'(z_n)}{Q_n(z_n)}
\]
(48)
Taking limits in (47) and (48) we obtain (16). Relation (17) follows from (14) and (16). □

4. The polynomial \( \Omega_n \)

Some basic properties of the zeros of \( \Omega_n \) follow directly from (1) and (2). For example, the multiplicity of the zeros of \( \Omega_n \) is at most \( \lambda_n \) a zero of multiplicity \( 3 \) is also a zero of \( \Omega_n \) and a zero of multiplicity \( 2 \) is a critical point of \( \Omega_n \). In the next lemma we prove conditions for the boundedness of the zeros of \( \Omega_n \) and determine their asymptotic behavior.

**Lemma 11.** Let \( \mu \in \mathcal{P}_m(\Delta) \) where \( m \in \mathbb{R} \) and define for \( z \in C \), \( D(z) = \sup_{n \to \infty} |z - x| \) if \( \{x_n\}_{n=1}^\infty \) is a sequence of complex numbers with limit \( x \in C \), then for every \( d > 1 \) there is a positive number \( N_d \), such that \( \{z \in C : |\Omega_n(z)| = 0\} \subset \{z \in C : |z| \leq D(z) + d\} \) whenever \( n > N_d \).
Proof. As $\Omega_0(z) = 0$ then $\tilde{\Omega}_0(z) = \tilde{\Omega}_0(\zeta)$. From the Gauss–Lucas theorem (cf. [16, Section 2.1.3]), it is known that the critical points of $\tilde{\Omega}_0$ are in the convex hull of its zeros and from (b) of Theorem 1 the zeros of the polynomials $\{\tilde{\Omega}_0\}_{n=0}^{\infty}$ accumulate on $\Delta$. Hence, from the bisector theorem (see [16, Section 5.5.7]) $|z| \leq D(\zeta) + 1$ and the lemma is established. □

We are now ready to prove Theorem 3.

Proof (Theorem 3). From Lemma 11 we have that the zeros of $\Omega_n$ are located in a compact set. From (15) the zeros of $\Omega_n$ satisfy the equation

$$\left|\hat{\Omega}_n(z)\right| = \left|\hat{\Omega}_n(\zeta)\right|^{\frac{1}{n}}. \quad (49)$$

If $z \in \mathbb{C} \setminus \Delta$, taking limit when $n \to \infty$, from Lemma 11, and using (17) of Theorem 2 on both sides of (49), we have that the zeros of the sequence of polynomials $\{\Omega_n\}_{n=0}^{\infty}$ cannot accumulate outside the set

$$\{z \in \mathbb{C} : |\hat{\Omega}(z)| = |\hat{\Omega}(\zeta)| \} \bigcup \Delta.$$

To verify the second statement of the theorem, note that if $z$ is a zero of $\Omega_n$, from (15) we get

$$\prod_{k=1}^{m} \frac{z - \zeta_{n,k}}{z - \zeta_{n,k}} = 1, \quad \text{where } \zeta_{n,k} \text{ are the zeros of } \tilde{\Omega}_n. \quad (50)$$

Let $V_\varepsilon(\Delta)$ be the $\varepsilon$-neighborhood of $\Delta$, defined as $V_\varepsilon(\Delta) = \{z \in \mathbb{C} : |z - \zeta| < \varepsilon\}$, as $\lim_{n \to \infty} \zeta_n = \zeta$, then for all $\varepsilon > 0$ there is a $N_\varepsilon > 0$ such that $|\hat{\Omega}(\zeta_n) - \hat{\Omega}(\zeta)| < \varepsilon$ whenever $n > N_\varepsilon$.

If $\hat{\Omega}(\zeta) > 2$, let us choose $\varepsilon = \frac{1}{2}(\hat{\Omega}(\zeta) - 2)$ and suppose that there is a $\zeta_0 \in V_{\varepsilon}(\Delta)$ such that $\Omega_n(\zeta_0) = 0$ for some $n > N_\varepsilon$. Hence

$$\prod_{k=1}^{m} \frac{\zeta_0 - \zeta_{n,k}}{\zeta_0 - \zeta_{n,k}} < \frac{2 + \varepsilon}{2|\hat{\Omega}(\zeta)|} < 1, \quad (51)$$

which is a contradiction with (50), hence $\{z \in \mathbb{C} : \Omega_n(z) = 0\} \cap V_{\varepsilon}(\Delta) = \emptyset$ for all $n > N_\varepsilon$, i.e. the zeros of $\Omega_n$ cannot accumulate on $V_{\varepsilon}(\Delta)$.

From (15) it is straightforward that a multiple zero of $\Omega_n$ is also a critical point of $\tilde{\Omega}_n$. But, from (b) of Theorem 1 and the Gauss–Lucas theorem the set of accumulation points of $\tilde{\Omega}_n$ is $\Delta$, where we have that for $n$ sufficiently large the zeros of $\Omega_n$ are simple. □

Proof (Theorem 4). 1. Let us prove first that

$$\frac{\Omega_n(z)}{\tilde{\Omega}_n(z)} = 1 - \frac{\tilde{\Omega}_n(\zeta)}{\hat{\Omega}_n(z)} \quad \text{uniformly on compact subsets } K \text{ of the set } \{z \in \mathbb{C} : |\hat{\Omega}(z)| > |\hat{\Omega}(\zeta)|\}. \quad (52)$$

In order to prove (52) it is sufficient to show that

$$\frac{\tilde{\Omega}_n(\zeta)}{\hat{\Omega}_n(z)} \quad \text{uniformly on } K. \quad (53)$$

By setting $z = \zeta$, we have

$$\frac{\tilde{\Omega}_n(\zeta)}{\hat{\Omega}_n(z)} = \frac{\tilde{\Omega}_n(\zeta)}{\hat{\Omega}_n(z)} \quad \text{uniformly on } K.$$
From [7] and Lemmas 8, 9, we have that for all \( n \) large enough it is possible to find constants \( c', c \) such that
\[
\varepsilon_{1} \leq \left|\frac{\partial \Phi_{n}(z)}{\partial z}\right| \leq c,
\]
uniformly on compact subsets of \( C \setminus \Delta_{c} \). Then we have
\[
\left|\frac{\delta_{n}(z)}{\delta_{n}} - 1\right| \leq 1 - n^{-\infty}
\]
uniformly on compact subsets \( K \) of the set \( \{ z \in C : |\Psi(z)| < |\Psi(\zeta)| \} \setminus \Delta_{c} \). Note that
\[
\frac{\partial \delta_{n}(z)}{\partial \delta_{n}(z)} = \frac{\partial \Phi_{n}(z)}{\partial \Phi_{n}(z)} \frac{\partial \Phi_{n}(z)}{\partial \Phi_{n}(z)} \frac{\partial \Phi_{n}(z)}{\partial \Phi_{n}(z)} \frac{\partial \Phi_{n}(z)}{\partial \Phi_{n}(z)} \frac{\partial \Phi_{n}(z)}{\partial \Phi_{n}(z)}
\]
Now, the first part of the assertion \( 2 \) is straightforward from (16) of Theorem 2 and (54).
If \( \delta(z) > 2 \), let \( V_{\epsilon}(\Delta_{c}) = \{ z \in C : |\Psi(z)| < \epsilon \} \) be an \( \epsilon \)-neighborhood of \( \Delta_{c} \), where \( \epsilon = \epsilon_{\zeta} = \frac{\epsilon_{1}}{2} - 1 \). By the same reasoning used to deduce (51) we get that
\[
\left|\frac{\delta_{n}(z)}{\delta_{n}(z)} \right| < \epsilon^{\infty}, \quad \text{for all } z \in V_{\epsilon}(\Delta_{c}), \quad \epsilon < 1.
\]
Hence from the first part of the assertion 2 and (56) we get the second part of the assertion 2. \( \square \)

5. A fluid dynamics model

In this section we show a hydrodynamical model for the zeros of the orthogonal polynomials with respect to the pair \((\mathcal{L}, \mu)\). In [4], we gave a hydrodynamic interpretation for the critical points of orthogonal polynomials with respect to a Jacobi differential operator.

Let us consider a flow of an incompressible fluid in the complex plane, due to a system of \( n \) different points \((n > 1)\) fixed at \( w_{1} \), \( 1 \leq i \leq n \). At each point \( w_{1} \) of the system there is defined a complex potential \( \Psi_{n} \) which for the Laguerre case equals to the sum of a source (sink) with a fixed strength \( 3\varepsilon_{1} \) plus a vortex with a fixed strength \( 3\varepsilon_{1} \) plus a uniform stream \( U_{c} \) at infinity. Here \( c \) and \( d \) are fixed complex numbers which depend on the position of the remaining points \( \{w_{i}\}^{n}_{i=1} \), see [10, Ch. VIII] for the terminology. We shall call \( n \) system to the set of the \( n \) points fixed at \( w_{1} \) with its respective potential of velocities.

Define the functions
\[
f(c, w_{1}, \ldots, w_{n}) = R_{n}^{c}(w_{1}) R_{n}^{c}(w_{1}) \quad i = 1, \ldots, n \quad \text{where } R_{n}(z) = \prod_{i=1}^{n} (z - w_{i}).
\]
The complex potentials $V_L$ (Laguerre case) or $V_H$ (Hermite case) at any point $z$ (see [6, Ch. 10]), by the principle of superposition of solutions, are given by

$$V_L(z) = \sum_{i=1}^{n} V_{L,i} = \sum_{i=1}^{n} \left( -z + (1 + z - w_i) \log(z - w_i) + (z + w_i \log(z - w_i)) f_i(w_1, \ldots, w_n) \right),$$

and

$$V_H(z) = \sum_{i=1}^{n} V_{H,i} = \sum_{i=1}^{n} \left( -z + \frac{1}{2} (f_i(w_1, \ldots, w_n) - 2w_i) \log(z - w_i) \right).$$

From a complex potential $V$, a complex velocity $V$ can be derived by differentiation ($V(z) = \frac{dz}{dt}$). A standard problem associated with the complex velocity is to find the zeros, that correspond to the set of stagnation points, i.e. points where the fluid has zero velocity.

We are interested in the problem: Build an $n$ system (location of the points $w_1, \ldots, w_n$) such that the stagnation points are at preassigned points with nice properties. As it is well known, the zeros of the orthogonal polynomials with respect to a finite positive Borel measures on $\mathbb{R}$ have a rich set of nice properties (cf. in [18, Ch. VI]), and will be taken as preassigned stagnation points.

Here we consider $\mu \in \mathcal{P}_1[\mathbb{R}]$ or $\mu \in \mathcal{P}_2[\mathbb{R}]$. In the next paragraph we establish the statement of the problem for both Laguerre and Hermite cases.

**Problem.** Let $\{x_1, \ldots, x_n\}$ be the set of zeros of the $n$th orthogonal polynomial $P_n (n > 1)$ for the Laguerre case and $n > 2$ for the Hermite) with respect to a measure $\mu \in \mathcal{P}_1[\mathbb{R}]$ or $\mu \in \mathcal{P}_2[\mathbb{R}]$.

Suppose a flow is given, with complex potential $V_L$ (Laguerre case) or $V_H$ (Hermite). Build an $n$ system (location of the points $w_1, \ldots, w_n$) such that the stagnation points are attained at the points $z = x_k$, with $k = 1, 2, \ldots, n$.

Consider first the Laguerre case. If $x_k$ ($k = 1, \ldots, n$) are stagnation points then

$$\frac{\partial V_L}{\partial x}(x_k) = (1 + \alpha - x_k) \sum_{i=1}^{n} \frac{1}{x_k - w_i} + x_k \sum_{j=1}^{n} \frac{R_n'(w_j)}{R_n(x_k - w_j)} = 0. \quad (59)$$

We are looking for a solution $R_n(z) = \prod_{i=1}^{n} (z - w_i)$, with $w_i \neq w_j \neq x_k, \forall i, j, k, j \neq j$, such that (59) holds. This assumption implies that the sum in the second term of the left hand side of expression (59) is the partial-fraction decomposition of $\frac{R_n'}{R_n}$ evaluated at $x = x_k$. Therefore, (59) is equivalent to

$$x_k R_n'(x_k) + (1 + \alpha - x_k) R_n(x_k) = 0, \quad k = 1, 2, \ldots, n.$$ 

Note that $x_k R_n'(x_k) + (1 + \alpha - x_k) R_n(x_k)$ is a polynomial of degree $n$, with leading coefficient $\lambda_n$ that vanishes at the zeros of $P_n$, i.e.

$$x_k R_n'(x_k) + (1 + \alpha - x_k) R_n(x_k) = \lambda_n P_n(x). \quad (60)$$

Observe that expression (60) is equivalent to (6). From Proposition 4, the zeros of $\tilde{Q}_L, \tilde{Q}_R$ are real, simple and $Q_n'(x_k) \neq 0$. Therefore, $R_n = \tilde{Q}_L$ is a solution. Hence, an answer to our problem yields the $n$ zeros as the $n$ zeros of the polynomial $Q_n$.
For the Hermite case we have a similar situation. Thus, if \( x_k \) is a stagnation point, \( \frac{\partial^2 V}{\partial z^2}(x_k) = 0 \), which gives
\[
x_k \sum_{j=1}^{\infty} \frac{1}{x_k - w_j} - \frac{1}{2} \sum_{j=1}^{\infty} \frac{R''_n(w_j)}{R'_n(w_j)(x_k - w_j)} = 0, \quad k = 1, 2, \ldots, n. \tag{61}
\]
Again, we can deduce that the expression (61) equals to
\[
\frac{1}{2} R''_n(x_k) - x R'_n(x_k) = 0, \quad k = 1, \ldots, n.
\]
Note that \( \frac{1}{2} R''_n(x_k) - x R'_n(x_k) \) is a polynomial of degree \( n \), with leading coefficient \( \lambda_n \) that vanishes at the zeros of \( \mathcal{Q}_n \), i.e.
\[
\frac{1}{2} R''_n(x_k) - x R'_n(x_k) = \lambda_n P_n(x). \tag{62}
\]
Therefore, the expression (62) is equivalent to (6). From Proposition 4, the zeros of \( \mathcal{Q}_n \) are real and simple and \( \mathcal{Q}_n'(x_k) \neq 0 \), which implies that \( P_n = \mathcal{Q}_n \) is a solution to our problem.

As a conclusion,

**Answer.** The flow of an incompressible two-dimensional fluid, due to \( n \) points with complex potential \( V_L \) or \( V_H \), located at the zeros of the \( n \)th orthogonal polynomial \( \mathcal{Q}_n \) with respect to \((L, \mu)\), with \( \mu \in \mathbb{P}_1[\mathbb{R}] \) or \( \mu \in \mathbb{P}_2[\mathbb{R}] \) has its \( n \) stagnation points at the \( n \) zeros of the \( n \)th orthogonal polynomial \( \mathcal{Q}_n \).

It would be interesting to consider the uniqueness of the solution obtained. In other words, what could be said about the solutions of the form \( \mathcal{Q}_n(z) = \mathcal{Q}_n(z) - \mathcal{Q}_n(z_n) \) and to extend this model to more general classes of measures \( \mu \). It would be also of interest to decide if these stagnation or equilibrium points are stable.

**Acknowledgments**

The authors thank the comments and suggestions made by the referees which helped improve the manuscript. First author’s research was partially supported by FAPESP of Brazil, under grant 2012/21042-0. First and second authors’ research was partially supported by Ministerio de Economía y Competitividad of Spain, under grant MTM2012-36732-C03-01.

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